Revista Matemática Iberoamericana
Vol. 4, n.º 2, 1988

H^p-Theory on Euclidean Space and the Dirac Operator

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To the memory of José Luis Rubio de Francia

Introduction

Classical H^p-theory is the study of the boundary regularity of one particular linear first-order differential operator: the Cauchy-Riemann $\bar{\partial}$-operator. This is an elliptic operator, but in applications to planar potential theory great use is made of its *analytic over-determinedness*, meaning that the imaginary part V of any solution $F = U + iV$ of $\bar{\partial}F = 0$ is essentially determined by its real part U (and vice-versa). Put another way, every analytic function F can be written locally as $\partial\Phi$ with Φ real-valued harmonic function. Such analytic over-determinedness is the basis for *real* H^p-theory. The close relationship to the Dirac operator that $\bar{\partial}$ and ∂ have been known to have for many years (cf. [20]) has been used in more recent times to develop a theory of analytic functions on Euclidean space of any dimension, complex-valued functions being replaced by Clifford algebra-valued functions ([6]). A routine application of Stokes theorem allows one to prove a Cauchy Integral Theorem, for instance, in this more general context. In [15], we then used the L^p-boundedness of the principal-value Cauchy integral operator associated with Lipschitz domains in $\mathbb{R}^{n+1}$ to establish a boundary value theory for H^p-spaces defined on such domains by a family of first-order differential operators said to be of *Dirac type*. These include not only the classical geometric differential operators, but also the rotation-invariant systems $\bar{\partial}_\tau$ introduced in [9]. Such Dirac type operators are virtually the only ones for which a Cauchy integral theorem can exist ([16]). One purpose of the present paper is to outline these ideas in their «ultimate form» by exploiting the theory of Clifford algebras and Clifford modules more fully than in the earlier paper. But more importantly we shall formulate a class of boundary value problems for these Hardy spaces modelled on the usual Riemann-Hilbert problem and its generalizations for $\bar{\partial}$. Just as the latter contains the prototypical Dirichlet, Neumann and Oblique-derivative problems, so this general class contains the well-known higher-dimensional versions of these problems as special cases (cf. [22]). The point to be emphasized is that the splitting $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ used previously to write a complex-valued function as $F = U + iV$ has a natural interpretation in the context of Clifford theory. Now classical H^p-theory, representation theory for classical Lie groups, and algebraic geometry questions concerning, say, compact Riemann surfaces all fall within complex analytic function theory. Hence if such operators of Dirac type as ∂_τ are to have a deeper meaning than being merely *analogues of* $(\bar{\partial}, \partial)$ *for higher-dimensional Euclidean space*, they should be defined on more general Riemannian manifolds, with elliptic boundary value problems and H^p-theory being just part of the analysis associated with them in the case of Euclidean space. Indeed, as Seeley has so clearly pointed out ([29]), the Riemann-Hilbert problem and its variants for $\bar{\partial}$ lead very naturally to elliptic singular integral equations whose solution is the forerunner of the Atiyah-Singer Index theorem for manifolds with or without boundary. These are the ideas basic to [14] and to the series of papers beginning with [9], [10] and [13]; but here we shall mention them only briefly. They have been formulated over a long period of time during which many conversations with Kathy Davis and Ray Kunze were immensely useful in helping fix the fundamental concepts. More recent conversations with John Ryan were very helpful too.

1. Classical Case

Let Ω be the interior of a Jordan curve in the complex plane or the region $\{z \in \mathbb{C}: \operatorname{Im}(z) > \phi(\operatorname{Re} z)\}$ above the graph of a function $\phi: \mathbb{R} \to \mathbb{R}$; the unit disk or upper half-plane are natural examples. We shall think of Ω as being «flat» when it is equipped with the Euclidean metric and «curved» when a Riemannian metric on it gives it non-zero curvature as in the case of the hyperbolic metric. More generally still, Ω could be any Riemann surface. Geometrically, the flat case arises from identifying $\mathbb{C}$ with the tangent space to a curved Ω at some base point $\omega_0 \in \Omega$. In the flat case the Cauchy-Riemann $\bar{\partial}$-operator has constant coefficients, whereas it will have non-constant coefficients when Ω has curvature. Our principal focus, of course, is on the flat case where the Hardy spaces form one natural family of spaces of analytic functions.

First let Ω be the upper half-plane in $\mathbb{C}$. The Hardy $H^p(\Omega)$-spaces here consist of all analytic functions on Ω such that

$$\|F\|_{H^p(\Omega)} = \sup_{y>0}\left(\int_{-\infty}^{\infty} |F(x+iy)|^p \, dx\right)^{1/p} \qquad (0 < p < \infty) \tag{1.1}$$

is finite. If $\mathfrak{H}$ is any finite-dimensional complex Hilbert space, then corresponding $H^p(\Omega)$-spaces of $\mathfrak{H}$-valued analytic functions can be defined in exactly the same way. All the H^p-theory for scalar-valued functions carries over component by component to the vector-valued case.

For $p > 1$ straightforward use of analyticity and weak*-compactness gives (1.2)

(i) (*characterization of boundary values*) each F in $H^p(\Omega)$ has boundary values $F^+(x)$ a.e. on $\partial\Omega \sim \mathbb{R}$ and the space

$$H^p(\partial\Omega) = \{F^+ : F \in H^p(\Omega)\}$$

coincides with the subspace of all complex-valued functions f in $L^p(\partial\Omega)$ such that $(I - i\mathfrak{H})f = 0$ where

$$\mathfrak{H}f(x) = P.V.\frac{1}{\pi}\int_{\partial\Omega} \frac{f(y)}{x-y}\, dy \qquad (x \in \partial\Omega)$$

is the Hilbert transform of f;

(ii) (*analytic overdeterminedness*) the Cauchy integral

$$f \to \mathcal{C}f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(x)}{x-z}\, dx \qquad (z \in \Omega)$$

is an isomorphism from the space of all real-valued functions f in $L^p(\partial\Omega)$ onto $H^p(\Omega)$.

In proving the existence of boundary values in $L^p(\partial\Omega)$, $p > 1$, harmonicity of F rather than analyticity would have been sufficient. But when $p = 1$, the requirement of analyticity is essential. Thus one *role of $\bar\partial$ is to improve on the boundary regularity of harmonic functions*; in fact, boundary values F^+ exist for every F in $H^p(\partial\Omega)$, $p > 0$. Stein-Weiss ([31]) showed that this fundamental boundary regularity property could be derived from the subharmonicity of $z \to |F(z)|^p$; indeed,

$$\Delta|F(z)|^p \geqslant \frac{1}{4}p^2|F(z)|^{p-2}|\partial F(z)|^2 \geqslant 0 \qquad (0 < p < 2) \tag{1.3}$$

whenever F is analytic and non-zero, as one easily calculates using the *factorization* $\Delta = \partial\bar\partial = \bar\partial\partial$.

For an arbitrary flat Ω having Lipschitz (or smoother) boundary $\partial\Omega$, there is an analogous definition of $H^p(\partial\Omega)$, replacing (1.1) with the usual non-tangential maximal function norm

$$\|F\|_{H^p(\Omega)} = \left(\int_{\partial\Omega} N(F)(s)^p\, ds\right)^{1/p} \tag{1.4}$$

obtained by integrating along $\partial\Omega$ with respect to arc-length measure. But now the boundary regularity theory becomes decidedly more difficult as the smoothness of $\partial\Omega$ decreases. In his thesis Kenig used conformal mapping techniques to reduce the study of a general $H^p(\Omega)$-space to that of a weighted H^p-space on the upper half-plane ([21]), but the L^p-boundedness of the principal-value Cauchy integral operator on $\partial\Omega$ can also be used (cf. Section 3). Using such techniques it can be shown for each Ω there is a constant $p(\Omega)$, $1 \leqslant p(\Omega) < 2$, such that properties (i) and (ii) in (1.2) still hold provided $p > p(\Omega)$. If $\partial\Omega$ is C^1-smooth, then $p(\Omega) = 1$ as in the case of the upper half-plane, but $p(\Omega) > 1$ if $\partial\Omega$ has only Lipschitz smoothness. Results of Kenig and Pipher suggest that there should be an atomic type decomposition of $H^p(\partial\Omega)$ for $p \leqslant p(\Omega)$ and Lipschitz $\partial\Omega$ ([23]).

One way of discovering the relationship between $(\bar\partial, \partial)$ and the Dirac operator for $\mathbb{R}^2$ is to identify the Clifford algebra for $\mathbb{R}^2$ with the quaternions

$$\mathbb{H} = \{x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} : x_j \in \mathbb{R}\} \tag{1.5}$$

where

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \qquad \boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{ji} \tag{1.6}$$

Thus $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are imaginary units; in particular,

$$\mathbb{C} = \{x_0 + \boldsymbol{j}x_2 : x_j \in \mathbb{R}\}, \qquad \mathbb{H} = \{z + \boldsymbol{i}w : z, w \in \mathbb{C}\}. \tag{1.7}$$

Under the embedding of $\mathbb{R}^2$ into $\mathbb{H}$ given by

$$(x, y) \to x\boldsymbol{i} + y\boldsymbol{k} = \boldsymbol{i}(x + \boldsymbol{j}y),$$

the standard Dirac operator takes the form

$$D = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{k}\frac{\partial}{\partial y} = \boldsymbol{i}\bar\partial = \partial\boldsymbol{i}, \tag{1.8}$$

setting

$$\bar\partial = \frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y}, \qquad \partial = \frac{\partial}{\partial x} - \boldsymbol{j}\frac{\partial}{\partial y}. \tag{1.8$'$}$$

Now let $F = F(x, y) = \Phi + \boldsymbol{i}\Psi$ be an $\mathbb{H}$-valued function on Ω, where Φ, Ψ are complex-valued functions determined by the decomposition $\mathbb{H} = \mathbb{C} \oplus \boldsymbol{i}\mathbb{C}$ of $\mathbb{H}$. Then

$$DF = \boldsymbol{i}\frac{\partial F}{\partial x} + \boldsymbol{k}\frac{\partial F}{\partial y} = \boldsymbol{i}\bar\partial\Phi - \partial\Psi. \tag{1.9}$$

Hence $DF = 0$ on Ω if and only if $\bar{\partial}\Phi = 0$, $\partial\Psi = 0$; and so $\bar{\partial}$ (resp. ∂) arises from restricting the Dirac operator to $\mathbb{H}$-valued functions having range in $\mathbb{C}$ (resp. $\boldsymbol{i}\mathbb{C}$). Notice that these range spaces $\mathbb{C}$ and $\boldsymbol{i}\mathbb{C}$ are just the respective $+1$ and -1 eigenspaces of the involution $z + \boldsymbol{i}w \to z - \boldsymbol{i}w$ on $\mathbb{H}$; notice also that if Φ is $\mathbb{C}$-valued, then $D\Phi$ is $\boldsymbol{i}\mathbb{C}$-valued, whereas $D\Phi$ is $\mathbb{C}$-valued when Φ is $\boldsymbol{i}\mathbb{C}$-valued, so that the effect of D is to interchange the eigenspaces of an involution. This is exactly how the important geometric differential operators associated with the Dirac operator on more general manifolds are defined (cf. [14, Chap. V]; [19]).

There is another interesting way of obtaining $(\bar{\partial}, \partial)$ from the Dirac operator by realizing $\mathbb{H}$ as the real sub-algebra

$$(1.10) \qquad \mathbb{H} = \left\{ \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} : z_1, z_2 \in \mathbb{C} \right\}$$

of $\mathbb{C}^{2\times 2}$ where

$$\boldsymbol{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{j} = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \qquad \boldsymbol{k} = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}.$$

For then

$$(1.11) \qquad D = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{k}\frac{\partial}{\partial y} = \begin{bmatrix} 0 & -\partial \\ \bar{\partial} & 0 \end{bmatrix},$$

and if $F = \left[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}\right]$ is a column vector of complex-valued functions, then

$$DF = \begin{bmatrix} 0 & -\partial \\ \bar{\partial} & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} -\partial\Psi \\ \bar{\partial}\Phi \end{bmatrix}.$$

Hence $DF = 0$ if and only if $\bar{\partial}\Phi = 0$, $\partial\Psi = 0$; once again the effect of D is to interchange the eigenspaces of an involution, this time the involution being defined by

$$(1.12) \qquad \begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}.$$

Generalizations of this derivation of $(\bar{\partial}, \partial)$ from D use the so-called Spin representations of Clifford algebras and are valid on arbitrary spin manifolds ([14, chap. V]; [5, Section 5]).

As a constant coefficient operator in the flat case, $(\bar{\partial}, \partial)$ commute with translations; but, more importantly, they also commute with the rotation operators $F \mapsto F_\theta$, $F_\theta(z) = F(e^{i\theta}z)$, in the sense that

$$(1.13) \qquad \bar{\partial}(F_\theta) = e^{i\theta}(\bar{\partial}F)_\theta, \qquad \partial(F_\theta) = e^{-i\theta}(\partial F)_\theta.$$

These properties of invariance under the rigid motions of the complex plane are the reason why analytic function theory can be incorporated so successfully into Fourier series and Fourier Transform theory, and ultimately therefore for the importance of H^p-theory. On the other hand, when Ω has curvature, the Cauchy-Riemann operator is invariant with respect to the semi-simple automorphism group G of Ω, and hence is the reason for the extensive use of analytic function theory in realizing representations of such G. Our emphasis on the invariance properties of operators of Dirac type is thus well-founded.

Let us now study in greater detail the elliptic boundary value problems associated with $H^p(\Omega)$ for a domain in $\mathbb{C}$. Qualitatively, the classical Riemann-Hilbert problem can be formulated as

Riemann-Hilbert Problem 1.14. *Given real-valued functions a, b, and f on $\partial\Omega$, find a function F in $\Omega \cup \partial\Omega$ such that*

(i) $\bar{\partial} F = 0$ *on* Ω
(ii) $L_0(F) = \mathrm{Re}\,((a - ib)F) = f$ *on* $\partial\Omega$.

