Revista Matemática Iberoamericana
Vol. 4, n.º 2, 1988

BMO Harmonic Approximation in the Plane and Spectral Synthesis for Hardy-Sobolev Spaces

Joan Mateu and Joan Verdera

En record de José Luis Rubio de Francia

Introduction

The spectral synthesis theorem for Sobolev spaces of Hedberg and Wolff [7] has been applied in combination with duality, to problems of L^q approximation by analytic and harmonic functions. In fact, such applications were one of the main motivations to consider spectral synthesis problems in the Sobolev space setting. In this paper we go the opposite way in the context of the *BMO*-H^1 duality: we prove a *BMO* approximation theorem by harmonic functions and then we apply the ideas in its proof to produce a spectral synthesis result for variants of Sobolev spaces involving the Fefferman-Stein Hardy space H^1. It should be mentioned that our techniques work only in dimension 2, one of the reasons being that the fundamental solution for the Laplacian is in BMO only in the plane (see below for a more detailed discussion of this fact). Our main result reads as follows.

Theorem. *Let $X \subset \mathbb{C}$ be compact and let $f \in VMO(\mathbb{C})$ be harmonic on $\overset{\circ}{X}$. Then we can find a sequence (f_n) in $VMO(\mathbb{C})$, each f_n being harmonic on some neighbourhood of X, such that*

$$f_n \to f \quad in \quad BMO(\mathbb{C}).$$

There is a more conventional way of stating the above theorem, which we now proceed to explain. To each measurable subset E of $\mathbb{R}^n$ one associates the restriction spaces

$$BMO(E) = BMO(\mathbb{R}^n)|_E \simeq BMO(\mathbb{R}^n)/I(E)$$

and

$$VMO(E) = VMO(\mathbb{R}^n)|_E \simeq VMO(\mathbb{R}^n)/J(E),$$

where

$$I(E) = \{f \in BMO(\mathbb{R}^n): f = 0 \text{ a.e. on } E\}$$

and

$$J(E) = \{f \in VMO(\mathbb{R}^n): f = 0 \text{ a.e. on } E\}.$$

Endowed with the quotient norm, $BMO(E)$ and $VMO(E)$ become Banach spaces (modulo constants). T. Wolff has identified $BMO(E)$ with the set of measurable functions on E which satisfy the conclusions of the John-Nirenberg Theorem [4, p. 440]. As a result, functions in $BMO(E)$ (and their norms) can be described in a very concrete way in terms of their values on E. The analogous task for $VMO(E)$ has been achieved by Holden [8].

Consider now a compact $X \subset \mathbb{C}$ and let $H(X)$ be the closure in $BMO(X)$ or those functions which are harmonic on neighbourhoods of X. Holden's extension theorem implies that $H(X)$ is contained in $VMO(X)$, and consequently in

$$h(X) = VMO(X) \cap \{f: \Delta f = 0 \text{ on } \mathring{X}\}.$$

We can now restate our main result.

Theorem 1. *$H(X) = h(X)$, for any compact $X \subset \mathbb{C}$.*

The main point is, of course, than the compact X is arbitrary. This is in strong contrast with the well-known results on uniform harmonic approximation due to Keldysh [10] and Deny [3]. Using them, one can easily construct examples of compacta X such that not all continuous functions on X which are harmonic on $\mathring{X}$ can be uniformly approximated on X by functions harmonic on neigbourhoods of X. One of the reasons for this difference is that the fundamental solution for the Laplacian in the plane is in BMO, but not in L^∞. But this fact alone cannot provide a full explanation for Theorem 1, as becomes clear when one considers the L^p harmonic approximation problem: $\log|z|$ is still (locally) in L^p, but the natural L^p version of Theorem 1 does not hold for $2 \leqslant p$, according to a clever example of Hedberg [6, p. 77]. It turns out that the gradient of the fundamental solution plays a role too. As shown in [17], the capacity associated to the kernel $1/z = \nabla(\log|z|)$ and to *VMO* vanishes on sets of σ-finite length, and this is the reason why Hedberg's construction fails to give a counter-example in the *BMO* case.

As far as techniques are concerned, we use a refinement of Vitushkin localization and matching coefficients methods and a final duality argument, which takes full advantage of an old result of Kolmogorov and Vercenko in geometric measure theory, via a differentiability theorem for functions with second derivatives in $H^1(\mathbb{R}^2)$ due to J. Dorronsoro.

We come now to spectral synthesis. We wish to consider functions on $\mathbb{R}^n$ with all derivatives of order s in $H^1(\mathbb{R}^n)$. A convenient way of doing that is to define, for $0 < s \leqslant n$,

$$I_s H^1(\mathbb{R}^n) = \{I_s * h : h \in H^1(\mathbb{R}^n)\},$$

where I_s is the Riesz potential of order s, that is, $I_s(x) = |x|^{-n+s}$ if $0 < s < n$ and $I_s(x) = \log|x|$ if $s = n$. We endow $I_s H^1(\mathbb{R}^n)$ with the Banach space norm $\|f\| = \|h\|_{H^1}$, $f = I_s * h$. Recent results of Adams [1] show that functions in $I_s H^1(\mathbb{R}^n)$ can be well defined except for sets of zero $n - s$ dimensional Hausdorff measure (they are continuous for $s = n$), and when so strictly defined they enjoy some continuity properties measured by the $n - s$ dimensional Hausdorff content M^{n-s}. Assume now that a closed set E in $\mathbb{R}^n$ is fixed and a function $f \in I_s H^1(\mathbb{R}^n)$ is given with the property that for some $\varphi_j \in C_0^\infty(E^c)$, $\varphi_j \to f$ in $I_s H^1(\mathbb{R}^n)$. Then it is not difficult to show that

$$(1) \qquad \partial^\alpha f(x) = 0, \quad \text{for} \quad M^{n-s+|\alpha|} - \text{almost all } x \in E, \ |\alpha| \leqslant s - 1.$$

We believe that the following spectral synthesis theorem is true.

Conjecture. Let $f \in I_s H^1(\mathbb{R}^n)$ safisfy (1). Then there exist $\varphi_j \in C_0^\infty(E^c)$ with

$$\varphi_j \to f \quad \text{in} \quad I_s H^1(\mathbb{R}^n).$$

For $s = 1$ and any n the above statement has been shown to be true very recently by J. Orobitg [11] using truncation results of S. Janson for $I_1 H^1(\mathbb{R}^n)$. Our contribution here is to verify the conjecture for the first case in which truncation is not available, that is, for $n = s = 2$.

Theorem 2. *Let $E \subset \mathbb{R}^2$ be closed and let $f \in I_2 H^1(\mathbb{R}^2)$ satisfy*

$$f(x) = 0, \qquad x \in E,$$

and

$$\nabla f(x) = 0, \quad \textit{for } M^1 - \textit{almost all } x \in E.$$

Then there is a sequence (φ_j) *in* $C_0^\infty(E^c)$ *such that*

$$\varphi_j \to f \quad in \quad I_2 H^1(\mathbb{R}^2).$$

Again our techniques are a combination of the constructive methods of Vitushkin with duality arguments exploiting ideas coming from the proof of Theorem 1.

