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Almost-Everywhere Convergence of Fourier Integrals for Functions in Sobolev Spaces, and an L^2-Localisation Principle

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In memory of José Luis Rubio de Francia

1. Introduction

For a function f defined on $\mathbb{R}^n$, we associate its Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx,$$

and the partial sum operators

$$(1.1) \qquad S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi$$

whenever these make sense. One of the most classical problems in Fourier analysis is that of pointwise convergence of $S_R f(x)$ to $f(x)$ as $R \to \infty$. That is, for which spaces of functions f do we have $S_R f(x) \to f(x)$ at a given point, almost-everywhere, everywhere, or uniformly? One may consider also the corresponding lacunary problem, *i.e.* when does $S_{2^k} f(x) \to f(x)$ as $k \to \infty$, $k \in \mathbb{N}$?

The natural spaces to consider for uniform convergence are (spaces of) continuous functions, and the natural spaces to consider for almost-everywhere convergence are the L^p-spaces, $1 \leqslant p < \infty$. Even for $n = 1$ and $p = 2$, the almost-everywhere question has only relatively recently been answered by Carleson [4], and historically progress has been made by considering easier problems. One such easier problem arises when we replace the rough Fourier multiplier $\chi_{|\xi| \leq R}$ in (1.1) by a smoother cousin, for example $(1 - |\xi|^2/R^2)_+^\alpha$ with $\alpha > 0$. This family of multipliers, the so-called Bochner-Riesz multipliers, gives rise to a summability method which has been much studied in the last 20 years-see for example [1, 2, 5, 7, 8, 9, 10, 12]. As an alternative way to make things simpler, one may, instead of insisting upon results, say, for L^p spaces, look for results for (large) subspaces of L^p which consist of functions having some smoothness. Of course, the classical tests for pointwise convergence of Fourier series are stated in terms of smoothness hypotheses on the functions, but this point of view seems not yet to have been treated from a modern perspective. The aim of this article is to lay down a framework within which this question may be considered. This turns out to be possible due to the progress made in the theory of Bochner-Riesz multipliers, although there is not apparently any formal link between smoothing the mutliplier in (1.1) and multiplying $\hat{f}(\xi)$ by a power of $|\xi|$.

To place our results in context, let us recall some classical results on the unit circle $\boldsymbol{T}$, where S_N denotes the N^{th} partial sum operator.

(i) For some $f \in L^1(\boldsymbol{T})$, $S_{2^k} f \to f$ a.e. fails (Kolmogorov's example).
(ii) If, however, $f \in L^1(\boldsymbol{T})$ and f vanishes in some open set Ω, then $S_N f \to 0$ on Ω (Riemann's localisation principle).
(iii) If $f \in L^p(\boldsymbol{T})$, $1 < p \leqslant \infty$, then $S_{2^k} f \to f$ a.e. (Littlewood-Paley).
(iv) For some continuous f, $S_N f \to f$ uniformly on $\boldsymbol{T}$ fails (du Bois Reymond).
(v) If, however, $\omega(t)$ denotes the L^∞ modulus of continuity of f, and

$$\int_0^1 \omega(t) \frac{dt}{t} < \infty,$$

then $S_N f \to f$ uniformly on $\boldsymbol{T}$ (Dini's test).
(vi) If the Fourier coefficients of f satisfy $\sum |a_k|^2 \log |k| < \infty$, then $S_N f \to f$ a.e. (Kolmogorov-Seliverstov-Plessner).

(For all these results, see Zygmund [19].) Only in the 1960's was the situation on $\boldsymbol{T}$ finalised, the result *par excellence* being due to Carleson [4] and Hunt [13]:

(vii) If $f \in L^p(\boldsymbol{T})$, $1 < p \leqslant \infty$, then $S_N f \to f$ a.e.

In this paper we concentrate on the higher-dimensional case, and so from now on we work in $\mathbb{R}^n$, with $n \geqslant 2$.

Let $1 \leqslant p < \infty$, and let us consider the almost-everywhere convergence problem for $L^p(\mathbb{R}^n)$. If almost-everywhere convergence is to hold for $f \in L^p$, then $S_1 f$ should at the very least be defined as a tempered distribution. Hence by duality, $S_1 \colon \mathcal{S} \to L^{p'}(\mathbb{R}^n)$, ($\mathcal{S}$ denoting the Schwartz class), and, by choosing $\hat{f} \equiv 1$ on $\{|\xi| \leqslant 1\}$ we see that unless

$$K_1(x) = \int_{|\xi| \leq 1} e^{2\pi i x \cdot \xi}\, d\xi$$

belongs to $L^{p'}(\mathbb{R}^n)$, we cannot make sense of the problem. But of course

$$K_1(x) = \frac{J_{n/2}(2\pi|x|)}{|x|^{n/2}} \sim \frac{e^{2\pi i|x|}}{|x|^{(n+1)/2}} + \text{lower order terms at } \infty,$$

(see [17]). Thus we see that it is reasonable to consider only the almost-everywhere convergence problem for $f \in L^p(\mathbb{R}^n)$, $p < 2n/(n-1)$. (We owe this observation to José L. Rubio de Francia.) On the other hand, if $p < 2$, any sort of almost-everywhere convergence of $S_R f(x)$ for $f \in L^p$ would imply that the corresponding maximal operator would be weak type p, due to the abstract principle of Calderón and Stein-see for example [11]. This would then contradict C. Fefferman's disc multiplier theorem [9] or one of its variants.

Of course it will not be an easy task to obtain a positive a.e. convergence result for $S_R f$, $f \in L^p$, $2 \leqslant p < 2n/(n-1)$, since such a result would logically imply Carleson's Theorem. However, we can obtain a form of the localisation principle in this range.

Theorem 1. *If $n \geqslant 2$, $f \in L^p(\mathbb{R}^n)$ with $2 \leqslant p < 2n/(n-1)$ and $f \equiv 0$ on an open set Ω, then $S_R f \to 0$ a.e. on Ω.*

An earlier variant of Theorem 1 where f was assumed to have compact support was obtainded by Sjölin, [14]. It may be worth mentioning that the analogue of the Littlewood-Paley theorem for $\mathbb{R}^n$ has been obtained by Rubio de Francia, Vega and the first author in [2]:

Theorem A. *If $n \geqslant 2$ and $f \in L^p(\mathbb{R}^n)$ with $2 \leqslant p < 2n/(n-1)$, then $S_{2^k} f \to f$ a.e.*

The proof of Theorem 1 uses many of the ideas from [2].