The Dirichlet problem, of course, is the special case $a = 1$, $b = 0$. In real terms the general boundary condition $L_0(F) = f$ is

$$a(z)U(z) + b(z)V(z) = f(z) \qquad (z \in \partial\Omega)$$

when $F = U + iV$, and so (1.14)(ii) imposes a linear relation on the real and imaginary parts of F at each point of $\partial\Omega$. This was how Riemann first formulated the problem. Thus from a conceptual point of view, the problem must exploit the analytic over-determinedness of $\bar{\partial}$ if there is to be any hope of a solution existing and having some degree of uniqueness. By allowing more general boundary conditions in (1.14)(ii), we get other classical problems as special cases. For instance, given complex-valued functions A_0, A_1 on $\partial\Omega$, F can be required to satisfy the boundary condition

$$(1.14)(\mathrm{ii})' \qquad L_1(F) = \mathrm{Re}\,(\overline{A_0(z)}F(z) + A_1(z)\,\partial F(z)) = f(z) \qquad (z \in \partial\Omega)$$

on $\partial\Omega$; here ∂F denotes the boundary values of the derivative ∂F of F in Ω. The so-called Generalized Riemann-Hilbert problem imposes conditions such as (1.14)(ii)′ on F and ∂F, as well as possibly on higher order derivatives $\partial^2 F, \ldots, \partial^m F$ (and integral terms too) (cf. [12]; [26]).

To study the boundary condition $L_1(F) = f$ in detail, let $t = t(s)$ be the parameterization of $\partial\Omega$ by arc length, and let $\theta = \theta(s)$ be the angle between the positive tangent to $\partial\Omega$ at t and the x-axis. Thus $e^{i\theta(s)} = t'(s)$. In a famous study of the theory of tides, Poincaré considered the following problem.

Poincaré Problem 1.15. *Given real-valued functions ϕ, ψ, c and f on $\partial\Omega$, find a real-valued function U on $\Omega \cup \partial\Omega$ satisfying*

(i) $\Delta U = 0$ *on* Ω

(ii) $\phi(s)\dfrac{\partial U}{\partial s} + \psi(s)\dfrac{\partial U}{\partial n} + c(s)U = f$ *on* $\partial\Omega$

where $\partial U/\partial s$ is the tangential derivative and $\partial U/\partial n$ is the normal derivative of U at $t = t(s)$.

This contains both the Dirichlet and Neumann problems, as well as the Oblique-derivative problem, as special cases. But on setting

$$A_0(s) = c(s), \qquad A_1(s) = \frac{1}{2}e^{i\theta(s)}(\phi + i\psi)(s) \qquad (t(s) \in \partial\Omega),$$

it is easily checked that the Poincaré problem can be solved by finding an analytic function $F = U + iV$ satisfying $L_1(F) = f$ on $\partial\Omega$.

In the Russian literature (cf. [12]; [26]) the Riemann-Hilbert problem and its variants were most often studied quantitatively in the context of open sets Ω whose boundary is (possibly piecewise-) $C^{1+\epsilon}$, with the boundary functions being in Lip(α)-spaces. The extra smoothness on $\partial\Omega$ ensured that Fredholm theory could be applied and that the principal value Cauchy integral operator was bounded on Lip(α) (Privalov's theorem). As we now know, $C^{1+\epsilon}$-smoothness is not essential for this last property, but the Riemann-Hilbert problem does not appear to have been studied for general Lipschitz domains. For concreteness therefore, we shall assume that $\partial\Omega$ is $C^{1+\epsilon}$-smooth for some $\epsilon > 0$ and replace Lip(α)-spaces by boundary L^p-spaces, $1 < p < \infty$. We will however assume that $A = A(z)$ is a complex-valued function on $\partial\Omega$ which is in Lip(α) for some $\alpha > 0$. In this setting (1.14) becomes what we shall call

Hardy-Riemann-Hilbert Problem 1.16. *Given a real-valued function f in $L^p(\partial\Omega)$, find F in $H^p(\Omega)$ such that* $\mathrm{Re}\,(\overline{A(z)}F(z)) = f(z)$ *on* $\partial\Omega$.

In view of the boundary regularity theory for H^p-spaces, this problem amounts to solving the equation

$$\mathfrak{I}_A(F) = f$$

where

$$\mathfrak{I}_A : H^p(\partial\Omega) \to L^p(\partial\Omega, \mathbb{R}), \qquad (\mathfrak{I}_A F)(z) = \mathrm{Re}\,(\overline{A(z)}F(z)), \tag{1.17}$$

and $L^p(\partial\Omega, \mathbb{R})$ is the Lebesgue space of real-valued functions on $\partial\Omega$. From the well-known solution of (1.14) for Lip(α)-spaces (cf. [12]; [26]) we obtain

Theorem 1.18. *If $A = A(z)$ is non-vanishing on ∂Q, then $\mathfrak{J}_A : H^p(\partial\Omega) \to L^p(\partial\Omega, \mathbb{R})$ is a Fredholm operator with index*

$$\operatorname{Index}(\mathfrak{J}_A) = 2\omega_A + 1$$

where ω_A is the winding number of A.

Since both the kernel and co-kernel of $\mathfrak{J}_A$ are known explicitly, the Hardy-Riemann-Hilbert problem is thus completely solved. Extensions to less smooth domains and other function classes for A ($A \in BMO(\partial\Omega)$?) would clearly be of interest. The Riemann-Hilbert problem has been formulated and extensively studied for $\mathfrak{H}$-valued functions also (cf., for instance, [35]). On the other hand, the Riemann-Roch for compact Riemann surfaces relating the index of $\bar{\partial}$ to the genus of the surface is another of the classical index problems associated with analytic functions and was a particularly influential example in the development of the Atiyah-Singer Index Theorem.

The study of Hardy spaces and Bergman spaces of analytic functions is also the starting point for the realization of unitary representations of semi-simple Lie groups. Indeed the group

$$SL(2, \mathbb{R}) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \det(g) = 1 \right\}$$

of 2×2 real matrices having determinant 1 acts on the compactified plane $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto z \cdot g = \frac{b + zd}{a + cz} \qquad (z \in \mathbb{C}). \tag{1.19}$$

Its orbits consist of the upper and lower half-planes $\Omega_\pm$ in $\mathbb{C}$ together with their compactified mutual boundary $\mathbb{R} \cup \{\infty\}$. There is also an induced action of $SL(2, \mathbb{R})$ on the Hardy $H^2(\Omega_+)$-space and on the Lebesgue space $L^2(\mathbb{R})$ of complex-valued functions on $\mathbb{R}$:

$$(1.20)\ \text{(i)} \qquad (\pi(g)F)(z) = \frac{1}{a + zc} F(z \cdot g) \qquad (z \in \Omega_+)$$

defines a unitary representation of $SL(2, \mathbb{R})$ on $H^2(\Omega_+)$, while

$$(1.20)\ \text{(ii)} \qquad (\sigma(g)f)(v) = \frac{1}{a + vc} f(v \cdot g) \qquad (v \in \mathbb{R})$$

defines a unitary representation on $L^2(\mathbb{R})$. There are several implications of these results for Euclidean harmonic analysis:

(i) the familiar transformations of translation, «rotation» and dilation on $\mathbb{R}$ all arise as special cases of (1.19), as does inversion $v \to -1/v$;

(ii) the Cauchy integral

$$f \to F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(v)}{v - z} dv = \mathcal{C} f(z)$$

intertwines the representations in (1.20) in the sense that $\pi(g)F = \mathcal{C}(\sigma(g)f)$;

(iii) the Hilbert transform

$$\mathcal{H} f(x) = P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(v)}{x - v} dv$$

intertwines the representation in (1.20)(ii) in the sense that $\sigma(g)(\mathcal{H}f) = \mathcal{H}(\sigma(g)f)$.

Hence all the familiar and important results in H^2-theory have a group-theoretic interpretation in terms of representations of $SL(2, \mathbb{R})$. But $SL(2, \mathbb{R})$ is just the two-fold covering of the conformal group of $\mathbb{R}$; in addition, if Ω_+ is given the Poincaré metric, then $SL(2, \mathbb{R})$ is just the two-fold covering of the (identity component of the) isometry group of Ω_+. A discussion of H^2-theory in higher dimensions, therefore, should naturally include corresponding representations of the two-fold covering of the conformal group of $\mathbb{R}^n$ and of the isometry group of $(n + 1)$-dimensional hyperbolic space. This will be done in Section 6 as part of a general development of Hardy H^p-theory on Euclidean space. But first a detailed analytic and algebraic study of the higher-dimensional replacements for the Cauchy-Riemann $(\bar{\partial}, \partial)$-operators has to have been made. These are the Dirac $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ operators, and the dicussion in Sections 2, 3 and 4 show that they are intrinsically associated with Euclidean space. Then in Section 5 we show how the operators $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ lead to operator ∂_τ associated with any irreducible representation of the Euclidean rotation group, before finally turning to aspects of the general theory in Section 6.

2. The Dirac $\mathfrak{D}$-operator

Let $\mathfrak{H}$ be a finite-dimensional real or complex Hilbert space and let $\mathcal{L}(\mathfrak{H})$ be the usual C^*-algebra of bounded linear operators on $\mathfrak{H}$, the adjoint of an operator a in $\mathcal{L}(\mathfrak{H})$ being denoted by $\bar{a}$ (cf. [18]). The norm on $\mathfrak{H}$ will be denoted by $|\cdot|$, the inner product by $(\cdot, \cdot)$, and the operator norm by $\|\cdot\|$; thus

$$(2.1) \qquad (au, v) = (u, \bar{a}u), \qquad \|\bar{a}a\| = \|a\|^2$$

hold for all a in $\mathfrak{L}(\mathfrak{H})$ and u, v in $\mathfrak{H}$. Now fix skew-adjoint elements $e_1, \ldots, e_n$ in $\mathfrak{L}(\mathfrak{H})$ such that

$$e_j e_k + e_k e_j = -2\delta_{jk} e_0 \qquad (1 \leqslant j, k \leqslant n) \tag{2.2}$$

where e_0 ($= I$) is the identity operator on $\mathfrak{H}$. For instance, $\mathfrak{H}$ could be the complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$; then multiplication on $\mathbb{C}$ by the imaginary unit i ($= \sqrt{-1}$) or on $\mathbb{H}$ by its imaginary units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ defines skew-adjoint operators satisfying (2.2). More generally still, scalar multiplication

$$i: \xi \mapsto i\xi \qquad (\xi \in \mathfrak{H}) \tag{2.3}$$

by i ($= \sqrt{-1}$) on any finite-dimensional complex Hilbert space $\mathfrak{H}$ is skew-adjoint.

Definition 2.4. *The Dirac $\mathfrak{D}$-operator and its adjoint $\bar{\mathfrak{D}}$ are the first-order systems of differential operators on $C^\infty(\Omega, \mathfrak{H})$ defined by*

$$\mathfrak{D} = \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}, \qquad \bar{\mathfrak{D}} = \sum_{j=0}^{n} \bar{e}_j \frac{\partial}{\partial x_j}$$

for any open set Ω in $\mathbb{R}^{n+1}$.

When $\mathfrak{H}$ is any complex Hilbert space, for example, $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ are just the classical Cauchy-Riemann operators $\bar{\partial}$ and ∂ respectively, while $\mathfrak{D}$ is the Fueter operator when $\mathfrak{H} = \mathbb{H}$ ([6, Chap. 2]). As the notation and terminology suggest, the skew-adjointness of the e_j ensure that $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ are adjoint in the sense that

$$\int_\Omega (\mathfrak{D}F(x), \Phi(x))\, dx = \int_\Omega (F(x), \bar{\mathfrak{D}}\Phi(x))\, dx \tag{2.5}$$

when F, Φ are compactly supported functions in $C^\infty(\Omega, \mathfrak{H})$. On the other hand, since

$$\mathfrak{D}\bar{\mathfrak{D}} = \sum_{j=0}^{n} \left(\frac{\partial}{\partial x_j}\right)^2 = \bar{\mathfrak{D}}\mathfrak{D}, \tag{2.6}$$

every solution of $\mathfrak{D}F = 0$ or $\bar{\mathfrak{D}}F = 0$ has harmonic components. Consequently, $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ are Generalized Cauchy-Riemann (GCR-) systems as introduced by Stein-Weiss ([33, p. 231]); in particular, they are *injectively elliptic* meaning that the symbol mappings

$$\xi \to \left(\sum_{j=0}^{n} \lambda_j e_j\right)\xi, \qquad \xi \to \left(\sum_{j=0}^{n} \lambda_j e_j\right)^{-} \xi \qquad (\xi \in \mathfrak{H}) \tag{2.7}$$

are injective on $\mathfrak{H}$ for every non-zero $\lambda = (\lambda_0, \ldots, \lambda_n)$ in $\mathbb{R}^{n+1}$ (cf. [10, p. 79]; [33, p. 231]). In fact, as we shall see, the algebraic structure in (2.2) forces the mappings in (2.7) to be surjective as well as injective, and so $\mathfrak{D}$, $\bar{\mathfrak{D}}$ are *elliptic first-order systems* in the usual sense of the term elliptic. But the consequences of this algebraic structure go much deeper.