Section 1 contains the constructive part of the proof of Theorem 1: after some preliminary lemmas an approximation theorem by potentials of measures is proven. In Section 2 the duality argument completing the proof of Theorem 1 is presented. Theorem 2 is dealt with in Section 3.

We close this introductory section by establishing some notation and recalling a few well-known facts which will be used throughout the paper.

1. *Hausdorff content.* Let $h(t)$, $t > 0$, be a non-decreasing function. For $E \subset \mathbb{R}^n$ we set

$$M^h(E) = \inf \sum_j h(\sigma_j), \tag{2}$$

where the infimum is taken over all countable coverings of E by squares with sides of length σ_j and parallel to the coordinate axis. When $h(t) = t^\alpha$, $0 < \alpha$, one writes M^α instead of M^h. We will need also the dyadic version M_d^h of M^h, which is defined by the right-hand side of (2) where now the infimum is over all coverings of E by dyadic squares of side length σ_j. It is clear that for some constant C depending on n,

$$M^h(E) \leqslant M_d^h(E) \leqslant CM^h(E), \quad \text{for any} \quad E.$$

2. *Cauchy and Beurling transforms.* They are the operators defined respectively by

$$Cf = \frac{1}{z} * f \quad \text{and} \quad Bf = P.V. \frac{1}{z^2} * f,$$

where f is an appropriate function on the plane.

Then we have $B = -\partial C$, where ∂ is the conjugate of the Cauchy-Riemann operator $\bar{\partial}$.

3. For $f \in BMO(\mathbb{R}^2)$ we shall write

$$\|f\|_* = \sup \frac{1}{|\Delta|} \int_\Delta |f - f_\Delta|, \tag{3}$$

where $|\Delta|$ is the area of Δ, f_Δ is the mean value of f on Δ and the supremum is taken over all discs Δ. If D is a disc, we let $\|f\|_{*,D}$ stand for the right-hand side of (3), now the supremum being taken just on the discs Δ contained in D.

4. The letter C will denote a constant, which may be different at each occurence and which is independent of the relevant variables under consideration.

1. Approximation by Potentials of Measures

A family (E_j) of subsets of the plane is said to be *almost disjoint with constant* N whenever each $z \in \mathbb{C}$ belongs to at most N sets E_j.

Our first lemma was discovered in [2]. However, a (slightly different) proof of it will be presented here for the reader's convenience.

Lemma 1.1. *Let (Δ_j) be a finite family of open discs such that for some $\lambda > 1$ $(\lambda\Delta_j)$ is almost disjoint with constant N. Let $h_j \in BMO(\mathbb{C})$ be harmonic outside a compact subset of Δ_j and assume that*

$$h_j(z) = O(|z|^{-2}) \quad as \quad z \to \infty.$$

Then

$$\Big\| \sum_j h_j \Big\|_* \leqslant C \max_j \|h_j\|_* \tag{4}$$

for some positive constant $C = C(\lambda, N)$.

PROOF. We claim that for each j we can find $\chi_j \in BMO(\mathbb{C})$, with compact support contained in $\lambda\Delta_j$, such that

$$h_j = B(\chi_j) + \overline{B(\chi_j)} \quad \text{on} \quad (\lambda\Delta)^c$$

and

$$\|\chi_j\|_* \leqslant C \|h_j\|_*.$$

To prove the claim fix j and set $h = h_j$, $\Delta = \Delta_j$. Let us assume that Δ is centered at the origin at let δ be its radius. On $(\lambda\Delta)^c$ h has the expansion

$$h(z) = \sum_{n=2}^{\infty} a_n z^{-n} + \sum_{n=2}^{\infty} \bar{a}_n \bar{z}^{-n},$$

where $a_n = n^{-1}\langle \Delta h, z^n \rangle$, the bracket meaning the action of the compactly supported distribution Δh on the function z^n. Set

$$\lambda_1 = 1 + \frac{1-\lambda}{3} \quad \text{and} \quad \lambda_2 = 1 + \frac{2(1-\lambda)}{3}.$$

We need now the estimate

$$|a_n| \leqslant C\lambda_1^n \delta^n \|h\|_* . \tag{5}$$

A quick argument to get (5) is the following. Let $\varphi \in C_0^\infty(\lambda_1 \Delta)$, $\varphi = 1$ on Δ, $|\partial^\alpha \varphi| \leqslant C\delta^{-|\alpha|}$, $0 \leqslant |\alpha| \leqslant 2$.

We have

$$\begin{aligned} n|a_n| &= |\langle \Delta h, \varphi(z) z^n \rangle| \\ &= |\langle h, \Delta(\varphi(z) z^n)\rangle| \\ &= |\langle h - h_{\lambda_1 \Delta}, \Delta(\varphi(z) z^n)\rangle| \\ &\leqslant \int_{\lambda_1 \Delta} |h - h_{\lambda_1 \Delta}|\, |\Delta(\varphi(z) z^n)| \\ &\leqslant Cn\lambda_1^n \delta^n \|h\|_* , \end{aligned}$$

which is (5).

Take now $\varphi \in C^\infty(\mathbb{C})$, $\varphi = 0$ on $\lambda_2 \Delta$, $\varphi = 1$ on $(\lambda\Delta)^c$, $|\nabla \varphi| \leqslant C\delta^{-1}$. Then

$$\frac{\varphi(z)}{z^{n-1}} = C\left(\frac{\bar\partial \varphi}{z^{n-1}}\right), \qquad n = 2, 3, \ldots ,$$

and so

$$\frac{1}{z^{n-1}} = C\left(\frac{\bar\partial \varphi}{z^{n-1}}\right) \quad \text{on} \quad (\lambda\Delta)^c. \tag{6}$$

Differentiating with respect to z both sides of (6) we obtain

$$\frac{1}{z^n} = B(\psi_n) \quad \text{on} \quad (\lambda\Delta)^c,$$

where $\psi_n = (n-1)^{-1} z^{-(n-1)} \bar\partial \varphi$. It is clear that

$$\|\psi_n\|_* \leqslant 2\|\psi_n\|_\infty \leqslant Cn^{-1}(\lambda_2 \delta)^{-n}.$$

Set

$$\chi = \sum_{n=2}^{\infty} a_n \psi_n ,$$

so that

$$\|\chi\|_* \leqslant \sum_{n=2}^{\infty} C(\lambda_1 \delta)^n \|h\|_* n^{-1} (\lambda_2 \delta)^{-n} = C\|h\|_* ,$$

and $h(z) = B(\chi) + \overline{B(\chi)}$ on $(\lambda\Delta)^c$. The claim is then proven.