We now turn to see how matters may be improved if we add a little smoothness to our L^p-spaces. For $1 \leqslant p < \infty$ and $\alpha > 0$, we let $L^p_\alpha(\mathbb{R}^n)$ be the Sobolev space of Bessel potentials of order α, *i.e.* the closure of $\{f \in \mathcal{S}: [(1 + |\xi|^2)^{\alpha/2}\hat{f}(\xi)]^\vee \in L^p\}$ under the obvious norm. (When $\alpha \in N$ and $1 < p < \infty$ this coincides with the usual Sobolev space of functions whose derivatives of order up to α are in L^p. For this and all the elementary facts about Sobolev spaces, see [16, Chap. 5].) If we now consider the a.e. convergence problem for L^p_α, we observe that inserting a power of $(1 + |\xi|^2)^{1/2}$ in (1.1) does not affect the nature of the singularity of $\chi_{|\xi| \leq R}$; and so the argument above shows us that *no matter how large α is*, we should only consider the problem for $p < 2n/(n-1)$.

Theorem 2. *If $n \geqslant 2$, $f \in L^p_\alpha(\mathbb{R}^n)$ with $\alpha > 0$ and $2 \leqslant p < 2n/(n-1)$, then $S_R f \to f$ a.e.*

In fact the proof of Theorem 2 will show us that a stronger statement holds.

Theorem 3. *Let $n \geqslant 2$, $2 \leqslant p < 2n/(n-1)$ and $f \in L^p(\mathbb{R}^n)$. If for all $\phi \in C_c^\infty(\mathbb{R}^n)$ the function $g = \phi f$ satisfies*

$$\int |\hat{g}(\xi)|^2 (1 + \log^+ |\xi|)^2 \, d\xi < \infty,$$

then $S_R f \to f$ a.e.

Theorem 3 is a weak form of a Kolmogorov-Seliverstov-Plessner-type theorem, and is formally similar to the Rademacher-Menšov theorem (see [19]) for general orthogonal series. It would therefore be interesting to be able to replace the factor $(1 + \log^+ |\xi|)^2$ by $(1 + \log^+ |\xi|)$ (or something smaller!) in the statement of this theorem.

To conclude the discussion of the case $p \geqslant 2$, let us state a result which might be regarded as an analogue for $\mathbb{R}^n$ of Dini's test:

Theorem 4. *Let $n \geqslant 2$. If $f \in L^{2n/(n-1),1}_{(n-1)/2}(\mathbb{R}^n)$, then $S_R f \to f$ uniformly.*

(Here, $L^{p,1}_\alpha$ is the space of functions whose derivatives up to order α are in the Lorentz space $L^{p,1}$; $L^{2n/(n-1),1}_{(n-1)/2}(\mathbb{R}^n)$ embeds in $C_0(\mathbb{R}^n)$.)

We now consider what may happen when $1 \leqslant p < 2$. In this situation, any positive result for L^p_α, $p > 1$ implies a corresponding positive result for Bochner-Riesz means of order $\beta > \alpha$ on L^p. For example, a «lacunary» localisation principle for L^p_α would imply that $(1 - |\xi|^2)^\beta_+$ is a Fourier multiplier of L^p for each $\beta > \alpha$, and a «full» localisation principle would imply that

$$\left\| \sup_{1 \leq t \leq 2} |[(1 - |\xi|^2/t^2)^\beta_+ \hat{f}(\xi)]^\vee| \right\|_p \leqslant c \|f\|_p \tag{1.2}$$

for each $\beta > \alpha$. Now a necessary condition for $(1 - |\xi|^2)^\beta_+$ to be a Fourier multiplier of L^p is

$$\frac{1}{p} < \frac{n + 1 + 2\beta}{2n}$$

(take $\hat{f} \in \mathbb{S}$, $\hat{f} \equiv 1$ on $|\xi| \leqslant 1$) and this has been shown to be sufficient (for $p \leqslant 2$) only when

$$\beta > \frac{n-1}{2(n+1)}$$

(see [10], [18]) or when $n = 2$ and $\beta > 0$ (see [5], [12], [10], [7]). The state of affairs for the maximal Bochner-Riesz operator is even worse: (1.2) has only been shown to hold in the cases corresponding to interpolation with the «trivial» L^1 and L^2 estimates: that is, (1.2) is only known to hold for $1 < p \leqslant 2$ and $\alpha > 0$ when

$$\frac{1}{p} < \frac{\alpha}{n-1} + \frac{1}{2}.$$

For the a.e. convergence problem, we have

Theorem 5. *Let $n \geqslant 2$. If $f \in L^1_{(n-1)/2}(\mathbb{R}^n)$, then $S_R f \to f$ a.e.*

Notice that Theorem 5 is false when $n = 1$, by Kolmogorov's example. Moreover the corresponding statement for Bochner-Riesz means of order $(n-1)/2$ is false, since Stein has shown (see [17]) that localisation fails for L^1 in this case. However Theorem 5 is very easy to prove, and is closely related to Theorem 4. A immediate consequence of Theorems 2 and 5 is

Theorem 6. *Let $n \geqslant 2$. If $f \in L^p_\alpha(\mathbb{R}^n)$ with*

$$1 < p \leqslant 2, \ \alpha > 0 \quad \textit{and} \quad \frac{1}{p} < \frac{\alpha}{n-1} + \frac{1}{2},$$

then $S_R f \to f$ a.e.

Finally, for lacunary convergence, we can improve Theorem 6 in some special cases.

Theorem 7. *If $n = 2$ and $\alpha > 0$, or if $n \geqslant 3$ and $\alpha > \dfrac{n-1}{2(n+1)}$, then $S_{2^k} f \to f$ a.e. for $f \in L^p_\alpha$, $\dfrac{1}{2} \leqslant \dfrac{1}{p} < \dfrac{n+1+2\alpha}{2n}$.*

In view of the recent work of M. Christ on weak type estimates for Bochner-Riesz operators (see [6]) it would be interesting to establish Theorem 7 for $1/p = (n+1+2\alpha)/2n$. We hope to return to this matter in a forthcoming article.

The paper is organized as follows. Theorem 1 is proved in Section 2, Theorems 3 and 4 in Section 3, and Theorems 5 and 7 in Section 4. The proof of Theorem 7 requires some $L^p - L^q$ estimates for the operator S_1, and more generally one can consider $L^p - L^q$ estimates for Bochner-Riesz means of order $\alpha \leqslant 0$. These have previously been studied by Sogge, [15], and we make some improvements on his estimates in Section 5. (Although for $\alpha < 0$ these estimates have no relation with the rest of the paper, it seems convenient to include them here.)