Denote by $\nu: \mathbb{R}^{n+1} \to \mathfrak{L}(\mathfrak{H})$ the linear embedding

$$(2.8) \qquad \nu: x = (x_0, \ldots, x_n) \to \sum_{j=0}^{n} x_j e_j \qquad (x \in \mathbb{R}^{n+1})$$

of $\mathbb{R}^{n+1}$ into $\mathfrak{L}(\mathfrak{H})$. Then, because of (2.2),

$$\nu(x)^- \nu(x) = |x|^2 e_0 = \nu(x)\nu(x)^- \qquad (x \in \mathbb{R}^{n+1})$$

where $|x| = (\Sigma_j |x_j|^2)^{1/2}$ is the Euclidean length of x; on the other hand,

$$(2.9) \qquad \|\nu(x)\|^2 = \|\nu(x)^- \nu(x)\| = |x|^2 \|e_0\| = |x|^2$$

because of the C^*-algebra norm property. Hence $\nu: \mathbb{R}^{n+1} \to \mathfrak{L}(\mathfrak{H})$ is an *isometric embedding of* $\mathbb{R}^{n+1}$ *into* $\mathfrak{L}(\mathfrak{H})$; we shall therefore identify $\mathbb{R}^{n+1}$ with a subspace of $\mathfrak{L}(\mathfrak{H})$, regarding $e_0, \ldots, e_n$ simply as an orthonormal basis for $\mathbb{R}^{n+1}$ and omitting any mention of the mapping ν in (2.8). With this convention, let $\mathfrak{A}(\mathfrak{H})$ be the (real) $\mathfrak{C}^*$-subalgebra of $\mathfrak{L}(\mathfrak{H})$ generated by $\mathbb{R}^{n+1}$ ([18]). Now, in the special cases of $\mathfrak{H} = \mathbb{C}$ or $\mathbb{H}$,

$$(2.10) \qquad \mathfrak{A}(\mathfrak{H}) = \mathbb{C} \sim \mathbb{R}^2, \qquad \mathfrak{A}(\mathfrak{H}) = \mathbb{H} \sim \mathbb{R}^4$$

and

$$(2.11) \qquad \|wz\| = \|w\| \, \|z\| = |w| \, |z| \qquad (w, z \in \mathfrak{A}(\mathfrak{H}));$$

consequently, in addition to the linear structure, there is a multiplicative structure on $\mathbb{R}^2$ and $\mathbb{R}^4$ with respect to which the metric is multiplicative, *i.e.*, satisfies (2.11). Although this property characterizes $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ as C^*-algebras ([18, p. 78]), it will be exceedingly important and highly instructive to think of $\mathfrak{A}(\mathfrak{H})$ as being a sufficiently strong substitute for any $\mathfrak{H}$. Indeed, by (2.2),

$$(2.12)\ \text{(i)} \qquad \|wz\| = \|(wz)^-(wz)\| = |w| \, |z| \qquad (w, z \in \mathbb{R}^{n+1}),$$

and more generally,

$$(2.12)\ \text{(ii)} \qquad \|a \cdots wz\| = |a| \cdots |w| \, |z|$$

for any finite product $a \cdots wz$ in $\mathfrak{A}(\mathfrak{H})$ of elements $a, \ldots, w, z$ in $\mathbb{R}^{n+1}$. What fails in general is the multiplicativity on linear combinations of such finite products. Nonetheless, all of this and Adams' solution of the vector fields on spheres problem are much more than is needed to establish the following results.

Theorem 2.13. *The Dirac $\mathfrak{D}$- and $\bar{\mathfrak{D}}$-operators are first-order elliptic systems of differential operators on $C^\infty(\Omega, \mathfrak{H})$ for any open set Ω in $\mathbb{R}^{n+1}$; in particular, if $n = 8d + r$, $0 \leqslant r \leqslant 7$, then*

(i) $\dim_{\mathbb{R}}(\mathfrak{H}) \geqslant (r+1)16^d$
(ii) $\dim_{\mathbb{C}}(\mathfrak{H}) \geqslant 2^{[r/2]}16^d$.

PROOF. Since

$$\left(\sum_{j=0}^{n} \lambda_j e_j\right)\left(\sum_k \lambda_k e_k\right)^- = |\lambda|^2,$$

it is clear that the symbol mappings

$$\xi \to \left(\sum_{j=0}^{n} \lambda_j e_j\right)\xi, \qquad \xi \to \left(\sum_{j=0}^{n} \lambda_j e_j\right)^- \xi$$

are invertible on $\mathfrak{H}$ for each $\lambda \neq 0$. Hence $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ are elliptic. As a consequence of this ellipticity, the dimension of $\mathfrak{H}$ cannot be too small in relation to n (cf. [1], [30]). More precisely, if

$$\dim_{\mathbb{F}}(\mathfrak{H}) = (2k+1)2^c 16^\delta \qquad (0 \leqslant c \leqslant 3, k \geqslant 0),$$

then

(i) $n + 1 \leqslant 8\delta + 2^c \quad (\mathbb{F} = \mathbb{R})$,
(ii) $n + 1 \leqslant 8\delta + 2c + 2 \quad (\mathbb{F} = \mathbb{C})$.

Now suppose $n = 8d + r$, $0 \leqslant r \leqslant 7$. Then

$$\dim_{\mathbb{R}}(\mathfrak{H}) \geqslant (r+1)16^d, \qquad \dim_{\mathbb{C}}(\mathfrak{H}) \geqslant 2^{[r/2]}16^d,$$

completing the proof. □

The significance of these estimates for $\dim_{\mathbb{R}}(\mathfrak{H})$ will become clear later. Let

$$C(x) = \frac{1}{\omega_{n+1}}\left(\frac{\bar{x}}{|x|^{n+1}}\right) \qquad (x \in \mathbb{R}^{n+1}, x \neq 0) \tag{2.14}$$

be the Cauchy kernel on $\mathbb{R}^{n+1}$. It is well-defined as an $\mathfrak{A}(\mathfrak{H})$-valued solution of $\mathfrak{D}\Phi = 0$ on $\mathbb{R}^{n+1}\backslash\{0\}$ since

$$C(x) = \frac{1}{(1-n)\omega_{n+1}}\bar{\mathfrak{D}}\left(\frac{1}{|x|^{n-1}}\right)$$

expresses the Cauchy kernel as a constant multiple of $\bar{\mathfrak{D}}(\Gamma_{n+1})$, where Γ_{n+1} is the fundamental solution of the Laplacian on $\mathbb{R}^{n+1}$. Similarly,

$$d\sigma(x) = \sum_{j=0}^{n} (-1)^j e_j \, dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \tag{2.15}$$

is an $\mathfrak{A}(\mathfrak{H})$-valued n-form on $\mathbb{R}^{n+1}$. The usual argument using Stokes theorem (cf., for intance, [6, p. 51]; [14, Chap. II, Section 3]) then gives

Theorem 2.16. *Let M be a compact set in Ω having suitably smooth boundary ∂M. Then the Cauchy integral*

$$f \to \mathcal{C}f(x) = \frac{1}{\omega_{n+1}} \int_{\partial M} \frac{\bar{y} - \bar{x}}{|y - x|^{n+1}} \, d\sigma(y) f(y)$$

of any f in $C(\partial M, \mathfrak{H})$ satisfies $\mathfrak{D}(\mathcal{C}f) = 0$ on $\mathbb{R}^{n+1} \backslash \partial M$, while

$$\frac{1}{\omega_{n+1}} \int_{\partial M} \frac{\bar{y} - \bar{x}}{|y - x|^{n+1}} \, d\sigma(y) F(y) = \begin{cases} F(x) & (x \in M \backslash \partial M) \\ 0 & (x \in \Omega \backslash M) \end{cases}$$

whenever $\mathfrak{D}F = 0$ on Ω.

3. Hardy Spaces

In complete analogy with the classical case, the Cauchy integral results lead naturally to Hardy spaces on domains in $\mathbb{R}^{n+1}$.

Let Ω be a domain in $\mathbb{R}^{n+1}$ which is either a bounded Lipschitz domain or a special Lipschitz domain in the unbounded case. Thus for each x in $\partial\Omega$ having inward unit normal $\eta(x)$, there is a cone

$$\Gamma_\alpha(x) = \{ z \in \mathbb{R}^{n+1} : |z - x| < \delta, (z - x, \eta(x)) > \alpha |z - x| \} \tag{3.1}$$

having vertex at x and lying wholly inside Ω for some choice of α, δ independent of x. Now denote by $\mathcal{H}^p(\Omega, \mathfrak{H})$ the *harmonic Hardy space* of all harmonic functions in $C^\infty(\Omega, \mathfrak{H})$ whose non-tangential maximal function

$$N(F)(x) = \sup_{z \in \Gamma_\alpha(x)} |F(z)| \qquad (x \in \partial\Omega) \tag{3.2}$$

is L^p-integrable on $\partial\Omega$. Since the Laplacian is elliptic, $\mathcal{H}^p(\Omega, \mathfrak{H})$ is a Banach space under the norm

$$\|F\|_p = \left(\int_{\partial\Omega} N(F)(x)^p \, dS \right)^{1/p} \qquad (1 \leqslant p < \infty) \tag{3.3}$$

defined with respect to scalar surface measure on $\partial\Omega$.

Definition 3.4. *The Hardy $H^p(\Omega, \mathfrak{H})$ consists of all solutions of $\mathfrak{D}F = 0$ in $C^\infty(\Omega, \mathfrak{H})$ for which* (3.3) *is finite.*

There are corresponding spaces $\mathcal{H}^p(\Omega, \mathfrak{H})$ and $H^p(\Omega, \mathfrak{H})$ when $0 < p < 1$, though they are not Banach spaces of course. Ellipticity and GCR-properties of $\mathfrak{D}$ ensure that $H^p(\Omega, \mathfrak{H})$ is a closed subspace of $\mathcal{H}^p(\Omega, \mathfrak{H})$.

The key to studying Hardy spaces is the L^p-boundedness of the principal-value Cauchy integral

$$\text{(3.5)} \qquad \mathcal{H}f(x) = P.V.\left(\frac{2}{\omega_{n+1}} \int_{\partial\Omega} \frac{\bar{y} - \bar{x}}{|y - x|^{n+1}} d\sigma(y) f(y)\right) \qquad (x \in \partial\Omega)$$

for $\mathfrak{H}$-valued functions on $\partial\Omega$. In the seminal work of Coifman, McIntosh and Meyer ([7]) establishing this boundedness for *complex-valued* functions through use of P_t, Q_t-operator techniques, essential though unconscious use is made of the multiplicativity property $|wz| = |w|\,|z|$ for all w, z in $\mathbb{C}$ in controlling the operator norm of iterated powers of P_t and Q_t. The fundamental multiplicative norm property (2.12) for arbitrary $\mathfrak{H}$ maintains the same control on the corresponding operators in the general case; one then proves (cf. [14])

Theorem 3.6. *Let Ω be a special Lipschitz domain in $\mathbb{R}^{n+1}$. Then for each p, $1 < p < \infty$,*

(i) *the principal-value Cauchy integral operator $f \to \mathcal{H}f$ is bounded on $L^p(\partial\Omega, \mathfrak{H})$;*
(ii) *the Cauchy integral $f \to \mathcal{C}f$ maps $L^p(\partial\Omega, \mathfrak{H})$ boundedly into $H^p(\Omega, \mathfrak{H})$;*
(iii) *the Cauchy integral $\mathcal{C}f$ of any f in $L^p(\partial\Omega, \mathfrak{H})$ has non-tangential boundary values on $\partial\Omega$ such that*

$$\lim_{z \to x} \mathcal{C}f(z) = \frac{1}{2}(f(x) + \mathcal{H}f(x))$$

almost everywhere on $\partial\Omega$.

Because of the GCR-property enjoyed by $\mathfrak{D}$, the solution of the Dirichlet problem for Ω with scalar-valued L^p-boundary data can be applied componentwise to the Hardy spaces in studying the boundary value behaviour of functions in $H^p(\Omega, \mathfrak{H})$ (cf. [22]). Let $p(\Omega)$ be the critical index for the scalar Dirichlet problem with L^p-data, *i.e.*, for each real-valued function f in $L^p(\partial\Omega)$, $p > p(\Omega)$, there is a function U so that

$$\text{(3.7)} \quad \text{(i)} \qquad \Delta U = 0 \quad \text{in} \quad \Omega$$

$$\text{(ii)} \qquad U|_{\partial\Omega} = f.$$

In the case of any C^1-domain, $p(\Omega) = 1$; but in general, $p(\Omega) > 1$. By a simple adaptation of the proof of Theorem 3.19 in Chapter 2 in [33], one readily sees that each F in $H^p(\Omega, \mathfrak{H})$, $p > p(\Omega)$, has non-tangential boundary values F^+ in $L^p(\partial\Omega, \mathfrak{H})$.

Definition 3.8. *Denote by* $H(\partial\Omega, \mathfrak{H})$ *the subspace in* $L^p(\partial\Omega, \mathfrak{H})$ *of the non-tangential boundary values* F^+ *of all functions* F *in* $H^p(\Omega, \mathfrak{H})$, $p > p(\Omega)$.

In [15] we proved the following result characterizing this space $H^p(\partial\Omega, \mathfrak{H})$ of boundary values.

Theorem 3.9. *Let* Ω *be a special Lipschitz domain in* $\mathbb{R}^{n+1}$. *Then, for each* p, $p(\Omega) < p < \infty$,

(i) *the Cauchy integral* $f \to \mathcal{C}f$ *is a bounded mapping from* $L^p(\partial\Omega, \mathfrak{H})$ *onto* $H^p(\Omega, \mathfrak{H})$;

(ii) *every* F *in* $H^p(\Omega, \mathfrak{H})$ *is the Cauchy integral* $\mathcal{C}F^+$ *of its non-tangential boundary values* F^+ *in* $H^p(\partial\Omega, \mathfrak{H})$;

(iii) *a function* f *in* $L^p(\partial\Omega, \mathfrak{H})$ *is in* $H^p(\partial\Omega, \mathfrak{H})$ *if and only if* $f = \mathcal{K}f$.