Define

$$b_j = h_j - B(\chi_j) - \overline{B(\chi_j)}.$$

Therefore $\|b_j\|_* \leqslant C\|h_j\|_*$ and $\operatorname{supp}(b_j) \subset \lambda\Delta_j$. Since

$$\begin{aligned}\left\|\sum h_j\right\|_* &\leqslant \left\|\sum b_j\right\|_* + 2\left\|B\left(\sum \chi_j\right)\right\|_* \\ &\leqslant \left\|\sum b_j\right\|_* + C\left\|\sum \chi_j\right\|_*,\end{aligned}$$

it suffices to prove Lemma 1.1 under the additional assumption that $\operatorname{supp}(h_j) \subset \lambda\Delta_j$ for each j, which is not difficult (see [2] for details). □

The next lemma is a variant of Lemma 1.1, in which we require less decay of the h_j at infinity but we have a packing condition on the family of discs Δ_j.

Lemma 1.2. *Let $\omega(t)$, $t > 0$, a non-decreasing function satisfying $\omega(2t) \leqslant C\omega(t)$, $t > 0$.* Let *(Δ_j) be a finite family of open discs of radii δ_j with the following properties.*

(i) *For some $\lambda > 1$ $(\lambda\Delta_j)$ is an almost disjoint family.*
(ii) *For any disc Δ of radius δ*

$$\sum_{\Delta_j \subset \Delta} \delta_j \omega(\delta_j) \leqslant C\delta\omega(\delta).$$

Let $h_j \in BMO(\mathbb{C})$ be harmonic outside a compact subset of Δ_j, $\|h_j\|_* \leqslant \omega(\delta_j)$ and $h_j(z) = O(|z|^{-1})$ as $z \to \infty$.

Then

$$\left\|\sum_j h_j\right\|_* \leqslant C\omega(d),$$

d being the diameter of $\bigcup_j \Delta_j$.

Remark. Notice that when $\omega \equiv 1$ the conclusion of the lemma is essentially (4) and the hypothesis (ii) can be regarded as a linearity condition on the family (Δ_j).

PROOF. We are going to perform a reduction to Lemma 1.1. To this aim expand h_j at ∞

$$h_j(z) = \frac{a_j}{z} + \frac{\bar{a}_j}{\bar{z}} + O(|z|^{-2}),$$

and notice that

$$|a_j| \leqslant C\delta_j \|h_j\|_* \leqslant C\delta_j \omega(\delta_j).$$

Set

$$\mu_j = a_j \left| \frac{1}{2} \Delta_j \right|^{-1} \chi_{\frac{1}{2}\Delta_j}$$

and

$$P_j = C(\mu_j) + \overline{C(\mu_j)},$$

so that

$$h_j = P_j + O(|z|^{-2}).$$

Since

$$\left\| \sum h_j \right\|_* \leqslant \left\| \sum_j h_j - P_j \right\|_* + \left\| \sum P_j \right\|_*,$$

by Lemma 1.1 it is clearly enough to show that

$$\|P_j\|_* \leqslant C \|h_j\|_* \tag{7}$$

and

$$\left\| \sum P_j \right\|_* \leqslant C\omega(d). \tag{8}$$

Now, (7) is a consequence of the well known inequality [9]

$$\|C(\mu)\|_* \leqslant C \sup \{ |\mu|(\Delta(z,r)r^{-1}) : z \in \mathbb{C}, r > 0 \}, \tag{9}$$

μ being any locally finite measure, and (8) follows also from (9) provided we ascertain that for each disc $\Delta(z,r)$

$$\sum_j |\mu_j|(\Delta(z,r)) \leqslant C\omega(d)r.$$

This can be done as follows.

$$\sum |\mu_j|(\Delta(z,r)) = \sum_{r \leq \delta_j} + \sum_{r > \delta_j} \leqslant C\omega(d) \sum_{r \leq \delta_j} r^{-1} |\Delta(z,r) \cap \Delta_j| + C \sum_{\Delta_j \subset \Delta(z,3r)} \delta_j \omega(\delta_j)$$

$$= I + II.$$

Clearly $I \leqslant C\omega(d)r$. If $r \leqslant d$ we estimate II by $Cr\omega(r) \leqslant Cr\omega(d)$. Otherwise, given any $w \in \bigcup \Delta_j$, II can be estimated by

$$\sum_j \delta_j \omega(\delta_j) = \sum_{\Delta_j \subset \Delta(w,d)} \delta_j \omega(\delta_j) \leqslant C d\omega(d) \leqslant Cr\omega(d). \quad \square$$

We come now to a result which gives a simple but useful device to produce functions with a double zero at ∞, as required by Lemma 1.1.

Lemma 1.3. *Let $\delta > 0$ and $0 < p \in \mathbb{Z}$. Let z_1, z_2, z_3 be three points satisfying*

$$|z_j - z_k| \leqslant C\delta, \qquad j, k = 1, 2, 3,$$

and

$$\min\{|z_1 - z_2|, d(z_3, l(z_1, z_2))\} \geqslant \delta/p,$$

where $l(z_1, z_2)$ is the straight line joining z_1 and z_2. Assume $a \in \mathbb{R}$, $b \in \mathbb{C}$ and

$$|a| \leqslant CN, \qquad |b| \leqslant C\delta N$$

for some $N > 0$. Then there exists a function g satisfying $\Delta g = 0$ on $\mathbb{C}\setminus\{z_1, z_2, z_3\}$,

$$g(z) = a \log|z| + \frac{b}{z} + \frac{\bar{b}}{\bar{z}} + O(|z|^{-2}), \quad \text{as} \quad z \to \infty$$

and

$$\|g\|_* \leqslant Cp^2N.$$

PROOF. Without loss of generality we can assume $z_1 = 0$, $z_2 = r > 0$. Set

$$g = \log|z| * \mu, \qquad \mu = \lambda_1\delta_0 + \lambda_2\delta_r + \lambda_3\delta_{z_3},$$

where δ_z is the Dirac measure at the point z and the λ_j are complex numbers to be determined. The function g has the required expansion at ∞ if

$$\begin{aligned} a &= \lambda_1 + \lambda_2 + \lambda_3 \\ -b &= \lambda_2 r + \lambda_3 z_3 \\ -\bar{b} &= \lambda_2 r + \lambda_3 \bar{z}_3. \end{aligned}$$

Solving for the λ_j, we get the estimates

$$|\lambda_j| \leqslant Cp^2N, \qquad j = 1, 2, 3,$$

which give the desired bound for $\|g\|_*$. □

The next lemma is the bulk of our technical arsenal. It deals with a covering property which seems to be of interest by itself (see the remark below).