We were privileged to discuss all stages of this work with José Luis Rubio de Francia. As always, he freely shared his ideas with us, and we feel this work owes much to his insight. The first author enjoyed the generous and gracious hospitality of José Luis and his family during several delightful visits to Madrid. We will miss him a great deal, and so it is with great sadness that we dedicate this work to his memory.

2. A Localisation Principle

We begin with our localisation principle.

Theorem 2.1. *Suppose* $2 \leqslant p < 2n/(n-1)$ *and suppose* $f \in L^p(\mathbb{R}^n)$. *Then, almost everywhere off* $\operatorname{supp} f$ *we have* $S_R f(x) \to 0$.

PROOF. By translation and dilation invariance, we may assume without loss of generality that f is supported in $\{|x| \geqslant 3\}$. We must show that for every open ball B contained in $\{|x| < 3\}$ we have $S_R f(x) \to 0$ a.e. in B. We shall do this for $B = \{x: |x| < 1\}$; for any other ball $\{|x| < r\}$ with $r < 3$ the proof is similar.

It suffices, as usual, to obtain an appropriate estimate for the maximal operator

$$S_* f(x) = \sup_{1 < R < \infty} |S_R f(x)|;$$

in fact, for the L^2 case we shall prove

$$\int_{|x| \leq 1} |S_* f(x)|^2 \, dx \leqslant c \int_{|x| \geq 3} |f|^2. \tag{2.1}$$

In order to make use of the support restrictions in (2.1) we split up both f and

$$K_R(x) = \int_{|\xi| \leq R} e^{2\pi i x \cdot \xi} \, d\xi = R^n K_1(Rx)$$

into dyadic pieces. Let ϕ be a smooth radial function with $\chi_{|x| \leq 1} \leqslant \phi \leqslant \chi_{|x| \leq 2}$. Then

$$1 = \phi(x) + \sum_{j=1}^{\infty} [\phi(x/2^j) - \phi(x/2^{j-1})] = \phi(x) + \sum_{j=1}^{\infty} \psi(x/2^j) = \phi(x) + \sum_{j=1}^{\infty} \psi_j(x),$$

where $\psi(x) = \phi(x) - \phi(2x)$. Let $K_R^j = K_R \psi_j$, and $f_j = f\psi_j$. Then

$$S_R f = K_R * f = \sum_{j,k\geq 1} K_R^j * f_k + (K_R \phi) * f,$$

since $\phi f \equiv 0$. Now if $|j-k| \geqslant 3$, then $\operatorname{supp}(K_R^j * f_k) \cap \{|x| \leqslant 1\} = \varnothing$, and $\operatorname{supp}(K_R \phi * f) \cap \{|x| \leqslant 1\} = \varnothing$ since $\operatorname{supp} f \subseteq \{|x| \geqslant 3\}$. Thus to obtain (2.1) it suffices to show

$$\int \sup_{R>1} \Big| \sum_{j\geq 1} K_R^j * f_{j+s} \Big|^2 dx \leqslant c \int |f|^2 \tag{2.2}$$

for $|s| < 3$; we shall deal only with the diagonal case $s = 0$, the other cases being exactly similar. The inequality (2.2) is now the special case $0 < \alpha = \gamma < 1/2$ of the following theorem:

Theorem 2.2. *Let $\{g_j\}_{j=1}^{\infty}$ be an arbitrary sequence of test functions. Suppose $0 \leqslant \alpha < 1/2$ and $\gamma > 0$. Then*

$$\int_{\mathbb{R}^n} \sup_{R>1} \Big| \sum_{j\geq 1} K_R^j * g_j \Big|^2 \frac{dx}{|x|^{2\alpha}} \leqslant c_{\alpha,\gamma} \sum_{j\geq 1} 2^{2j\gamma} \int_{\mathbb{R}^n} |g_j|^2 \frac{dx}{|x|^{2\alpha}}.$$

Remark 1. If we set $g_j = f_j = f\psi_j$ with $\operatorname{supp} f \subseteq \{|x| \geqslant 3\}$, Theorem 2.2 shows that

$$\int_{|x|\leq 1} |S_* f(x)|^2 dx \leqslant c_{\alpha,\gamma} \int_{|x|\geq 3} |f|^2 \frac{dx}{|x|^{\eta}}$$

with $\eta = 2(\alpha - \gamma) < 1$ but arbitrary. In particular since $L^p(|x| \geqslant 3, dx) \subseteq L^2(dx/|x|^{\eta})$ for $n(1-2/p) < \eta$, we see that Theorem 2.1 follows from Theorem 2.2. It is easy to see that Theorem 2.2 with parameters $\alpha = \gamma = 0$ is false, and thus even to obtain the L^2 case of Theorem 2.1 one is forced to consider weighted L^2 inequalities.

Remark 2. The results of this paper were announced in [3], where the statement of Theorem 2 is slightly incorrect. The correct version is the present Theorem 2.2.

The proof of Theorem 2.2 requires several lemmas. Let H_α be the (homogeneous) Sobolev space $\left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \int |\hat{f}(\xi)|^2 |\xi|^{2\alpha} d\xi < \infty \right\}$.

Lemma 2.3. *If $0 \leqslant \alpha < 1/2$, $t > 0$ and $\delta < 2t$, then*

$$\int_{||\xi|-t|\leq\delta} |h(\xi)|^2 d\xi \leqslant c_\alpha \delta^{2\alpha} \|h\|_{H_\alpha}^2$$

with c_α independent of t and δ.

PROOF. If $\delta < t/2$, the inequality is merely a rescaled version of Lemma 3 in [2]; if $t/2 \leqslant \delta < 2t$,

$$\int_{||\xi| - t| \leq \delta} |h|^2 \leqslant \int_{|\xi| \leq 3t} |h|^2 \leqslant c_\alpha t^{2\alpha} \|h\|_{H_\alpha}^2,$$

as may be seen by chopping up the ball $\{|\xi| \leqslant 3t\}$ into dyadic annuli.

Let $m_t^j(\xi) = (K_t^j)^\wedge(\xi)$ for $j \geqslant 1$ and $t > 0$.

Lemma 2.4. *Let $\beta \in \mathbb{N}$, $\lambda \geqslant 0$, and $t > 0$. Then*

$$\left|\left(\frac{d}{dt}\right)^\beta m_t^j(\xi)\right| \leqslant c_{\lambda,\beta} 2^{j\beta}/(1 + 2^j||\xi| - t|)^\lambda$$

with $c_{\lambda,\beta}$ independent of t and j.

PROOF. By a change of variables it suffices to consider the case $t = 1$ and we shall consider only the case $\beta = 0$, the cases $\beta \geqslant 1$ being handled similarly.