Two results are conspicuously absent from (3.9): the first dealing with characterizations for $p \leqslant p(\Omega)$, and the second with an analytic over-determinedness characterization such as (1.2)(ii). Carlos Kenig has remarked to us that the ideas in his joint work with Jill Pipher ([23]) may well yield characterizations of $H^p(\partial\Omega, \mathfrak{H})$ below the critical index $p(\Omega)$. As regards the second omission, let us assume that for each imaginary unit j in $\mathbb{R}^{n+1}$ $(\subseteq \mathfrak{A}(\mathfrak{H}))$ there is a splitting $\mathcal{K} = \mathfrak{H}_0 \oplus j\mathfrak{H}_0$ of $\mathfrak{H}$ for some subspace $\mathfrak{H}_0$ of $\mathfrak{H}$. We shall then say that $\mathfrak{H}$ has the *splitting property*. From the structural characterizations of $\mathfrak{H}$ described in the next section, it likely follows that every $\mathfrak{H}$ has this property. Notice that each such splitting determines an involution $x + jy \to \overline{x - jy}$ on $\mathfrak{H}$. The prototypical example of course is the splitting $\mathbb{C} = \mathbb{R} \oplus \sqrt{-1}\,\mathbb{R}$ where the involution is complex conjugation. For this reason we shall denote by Re, Im: $\mathfrak{H} \to \mathfrak{H}_0$ the linear mappings

$$(3.10) \qquad z \to x = \mathrm{Re}(z), \qquad z \to y = \mathrm{Im}(z), \qquad z = x + jy \qquad (x, y \in \mathfrak{H}_0)$$

that a splitting $\mathfrak{H} = \mathfrak{H}_0 \oplus j\mathfrak{H}_0$ determines.

Conjecture 3.11. (Real H^p-theory.) *Suppose* $\mathfrak{H} = \mathfrak{H}_0 \oplus j\mathfrak{H}_0$ *is a splitting of* $\mathfrak{H}$. *Then, for each* $p, p(\Omega) < p < \infty$,

(i) *the Cauchy integral* $f \to \mathcal{C}f$ *is an isomorphism from* $L^p(\partial\Omega, \mathfrak{H}_0)$ *onto* $H^p(\Omega, \mathfrak{H})$, *and conversely*,

(ii) *the boundary operator* $F \to \mathrm{Re}(F|_{\partial\Omega})$ *is continuous from* $H^p(\Omega, \mathfrak{H})$ *onto* $L^p(\partial\Omega, \mathfrak{H}_0)$.

A Fourier Transform argument confirms this conjecture for special choices of $\mathfrak{H}$ when Ω is the upper half-space

$$\Omega_+ = \{x + jy : x \in \mathbb{R}^n, y > 0\} \tag{3.12}$$

in $\mathbb{R}^{n+1}$ (cf. [14, Chap. II]).

4. Clifford Algebras and $\mathfrak{A}(\mathfrak{H})$

The definition of $\mathfrak{A}(\mathfrak{H})$ ensures that it is always a C^*-algebra representation of the Universal Clifford algebra $\mathfrak{A}_n$ for $\mathbb{R}^n$; algebraically, therefore, $\mathfrak{H}$ is simply an ungraded Clifford module ([4]). This makes available all the algebraic constructions associated with both of these classes of algebras, and we shall use freely results for these algebras established in [14] and [18] to which the reader is referred for all unsubstantiated assertions (and omitted references). The culmination of this section will be characterizations of $\mathfrak{A}(\mathfrak{H})$ and $\mathfrak{H}$, but the algebraic details should not be allowed to obscure the fact that these characterizations show that $\mathfrak{A}(\mathfrak{H})$ and $\mathfrak{H}$ are intrinsically associated with $\mathbb{R}^n$.

Recall that a Clifford algebra for $\mathbb{R}^n$ consists of a pair $(\mathbb{A}_n, \beta)$ where $\mathbb{A}_n$ is an associative algebra over $\mathbb{R}$ with identity 1 and $\beta: \mathbb{R}^n \to \mathbb{A}_n$ is a linear embedding of $\mathbb{R}^n$ into $\mathbb{A}_n$ such that

(4.1)
(i) $\mathbb{A}_n$ *is generated by* $\{\beta(x) : x \in \mathbb{R}^n\}$ *and* $\{\lambda 1 : \lambda \in \mathbb{R}\}$,
(ii) $\beta(x)^2 = -|x|^2 1 \quad (x \in \mathbb{R}^n)$.

The algebra $\mathfrak{A}(\mathfrak{H})$, for instance, always has these properties because of hypothesis (2.2). Clearly,

$$\dim_{\mathbb{R}} (\mathbb{A}_n) \leqslant 2^n. \tag{4.2}$$

Let $(\mathfrak{A}_n, \alpha)$ be a Clifford algebra for $\mathbb{R}^n$ having maximal dimension 2^n. It is universal in the sense that for every $(\mathbb{A}_n, \beta)$ there is a unique algebra homomorphism $\pi: \mathfrak{A}_n \to \mathbb{A}_n$ such that $(\pi \circ \alpha)(x) = \beta(x)$, $x \in \mathbb{R}^n$. Hence $\mathfrak{A}_n$, the so-called Universal Clifford algebra for $\mathbb{R}^n$, is unique up to isomorphism; we shall regard $\mathbb{R}$ and $\mathbb{R}^n$ as subspaces of $\mathfrak{A}_n$, identifying $\mathbb{R}^{n+1}$ with $\mathbb{R} \oplus \mathbb{R}^n$. Universality also ensures that the mapping $x \to -x$ on $\mathbb{R}^n$ extends to a unique automorphism $a \to a'$ of $\mathfrak{A}_n$, *the principal automorphism* of $\mathfrak{A}_n$, and a unique anti-automorphism $a \to \bar{a}$ of $\mathfrak{A}_n$, *conjugation* on $\mathfrak{A}_n$, such that

$$x' = -x = \bar{x} \qquad (x \in \mathbb{R}^n) \tag{4.3}$$

(cf. [14, Chap. I]; [27, p. 245]). The composition $a \to a^* = (a')^- = (\bar{a})'$ of conjugation and the principal automorphism is called the *principal anti-*

automorphism of $\mathfrak{A}_n$. Known basic structural properties of $\mathfrak{A}(\mathfrak{H})$ are collected together in the next theorem. We include its proof for convenience and clarity of exposition.

Theorem 4.4. *The algebra $\mathfrak{A}(\mathfrak{H})$ is a Clifford algebra for $\mathbb{R}^n$ such that the homomorphism $\pi: \mathfrak{A}_n \to \mathfrak{A}(\mathfrak{H})$ satisfies*

(i) *π is always an isomorphism when $n \neq 4l + 3$, $l \geq 0$;*
(ii) *π fails to be an isomorphism if and only if $e_1 \cdots e_n = \pm I$;*
(iii) *if π is not an isomorphism, then there is a central idempotent E in $\mathfrak{A}_n$ such that π is an isomorphism from the ideal $\{aE: a \in \mathfrak{A}_n\}$ in $\mathfrak{A}_n$ onto $\mathfrak{A}(\mathfrak{H})$, and* $\ker \pi = \{a(I - E): a \in \mathfrak{A}_n\}$.

Furthermore, whether π is an isomorphism or not,

(iv) $\pi(a)^- = \pi(\bar{a}) \quad (a \in \mathfrak{A}_n)$.

Since algebraically-*isomorphic C^*-algebras are always isometrically-*isomorphic ([18, p. 87]), the C^*-algebras $\mathfrak{A}(\mathfrak{H})$ are all isometrically-*isomorphic when $n \neq 4l + 3$, $l \geq 0$, or when $n = 4l + 3$ and the generators $e_1, \ldots, e_n$ satisfy $e_1 \cdots e_n \neq \pm I$. *In fact, they are all C^*-algebra realizations of $\mathfrak{A}_n$ in which the adjoint operation on $\mathfrak{A}(\mathfrak{H})$ coincides with conjugation on $\mathfrak{A}_n$.* The example of $\mathfrak{A}(\mathfrak{H}) = \mathfrak{H} = \mathbb{H}$, with e_1, e_2, e_3 multiplication by the imaginary units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, is a case where $e_1 e_2 e_3 = -I$.

PROOF OF THEOREM 4.4. Let $\epsilon_1, \ldots, \epsilon_n$ be generators for $\mathfrak{A}_n$ such that

$$\epsilon_j \epsilon_k + \epsilon_k \epsilon_j = -2\delta_{jk}, \qquad \pi(\epsilon_j) = e_j \qquad (1 \leq j \leq n)$$

with respect to the homomorphism $\pi: \mathfrak{A}_n \to \mathfrak{A}(\mathfrak{H})$. Now the kernel of π is a two-sided ideal in $\mathfrak{A}_n$, so $\pi: \mathfrak{A}_n \to \mathfrak{A}(\mathfrak{H})$ will be an isomorphism if and only if $\mathfrak{A}_n$ is simple. But $\mathfrak{A}_n$ is always simple when $n \neq 4l + 3$; on the other hand, when $n = 4l + 3$, it is simple if and only if $\epsilon_1 \cdots \epsilon_n \notin \mathbb{R}$ ([14, Chap. I, Corollary 3.6, Theorem 3.19]). This proves (i). It also proves the necessity half of (ii) because

$$e_1 \cdots e_n = \pi(\epsilon_1 \cdots \epsilon_n) = \lambda I \qquad (\lambda \in \mathbb{R})$$

if $\epsilon_1 \cdots \epsilon_n \in \mathbb{R}$; note that $\lambda = \pm 1$ since

$$(e_1 \cdots e_n)^2 = (e_1 \cdots e_n)(e_1 \cdots e_n) = (-1)^{n(n+1)/2} = 1$$

when $n = 4l + 3$. On the other hand,

$$\dim_{\mathbb{R}} (\mathfrak{A}(\mathfrak{H})) < 2^n = \dim_{\mathbb{R}} (\mathfrak{A}_n)$$

when $e_1 \cdots e_n = \pm I$, for then $e_n = \mp e_1 \cdots e_{n-1}$; this ensures that it is not an isomorphism, completing the proof of (ii).

To establish (iii) set

$$E = \tfrac{1}{2}(I + \epsilon_1 \cdots \epsilon_n) \quad \text{or} \quad E = \tfrac{1}{2}(I - \epsilon_1 \cdots \epsilon_n) \tag{4.5}$$

according as $e_1 \cdots e_n = I$ or $-I$. In either case, E is a central idempotent in $\mathfrak{A}_n$ and

$$\mathfrak{A}_n = \{aE + a(I - E) : a \in \mathfrak{A}_n\}$$

where $I_+ = \{aE : a \in \mathfrak{A}_n\}$ and $I_- = \{a(I - E) : a \in \mathfrak{A}_n\}$ are two-sided ideals in $\mathfrak{A}_n$ such that $\mathfrak{A}_n = I_+ \oplus I_-$ (cf. [14, Chap. I]). Now by the definition of E,

$$\pi(aE) = \pi(a), \qquad \pi(a(I - E)) = 0.$$

Consequently, π maps I_+ onto $\mathfrak{A}(\mathfrak{H})$, while $\ker(\pi) \supseteq I_-$. But I_+ and I_- are the only non-trivial, proper, two-sided ideals in $\mathfrak{A}_n$ ([14, Chap. I, Theorem 3.19]). Hence $\pi : I_+ \to \mathfrak{A}(\mathfrak{H})$ is an isomorphism and $\ker(\pi) = I_-$. This proves (iii).

To prove (iv) observe first that

$$(\epsilon_{j_1} \cdots \epsilon_{j_k})^- = (-1)^{k(k+1)/2} \epsilon_{j_1} \cdots \epsilon_{j_k}$$

(cf. [14, Chap. I (2.25)]); a similar result

$$(e_{j_1} \cdots e_{j_k})^- = (-1)^{k(k+1)/2} e_{j_1} \cdots e_{j_k}$$

holds in $\mathfrak{A}(\mathfrak{H})$ since the adjoint operation is an anti-automorphism such that ${}^- : e_j \to -e_j$. Thus

$$\pi((\epsilon_{j_1} \cdots \epsilon_{j_k})^-) = (\pi(\epsilon_{j_1} \cdots \epsilon_{j_k}))^-$$

holds whether or not π is an isomorphism. Now the products $\epsilon_{j_1} \cdots \epsilon_{j_k}$, $1 \leqslant j_1 < \cdots < j_k \leqslant n$ together with the identity are a basis for $\mathfrak{A}_n$. Hence this last equality extends linearly to all of $\mathfrak{A}_n$, *i.e.*, $\pi(\bar{a}) = \pi(a)^-$, when π is an isomorphism. If, however, π is not an isomorphism, then $n = 4l + 3$ and

$$(\epsilon_1 \cdots \epsilon_n)^- = (-1)^{n(n+1)/2} \epsilon_1 \cdots \epsilon_n = \epsilon_1 \cdots \epsilon_n,$$

ensuring that $\bar{E} = E$. Consequently,

$$\pi((a - aE)^-) = \pi(\bar{a} - \bar{a}E) = 0 \qquad (a \in \mathfrak{A}_n).$$

Hence the equality $\pi(\bar{a}) = \pi(a)^-$ continues to hold on all of $\mathfrak{A}_n$ even if π is not an isomorphism. This completes the proof of (iv). $\square$

A failure of $\mathfrak{A}(\mathfrak{H})$ to be isomorphic to $\mathfrak{A}_n$ when $n = 4l + 3$ is not serious because $\mathfrak{A}(\mathfrak{H})$ can then be embedded in a larger C^*-algebra which is isomorphic to $\mathfrak{A}_n$. For suppose $e_1 \cdots e_n = -I$, say, and regard $\mathfrak{H} \oplus \mathfrak{H}$ as the Hilbert space

(4.6) (i) $$\mathfrak{H} \oplus \mathfrak{H} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : u, v \in \mathfrak{A} \right\}$$

of column vectors under the inner product

(4.6) (ii) $$\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right) = (u_1, u_2) + (u_1, v_2)$$

derived from that on $\mathfrak{H}$. Then

(4.7) $$\gamma_1 = \begin{bmatrix} e_1 & 0 \\ 0 & -e_1 \end{bmatrix}, \ldots, \gamma_n = \begin{bmatrix} e_n & 0 \\ 0 & -e_n \end{bmatrix}$$

are skew-adjoint operators on $\mathfrak{H} \oplus \mathfrak{H}$ such that

$$\text{(i) } \gamma_j \gamma_k + \gamma_k \gamma_j = -2\delta_{jk} I, \qquad \text{(ii) } \gamma_1 \cdots \gamma_n = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},$$

since $n = 4l + 3$. Criterion (4.4) (ii) thus ensures that $\mathfrak{A}_n$ is isomorphic to the C^*-subalgebra generated in $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{H})$ by $\gamma_1, \ldots, \gamma_n$. This clearly is

(4.8) $$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathfrak{A}(\mathfrak{H}) \right\} \cong \mathfrak{A}(\mathfrak{H}) \oplus \mathfrak{A}(\mathfrak{H}),$$

acting on the Hilbert space (4.6).