Main Lemma 1.4. *Let $h(t) = t\omega(t)$ be a measure function with ω non-decreasing and satisfying $\omega(2t) \leqslant C\omega(t)$. Then for any compact set $K \subset \mathbb{C}$ there exists a finite family of discs (Δ_j) which can be divided into two sub-families (Δ_j^g) and (Δ_j^b) (the superscripts g and b stand for good and bad) in such a way that the following holds.*

(a) $K \subset \bigcup_j \Delta_j$.
(b) *For some* $\lambda = \lambda(h) > 1$, $(\lambda \Delta_j)$ *is an almost disjoint family with constant depending on* h *but not on* K.
(c) *Each* Δ_j^g *has the three points property, that is, there exist* $z_1, z_2, z_3 \in \Delta_j^g \cap K$ *such that*

$$\min \{ |z_1 - z_2|, d(z_3, l(z_1, z_2)) \} > \eta \delta_j^g,$$

where δ_j^g *is the radius of* Δ_j^g, $\eta = \eta(h) > 0$ *and* $l(z_1, z_2)$ *is the straight line through* z_1 *and* z_2.
(d) $\sum_j h(\delta_j^b) \leqslant CM^h(K)$, *where* C *depends on* h *but not on* K.
(e) *For each disc* Δ *of radius* δ

$$\sum_{\Delta_j^b \subset \Delta} h(\delta_j^b) \leqslant Ch(\delta),$$

where C *depends on* h *but not on* K.

Remark. We do not know, even for $h(t) = t$, when a family (Δ_j) can be constructed satisfying (a), (b) and (d), (e) with Δ_j^b replaced by Δ_j. This is true when $h(t)$ tends to zero fast enough, for example for $h(t) = t^{1+\epsilon}$, $\epsilon > 0$.

PROOF OF THE MAIN LEMMA. The first step consists in finding a family of discs D_j of radius r_j such that

(i) $K \subset \bigcup_j D_j$
(ii) $\sum_j h(r_j) \leqslant CM^h(K)$
(iii) $\sum_{D_j \subset D} h(r_j) \leqslant Ch(r)$, for each disc D of radius r.

To construct the D_j we consider a family (Q_j) of dyadic squares of side length σ_j satisfying $K \subset \bigcup_j Q_j$ and $\sum_j h(\sigma_j) \leqslant 2M_d^h(K)$. It is easy to modify that family so that in addition one has

$$\sum_{Q_j \subset Q} h(\sigma_j) \leqslant h(\sigma), \tag{10}$$

for each dyadic square Q of side length σ. In fact, if for such a Q (10) fails then we remove the Q_j contained in Q from our family and we put Q in it. Take now as D_j the disc circumscribed to Q_j.

Unfortunately property (b) with Δ_j replaced by D_j does not necessarily hold. The idea is to construct a «halo» around each D_j, consisting of smaller discs which will be called «defenses» because their effect is to prevent too much overlapping. The difficulty will be then to preserve properties (ii) and (iii).

Generation of defenses: step 1. Let ϵ be a small positive number to be determined later, such that ϵ^{-1} is an integer. Choose one of the discs D_j of maximal size. After relabelling we can assume it is D_1. We distinguish two cases.

Case 1. D_1 has the three points property, that is, (*c*) holds with Δ_j^b replaced by D_1 and $\eta = \epsilon^4/2$. D_1 will be a member of the family (Δ_j^g) and it will be surrounded by some defenses which now we proceed to define. Consider ϵ^{-3} points equally spaced on the boundary of D_1, and define the defenses of D_1 to be the discs of radii $\epsilon^2 r_1$ centered at these points. Then the defenses of D_1 form an almost disjoint family with constant comparable to ϵ^{-1}.

We now define two new families of discs, $\mathfrak{F}_1^d$ (the superscript d is for definitive) and $\mathfrak{F}_1^p$ (the superscript p is for process). Only the family $\mathfrak{F}_1^p$ will be subject to a new process of generation of defenses. $\mathfrak{F}_1^d$ consists of the single disc D_1 and $\mathfrak{F}_1^p$ consists of the defenses of D_1 plus the discs D_j which are not contained in the union of D_1 and its defenses.

Case 2. D_1 has not the three points property. In this case $K \cap D_1$ is contained in a strip S of width $\epsilon^4 r_1$.

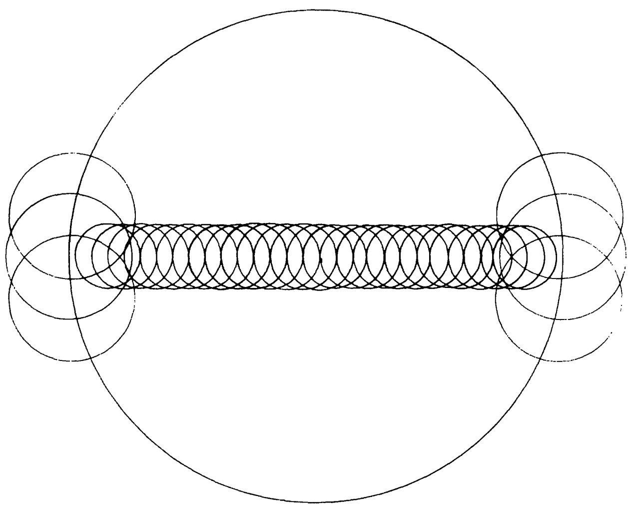

Figure 1

Observe that the length of $S \cap \partial D_1$ is at most $C\epsilon^2 r_1$. Cover $S \cap D_1$ by discs of radii $\epsilon^4 r_1$ in such a way that they form an almost disjoint family with some numerical constant. These discs will form the family $\mathfrak{F}_1^d$. Consider, as in Case 1, ϵ^{-3} equally spaced discs of radii $\epsilon^2 r_1$ centered on ∂D_1, and take as defenses of D_1 those that intersect $S \cap \partial D_1$. In the family $\mathfrak{F}_1^p$ we put the defenses of D_1 plus the discs D_j different from D_1 which are not contained in the union of the defenses of D_1 and of the discs in $\mathfrak{F}_1^d$.

Generation of defenses: step 2. We now examine the family $\mathfrak{F}_1^p$. We take a disc D in $\mathfrak{F}_1^p$ of maximal size and distinguish two cases.

Case 1. D has the three points property. If D is one of the discs D_j we proceed as in step 1, but this time, we only use those defenses which are not already contained in the union of the discs in $\mathfrak{F}_1^d \cup \mathfrak{F}_1^p$.

If D is not a D_j then D is a defense of D_1. In this case let C be the circumference passing through the points of intersection of the boundaries of pairs of defenses of D_1 (see figure 2). If r is the radius of D we consider a family $\mathfrak{D}$ of discs of radii $\epsilon^2 r$ with equally spaced centers on C, which is almost disjoint with constant comparable to ϵ^{-1}. As shown in figure 2 just an arc γ of ∂D of length comparable to ϵr is in the exterior of the disc bounded by C.

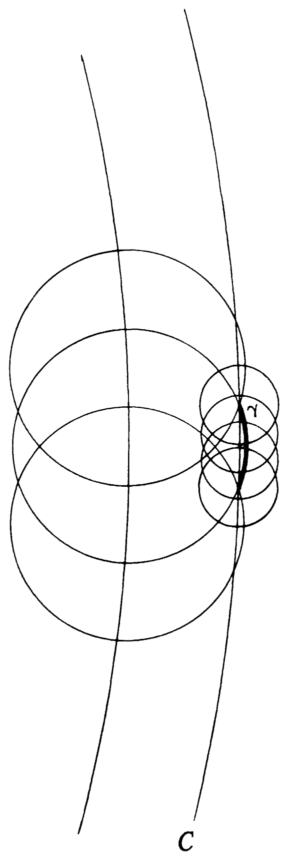

Figure 2

The defenses of D are the discs in the family $\mathfrak{D}$ which intersect γ. The family $\mathfrak{F}_2^d$ consists of $\mathfrak{F}_1^d$ plus the disc D, and the family $\mathfrak{F}_2^p$ consists of the defenses of D and the discs in $\mathfrak{F}_1^p$ which are not contained in the union of $\mathfrak{F}_2^d$ and the defenses of D.