First suppose $|\xi| < 1$. Then, since $\psi(0) = 0$,

$$\begin{aligned}
|m_1^j(\xi)| &= |\chi_B * (\hat{\psi})_{2^j}(\xi)| \\
&= \left| \int (\hat{\psi})_{2^j}(\eta)[\chi_B(\xi - \eta) - 1]\, d\eta \right| \\
&= \left| \int_{|\xi - \eta| \geq 1} (\hat{\psi})_{2^j}(\eta)\, d\eta \right| \\
&\leqslant \int_{|\eta| \leq ||\xi| - 1|} |(\hat{\psi})_{2^j}(\eta)|\, d\eta \\
&\leqslant \int_{|u| \geq 2^j/||\xi| - 1|} |\hat{\psi}(u)|\, du \\
&\leqslant c_\lambda/(1 + 2^j||\xi| - 1|)^\lambda \quad \text{for any} \quad \lambda \geqslant 0.
\end{aligned}$$

Similarly, if $|\xi| > 1$,

$$\begin{aligned}
|m_1^j(\xi)| &= \left| \int_{|\xi - \eta| \leq 1} (\hat{\psi})_{2^j}(\eta)\, d\eta \right| \\
&\leqslant \int_{|\eta| \geq ||\xi| - 1|} |(\hat{\psi})_{2^j}(\eta)|\, d\eta \\
&\leqslant c_\lambda/(1 + 2^j||\xi| - 1|)^\lambda \quad \text{for any} \quad \lambda \geqslant 0.
\end{aligned}$$

Fix a smooth function $\gamma: \mathbb{R} \to \mathbb{R}$ such that $\gamma(t) \equiv 0$ for $t \leqslant 1/2$ and $\gamma(t) \equiv 1$ for $t \geqslant 1$. Let $L_t^j = \gamma(t) K_t^j$ for $t \in \mathbb{R}$, and let D^β be the fractional differentiation operator of order β on $\mathbb{R}$, that is $(D^\beta h)^\wedge(v) = |v|^\beta \hat{h}(v)$.

Lemma 2.5. *For $\beta \geqslant 0$ and $0 \leqslant \alpha < 1/2$ we have*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} |D^\beta L_t^j * f|^2\, dt \frac{dx}{|x|^{2\alpha}} \leqslant c 2^{-j} 2^{2j\beta} \int |f|^2 \frac{dx}{|x|^{2\alpha}} \tag{2.4}$$

(*where D^β acts with respect to the t-variable*).

PROOF. The Lemma says the operator $f \mapsto L_t^j * f$ is bounded into certain (Hilbert space valued) L^2-Sobolev spaces on $\mathbb{R}$. If we can establish this boundedness (with the correct constants) for integral values of the derivative parameter β, then we shall have the lemma in general by interpolation. By Plancherel's theorem on $\mathbb{R}$ (in the Hilbert space valued setting), it suffices to replace D^β in (2.4) by the usual derivative $(d/dt)^\beta$ when $\beta \in \mathbb{N}$. Thus it suffices to prove

$$\int_{\mathbb{R}^n} \int_0^\infty \left| \left(\frac{d}{dt}\right)^\beta K_t^j * f \right|^2 dt \frac{dx}{|x|^{2\alpha}} \leqslant c 2^{-j} 2^{2j\beta} \int |f|^2 \frac{dx}{|x|^{2\alpha}} \tag{2.5}$$

for $0 \leqslant \alpha < 1/2$, $\beta \in \mathbb{N}$, and we shall do so only in the case $\beta = 0$, the other cases being exactly the same.

We first notice that the operator $f \to K_t^j * f$ is a local one (in x-space); that is, if f is supported in a cube Q of side 2^j, then $K_t^j * f$ is supported in a larger cube Q^* of comparable volume. Thus it suffices to prove (2.5) for f supported in such a cube, and we consider two cases, that when $\operatorname{dist}(Q^*, 0) \geqslant 2^j$ and when $\operatorname{dist}(Q^*, 0) < 2^j$.

When $\operatorname{dist}(Q^*, 0) \geqslant 2^j$, the factors $|x|^{-2\alpha}$ in the integrals in (2.5) are essentially constants, so we may ignore them. Hence it is enough to show that

$$\int_{\mathbb{R}^n} \int_0^\infty |K_t^j * f|^2 \, dt \, dx \leqslant c 2^{-j} \int |f|^2 \, dx,$$

or, by Plancherel's theorem, that

$$\sup_{\xi \in \mathbb{R}^n} \int_0^\infty |m_t^j(\xi)|^2 \, dt \leqslant c 2^{-j}.$$

But this is an easy consequence of Lemma 2.4.

When $\operatorname{dist}(Q^*, 0) \leqslant 2^j$, it suffices to show

$$\int_{\mathbb{R}^n} \int_0^\infty |K_t^j * f|^2 \, dt \frac{dx}{|x|^{2\alpha}} \leqslant c 2^{-2j\alpha} 2^{-j} \int_Q |f|^2 \, dx \tag{2.6}$$

since

$$\int_Q |f|^2 \, dx \leqslant c 2^{2j\alpha} \int_Q |f|^2 \frac{dx}{|x|^{2\alpha}}.$$

By duality, (2.6) is equivalent to

$$\int_{\mathbb{R}^n} \left| \int_0^\infty K_t^j * f_t \, dt \right|^2 dx \leqslant c 2^{-2j\alpha} 2^{-j} \iint_0^\infty |f_t(x)|^2 \, dt \, |x|^{2\alpha} \, dx,$$

or, by Plancherel's theorem, to

$$\int_{\mathbb{R}^n} \left| \int_0^\infty m_t^j(\xi) g_t(\xi) \, dt \right|^2 d\xi \leqslant c 2^{-2j\alpha} 2^{-j} \int_0^\infty \| g_t \|_{H_\alpha}^2 \, dt. \tag{2.7}$$