Now every finite-dimensional real C^*-algebra is the finite direct sum of full matrix rings $M(k, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Since each is of these is simple, it follows that

(4.9) (i) $$\mathfrak{A}_n \cong M(k, \mathbb{K}) \qquad (n \neq 4l + 3)$$

for some choice of $(k, \mathbb{K})$, while

(4.9) (ii) $$\mathfrak{A}_n \cong M(k, \mathbb{K}) \oplus M(k, \mathbb{K}) \qquad (n = 4l + 3).$$

This is consistent with the well-known realizations

(4.10) $$\mathfrak{A}_0 \cong \mathbb{R}, \qquad \mathfrak{A}_1 \cong \mathbb{C}, \qquad \mathfrak{A}_2 \cong \mathbb{H}, \qquad \mathfrak{A}_3 \cong \mathbb{H} \oplus \mathbb{H}$$

and the less well-known

(4.10) (i) $$\mathfrak{A}_4 \cong \mathbb{H}^{2 \times 2}, \quad \mathfrak{A}_5 \cong \mathbb{C}^{4 \times 4}, \quad \mathfrak{A}_6 \cong \mathbb{R}^{8 \times 8}, \quad \mathfrak{A}_7 \cong \mathbb{R}^{8 \times 8} \oplus \mathbb{R}^{8 \times 8}.$$

The following fundamental Periodicity Theorem completes the characterization of $\mathfrak{A}_n$ for every n.

Theorem 4.10. (ii) *For each $n \geqslant 0$ there is a realization of $\mathfrak{A}_{n+8}$ as the algebra*

$$\mathfrak{A}_{n+8} = M(16, \mathfrak{A}_n)$$

of all 16×16 matrices having entries from $\mathfrak{A}_n$.

Consequently, when $n = 8d + r$, $0 \leqslant r < 8$, the choice of $\mathbb{K}$ in (4.9) is given by

(4.11) (i) $\mathbb{K} = \mathbb{R} \quad (r = 0, 6, 7), \qquad \mathbb{K} = \mathbb{C} \quad (r = 1, 5), \qquad \mathbb{K} = \mathbb{H} \quad (r = 2, 3, 4),$

while k is determined by

(4.11) (ii) $k^2 \dim_{\mathbb{R}}(\mathbb{K}) = 2^{8d+r} \ (r \neq 3, 7), \qquad k^2 \dim_{\mathbb{R}}(\mathbb{K}) = 2^{8d+r-1} \ (r = 3, 7)$

since $\dim_{\mathbb{R}}(\mathfrak{A}_n) = 2^n$.

In summarizing these results it will be convenient to introduce the following definition.

Definition 4.12. *A finite-dimensional Hilbert space $\mathfrak{H}$ is said to be an $\mathfrak{A}_n$-module when there exist skew-adjoint operators $e_1, \ldots, e_n$ in $\mathfrak{L}(\mathfrak{H})$ such that*

$$e_j e_k + e_k e_j = -2\delta_{jk} e_0 \qquad (1 \leqslant j, k \leqslant n),$$

where e_0 $(= I)$ is the identity operator.

As before, the real C^*-algebra of $\mathfrak{L}(\mathfrak{H})$ generated by $e_1, \ldots, e_n$ will be denoted by $\mathfrak{A}(\mathfrak{H})$. Then for each n there is a real Hilbert space $\mathfrak{K}_n \cong \mathbb{K}^k$, where $(k, \mathbb{K})$ are determined by (4.11), such that

(4.13) (i) *$\mathfrak{A}(\mathfrak{H})$ is isometrically-*isomorphic to the real C^*-algebra $\mathfrak{L}(\mathfrak{K}_n)$ when $n \neq 4l + 3$,*

(4.13) (ii) *$\mathfrak{A}(\mathfrak{H})$ is isometrically-*isomorphic to $\mathfrak{L}(\mathfrak{K}_n)$ or to $\mathfrak{L}(\mathfrak{K}_n) \oplus \mathfrak{L}(\mathfrak{K}_n)$ when $n = 4l + 3$.*

This space $\mathfrak{K}_n$ is called the space of *real spinors*. By identifying $\mathfrak{A}_n$ with $\mathfrak{K}_n \otimes \mathfrak{K}_n$ when $n \neq 4l + 3$, or with $\mathfrak{K}_n \otimes \mathfrak{K}_n \oplus \mathfrak{K}_n \otimes \mathfrak{K}_n$ when $n = 4l + 3$, the Universal Clifford algebra itself is an $\mathfrak{A}_n$-module under left multiplication. Thus Clifford analyticity, the study of $\mathfrak{A}_n$-valued solutions of $\mathfrak{D}F = 0$, is one special case of the theory of $\mathfrak{H}$-valued solutions. On the other hand, every real $M(k, \mathbb{K})$-module is isomorphic to the real tensor product $\mathbb{K}^k \otimes \mathbb{R}^m = \mathbb{K}^{k \times m}$ for some $m \geqslant 1$. Since the latter becomes a real Hilbert space under the inner product

(4.14) $$(w, z) = \operatorname{tr}(\bar{z}w) \qquad (w, z \in \mathbb{K}^{k\times m})$$

where $\bar{z} = [\bar{z}_{sr}] \in \mathbb{K}^{m\times k}$ is the conjugate of $z = [z_{rs}] \in \mathbb{K}^{k\times m}$, we obtain the following fundamental characterization familiar from real K-theory.

Theorem 4.15. *When $n = 8d + r$, $0 \leqslant r < 8$, and $(k, \mathbb{K})$ are specified by (4.11), then every $\mathbb{K}^{k\times m}$, $m \geqslant 1$, is an $\mathfrak{A}_n$-module under matrix multiplication of the left by $M(k, \mathbb{K})$. Conversely, every real Hilbert space that is also an $\mathfrak{A}_n$-module is isomorphic to some $M(k, \mathbb{K})$-module $\mathbb{K}^{k\times m}$ or to the direct sum of two such modules; this last case can occur only if $r = 3$ or 7.*

Thus real $\mathfrak{A}_n$-modules have a rigid structure with a very remarkable «period 8» property. The complex $\mathfrak{A}_n$-modules have a «period 2» property, and so have an even more rigid structure. To be more precise, let $\mathfrak{C}(\mathfrak{H})$ be the complex C^*-algebra generated in $\mathfrak{L}(\mathfrak{H})$ by $e_1, \ldots, e_n$ when $\mathfrak{H}$ is a complex $\mathfrak{A}_n$-module. Then $\mathfrak{C}(\mathfrak{H})$ is isometrically-*isomorphic to $M(2^m, \mathbb{C})$ when $n = 2m$ or to one of $M(2^m, \mathbb{C})$, $M(2^m, \mathbb{C}) \oplus M(2^m, \mathbb{C})$ when $n = 2m + 1$. Thus for each n there is a complex Hilbert space $\mathfrak{S}_n \cong \mathbb{C}^k$, $k = 2^{[n/2]}$, such that $\mathfrak{C}(\mathfrak{H})$ is isometrically-*isomorphic to the complex C^*-algebra $\mathfrak{L}(\mathfrak{S}_n)$ when $n = 2l$ or to one of $\mathfrak{L}(\mathfrak{S}_n)$, $\mathfrak{L}(\mathfrak{S}_n) \oplus \mathfrak{L}(\mathfrak{S}_n)$ when $n = 2l + 1$. This space $\mathfrak{S}_n$ is called the space of *complex spinors*. Every complex $\mathfrak{A}_n$-module, therefore, is isomorphic to an $M(k, \mathbb{C})$-module $\mathbb{C}^{k\times m}$ or to the direct sum of two such modules; this last case can occur only if n is odd.

To complete this section we return to the C^*-algebra representations of $\mathfrak{A}_n$ on an $\mathfrak{A}_n$-module $\mathfrak{H}$. There are always two such representations:

(4.16) (i) *the representation $\pi: \mathfrak{A}_n \to \mathfrak{A}(\mathfrak{H})$ derived from the universal property of $\mathfrak{A}_n$, and*

(4.16) (ii) *the representation $\sigma: \mathfrak{A}_n \to \mathfrak{A}(\mathfrak{H})$ defined by $\sigma(a) = \pi(a')$ where $a \to a'$ is the principal automorphism on $\mathfrak{A}_n$.*

Notice that

$$\pi(\bar{a}) = \pi(a)^-, \qquad \sigma(a^*) = \sigma(a)^- .$$

These will become crucial in the next section in realizing all the irreducible representations of the Euclidean rotation group.

5. Hardy Spaces $H^p(\Omega, \mathfrak{B})$

A large and very important class of Hardy spaces arise as subspaces of $H^p(\Omega, \mathfrak{H})$ by considering only those functions having range in a fixed subspace of an $\mathfrak{A}_n$-module $\mathfrak{H}$. Let $\mathfrak{V}$ be such a subspace and regard $\mathfrak{D}$ as a first-order system of differential operators

$$\mathfrak{D}: C^\infty(\Omega, \mathfrak{V}) \to C^\infty(\Omega, \mathfrak{H}). \tag{5.1}$$

Such an operator is said to be of *Dirac type*. It inherits many of the properties that $\mathfrak{D}$ has as an operator on $C^\infty(\Omega, \mathfrak{H})$. It obviously has the GCR-property, for instance; and the Cauchy integral representation (2.16) is still valid for solutions in $C^\infty(\Omega, \mathfrak{V})$ of $\mathfrak{D}F = 0$. On the other hand, the Cauchy integral of a function f in $C(\partial\Omega, \mathfrak{V})$ need not be $\mathfrak{V}$-valued unless each e_j maps $\mathfrak{V}$ into $\mathfrak{V}$, *i.e.,* $\mathfrak{V}$ is an $\mathfrak{A}(\mathfrak{H})$-submodule of $\mathfrak{H}$; so in general the Cauchy integral cannot be used directly to construct $\mathfrak{V}$-valued solutions of $\mathfrak{D}F = 0$. But most importantly, the first-order system (5.1) is usually an *over-determined* elliptic system in the sense that its symbol mapping is injective without being surjective; the term *injectively elliptic* used in Section 2 would seem to be an appropriate one to emphasize the distinction with standard usage of the term elliptic. The operator $\mathfrak{D}$ will still be elliptic on $C^\infty(\Omega, \mathfrak{V})$ in this standard sense, however, when $\mathfrak{V}$ is an $\mathfrak{A}(\mathfrak{H})$-submodule of $\mathfrak{H}$.

Definition 5.2. *If $\mathfrak{V}$ is a subspace of $\mathfrak{H}$, the Hardy space $H^p(\Omega, \mathfrak{V})$, $0 < p < \infty$, is the set of all solutions in $C^\infty(\Omega, \mathfrak{V})$ of $\mathfrak{D}F = 0$ for which*

$$\|F\|_p = \left(\int_{\partial\Omega} N(F)(x)^p \, dS\right)^{1/p}$$

is finite.

The most important subspaces $\mathfrak{V}$ arise from representations of the Euclidean rotation group. Let

$$\Lambda_n = \{a \cdots wz : a, \ldots, w, z \in \mathbb{R}^{n+1}\} \tag{5.3}$$

be the multiplicative semi-group in $\mathfrak{A}_n$ of all finite products of elements $a, \ldots, w, z$ in $\mathbb{R}^{n+1}$; this is the subset of $\mathfrak{A}_n$ on which the C^*-algebra norm $\|\cdot\|$ is multiplicative. Thus the unit sphere

$$\{k \in \Lambda_n : \|k\| = 1\} \tag{5.4}$$

in Λ_n is a multiplicative group, the so-called Spin group Spin $(n+1)$. On the other hand, to each a in $\mathbb{R}^{n+1}$, $a \neq 0$, corresponds a «twisted» similarity transformation

$$\sigma(a): v \to av(a')^{-1} \qquad (v \in \mathbb{R}^{n+1}) \tag{5.5}$$

of $\mathbb{R}^{n+1}$ which can be shown to be the product of two reflections. Consequently, $\sigma(a)$ is a proper rotation of $\mathbb{R}^{n+1}$, *i.e.*, $\sigma(a) \in SO(n+1)$ for each non-zero a in $\mathbb{R}^{n+1}$, and σ extends to a covering homomorphism

$$\sigma\colon \mathrm{Spin}\,(n+1) \to SO(n+1), \qquad \sigma(k)\colon v \to kv(k')^{-1} \tag{5.6}$$

from Spin $(n+1)$ onto $SO(n+1)$. The restriction to k in the unit sphere of Λ_n ensures that σ is a two-fold covering. In the special cases of $\mathfrak{A}_1 = \mathbb{C}$ and $\mathfrak{A}_2 = \mathbb{H}$, Spin $(n+1)$ reduces respectively to the unit circle $\{e^{i\theta/2}: 0 \leqslant \theta < 4\pi\}$ in $\mathbb{C}$ and the group $SU(2)$ regarded as the unit sphere in $\mathbb{H}$. It is well-known that these groups are two-fold coverings of $SO(2)$ and $SO(3)$. Hence, once more, we see how the Clifford theory contains the higher-dimensional analogues of classical low-dimensional results; this time it is the algebraic and geometric structure of the Euclidean rotation group, as expressed through (5.3), . . . , (5.6), that is contained in Clifford theory. Even more basic to representation theory is the fact that every finite-dimensional Spin $(n+1)$-module, and hence also any $SO(n+1)$-module, can be realized canonically as a subspace of an appropriate $\mathfrak{A}_n$-module $\mathfrak{H}$. These modules are then natural candidates for the $\mathfrak{V}$ in (5.2).