Case 2. D has not the three points property. We proceed as in step 1, the only difference being that the defenses of D are now centered on C when D itself is a defense of D_1 (see figure 3).

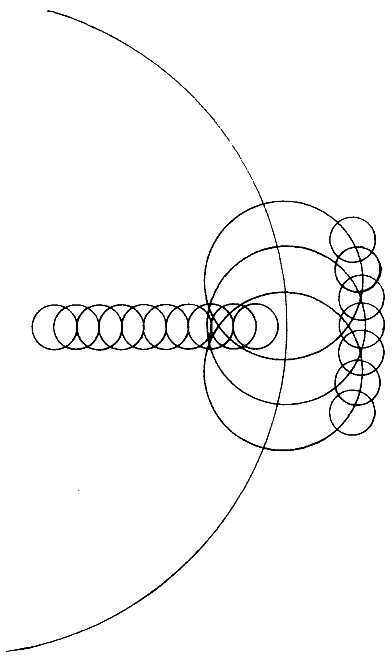

Figure 3

Proceeding inductively we produce at the n-th step families $\mathfrak{F}_n^d$ and $\mathfrak{F}_n^p$. We stop when all discs D_j have been either rejected or subject to a process of generation of defenses, that is, we stop at the N-th step provided $\mathfrak{F}_N^p$ does not include any D_j. We claim that $\mathfrak{F}_N^d = \{\Delta_j\}$ is the family we are looking for. We say that a disc $\Delta_j \in \mathfrak{F}_N^d$ is a good disc, that is, a disc Δ_j^g, if it has the three points property. Otherwise it is a bad disc Δ_j^b. Therefore, properties (*a*) and (*c*) clearly hold. Let us proceed to prove (*b*).

Claim. Let $\lambda = 1 + \epsilon^2/4$. Then for ϵ small enough $\{\lambda\Delta_j\}$ is an almost disjoint family with constant depending on ϵ.

PROOF OF THE CLAIM. In the typical case of generation of defenses we start with a disc D_j of radius δ. At the n-th step we will produce defenses of radii $\epsilon^{2n}\delta$ centered on a circumference C_{n-1} concentric with D_j. Let L_n be the boundary of the smallest disc concentric with D_j containing the dilates with factor λ of the defenses in the n-th generation.

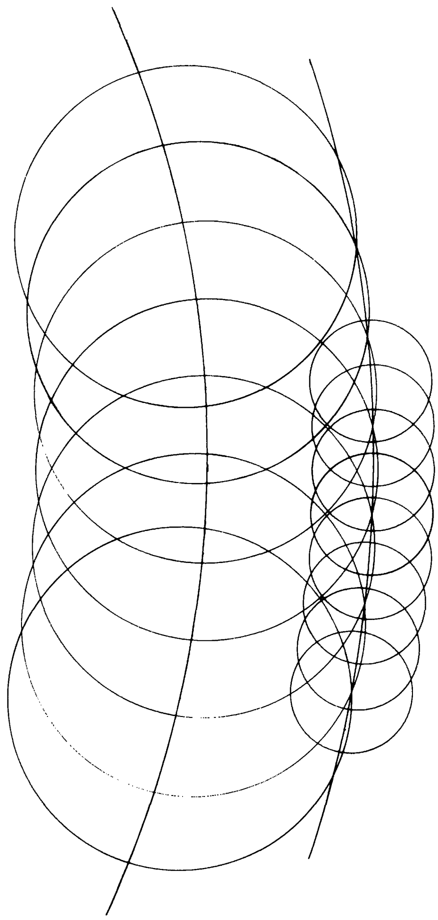

Figure 4

An easy but cumbersome computation shows that for ϵ small enough

$$\text{radius } L_n < \text{radius } C_n + \delta\epsilon^{2n+2}/2.$$

This inequality guarantees that to travel from the exterior of the disc bounded by C_{n+1} to a dilate of a disc D in the n-th generation one has to cover a distance not less than $C\epsilon^2$ radius D.

Let $z \in \mathbb{C}$ and assume that $z \in \lambda\Delta_j$ and that Δ_j is of maximal size among the discs with this property. The choice of ϵ (precisely, the just mentioned travelling property) shows that if $z \in \lambda\Delta_k$ then

$$\text{radius } \Delta_k \geqslant C\epsilon^2 \text{ radius } \Delta_j .$$

Thus there is a finite number $p = p(\epsilon)$ of possible values for the radii of the Δ_k such that $z \in \lambda\Delta_k$. Since the discs $\lambda\Delta_k$ of a fixed size form an almost disjoint family with constant depending on ϵ, the claim is proven.

We turn now to the proof of (d). We wish to show that

$$\sum_j h(\delta_j^b) \leqslant C \sum_k h(r_k), \tag{11}$$

but before it will be convenient to introduce some terminology to simplify the exposition of the argument.

Assume that the process of generation of defenses has been applied to some disc D. The first generation descendants of D are either the defenses of D or the discs which have been used to cover the strip S in the case that D has not the three points property. We define inductively the i-th generation descendants of D, $2 \leqslant i$, in the obvious way. The halo of D, Halo (D), is the set of all descendants of D. A disc D is an ancestor of a disc D^* if $D^* \in \text{Halo}(D)$. For $D \in \{\Delta_j\}$ and a non-negative integer n, $A_n(D)$ is the set of bad discs Δ_j^b which have got exactly n bad ancestors (a bad ancestor is an ancestor without the three points property) different from D in the halo of D.

Fix now $D \in \{\Delta_j\}$ and let r be its radius. We claim that

$$\sum_{\Delta_j^b \in A_0(D)} h(\delta_j^b) \leqslant C\epsilon^{-1} h(r), \quad \text{if } D \text{ is good,} \tag{12}$$

and

$$\sum_{\Delta_j^b \in A_0(D)} h(\delta_j^b) \leqslant C\epsilon h(r), \quad \text{if } \quad D \text{ is bad.} \tag{13}$$

To show (12) observe first that the halo of D is contained in $2D$. Consider a disc Δ_j^b in $A_0(D)$. Project the part of $\partial\Delta_j^b$ which will be eventually covered by defenses of Δ_j^b onto an interval I_j on $\partial(2D)$ as shown in figure 5, and notice that the length of I_j is comparable to $\epsilon\delta_j^b$.