Now, by Lemma 2.4 we may write, for any $\lambda \geqslant 0$,

$$|m_t^j(\xi)| \leqslant c_\lambda \sum_{k=0}^{\infty} 2^{-k\lambda} \chi_{\{\xi: 2^j ||\xi| - t| \le 2^k\}},$$

and so we consider

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \int_0^\infty g_t(\xi) \chi_{\{t: ||\xi| - t| \le 2^{k-j}\}} \, dt \right|^2 d\xi \\
&\qquad \leqslant \int_0^\infty \int_{\mathbb{R}^n} |g_t(\xi)|^2 \chi_{\{\xi: ||\xi| - t| \le 2^{k-j}\}} \, d\xi \, dt \sup_\xi |\{t > 0: ||\xi| - t| \leqslant 2^{k-j}\}| \\
&\qquad \leqslant c 2^{k-j} \int_0^\infty \int_{\{\xi: ||\xi| - t| \le 2^{k-j}\}} |g_t(\xi)|^2 \, d\xi \, dt \\
&\qquad \leqslant c 2^{k-j} 2^{2(k-j)\alpha} \int_0^\infty \| g_t \|_{H_\alpha}^2 \, dt
\end{aligned}$$

if $0 \leqslant \alpha < 1/2$, by Lemma 2.3. Thus, if $\lambda > \alpha + 1/2$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} \left| \int_0^\infty m_t^j(\xi) g_t(\xi) \, dt \right|^2 d\xi \right)^{1/2} &\leqslant c_\lambda \sum_{k=0}^{\infty} 2^{-k\lambda} 2^{(k-j)/2} 2^{(k-j)\alpha} \left(\int_0^\infty \| g_t \|_{H_\alpha}^2 \, dt \right)^{1/2} \\
&\leqslant c 2^{-j\alpha} 2^{-j/2} \left(\int_0^\infty \| g_t \|_{H_\alpha}^2 \, dt \right)^{1/2},
\end{aligned}$$

which is (2.7).

PROOF OF THEOREM 2.2. By the Sobolev embedding theorem on $\mathbb{R}$,

$$\begin{aligned}
\sup_{R \ge 1} \left| \sum_{j \ge 1} K_R^j * g_j \right|^2 &\leqslant \sup_{t \in \mathbb{R}} \left| \sum_{j \ge 1} L_t^j * g_j \right|^2 \\
&\leqslant c_\beta \int_{\mathbb{R}} \left| \sum_{j \ge 1} D^\beta L_t^j * g_j \right|^2 dt
\end{aligned}$$

provided $\beta > 1/2$, and so

$$\begin{aligned}
\left(\int \sup_{R \ge 1} \left| \sum_{j \ge 1} K_R^j * g_j \right|^2 \frac{dx}{|x|^{2\alpha}} \right)^{1/2} &\leqslant \left(c_\beta \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left| \sum_{j \ge 1} D^\beta L_t^j * g_j \right|^2 dt \frac{dx}{|x|^{2\alpha}} \right)^{1/2} \\
&\leqslant c_\beta \sum_{j \ge 1} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}} |D^\beta L_t^j * g_j|^2 \, dt \frac{dx}{|x|^{2\alpha}} \right)^{1/2} \\
&\leqslant c_\beta \sum_{j \ge 1} 2^{-j/2} 2^{j\beta} \left(\int |g_j|^2 \frac{dx}{|x|^{2\alpha}} \right)^{1/2}
\end{aligned}$$

(by Lemma 2.5 if $0 \leqslant \alpha < 1/2$)

$$\leqslant c_\beta \left(\sum_{j \ge 1} 2^{-j} 2^{2j\beta} 2^{-2j\gamma} \right)^{1/2} \left(\sum_{j \ge 1} 2^{2j\gamma} \int |g_j|^2 \frac{dx}{|x|^{2\alpha}} \right)^{1/2}$$

$$\leqslant c_{\alpha,\gamma}\left(\sum_{j\geq 1} 2^{2j\gamma}\int_{\mathbb{R}^n}|g_j|^2\frac{dx}{|x|^{2\alpha}}\right)^{1/2}$$

provided $\gamma > \beta - 1/2$ for some $\beta > 1/2$; that is, provided $\gamma > 0$. This concludes the proof of Theorems 2.2 and 2.1.

3. L^p_α Estimates, $p \geqslant 2$

In this section we first use the result of Section 2 to prove Theorem 3 of the introduction, and then establish an analogue of Dini's test for $\mathbb{R}^n$.

Theorem 3.1. *Let $n \geqslant 2$, $2 \leqslant p < 2n/(n-1)$ and $f \in L^p(\mathbb{R}^n)$. If for each $\phi \in C_c^\infty$ the function $g = \phi f$ satisfies*

$$\int |\hat{g}(\xi)|^2(1 + \log^+|\xi|)^2\, d\xi < \infty,$$

then $S_R f \to f$ a.e.

PROOF. We shall show $S_R f \to f$ a.e. on $\{|x| < 1\}$; by the localisation principle (Theorem 2.1) it suffices to assume that f is supported in $\{|x| \leqslant 2\}$, and satisfies

$$\int |\hat{f}(\xi)|^2(1 + \log^+|\xi|)^2\, d\xi < \infty.$$

By the Hardy-Littlewood maximal theorem we may assume that we are dealing with modified partial sum operators $\tilde{S}_R$ with multipliers $\tilde{m}_R = \chi_{|\xi|\leq R} - \tau(|\xi|/R)$ with $\tau \in C^\infty$ and $\chi_{|\xi|\leq 1/2} \leqslant \tau \leqslant \chi_{|\xi|\leq 3/4}$.

We shall again consider a maximal operator

$$\tilde{S}_* f(x) = \sup_{R>1} |\tilde{S}_R f(x)| \quad \text{for} \quad |x| \leqslant 1,$$

and for the moment fix an $N \in \mathbb{N}$ and consider $\tilde{S}_R f(x)$ with $R \sim 2^N$. Now

$$\begin{aligned}\tilde{S}_R f(x) &= \tilde{K}_R * f(x)\\ &= (\tilde{K}_R \phi) * f(x) \quad \text{if} \quad |x| \leqslant 1,\end{aligned}$$

where $\phi \in \mathcal{S}$, and $\phi \equiv 1$ on $\{|x| \leqslant 30\}$. Thus

$$\begin{aligned}\tilde{S}_R f(x) &= [\tilde{K}_1 \phi(\cdot/2^N)]_R * f(x)\\ &= \left[\sum_{j=1}^{N} \tilde{K}_1(\cdot)\psi(\cdot/2^j) + \tilde{K}_1(\cdot)\phi(\cdot)\right]_R * f(x)\end{aligned}$$

where $\psi(x) = \phi(x) - \phi(2x)$. Hence,

$$\sup_{R \sim 2^N} |\tilde{S}_R f(x)| \leqslant CMf(x) + \sup_{R \sim 2^N} \left| \sum_{j=1}^{N} (\tilde{K}^j)_R * f(x) \right|$$

$$\leqslant CMf(x) + \sum_{j=1}^{N} \sup_{R \sim 2^N} |(\tilde{K}^j)_R * f(x)|,$$

(where M denotes the Hardy-Littlewood maximal function and

$$\tilde{K}^j(\bullet) = \tilde{K}_1(\bullet)\psi(\bullet/2^j)).$$

Lemma 3.2. *There is a c independent of j so that*

$$\int \sup_{1<t<2} |(\tilde{K}^j)_t * h|^2 \, dx \leqslant c \int |h|^2.$$

Accepting the truth of this lemma for the moment, we see that

$$\left(\int_{|x| \leq 1} \sup_{R \sim 2^N} |\tilde{S}_R f|^2 \right)^{1/2} \leqslant C \|f\|_2 + \sum_{j=1}^{N} \left(\int_{|x| \leq 1} \sup_{t \sim 2^N} |(\tilde{K}^j)_t * f|^2 \, dx \right)^{1/2}$$

$$\leqslant C(1+N)\|f\|_2 .$$

Observe that this estimate is true, by the localisation Principle, for any function in L^2. This fact will be used later. We are indebted to Pascal Auscher, M.ª Jesús Carro and Guido Weiss, from Washington University, for the observation leading to this estimate. Our original arguments for the proof of the theorem were incorrect.