To be more precise, let $\mathfrak{K}_n$ be the Spinor module described in the previous section. By restricting to Spin $(n+1)$ the two spin representations of $\mathfrak{A}_n$ on $\mathfrak{K}_n$ we obtain representations S_+ and S_-,

$$S_+(k)\colon x \to kx, \qquad S_-(k)\colon x \to k'x \tag{5.7}$$

of Spin $(n+1)$ on $\mathfrak{K}_n$ (cf. (4.16)). Furthermore, these are irreducible representations of Spin $(n+1)$ because the spin representation of $\mathfrak{A}_n$ on $\mathfrak{K}_n$ is irreducible and the linear span of Spin $(n+1)$ is all of $\mathfrak{A}_n$. It will be convenient to denote these representations of Spin $(n+1)$ by $(\mathfrak{K}_n^+, S_+)$ and $(\mathfrak{K}_n^-, S_-)$ respectively where, of course, $\mathfrak{K}_n^\pm = \mathfrak{K}_n$ and S_+, S_- are defined by (5.7). When $(n+1)$ is even, S_+ and S_- are inequivalent representations, but they are equivalent when $(n+1)$ is odd. Taking any finite tensor product (over $\mathbb{R}$)

$$\mathfrak{H} = \mathfrak{K}_n^{\epsilon_1} \otimes \cdots \otimes \mathfrak{K}_n^{\epsilon_r} \qquad (\epsilon_j = \pm 1) \tag{5.8}$$

we thus obtain an $\mathfrak{A}_n$-module on which $S_{\epsilon_1} \otimes \cdots \otimes S_{\epsilon_r}$ defines a reducible representation of Spin $(n+1)$. From (5.6) and the Cartan-Weyl theory now follow the following fundamental result (cf. [14, Chap. III]).

Theorem 5.9. *Each irreducible representation of* Spin $(n+1)$ *can be realized by restricting* $S_{\epsilon_1} \otimes \cdots \otimes S_{\epsilon_r}$ *to a canonically specified subspace of* $\mathfrak{K}_n^{\epsilon_1} \otimes \cdots \otimes \mathfrak{K}_n^{\epsilon_r}$ *for an appropriate choice of* $(\epsilon_1, \ldots, \epsilon_r)$.

If $\tau = (m_1, m_2, \ldots)$ is the signature of an irreducible representation of Spin $(n+1)$, we shall denote by $\mathfrak{H}_r$ the $\mathfrak{A}_n$-module $\mathfrak{K}_n^{\epsilon_1} \otimes \cdots \otimes \mathfrak{K}_n^{\epsilon_r}$ and by $\mathfrak{V}_\tau$ the subspace of $\mathfrak{H}_\tau$ on which $S_{\epsilon_1} \otimes \cdots \otimes S_{\epsilon_r}$ realizes this representation. The detailed prescription of $(\epsilon_1, \ldots, \epsilon_r)$ and the subspace $\mathfrak{V}_\tau$ of $\mathfrak{H}_\tau$ are given in [14, Chap. III]. To realize any finite-dimensional Spin $(n+1)$-module $\mathfrak{X}$ in an $\mathfrak{A}_n$-module, let $\mathfrak{X} = \mathfrak{X}_{\tau_1} \oplus \mathfrak{X}_{\tau_2} \oplus \cdots$ be the decomposition of $\mathfrak{X}$ into its irreducible Spin $(n+1)$-components labelled by their signature. Then $\mathfrak{X}$ is equivalent as a Spin $(n+1)$-module to the subspace $\mathfrak{V}_{\tau_1} \oplus \mathfrak{V}_{\tau_2} \oplus \cdots$ of $\mathfrak{H}_{\tau_1} \oplus \mathfrak{H}_{\tau_2} \oplus \cdots$. The fundamental representations of Spin $(n+1)$, for example, all arise from the representations $(S_+, \mathfrak{K}_+)$, $(S_-, \mathfrak{K}_-)$ and from restrictions of $S_{\epsilon_1} \otimes S_{\epsilon_2}$ to subspaces of $\mathfrak{K}_{\epsilon_1} \otimes \mathfrak{K}_{\epsilon_2}$, $\epsilon_j = \pm 1$. Again details are given more fully in [14, Chap. III].

6. $H^p(\Omega, \mathfrak{V})$-theory

In this section we shall describe how the fundamental properties of the classical Hardy spaces carry over to the Hardy spaces $H^p(\Omega, \mathfrak{V})$, $\Omega \subseteq \mathbb{R}^{n+1}$, associated with a Spin $(n+1)$-submodule $\mathfrak{V}$ of $\mathfrak{H}$. Some aspects of the theory have been examined in detail already, others less so, and some not at all. The complete picture has yet to be painted, though the broad outlines can now be sketched in. Much as in Section 1, the fundamental properties are divided into four broad categories: boundary values, first-order differential operators, boundary characterizations, and representation theory.

(*a*) Boundary theory. The L^p-boundary regularity was established for $p > p(\Omega)$ by using the solution of the scalar-valued Dirichlet problem for Ω. Since this corresponds in the classical case to treating an analytic function simply as a harmonic function, it means that the condition $\mathfrak{D}F = 0$ did not improve upon properties of harmonic functions; instead, it imposed algebraic restrictions on the boundary functions through the characterization of $H^p(\partial\Omega, \mathfrak{H})$ as the subspace of functions in $L^p(\partial\Omega, \mathfrak{H})$ left fixed by the principal-value Cauchy integral operator $\mathfrak{K}$. Clearly this is a major omission and would be a major weakness of the theory if extra regularity did not occur. The stumbling block to progress is knowing how to relate existence of boundary values on $\partial\Omega$ with the algebraic structure of $\mathfrak{D}$ and the geometry of $\partial\Omega$. The only technique currently known for overcoming this problem establishes first the subharmonicity properties that solutions of $\mathfrak{D}F = 0$ have. For instance, a general result of Calderón ensures that to each $\mathfrak{A}_n$-module $\mathfrak{H}$ there corresponds a value of p_0, $0 < p_0 < 1$, so that $\Delta|F|^p \geqslant 0$ holds on Ω for every $p > p_0$ and every solution of $\mathfrak{D}F = 0$ in $C^\infty(\Omega, \mathfrak{H})$ ([33, p. 233]). In fact one can be more precise (cf. [14, Chap. IV]).

Theorem 6.1. *The subharmonicity condition $\Delta|F|^p \geqslant 0$ holds on Ω for every $p > (n-1)/n$ and every solution of $\mathfrak{D}F = 0$ in $C^\infty(\Omega, \mathfrak{H})$ whatever the choice of $\mathfrak{A}_n$-module and open set Ω.*

In the case $n = 1$ this result reduces to the fact that $\Delta|F|^p \geqslant 0$ holds for every $p > 0$ when F is analytic. Now, if F is merely a harmonic function in $C^\infty(\Omega, \mathfrak{H})$, then the best one could say is that $\Delta|F|^p \geqslant 0$ holds with $p \geqslant 1$. Thus, just as in the classical case, the extra condition $\mathfrak{D}F = 0$ on a harmonic function decreases the lower bound on the allowed range of p. But it is natural to ask if this lower bound can be decreased still further by considering only those solutions of $\mathfrak{D}F = 0$ taking values in some fixed subspace $\mathfrak{B}$ of $\mathfrak{H}$. This idea goes back to an important paper of Stein-Weiss ([32]), and is one of the principal reason for introducing the Hardy spaces $H^p(\Omega, \mathfrak{B})$.

Definition 6.2. *Let*

$$\mathfrak{D}: C^\infty(\Omega, \mathfrak{B}_\tau) \to C^\infty(\Omega, \mathfrak{H}_\tau)$$

be the operator of Dirac type associated with the irreducible representation of Spin $(n + 1)$ *having signature* τ. *Then its critical index* p_τ *is the smallest value of* p_0 *so that* $\Delta|F|^p \geqslant 0$ *holds in* Ω *for every* $p > p_0$ *and every solution of* $\mathfrak{D}F = 0$ *in* $C^\infty(\Omega, \mathfrak{B}_\tau)$.

Obviously $0 < p_\tau \leqslant (n-1)/n$, but the value of p_τ depends only on the way $\mathfrak{B}_\tau$ «lies» in $\mathfrak{H}_\tau$; in particular, it is independent of Ω ([14, Chap. IV]). By identifying particular examples of these operators of Dirac type with the operators ∂_τ in [9] (cf. also part (*b*) of this section), and hence with the operators introduced by Stein-Weiss, estimates for some p_τ can be deduced from the calculations in [32]. It would be very useful to have estimates for other p_τ (cf. [14, Chap. IV]).

The point of the condition $\Delta|F|^p \geqslant 0$ is that it can be used to establish L^p-boundary regularity, at least when Ω is an upper half-space, in complete analogy to the first half of (1.2) (i) (cf. [33, Chaps. II, IV]).

Theorem 6.3. *Let*

$$\Omega_+ = \{x + jy: x \in \mathbb{R}^n, y > 0\}$$

be an upper half-space in $\mathbb{R}^{n+1}_+$. *Then each* F *in* $H^p(\Omega_+, \mathfrak{B}_\tau)$, $p > p_\tau$, *has non-tangential boundary values* $F^+(x)$ *on* $\partial\Omega$ *for almost all x; furthermore*

$$\lim_{y \to 0^+} \int_{\mathbb{R}^n} |F(x + jy) - F^+(x)|^p \, dx.$$

With simple adaptations, the Stein-Weiss proof can be carried over to $H^p(\Omega, \mathfrak{B}_\tau)$ when Ω is the unit ball in $\mathbb{R}^{n+1}$, and with greater subtlety it may be possible to carry it over to C^1-domains. The possibility of extending it to any Lipschitz domain is a particularly interesting one because the proof depends on being able to solve the Dirichlet problem in order to construct least harmonic majorants of subharmonic functions. Formally, we shall denote by

$$H^p(\partial\Omega, \mathfrak{V}_\tau) = \{F|_{\partial\Omega} : F \in H^p(\Omega, \mathfrak{V}_\tau)\} \qquad (p > p_\tau)$$

the space of boundary values, restricting Ω if necessary to ensure that they exist. For particular Ω and $\mathfrak{V}_\tau$, known atomic and molecular characterizations from real H^p-theory should provide corresponding characterizations of $H^p(\partial\Omega, \mathfrak{V}_\tau)$ (cf. [8], [11]) but little is known at present.

***(b)* First-order differential operators.** Much is known about the algebraic structure of $(\mathfrak{D}, \bar{\mathfrak{D}})$-operators and Dirac type operators, as well as about their position within the class of all first-order differential operators. For instance, the $(\mathfrak{D}, \bar{\mathfrak{D}})$-operators are as easily derived from a standard Dirac operator as the Cauchy-Riemann operators were. In the formalism of Section 2, let $\mathfrak{H}$ be a finite-dimensional Hilbert space on which $e_1, \ldots, e_n$ are skew-adjoint operators satisfying (2.2). Then

$$\gamma_0 = \begin{bmatrix} 0 & -e_0 \\ e_0 & 0 \end{bmatrix}, \qquad \gamma_j = \begin{bmatrix} 0 & e_j \\ e_j & 0 \end{bmatrix} \quad (1 \leqslant j \leqslant n) \tag{6.4}$$

are skew-adjoint operators on the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$ of (4.6), and

$$\gamma_j\gamma_k + \gamma_k\gamma_j = -2\delta_{jk}I \qquad (0 \leqslant j, k \leqslant n). \tag{6.4$'$}$$

Consequently, $\mathfrak{H} \oplus \mathfrak{H}$ is an $\mathfrak{A}_{n+1}$-module. The first-order system defined by

$$DF = \sum_{j=0}^{n} \gamma_j \frac{\partial F}{\partial x_j} \qquad (F \in C^\infty(\Omega, \mathfrak{H} \oplus \mathfrak{H})) \tag{6.5}$$

for any open set Ω in $\mathbb{R}^{n+1}$ is then a standard Dirac operator as the term was used in [15]: because of (6.4), D is an elliptic self-adjoint differential operator such that

$$D = \begin{bmatrix} 0 & -\bar{\mathfrak{D}} \\ \mathfrak{D} & 0 \end{bmatrix}, \qquad D^2 = -\Delta. \tag{6.6}$$

Thus

$$DF = \begin{bmatrix} 0 & -\bar{\mathfrak{D}} \\ \mathfrak{D} & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} -\bar{\mathfrak{D}}\Psi \\ \mathfrak{D}\Phi \end{bmatrix}$$

on any $\mathfrak{H} \oplus \mathfrak{H}$-valued function

$$F = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix},$$

so $(\mathfrak{D}, \bar{\mathfrak{D}})$ arise from restricting D to $\mathfrak{H}$-valued functions.