The intervals I_j are disjoint because any disc Δ_j intersecting the shaded region R is not in $A_0(D)$. Thus

$$\sum_{\Delta_j^b \in A_0(D)} h(\delta_j^b) \leqslant C\epsilon^{-1}\omega(r) \sum_j \text{length}\,(I_j) \leqslant C\epsilon^{-1} r\omega(r).$$

The proof of (13) is similar, the only difference being that if D is bad then the halo of D emanates from an arc of ∂D of length comparable to $\epsilon^2 r$, and so $\sum_j$ length $(I_j) \leqslant C\epsilon^2 r$.

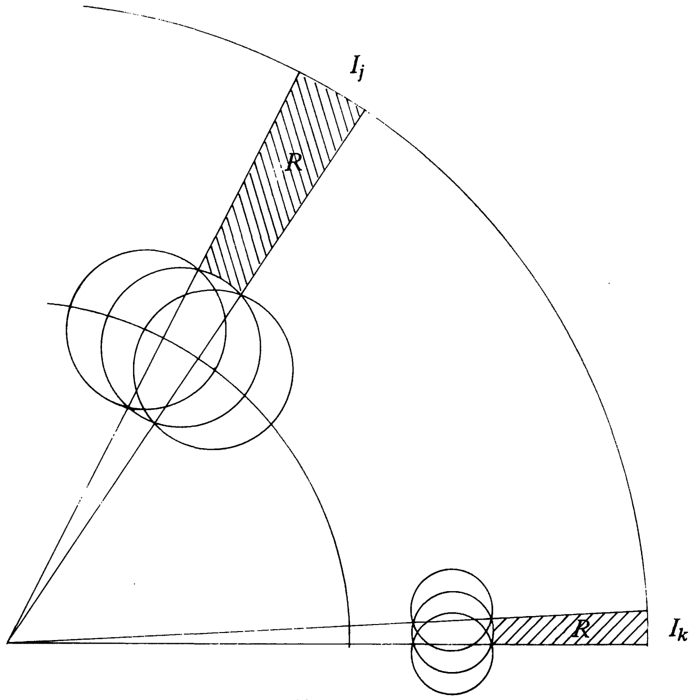

Figure 5

Fix now a D_k. One has

$$\sum_{\Delta_j^b \in \mathrm{Halo}(D_k)} h(\delta_j^b) = \sum_{n=0}^{\infty} \sum_{\Delta_j^b \in A_n(D_k)} h(\delta_j^b) \leqslant \sum_{n=0}^{\infty} (C\epsilon)^n \sum_{\Delta_j^b \in A_0(D_k)} h(\delta_j^b)$$
$$\leqslant C\epsilon^{-1}(1 - C\epsilon)^{-1} h(r_k),$$

where in the first inequality we applied (13) n times and in the second (12) once. The above inequality completes the proof of (11) provided ϵ is small enough so that $C\epsilon < 1$.

We are left with property (*e*). Fix a Δ_j^b. We wish to associate to it a certain ancestor $A(\Delta_j^b)$. We distinguish two cases.

Case 1. All ancestors of Δ_j^b are contained in 2Δ, where Δ is the test disc in (*e*). Then $A(\Delta_j^b)$ is, by definition, the oldest ancestor of Δ_j^b.

Case 2. At least one ancestor of Δ_j^b is not contained in 2Δ. In this case there is a youngest ancestor not contained in Δ, say D. If the center of D is not in Δ, then we set $A(\Delta_j^b) = D$. Otherwise the «father» D^* of D intersects Δ and its center is outside. We set $A(\Delta_j^b) = D^*$.

Let I and II be the families of ancestors so constructed which are in cases 1 and 2 respectively. We get then

$$\sum_{\Delta_j^b \subset \Delta} h(\delta_j^b) = \Big(\sum_{A \in I} \sum_{A(\Delta_j^b)=A}\Big) h(\delta_j^b) + \sum_{A \in II} \sum_{A(\Delta_j^b)=A} h(\delta_j^b) = S_1 + S_2 .$$

As we know, S_1 can be estimated by

$$\sum_{A \in I} Ch(\text{radius}\, A) \leqslant C \sum_{D_k \subset 2\Delta} h(r_k) \leqslant Ch(\delta).$$

To estimate S_2 fix $A \in II$ and let $\sigma(A) = \partial A \cap \Delta$ and $\sigma(\Delta) = \partial\Delta \cap A$. The halo of A contained in Δ emanates from $\sigma(A)$, and so

$$S_2 \leqslant C\omega(\delta) \sum_{A \in II} \text{lenght}\, \sigma(A) \leqslant C\omega(\delta) \sum_{A \in II} \text{length}\, \sigma(\Delta) \leqslant C\delta\omega(\delta),$$

where in the last inequality we used the fact that the family (Δ_j) is almost disjoint, and in the next to the last that $\sigma(A)$ and $\sigma(\Delta)$ have comparable lengths owing to the relative positions of A and Δ. □

We still need another lemma. To any $\varphi \in C_0^\infty(\mathbb{C})$ one associates the Vitushkin localization operator (see [18, p. 168])

$$V_\varphi f = \frac{1}{2\pi} \log|z| * \varphi\, \Delta f,$$

where f is any distribution in the plane.

Lemma 1.5. *Let $\varphi \in C_0^\infty(\Delta)$, Δ a disc of radius δ. We then have for any $f \in BMO(\mathbb{C})$*

$$\| V_\varphi f \|_* \leqslant C(\varphi) \| f \|_{*, 3\Delta},$$

where

$$C(\varphi) = C \sum_{|\alpha| \leq 2} \delta^{|\alpha|} \| \partial^\alpha \varphi \|_\infty .$$

PROOF. Using

$$\varphi \partial\bar\partial f = \partial\bar\partial(\varphi f) - (f \partial\bar\partial\varphi + \partial\varphi\bar\partial f + \bar\partial\varphi\partial f)$$

we readily get

$$V_\varphi f = \varphi f - \frac{1}{\pi} \frac{1}{z} * f \bar\partial\varphi - \frac{1}{\pi} \frac{1}{\bar z} * f \partial\varphi + \frac{1}{2\pi} \log|z| * f \Delta\varphi.$$

The BMO-norm estimate for the first three terms can be found in [17, 3.2. p. 290]. The BMO-norm of the last one is less than

$$C\int |f|\,|\Delta\varphi| \leqslant C\frac{1}{|\Delta|}\int |f|.$$

Since $V_\varphi f = V_\varphi(f - f_\Delta)$, we can replace f by $f - f_\Delta$ in the previous inequality, and this completes the proof of the lemma.

Given a compact $X \subset \mathbb{C}$ let $P(X)$ be the linear span in $BMO(X)$ of the functions of the type $\log|z-a|$, $a \notin X$, and $Re\,((1/z)*\mu)$, where μ is a complex Borel measure with compact support disjoint from $\overset{\circ}{X}$, and satisfying the growth condition

$$|\mu|(\Delta(z,r)) \leqslant \epsilon(r)r, \qquad z \in \mathbb{C}, r > 0,$$

for some $\epsilon(r) \to 0$ as $r \to 0$. Since $(1/z)*\mu$ is in $VMO(\mathbb{C})$ [9], $P(X)$ is a subspace of $h(X)$.