Now, if we set $\hat{g}(\xi) = (1 + \log^+ |\xi|)\hat{f}(\xi)$, $g \in L^2$, and we have

$$[(\tilde{K})_R * f]^\wedge(\xi) = (\tilde{K})^\wedge(\xi/R)\hat{f}(\xi)$$

$$= (\tilde{K}_1)^\wedge(\xi/R)\hat{g}(\xi)(1 + \log^+ |\xi|)^{-1}$$

$$= \frac{1}{N} (\tilde{K}_1)^\wedge(\xi/R)\widehat{P_N g}(\xi)$$

where

$$\widehat{P_N g}(\xi) = N(1 + \log^+ |\xi|)^{-1}\sigma(\xi/2^N)\hat{g}(\xi),$$

with σ a suitable smooth bump function of compact support away from 0 around the unit annulus. We can do this because the modified Fourier multiplier $(\tilde{K}_1)^\wedge$ has support in an annulus not containing the origin. Hence,

$$\int_{|x|\le 1} \sup_{R\ge 1} |\tilde{S}_R f|^2 \leqslant \sum_{N=1}^{\infty} \frac{1}{N^2} \int_{|x|\le 1} \sup_{R\sim 2^N} |\tilde{S}_R p_N g|^2$$

$$\leqslant C \sum_{N=1}^{\infty} \| p_N g\|_2^2$$

$$\leqslant C\|g\|_2^2,$$

which finishes the proof of the theorem.

Proof of Lemma 3.2. This lemma is entirely standard. We have

$$\sup_{1<t<2} |(\tilde{K}^j)_t * h(x)|^2 \leqslant 2\int_1^2 |(\tilde{K}^j)_t * h(x)| \left|\left(\frac{d}{dt}\right)(\tilde{K}^j)_t * h(x)\right| dt + O.K.$$

$$\leqslant 2\left(\int_1^2 |(\tilde{K}^j)_t * h|^2 \frac{dt}{t}\right)^{1/2} \left(\int_1^2 \left| t\left(\frac{d}{dt}\right)(\tilde{K}^j)_t * h\right|^2 \frac{dt}{t}\right)^{1/2}$$

and so

$$\int \sup_{1<t<2} |(\tilde{K}^j)_t * h(x)|^2\, dx$$

$$\leqslant c\left(\int_1^2 \int |(\tilde{K}^j)_t * h|^2\, dx \frac{dt}{t}\right)^{1/2} \left(\int_1^2 \int \left| t\frac{d}{dt}(\tilde{K}^j)_t * h\right|^2 dx \frac{dt}{t}\right)^{1/2}$$

$$\leqslant c\|h\|_2^2$$

provided we can show that

$$\left(\sup_{\xi} \int_1^2 |(\tilde{K}^j)^\wedge(\xi/t)|^2 \frac{dt}{t}\right)\left(\sup_{\xi} \int_1^2 \left| t\frac{d}{dt}(\tilde{K}^j)^\wedge(\xi/t)\right|^2 \frac{dt}{t}\right) \leqslant c \tag{3.1}$$

independently of j. By a calculation similar to that of Lemma 2.4 we have

$$\left|\left(\frac{d}{dt}\right)^{\beta}_{t=1} (\tilde{K}^j)^\wedge(\xi/t)\right| \leqslant c_{\lambda,\beta} 2^{j\beta}/(1+2^j||\xi|-1|)^{\lambda},$$

so the first term in the product in (3.1) is dominated by 2^{-j}, while the second is dominated by 2^j. Thus (3.1) holds and so does Lemma 3.2.

If we assume significantly more smoothness of our function f, we can get uniform convergence of $S_R f$ to f.

Proposition 3.3. *Let $n \geqslant 2$. If $f \in L^{2n/(n-1),1}_{(n-1)/2}(\mathbb{R}^n)$, then $S_R f$ converges to f uniformly on $\mathbb{R}^n$.*

PROOF. Suppose $g \in L^2_\alpha(\mathbb{R}^n)$ with $\alpha > n/2$; then $S_R g \to g$ uniformly since

$$\begin{aligned}|[S_R g - S_T g](x)| &= \left| \int_{R \le |\xi| \le T} \hat{g}(\xi) e^{2\pi i x \cdot \xi} \, d\xi \right| \\ &\leqslant \left(\int_{R \le |\xi| \le T} |\hat{g}(\xi)|^2 (1 + |\xi|^2)^\alpha \, d\xi \right)^{1/2} \left(\int \frac{d\xi}{(1 + |\xi|^2)^\alpha} \right)^{1/2} \\ &\to 0 \quad \text{as} \quad R, T \to \infty.\end{aligned}$$

Note that L^2_α is a dense subclass of $L^{2n/(n-1),1}_{(n-1)/2}$ for $\alpha > n/2$, so if $f \in L^{2n/(n-1),1}_{(n-1)/2}$ and $\epsilon > 0$ we may pick a $g \in L^2_\alpha$, $\alpha > n/2$, with $\|f - g\|_{L^{2n/(n-1),1}_{(n-1)/2}} < \epsilon$. Hence

$$\begin{aligned}\|S_R f - f\|_\infty &\leqslant \|S_R(f - g)\|_\infty + \|S_R g - g\|_\infty + \|g - f\|_\infty \\ &\leqslant \|S_R\|_{L^{2n/(n-1),1}_{(n-1)/2}, L^\infty} \|f - g\|_{L^{2n/(n-1),1}_{(n-1)/2}} + c\epsilon\end{aligned}$$