An alternative way of describing the Dirac type operators

$$\mathfrak{D}: C^\infty(\Omega, \mathfrak{V}) \to C^\infty(\Omega, \mathfrak{H}), \qquad \mathfrak{D}F = \sum_{j=0}^{n} e_j \frac{\partial F}{\partial x_j}, \tag{6.7}$$

helps bring out their algebraic structure more clearly. Since this second approach essentially defines a differential operator through its symbol, it is also a convenient one to use on more general manifolds. Let $\mathfrak{A}$, $\mathfrak{B}$ be finite dimensional real Hilbert spaces, and let $\mathfrak{A} \otimes \mathbb{R}^{n+1}$ be the usual Hilbert space tensor product. If $\{e_j\}_{j=0}^n$ is the standard basis for $\mathbb{R}^{n+1}$ this tensor product consists of the elements

$$\xi = \sum_{j=0}^{n} \lambda_j (a_j \otimes e_j) \qquad (\lambda_j \in \mathbb{R}, a_j \in \mathfrak{A}) \tag{6.8}$$

having Hilbert-Schmidt norm

$$\|\xi\| = \left(\sum_{j=0}^{n} \lambda_j^2 \|a_j\|_{\mathfrak{A}}^2 \right)^{1/2}. \tag{6.8$'$}$$

In particular, to each linear operator $A: \mathfrak{A} \otimes \mathbb{R}^{n+1} \to \mathfrak{B}$ there corresponds a family $A_0, \ldots, A_n$ of linear operators

$$A_j: \mathfrak{A} \to \mathfrak{B}, \qquad A_j(a) = A(a \otimes e_j) \qquad (0 \leqslant j \leqslant n).$$

Definition 6.9. *Let*

$$\mathfrak{d}_A F = (A \circ \nabla) F = \sum_{j=0}^{n} A_j \frac{\partial F}{\partial x_j}$$

be the differential operator obtained by composing the usual derivative

$$\nabla: C^\infty(\Omega, \mathfrak{A}) \to C^\infty(\Omega, \mathfrak{A} \otimes \mathbb{R}^{n+1}), \qquad \nabla F = \sum_{j=0}^{n} \frac{\partial F}{\partial x_j} \otimes e_j$$

of $\mathfrak{A}$-valued functions with a linear operator $A: \mathfrak{A} \otimes \mathbb{R}^{n+1} \to \mathfrak{B}$.

This defines for each A a linear first-order homogeneous constant-coefficient differential operator such that $\mathfrak{d}_A: C^\infty(\Omega, \mathfrak{A}) \to C^\infty(\Omega, \mathfrak{B})$; and every such operator can be put in this form for some choice of A. In (6.7), for instance,

$$A: \mathfrak{V} \otimes \mathbb{R}^{n+1} \to \mathfrak{H}, \qquad A: \sum_{j=0}^{n} v_j \otimes e_j \to \sum_{j=0}^{n} e_j v_j,$$

where $A_j: \mathfrak{V} \to \mathfrak{H}$, $1 \leqslant j \leqslant n$, is multiplication by the imaginary unit e_j, while $A_0: \mathfrak{V} \to \mathfrak{V}$ is the identity. In general, of course, the mapping

$$a \to A(a \otimes \lambda) = \sum_{j=0}^{n} \lambda_j (A_j a) \qquad (a \in \mathfrak{A}) \tag{6.10}$$

defined from $\mathfrak{A}$ to $\mathfrak{B}$ for each non-zero $\lambda = (\lambda_0, \ldots, \lambda_n)$ is just the symbol mapping of $\eth_A$. To understand the algebraic structure of $\eth_A$, therefore, we need to know how the properties of $A: \mathfrak{A} \otimes \mathbb{R}^{n+1} \to \mathfrak{B}$ determine those of $\eth_A: C^\infty(\Omega, \mathfrak{A}) \to C^\infty(\Omega, \mathfrak{B})$.

Let $\eth_B: C^\infty(\Omega, \mathfrak{A}) \to C^\infty(\Omega, \mathfrak{B}')$ be the differential operator determined by a linear operator $B: \mathfrak{A} \otimes \mathbb{R}^{n+1} \to \mathfrak{B}'$ having a possibly different range space $\mathfrak{B}'$ from that of $A: \mathfrak{A} \otimes \mathbb{R}^{n+1} \to \mathfrak{B}$. We shall say that $\eth_A$ is equivalent to $\eth_B$ if they have the same kernel, *i.e.*, $\eth_A F = 0$ if and only if $\eth_B F = 0$. Since the vanishing of $\eth_A F(x_0)$, $x_0 \in \Omega$, is a linear condition on the homogeneous first-order Taylor coefficients of F at x_0, it is easy to prove

Theorem 6.11. *The first-order differential operators $\eth_A$ and $\eth_B$* on $C^\infty(\Omega, \mathfrak{A})$ *are equivalent for every Ω in $\mathbb{R}^{n+1}$ if and only if A, B have the same kernel in $\mathfrak{A} \otimes \mathbb{R}^{n+1}$, i.e., $A\xi = 0$ if and only if $B\xi = 0$, $\xi \in \mathfrak{A} \otimes \mathbb{R}^{n+1}$.*

In other words, there is a $1 - 1$ correspondence between subspaces of $\mathfrak{A} \otimes \mathbb{R}^{n+1}$ and the classes of first-order differential operators determined by this notion of equivalence. In particular, an operator $\eth_A$ will be equivalent to the Dirac type operator (6.7) precisely when the kernel of A in $\mathfrak{V} \otimes \mathbb{R}^{n+1}$ satisfies

$$\ker A = \left\{ v \in \mathfrak{V} \otimes \mathbb{R}^{n+1} : v = \sum_{j=0}^{n} v_j \otimes e_j,\ \sum_{j=0}^{n} e_j v_j = 0 \right\}. \tag{6.12}$$

For the Spin $(n+1)$-modules $\mathfrak{V}_\tau$, group-theoretic methods were used in [10] to construct rotation-invariant operator $\eth_\tau$ whose symbol has property (6.12). Here again the idea goes back to the fundamental paper of Stein-Weiss ([32]). The crucial idea in the rotation-invariant case is to decompose the Spin $(n+1)$-module $\mathfrak{V}_\tau \otimes \mathbb{R}^{n+1}$ into its irreducible Spin $(n+1)$-submodules. The classical geometric differential operators are then the operators of Dirac type arising from the fundamental representations of Spin $(n+1)$, and the higher gradient operators introduced by Stein-Weiss are the Dirac type operators corresponding to representations on spherical harmonics. More details can be found in [10], [14].

A careful examination of the precise values of $(\mathbb{K}, k)$ in (4.11) explains the significance of the estimates of $\dim_{\mathbb{F}}(\mathfrak{H})$ in Theorem 2.13. For the results of Adams *et al.* ([1], [30]) show that if $\eth_A: C^\infty(\Omega, \mathfrak{A}) \to C^\infty(\Omega, \mathfrak{B})$ is any linear, elliptic, first-order system on an open set Ω in $\mathbb{R}^{n+1}$, $n = 8d + r$, $0 \leqslant r < 7$, then $\dim_{\mathbb{F}}(\mathfrak{A})$ satisfies the same estimates

$$\dim_{\mathbb{R}}(\mathfrak{A}) \geqslant (r+1)16^d, \qquad \dim_{\mathbb{C}}(\mathfrak{A}) \geqslant 2^{[r/2]}16^d,$$

as $\dim_{\mathbb{F}}(\mathfrak{H})$ did. Now the lower bounds on the dimension are attained for $\mathfrak{D}, \bar{\mathfrak{D}}$ on $\mathfrak{K}_n$- or S_n-valued functions. Hence in a very precise sense the Dirac $\mathfrak{D}$- and $\bar{\mathfrak{D}}$-operators are the «smallest» linear elliptic first-order systems of differential operators.

***(c)* Boundary value problems.** Using the known solutions of the various Riemann-Hilbert problems for analytic functions and the Dirichlet-Neumann problems in higher dimensions as a guide, we can formulate a series of boundary value problems for the Hardy spaces $H^p(\Omega, \mathfrak{H})$ and $H^p(\Omega, \mathfrak{V}_\tau)$. But apart from the special case when Ω is a half-space (or ball) in $\mathbb{R}^{n+1}$ where Fourier Transform or group-theoretic techniques can be exploited, few results have been established as yet, so the conjectural nature of what follows must be borne in mind. There are really two distinct difficulties:

(i) setting up the *algebraic* formalism of the boundary value problems for an arbitrary set Ω, especially *group-theoretic* aspects, and
(ii) handling the extra *analytic* difficulties created by the geometry of $\partial\Omega$, more particularly by its lack of smoothness.

To avoid problems of the second kind at this stage, we shall tacitly assume that $\partial\Omega$ is C^∞ when formulating a boundary value problem; for instance, the example of $\partial\Omega = \Sigma_n$, the unit sphere in $\mathbb{R}^{n+1}$, is a natural test case. Very intuitively, our basic philosophy will be that the boundary theory should reflect the *over-determinedness* of $(\mathfrak{D}, \bar{\mathfrak{D}})$. Given, say a Hardy space $H^p(\Omega, \mathfrak{H})$ or $H^p(\Omega, \mathfrak{V}_\tau)$, the problem is to exhibit a «smaller» Banach space $\mathfrak{X}$ of functions or distributions on $\partial\Omega$ isomorphic to this Hardy space, isomorphic meaning that there is a potential type operator $\mathcal{K}$ mapping the boundary space onto the Hardy space or a boundary type operator mapping the Hardy space onto the boundary space. A «smaller» Banach space would mean one where the range space of the functions or distributions in $\mathfrak{X}$ is a proper subspace of $\mathfrak{H}$ or $\mathfrak{V}_\tau$, or where they have to satisfy a boundary condition such as being fixed by the principal-value Cauchy integral operator or some such singular integral operator on $\partial\Omega$. These restrictions were present in the classical Riemann-Hilbert problem for analytic functions and in Theorem 3.9. To proceed in detail, however, a precise understanding of the meaning of over-determinedness is needed. Now $(\mathfrak{D}, \bar{\mathfrak{D}})$ and $(\bar{\partial}, \partial)$ are elliptic operators, and so are *determined* in the sense that the associated symbols are both surjective and injective. Nonetheless, the solutions of $\bar{\partial}F = 0$ are *analytically over-determined.* The splitting $\mathfrak{H} = \mathfrak{H}_0 \oplus j\mathfrak{H}_0$ was used in (3.11) to express the same analytic over-determinedness that $\mathfrak{D}$ is presumed to have as a differential operator on $C^\infty(\Omega, \mathfrak{H})$. Recall Re, Im: $\mathfrak{H} \to \mathfrak{H}_0$ were defined by

$$z = \operatorname{Re}(z) + j \operatorname{Im}(z) = x + jy \qquad (x, y \in \mathfrak{H}_0) \tag{6.13}$$

In an attempt to understand thoroughly the analytic over-determinedness of $\mathfrak{D}$, therefore, we introduce the following analogue of the Hardy-Riemann Hilbert problem for analytic functions (cf. (1.16)). Let

$$\Lambda(\mathfrak{H}) = \{a \cdots wz : a, \ldots, w, z \in \mathbb{R}^{n+1} \subseteq \mathfrak{H}\} \tag{6.14}$$

be the multiplicative semi-group in $\mathfrak{A}(\mathfrak{H})$ of all finite products of elements in $\mathbb{R}^{n+1}$ ($\subseteq \mathfrak{H}$). On this semi-group the C^*-algebra norm is multiplicative; in particular, if $A: \partial\Omega \to \Lambda(\mathfrak{H})$ is non-vanishing on $\partial\Omega$, then it is invertible.

(6.15) Hardy-Riemann-Hilbert Problem. *Let $A: \partial\Omega \to \Lambda(\mathfrak{H})$ be a sufficiently smooth function which is non-vanishing on $\partial\Omega$. Then, given f in $L^p(\partial\Omega, \mathfrak{H}_0)$, find F in $H^p(\partial\Omega, \mathfrak{H})$ such that* $\operatorname{Re}(\overline{A(z)}F(z)) = f(z)$ *on* $\partial\Omega$.

In view of the boundary regularity theory for H^p-spaces, this problem reduces to solving the equation $\mathfrak{I}_A(F) = f$ where

$$\mathfrak{I}_A : H^p(\partial\Omega, \mathfrak{H}) \to L^p(\partial\Omega, \mathfrak{H}_0), \qquad (\mathfrak{I}_A F)(z) = \operatorname{Re}(\overline{A(z)}F(z)), \tag{6.16}$$

just as in the classical case. The Dirichlet type problem in conjecture (3.11) is the special case $A = 1$. In the classical case one uses the Cauchy integral in an attempt to invert $\mathfrak{I}_A$, and presumably a similar attempt might solve (6.15). The availability of the Cauchy integral and all the related theory to solve boundary value problems such as (6.15) indicate the usefulness of a development of «analytic function theory» for higher dimensional Euclidean space. A topological characterization of the index of $\mathfrak{I}_A$, special case of the Index theorem for manifolds with boundary, would be particularly interesting also.

There are some natural geometric choices of A. Let $\eta(X)$ be the unit outward normal to $\partial\Omega$ at X. Then $X \to \eta(X)$ defines a function $\eta: \partial\Omega \to \Sigma_n$ ($\subseteq \Lambda(\mathfrak{H})$), the Gauss map, which is a natural candidate for A. For any such $A: \partial\Omega \to \Sigma_n$ the Hardy-Riemann-Hilbert is related to the classical Neumann or Oblique-derivative problem.