Theorem 1.6. *For each compact $X \subset \mathbb{C}$, $P(X)$ is dense in $h(X)$.*

PROOF. Let $f \in h(X)$. We can think that in fact $f \in VMO(\mathbb{C})$ and $\Delta f = 0$ on $\overset{\circ}{X}$. Replacing f by φf where $\varphi \in C^\infty(\mathbb{C})$ takes the value 1 on a neighbourhood of X, we can assume also that f is compactly supported. Fix a $\delta > 0$ and let (D_k, φ_k, f_k) be a δ-Vitushkin scheme for the approximation of f. This means (cf. [18, p. 168]) the following.

1. Each D_k is an open disc of radius δ and (D_k) is an almost disjoint covering of the plane.
2. $\varphi_k \in C_0^\infty(D_k)$, $\sum \varphi_k = 1$ on $\mathbb{C}$ and $|\partial^\alpha \varphi_k| \leqslant C\delta^{-|\alpha|}$, $0 \leqslant |\alpha| \leqslant 2$.
3. $f_k = \frac{1}{2\pi}\log|z| * \varphi_k \Delta f$, and so $f = \sum_k f_k$.

Notice that $f_k \equiv 0$ whenever $D_k \cap \operatorname{supp}(f) = \varnothing$, and thus only finitely many f_k do not vanish identically.

Choose a point $z_k \in D_k$ and expand f_k at ∞.

$$f_k(z) = a_k \log|z - z_k| + \frac{b_k}{z - z_k} + \frac{\bar{b}_k}{\bar{z} - \bar{z}_k} + O(|z|^{-2}).$$

Then we have $|a_k| \leqslant C\omega(\delta)$ and $|b_k| \leqslant C\delta\omega(\delta)$ where

$$\omega(\delta) = \sup\left\{\frac{1}{|\Delta|}\int_\Delta |f - f_\Delta| : \text{radius}\,\Delta \leqslant \delta\right\}$$

If we could improve the estimate for b_k to

$$|b_k| \leqslant CM^h(D_k \setminus \overset{\circ}{X}), \tag{14}$$

where $h(t) = t\omega(t)$, then the proof of Theorem 1.6 would be much simpler, as the reader will realize later. As a matter of fact (14) is not true. We get around this difficulty using a second localization for each f_k, whose purpose is to exploit the basic estimate $|b_k| \leqslant Ch(\delta)$ at a lower level, forcing in this way M^h to come on the scene.

To begin with the above program we claim that it suffices to construct $p_k \in P(X)$, p_k harmonic outside $5D_k$,

$$f_k = p_k + O(|z|^{-2}) \quad \text{as} \quad z \to \infty,$$

and

$$\| p_k \|_* \leqslant C\omega(\delta).$$

In fact, setting $p = \sum_k p_k \in P(X)$ we get by Lemma 1.1.

$$\| f - p \|_* = \left\| \sum f_k - p_k \right\| \leqslant C \max \| f_k - p_k \|_* \leqslant C\omega(\delta),$$

where the last inequality is a consequence of Lemma 1.5 and the well known fact that $\omega(2\delta) \leqslant C\omega(\delta)$ [14, p. 596]. To construct the p_k we proceed as follows. Since $\Delta f_k = \varphi_k \Delta f$, f_k is harmonic outside a compact subset K of $D_k \setminus \overset{\circ}{X}$. Let $\{\Delta_j\} = \{\Delta_j^g\} \cup \{\Delta_j^b\}$ be the family of discs given by Lemma 1.4 applied to K and $h(t) = t\omega(t)$. It is then clear that $\Delta_j \subset 5D_k$ for each j.

Observe now that, by construction, the sequence of different values of the radii of the Δ_j decreases as a geometric progression with ratio less than one. It is then clear that an appropriate variant of a Lemma of Harvey and Polking [5, 3.1, p. 43] can be applied to construct functions $\psi_j \in C_0^\infty(\lambda_1 \Delta_j)$, $\lambda_1 = (1 + \lambda)/2$, with $\sum_j \psi_j = 1$ on $\bigcup_j \Delta_j$ and $|\partial^\alpha \psi_j| \leqslant C\delta_j^{-|\alpha|}$, $0 \leqslant |\alpha| \leqslant 2$, where δ_j is the radius of Δ_j.

Set

$$f_{kj} = \frac{1}{2\pi} \log|z| * \psi_j \Delta f_k,$$

so that $f_k = \sum_j f_{kj}$. Fix now a j and distinguish two cases.

Case 1. $\Delta_j = \Delta_j^g$ is a good disc. Then one can find tree points as in (*c*) of Lemma 1.4. These points lie in $\Delta_j \setminus \overset{\circ}{X}$, but by a density argument they can be taken in $\Delta_j \setminus X$. Lemma 1.3. now produces a function $h_{kj}^g \in P(X)$, h_{kj}^g harmonic outside Δ_j^g, and such that $f_{kj}^g = h_{kj}^g + O(|z|^{-2})$, $\| h_{kj}^g \|_* \leqslant C \| f_{kj}^g \|_*$, where we have written f_{kj}^g instead to f_{kj}.

Case 2. $\Delta_j = \Delta_j^b$ is a bad disc. In this case we can suppose that $\Delta_j \backslash X$ is non-empty. Otherwise $\Delta_j \backslash \overset{\circ}{X}$ is also empty, and so Δ_j is not really necessary to cover K. Let $z_{kj} \in \Delta_j \backslash X$ and let f_{kj}^b stand for f_{kj}.

If

$$f_{kj}^b(z) = a_{kj} \log |z - z_{kj}| + O(|z|^{-1})$$

then we define

$$h_{kj}^b(z) = a_{kj} \log |z - z_{kj}|,$$

so that $h_{kj}^b \in P(X)$, h_{kj}^b is harmonic outside $\Delta_j^b, f_{kj}^b = h_{kj}^b + O(|z|^{-1})$ and

$$\| h_{kj}^b \|_* \leqslant C \| f_{kj}^b \|_* \leqslant C\omega(\delta_j^b),$$

δ_j^b being the radius of $\Delta_j = \Delta_j^b$. Let

$$F_k = \sum_j h_{kj}^g + \sum_j h_{kj}^b.$$

Then $F_k \in P(X)$, F_k is harmonic outside $5D_k$, and $f_k = F_k + O(|z|^{-1})$ as $z \to \infty$.