(if R is sufficiently large, since $\|g - f\|_\infty \leqslant c\|g - f\|_{L^{2n/(n-1),1}_{(n-1)/2}}$). It thus suffices to obtain the norm estimate

$$\|S_R f\|_\infty \leqslant c\|f\|_{L^{2n/(n-1),1}_{(n-1)/2}}. \tag{3.2}$$

We may again assume we are working with the modified operators $\tilde{S}_R$ (since $\|(S_R - \tilde{S}_R)f\|_\infty \leqslant c\|f\|_\infty$) and let $h \in L^{2n/(n-1),1}$ be such that

$$\hat{f}(\xi)(1 + |\xi|^2)^{(n-1)/4} = \hat{h}.$$

Then

$$\begin{aligned}(\tilde{S}_R f)^\wedge(\xi) &= \frac{\tilde{m}(\xi/R)}{(1 + |\xi|^2)^{(n-1)/4}} \hat{h}(\xi) \\ &= R^{-(n-1)/2} \tilde{m}(\xi/R) \widehat{\rho^R}(\xi) \hat{h}(\xi)\end{aligned}$$

with $\widehat{\rho^R}$ a smooth bump function adapted to the annulus $|\xi| \sim R$. So

$$\begin{aligned}\|\tilde{S}_R f\|_\infty &= \|R^{-(n-1)/2} \tilde{K}_R * \rho^R * h\|_\infty \\ &\leqslant R^{-(n-1)/2} \|\tilde{K}_R\|_{L^{2n/(n+1),\infty}} \|\rho^R * h\|_{L^{2n/(n-1),1}} \\ &\leqslant CR^{-(n-1)/2} \|\tilde{K}_R\|_{L^{2n/(n+1),\infty}} \|f\|_{L^{2n/(n-1),1}_{(n-1)/2}}.\end{aligned}$$

Now as is well-known,

$$K_1(x) \sim \frac{e^{2\pi i |x|}}{|x|^{(n+1)/2}} + \text{lower order terms},$$

so $\|\tilde{K}_R\|_{L^{2n/(n+1),\infty}} = O(R^{(n-1)/2})$.

4. L^p_α Estimates, $p < 2$

As noted in the introduction, the only general theorem available when $p < 2$ is Theorem 6, which is an immediate consequence of the results of the previous section and this next result (which is itself closely allied to Proposition 3.3).

Proposition 4.1. *Let $f \in L^1_{(n-1)/2}(\mathbb{R}^n)$ for $n \geqslant 2$. Then $S_R f \to f$ a.e.*

Proof. It is enough to see that the modified maximal operator $\tilde{S}_* f$ is finite almost everywhere for $f \in L^1_{(n-1)/2}$. As in the previous section,

$$\tilde{S}_R f(x) = R^{-(n-1)/2} \tilde{K}_R * \rho^R * h,$$

with $\|h\|_1 = \|f\|_{L^1_{(n-1)/2}}$, and, since $L^{2n/(n+1),\infty}$ is a Banach space when $n \geqslant 2$, we see that

$$\|\tilde{S}_* f\|_{L^{2n/(n+1),\infty}} \leqslant c \left\| \sup_{R>1} R^{-(n-1)/2} |\tilde{K}_R * \rho^R| \right\|_{L^{2n/(n+1),\infty}} \|f\|_{L^1_{(n-1)/2}}.$$

A moment's thought will convince the reader that $|\tilde{K}_R * \rho^R(x)| \leqslant c|\tilde{K}_R(x)|$ (since ρ^R is an averaging operator on a scale $1/R$), and, from the form of K_1, we see that indeed

$$\sup_{R>1} R^{-(n-1)/2} |\tilde{K}_R| \in L^{2n/(n+1),\infty}(\mathbb{R}^n).$$

If we instead consider the lacunary problem, we can say a little more.

Proposition 4.2. *If $n = 2$ and $\alpha > 0$, or if $n \geqslant 3$ and $\alpha > (n-1)/2(n+1)$, then $S_{2^k} f \to f$ a.e. for $f \in L^p_\alpha(\mathbb{R}^n)$, if $1/2 \leqslant 1/p < (n+1+2\alpha)/2n$.*

Proof. As in previous proofs, it suffices to see that

$$\left\| \sup_{k \geq 1} |\tilde{S}_{2^k} f| \right\|_q \leqslant c \|f\|_{L^p_\alpha},$$

for some $1/q < (n+1)/2n$ since

$$\left\| \sup_k |(S_{2^k} - \tilde{S}_{2^k}) f| \right\|_q \leqslant \|Mf\|_q \leqslant c\|f\|_q \leqslant c\|f\|_{L^p_\alpha}$$

if $1/q \geqslant 1/p - \alpha/n$. To do this, it suffices to show that

$$\left\| \sup_{k \geq 1} |2^{-k\alpha} \tilde{K}_{2^k} * \rho^{2^k} * h| \right\|_q \leqslant c\|h\|_p$$

(since $(\tilde{K})^\wedge$ has compact support in an annulus). Now the case $\alpha = 0$ of

Theorem 5.2 of Section 5 below states that

$$\|K_1 * f\|_q \leqslant c\|f\|_p$$

for $1/q < (n+1)/2n$ if $(n+3)/2(n+1) < 1/p \leqslant 1$, or if $n = 2$ and $3/4 < 1/p \leqslant 1$. By Young's inequality, the same holds for $\tilde{K}_1$ in place of K_1, and this now scales to

$$\|\tilde{K}_t * f\|_q \leqslant c t^{n(1/p - 1/q)} \|f\|_p.$$

Finally,

$$\left\| \sup_{k \geq 1} |2^{-k\alpha} \tilde{K}_{2^k} * \rho^{2^k} * h| \right\|_q \leqslant \sum_{k \geq 1} 2^{-k\alpha} 2^{kn(1/p - 1/q)} \|\rho^{2^k} * h\|_p$$
$$\leqslant c\|h\|_p$$

if $1/q < (n+1)/2n$, $\alpha > n(1/p - 1/q)$ and $(n+3)/2(n+1) < 1/p \leqslant 1$ ($3/4 < 1/p \leqslant 1$ when $n = 2$). Thus we obtain a.e. convergence of $S_{2^k} f$ for $1/2 \leqslant 1/p < (n+1+2\alpha)/2n$ and $\alpha > (n-1)/2(n+1)$ or, when $n = 2$, $\alpha > 0$.