By restricting to the Dirac type operator

$$\mathfrak{D}: C^\infty(\Omega, \mathfrak{B}_\tau) \to C^\infty(\Omega, \mathfrak{H}_\tau), \tag{6.17}$$

the differential operator becomes algebraically over-determined, *i.e.*, injectively elliptic, as well as analytically over-determined, and so algebraic restrictions now enter. Special cases for the unit ball were studied in [24], [25], and [28], but the only known results valid for general domains rise in studies of Dirichlet-Neumann Problem (cf. [22]), particularly in the work of Fabes-Kenig and Verchota ([11], [36]). This last case corresponds to the standard representation of $SO(n+1)$ on $\mathbb{R}^{n+1}$, so that $\mathfrak{B}_\tau = \mathbb{R}^{n+1}$ and the Dirac type operator (6.17) is equivalent to the div-curl system

(6.18) $\qquad \operatorname{div} F = 0 \qquad \operatorname{curl} F = 0, \qquad (F \in C^\infty(\Omega, \mathbb{R}^{n+1}))$

on Ω ([33, p. 234]). Now at each boundary point, the unit outward normal $\eta(X)$ determines a geometric splitting

(6.19) $\qquad \mathbb{R}^{n+1} = T_X \oplus N_X, \qquad v = x + \eta(X)y \qquad (x \perp \eta(X), y \in \mathbb{R})$

of each $v \in \mathfrak{V}_\tau$ into tangential and normal components. Hence each $\mathfrak{V}_\tau$-valued function $F = F(X)$ on $\partial\Omega$ admits a decomposition $F(X) = \Phi(X) + \eta(X)\Psi(X)$ with $\Phi(X) \perp \eta(X)$ and $\Psi = \Psi(X)$ a scalar-valued function on $\partial\Omega$; technically, $F = \Phi + \eta\Psi$ with Φ, $\eta\Psi$ sections of the tangent and normal bundles on $\partial\Omega$ respectively. Solving the Neumann problem for Ω with L^p-data is equivalent to finding F in $H^p(\Omega, \mathfrak{V}_\tau)$ so that $F(X) = \Phi(X) + \eta(X)\Psi(X)$ where $\Psi = \Psi(X)$ is a specified real-valued function in $L^p(\partial\Omega)$. Save possibly for a vanishing moment condition, the choice of Ψ is arbitrary. But any Φ arising as the tangential component of $F|_{\partial\Omega}$ is far from arbitrary (cf. [22], [36]).

To proceed in general, let $\mathfrak{H}_\tau = \mathfrak{H}_0 \oplus j\mathfrak{H}_0$ be a *fixed* splitting of $\mathfrak{H}_\tau$ and set

(6.20) $\qquad \mathfrak{T}_\tau = \{\operatorname{Re}(v) : v \in \mathfrak{V}_\tau\}, \qquad \mathfrak{N}_\tau = \{\operatorname{Im}(v) : v \in \mathfrak{V}_\tau\};$

we shall think of these as the respective *tangential* and *normal* components of $\mathfrak{V}_\tau$ analogous to the splitting of $\mathbb{R}^{n+1}$ in (6.19). But instead of splitting at every boundary point, we can compensate by introducing a function $A : \partial\Omega \to \Lambda(\mathfrak{H})$; for as we saw earlier, the function

$$f(z) = \operatorname{Re}(\overline{A(z)}F(z)) \qquad (z \in \partial\Omega)$$

would give the «normal component» of F at $z \in \partial\Omega$ for any appropriate choice of A, and a different choice of A would also give the «tangential component». Thus, let $A : \partial\Omega \to \Lambda(\mathfrak{H})$ be a sufficiently smooth non-vanishing function on $\partial\Omega$.

(6.21) Problem. *Determine the pseudo-differential operator $\mathcal{R}_\tau$ on $L^p(\partial\Omega, \mathfrak{N}_\tau)$ so that the equations*

$$\mathcal{R}_\tau f = 0, \qquad \operatorname{Re}(\overline{A(z)}F(z)) = f(z) \qquad (z \in \partial\Omega)$$

can be solved for F in $H^p(\Omega, \mathfrak{V}_\tau)$ and f in $L^p(\partial\Omega, \mathfrak{N}_\tau)$.

(*d*) Representation theory. Just as the hyperbolic plane H_2 can be realized as the upper half-plane or unit disk in $\mathbb{C}$, hence in the Clifford algebra $\mathfrak{A}_1$, so $(n+1)$-dimensional real hyperbolic space H_{n+1} can be realized as an upper half-space or unit ball in $\mathbb{R}^{n+1}$, hence in the Clifford algebra $\mathfrak{A}_n$. What is more, the algebra structure that becomes available because of this embedding makes it possible to carry over virtually intact from $SL(2, \mathbb{R})$ to $\operatorname{Spin}_0(n+1, 1)$ the algebraic, geometric, and analytic results associated with the group of isometries of H_{n+1}. The idea can be traced back to Vahlen and Maas, but it has been brought to the fore more recently by Ahlfors (cf. [3] for instance) and, independently, by Takahashi ([34]). Its significance as far as Hardy theory is concerned, arises for two reasons: when $\mathrm{Spin}_0(n+1,1)$ is the identity component of the two-fold covering of $SO_0(n+1,1)$, then

(i) $\mathrm{Spin}_0(n+1,1)$ *acts on* $\mathbb{R}^{n+1}$ *by fractional linear transformations each of which is a composition of translation, rotation, dilation and inversion*;
(ii) *there is an induced action of* $\mathrm{Spin}_0(n+1,1)$ *by unitary operators both on* $H^2(\Omega, \mathfrak{H})$, $\Omega = \mathbb{R}^{n+1}_+$, *and on the boundary space* $L^2(\mathbb{R}^n, \mathfrak{H})$, *that commutes with the Cauchy integral and the principal-value Cauchy integral operators.*

The representation-theoretic implications of these properties will be discussed elsewhere. Here we shall do little more than indicate how closely the results for $n > 1$ parallel those in the classical case $n = 1$ discussed in Section 1.

Fix an imaginary unit j in $\mathbb{R}^n$ ($\subseteq \mathfrak{A}_n$) and let $\mathfrak{A}_{n-1}$ be the Clifford subalgebra of $\mathfrak{A}_n$ generated by $e_1, \ldots, e_{n-1}$ where $\{e_1, \ldots, e_{n-1}, j\}$ is an orthonormal basis for $\mathbb{R}^n$. Then

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} \oplus j\mathfrak{A}_{n-1} \tag{6.22}$$

and

(6.23) (i)
$$\Omega_+ = \left\{ z : z = x + yj, x = \sum_{k=0}^{n-1} x_k e_k \in \mathbb{R}^n, y > 0 \right\}$$

is an upper half-space in $\mathbb{R}^{n+1}$ having the same boundary

(6.23) (ii)
$$\partial\Omega = \left\{ x : x = \sum_{k=0}^{n-1} x_k e_k \in \mathbb{R}^n \right\}$$

as the lower half-space

(6.23) (iii)
$$\Omega_- = \left\{ z : z = x + yj, x = \sum_{k=0}^{n-1} x_k e_k \in \mathbb{R}^n, y < 0 \right\}.$$

This imaginary unit j is the unit inward normal to Ω_+ and unit outward normal to Ω_-. Thus j plays exactly the same role for $\mathbb{R}^{n+1}$ as $i = \sqrt{-1}$ does classically for $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ when $n = 1$. In this last case, $\mathrm{Spin}_0(n+1,1)$ can be identified with $SL(2, \mathbb{R})$ acting on the compactified plane $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \to z \cdot g = \frac{b + zd}{a + cz} \qquad (z \in \mathbb{C})$$

as we saw in Section 1. In the general case, $\mathrm{Spin}_0(n+1,1)$ is realized as a multiplicative subgroup G of

$$M(2, \mathfrak{A}_{n-1}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathfrak{A}_{n-1} \right\}$$

acting on $\mathbb{R}^{n+1} \cup \{\infty\}$ by fractional linear transformations

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \to (a + zc)^{-1}(b + zd) \qquad (z \in \mathbb{R}^{n+1}), \tag{6.24}$$

so that the orbits of G in $\mathbb{R}^{n+1} \cup \{\infty\}$ consist of Ω_+, Ω_-, and their compactified mutual boundary $\partial\Omega \cup \{\infty\}$. Although there is an intrinsic definition of G (cf. [14, Chap. V]), it is more illuminating to proceed indirectly, constructing subgroups M, A and V of G with respect to which (6.24) reduces to rotations, dilations and translations respectively. Let Λ_{n-1} be the multiplicative semi-group generated in $\mathfrak{A}_{n-1}$ by $\partial\Omega$; when $n = 1$, Λ_{n-1} is just $\mathbb{R}$, of course. Set

$$M = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha' \end{bmatrix} : \alpha \in \Lambda_{n-1}, \|\alpha\| = 1 \right\}, \qquad A = \left\{ \begin{bmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{bmatrix} : \lambda > 0 \right\},$$

and

$$V = \left\{ \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} : v \in \partial\Omega_+ \right\}.$$

For these groups, (6.24) becomes

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha' \end{bmatrix} : z \to \alpha^{-1} z \alpha' = \sigma(a^{-1})v \qquad (z \in \mathbb{R}^{n+1}),$$

which is known to be a *rotation* of $\mathbb{R}^{n+1}$ fixing the $\boldsymbol{j}$-direction (cf. (5.5)), while

$$\begin{bmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{bmatrix} : z \to \lambda z \qquad (z \in \mathbb{R}^{n+1}),$$

defines *dilation* by λ, and

$$\begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} : z \to z + v \qquad (z \in \mathbb{R}^{n+1}),$$

defines *translation* by $v \in \partial\Omega$ in any hyperplane parallel to $\partial\Omega$. On the other hand, the fractional linear transformation determinaded by the Weyl group element

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is the inversion

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : z \to -1/z$$

which interchanges 0 and ∞. Finally, set

$$N = \omega V \omega^{-1} = \left\{ \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} : u \in \partial\Omega \right\}.$$

It is easy to check that MAN is a multiplicative group in $M(2, \mathfrak{A}_{n-1})$. Assuming that $\mathrm{Spin}_0(n+1, 1)$ has been realized as an intrisically defined group G in $M(2, \mathfrak{A}_{n-1})$, one shows

Theorem 6.25. (Bruhat decomposition.) *There is a uniquely determined coset-space decomposition*

$$G = MANV \cup MAN\omega$$

of G with respect to its subgroup MAN such that the left coset-space $MAN\backslash G$ is diffeomorphic to $\partial\Omega \cup \{\infty\}$, identifying V with $\partial\Omega$ and ω with $\{\infty\}$; furthermore, when G acts on $\mathbb{R}^{n+1} \cup \{\infty\}$ by fractional linear transformations

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \to z \cdot g = (a + zc)^{-1}(b + zd),$$

the orbits of G are Ω_+, Ω_-, and their compactified mutual boundary $\partial\Omega \cup \{\infty\}$.

With this coset-space decomposition for G it is not hard to show that $z \to z \cdot g$ preserves the usual hyperbolic metric on Ω_+ as a realization of H_{n+1}; and, conversely, every isometry in the connected group of isometries of H_{n+1} arises as such a fractional linear transformation. On the other hand, on the compactified boundary $\partial\Omega \cup \{\infty\}$, the fractional linear transformations $x \to x \cdot g$ are precisely the (sense-preserving) conformal transformations of $\mathbb{R}^n \cup \{\infty\}$ (cf. [2]). The geometric and algebraic properties of G are thus completely encoded in the transformations $z \to z \cdot g$ of $\mathbb{R}^{n+1} \cup \{\infty\}$. Analytic properties show up in Hardy H^2-spaces for Ω_+.

Let $\mathfrak{H}$ be an $\mathfrak{A}_n$-module, and let $\mathfrak{A}(\mathfrak{H})$ be the associated C^*-algebra realization of $\mathfrak{A}_n$. Then

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \to z \cdot g = (a + zc)^{-1}(b + zd) \qquad (z \in \mathbb{R}^{n+1}) \tag{6.26}$$

is well-defined solely within $\mathfrak{A}(\mathfrak{H})$, regarding z and the matriz entries of g as elements of $\mathfrak{A}(\mathfrak{H})$; in particular, any term such as $(a + zc)$ is an operator on $\mathfrak{H}$. Now let $H^2(\Omega_+, \mathfrak{H})$ be the Hardy space on Ω_+ whose norm is defined geometrically by the natural analogue

$$\|F\|_{H^2} = \sup_{y>0} \left(\int_{\partial\Omega} |F(x+jy)|^2 \, dx \right)^{1/2}$$

of (1.1).

Theorem 6.27. *For each g in G,*

$$(\pi_s(g)F)(z) = \frac{(a+zc)'}{|a+zc|^{n+1}} F(z \cdot g) \qquad (z \in \Omega_+)$$

is a unitary operator on $H^2(\Omega_+, \mathfrak{H})$, *and* $g \to \pi_s(g)$ *is a unitary representation of G on* $H^2(\Omega_+, \mathfrak{H})$.

As stated, Theorem 6.27 really hides the fundamental property of the Dirac $\mathfrak{D}$-operator that is needed. For $\pi(g)F$ must be a solution of $\mathfrak{D}(\pi(g)F) = 0$ if $\pi(g)F$ is to be in $H^2(\Omega_+, \mathfrak{H})$. This, however, follows immediately from the fact that

$$\mathfrak{D}(\pi_s(g)F(z \cdot g)) = \frac{1}{|a+zc|^2} \pi_s(g')(\mathfrak{D}F)(z \cdot g)$$

holds for any F in $C^\infty(\Omega_+, \mathfrak{H})$, an equivariance property which ensures that $\mathfrak{D}$ is invariant not only under translation and rotations, the transformations preserving the Euclidean distance, but also under transformations preserving hyperbolic distance on Ω_+, just as the Cauchy-Riemann operators were in the classical case.

To relate this action of G on $H^2(\Omega_+, \mathfrak{H})$ to one on spaces of boundary values, we prove

Theorem 6.28. *For each g in G,*

$$(\sigma_s(g)f)(x) = \frac{(a+xc)'}{|a+xc|^{n+1}} f(x \cdot g) \qquad (x \in \partial\Omega)$$

is a unitary operator on $L^2(\partial\Omega, \mathfrak{H})$ *and* $g \to \sigma_s(g)$ *is a unitary representation of G such that*

$$\text{(i) } \mathcal{H}(\sigma_s(g)f) = \sigma_s(g)(\mathcal{H}f) \qquad \text{(ii) } \mathcal{C}(\sigma_s(g)f) = \pi_s(g)(\mathcal{C}f)$$

where $\mathcal{H}$ *is the principal-value Cauchy integral operator on* $\partial\Omega$ *and* $\mathcal{C}$ *is the Cauchy integral operator.*

Details of these and other results will appear in [17].

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* Research partially supported by NSF grant DMS-8502352.
† Research partially supported by NSF grant DMS-8603234.