5. Bochner-Riesz Means of Negative Order

For $-(n+1)/2 \leqslant \alpha \leqslant 0$, we define for $f \in \mathcal{S}$

$$(T^\alpha f)^\wedge(\xi) = \Gamma(\alpha+1)^{-1}(1 - |\xi|^2)_+^\alpha \hat{f}(\xi),$$

so that T^α is convolution with $\pi^{-\alpha}|x|^{-n/2-\alpha} J_{n/2+\alpha}(2\pi|x|)$. In this section we examine for which $1 \leqslant p \leqslant q \leqslant \infty$, α and n we have

$$\|T^\alpha f\|_{L^q(\mathbb{R}^n)} \leqslant c\|f\|_{L^p(\mathbb{R}^n)}. \tag{5.1}$$

The case $\alpha = 0$ of this problem occurred in the preceding section, and the problem has also been previously studied by Börjeson and Sogge. By taking $\hat{f} \in \mathcal{S}$, $\hat{f} \equiv 1$ on $|\xi| \leqslant 1$, we see that necessary conditions for (5.1) to hold are $1/q < (n+1+2\alpha)/2n$, and, by duality, $1/p > (n-1-2\alpha)/2n$ (except when $\alpha = -(n+1)/2$, when we trivially have $\|T^{-(n+1)/2} f\|_\infty \leqslant c\|f\|_1$). Also, we can not expect, in general, (5.1) to hold unless $1/q \leqslant 1/p + 2\alpha/(n+1)$ since the optimal $L^p - L^{p'}$ theorem for T^{-1} is for $(n+3)/2(n+1) \leqslant 1/p \leqslant 1$-anything stronger would give a false $L^p - L^2$ restriction theorem for the Fourier transform.

For $-(n+1)/2 \leqslant \alpha \leqslant 0$, let

$$A_\alpha = \{(1/p, 1/q) \in [0,1]^2 : 1/q < (n+1+2\alpha)/2n,$$
$$1/p > (n-1-2\alpha)/2n,$$
$$1/q < 1/p + 2\alpha/(n+1)\}.$$

Sogge's results may be summarised as follows

Theorem [15].

(*a*) *If* $n = 2$ *and* $(1/p, 1/q) \in A_\alpha$, *then*

$$\|T^\alpha f\|_{L^q(\mathbb{R}^2)} \leqslant c\|f\|_{L^p(\mathbb{R}^2)}.$$

(*b*) *If* $n \geqslant 3$, $(1/p, 1/q) \in A_\alpha$ *and if* $\alpha \leqslant -1/2$, *then*

$$\|T^\alpha f\|_q \leqslant c\|f\|_p.$$

Thus a natural conjecture is that in all dimensions $(1/p, 1/q) \in A_\alpha$ is sufficient to imply (5.1) if $-(n+1)/2 < \alpha \leqslant 0$. It would also seem reasonable to conjecture that T^α is weak type $(p, 2n/(n+1+2\alpha))$ for $[(n+1+2\alpha)/2n] - [2\alpha/(n+1)] < 1/p \leqslant 1$ (true for $p = 1$!) and perhaps even weak type $(([n+1+2\alpha)/2n] - [2\alpha/(n+1)])^{-1}, 2n/(n+1+2\alpha))$ if $\alpha > 0$. This last conjecture would imply that T^α is strong type (p, q) on the line $1/q = 1/p + 2\alpha/(n+1)$, which is true for $q = p'$ (but false by Fefferman's theorem [8] if $\alpha = 0$ and $p \neq 2$).

Part (*b*) of Sogge's theorem is implied by the following lemma

Lemma [15]. *Let* $K^j(\cdot) = K_1(\cdot)\psi(\cdot/2^j)$ (*with* ψ *smooth of compact support in an annulus*). *Then*

$$\|K^j * f\|_q \leqslant c2^{-j/2}2^{jn(1/q-1/2)}\|f\|_p$$

if $1/q = (n+1)(n-1)^{-1}(1-1/p)$ *and* $0 \leqslant 1/q \leqslant 1/2$.

Here we notice that a stronger statement is true.

Lemma 5.1. *If* $(n+1)(n-1)^{-1}(1-1/p) \leqslant 1/q \leqslant 1/p$ *and* $(n+3)/2(n+1) \leqslant 1/p \leqslant 1$, *then*

$$\|K^j * f\|_q \leqslant c2^{-j/2}2^{jn(1/q-1/2)}\|f\|_p.$$

This lemma together with the observation that $T^\alpha: L^p \to L^{p'}$ if $1/p = 1/2 - \alpha/(n+1)$ immediately gives us:

Theorem 5.2. *If* $n \geqslant 3$, $(1/p, 1/q) \in A_\alpha$, *and* $(1/p, 1/q)$ *is strictly below both the lines joining*

$(2^{-1} - \alpha(n+1)^{-1}, 2^{-1} + \alpha(n+1)^{-1})$ *to* $((n+3)/2(n+1), (n+1+2\alpha)/2n)$
and $(2^{-1} - \alpha(n+1)^{-1}, 2^{-1} + \alpha(n+1)^{-1})$ *to*
$((n-1-2\alpha)/2n, (n-1)/2(n+1))$, *then* $\|T^\alpha f\|_q \leqslant c\|f\|_p$.

Curiously, Lemma 5.1 is a consequence of the (trivial) observation that $\|K^j * f\|_{p'} \leqslant c2^{-j(n+1)(1/p-1/2)}\|f\|_p$ although it is not obtainable by interpolation from this estimate. This came to light in conversations with Luis Vega.

Proof of Lemma 5.1. Only the case $q \leqslant 2$ is of interest, since we can interpolate this with the trivial $L^1 - L^\infty$ estimate to get the case $q > 2$. The case $q = 2$ follows immediately from the Stein-Tomas restriction theorem (and hence from $\|K^j * f\|_{p'} \leqslant c2^{-j(n+1)(1/p-1/2)}\|f\|_p$)-see Stein's argument in [10]. If now $p \leqslant q < 2$, we may assume that f is supported in a ball of radius 2^j; now

$$\|K^j * f\|_q \leqslant c\|K^j * f\|_2 2^{jn(1/q-1/2)}$$

by Hölder's inequality, and the case $q = 2$ gives the desired result.

For Section 4, we need part (*a*) of Sogge's theorem and Theorem 5.2, both in the case $\alpha = 0$,

$$1/q = (n+1)/2n - \epsilon,\ 1 \leqslant p \leqslant 4/3\ (n = 2),\ 1 \leqslant p \leqslant 2(n+1)/(n+3)\ (n \geqslant 3).$$

Theorem 5.2 has been obtained independently by A. Seeger, (personal communication).

We should like to thank C. Kenig and C. Meaney for pointing out to us, after this work was completed, that the analogue of Theorem 2 on the *n*-torus has been obtained by B. I. Golubov (Math USSR Sbornik **25**(1974), 177-197).

However, his methods do not apply to give our Theorem 3.

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