The *BMO*-norm of F_k is estimated by $\|F_k - f_k\|_* + \|f_k\|_*$ and

$$\|F_k - f_k\|_* \leqslant \left\| \sum_j h_{kj}^g - f_{kj}^g \right\|_* + \left\| \sum_j h_{kj}^b - f_{kj}^b \right\|_* = I + II.$$

Applying Lemma 1.1 we get $I \leqslant C \max_j \|f_{kj}^g\|_* \leqslant C\|f_k\|_*$. As for the term II we must resort to Lemma 1.2. We obtain

$$II \leqslant C\omega(\delta),$$

because the diameter of the union of the Δ_j is less than 10δ. Collecting the above estimates we finally get

$$\|F_k\|_* \leqslant C\omega(\delta).$$

Consider the expansions

$$f_k(z) - F_k(z) = \frac{b}{z - z_k} + \frac{\bar{b}}{\bar{z} - \bar{z}_k} + O(|z|^{-2}),$$

$$f_{kj}^b(z) - h_{kj}^b(z) = \frac{b_{kj}}{z - z_{kj}} + \frac{\bar{b}_{kj}}{\bar{z} - \bar{z}_{kj}} + O(|z|^{-2}).$$

Then

$$|b| = \left| \sum_j b_{kj} \right| \leqslant C \sum \delta_j^b \omega(\delta_j^b) \leqslant CM^h(K).$$

Let now μ be a positive measure supported on K satisfying

$$\mu(\Delta(z, r)) \leqslant h(r), \qquad z \in \mathbb{C}, \qquad r > 0,$$

and

$$\|\mu\| \geqslant CM^h(K).$$

If we set

$$p_k = F_k + \operatorname{Re}\left(\frac{1}{z} * b\|\mu\|^{-1}\mu\right),$$

it is clear that p_k satisfies all required properties, and so the proof of Theorem 1.6 is complete.

2. End of the Proof of Theorem 1

The proof of Theorem 1 will be completed by a duality argument combined with a result of Kolmogorov and Vercenko in geometric measure theory (see Lemma 2.2. below). The link between our problem and geometry is provided by the next lemma, due to J. Dorronsoro who proved it in answer to a question of the authors.

Lemma 2.1. (Dorronsoro.) *Let $F = \log|z| * h$, with $h \in H^1(\mathbb{R}^2)$. Then f has an ordinary differential at M^1-almost all $z \in \mathbb{R}^2$.*

PROOF. Clearly $\nabla f = (1/z) * h$, and so ∇f can be defined M^1 a.e., being a Riesz potential of order 1 of a function in $H^1(\mathbb{R}^2)$ [1]. We will show that, for M^1-almost all $a \in \mathbb{R}^2$ and for all $z \neq a$, we have

$$|f(z) - f(a) - \nabla f(a) \cdot (z - a)|\,|z - a|^{-1} \leqslant Tf(a), \tag{15}$$

where Tf is an operator satisfying the weak-type estimate

$$M^1(\{z: Tf(z) > \lambda\}) \leqslant C\lambda^{-1}\|h\|_{H^1}.$$

Since functions in $C_0^\infty(\mathbb{R}^2)$ with zero integral are dense in $H^1(\mathbb{R}^2)$ a standard argument will then complete the proof of the Lemma.

To prove (15) fix $z \neq a$ and set $\delta = |z - a|$, $\Delta = \Delta(a, \delta)$. The left hand-side of (15) is estimated by

$$\delta^{-1}|f(z) - f_\Delta| + \delta^{-1}|f(a) - f_\Delta| + |\nabla f(a)|.$$

An elementary argument involving a regularization of f now gives (cf. [16, p. 125-126])

$$|f(z) - f_\Delta| \leqslant C \int_{|\zeta| < 2\delta} |\zeta|^{-1} |\nabla f(z + \zeta)| \, dm(\zeta),$$

where m is planar Lebesgue measure. Let $0 \leqslant \varphi \in C_0^\infty(\Delta(z, 3\delta))$, $\varphi \equiv 1$ on $\Delta(z, 2\delta)$ and $|\nabla\varphi| \leqslant C\delta^{-1}$. Then

$$|f(z) - f_\Delta| \leqslant C(|\zeta|^{-1} * \varphi|\nabla f|)(z).$$

According to a remark of Stein [15, p. 385] the right hand-side of the above inequality is less than or equal to a constant times the norm of $\varphi|\nabla f|$ in the Lorenz space $L^{2,1}(\mathbb{R}^2)$, which in turn can be estimated by $C\|\nabla(\varphi\nabla f)\|_{L^1(\mathbb{R}^2)}$, owing to an imbedding theorem due to Poornima [12, Th. 1.4, p. 163]. Therefore

$$|f(z) - f_\Delta| \leqslant C|\Delta|^{-1} \int_{6\Delta} |\nabla f| + C \int_{6\Delta} |\nabla^2 f|.$$

The same inequality is, of course, true for $z = a$, and so the left hand-side of (15) is estimated by

$$CM(\nabla f)(a) + CM_1(\nabla^2 f)(a) + |\nabla f(a)| = I + II + III,$$

where M is the Hardy-Littlewood maximal operator and

$$M_1 g(a) = \sup_\delta \delta^{-1} \int_{|\zeta| < \delta} |g(a + \zeta)| \, dm(\zeta).$$

The required weak-type estimates for I and III follow from results of Adams [1, Theorem B] and they are obviously satisfied by II. □

To state the result of Kolmogorov and Vercenko mentioned above we need a definition. Let E be any subset of the plane and $a \in E$. A half-line $\{a + \rho v : \rho > 0\}$, $|v| = 1$, issuing from a is called an intermediate half-tangent of E at a if there exists a sequence $z_n \in E$, $z_n \neq a$, such that $z_n \to a$ and $|z_n - a|^{-1}(z_n - a) \to v$.

We have [13, 3.6, p. 266]

Lemma 2.2. (Kolmogorov and Vercenko.) *Let E be a subset of the plane with the property that for each $a \in E$ there exists some half-line issuing from a which is not an intermediate half-tangent of E at a. Then E has σ-finite length.*

Lemmas 2.1 and 2.2 will give now the result we really need.

Lemma 2.3. *Let $f = \log|z| * h$, $h \in H^1(\mathbb{R}^2)$. Let E be the set of points at which f vanishes and f has a non-zero ordinary differential. Then E has σ-finite length.*

PROOF. We are going to check that E satisfies the hypothesis of 2.2. Let $a \in E$ and assume without loss of generality that $a = 0$, $\partial f(0)/\partial x = 0$ and $\partial f(0)/\partial y = \rho > 0$. The definition of differential at 0 gives ($z = x + iy$)

$$f(z) = \rho y + \epsilon(z)|z|,$$

with $\epsilon(z) \to 0$ as $z \to 0$. If $\delta > 0$ is small enough, then $|\epsilon(z)| < \rho/2$ if $|z| < \delta$. Since f vanishes on E, we get

$$|y| < |x|, \quad \text{for} \quad 0 \neq z \in E \quad \text{and} \quad |z| < \delta.$$

Therefore any of the two imaginary semi-axis is not an intermediate half-tangent of E at 0. □

END OF THE PROOF OF THEOREM 1. The argument at the beginning of the proof of Theorem 1.6 shows that

$$VMO(X) = CMO(\mathbb{C})/K(X),$$

where $CMO(\mathbb{C})$ is the closure in $BMO(\mathbb{C})$ of $C_0^\infty(\mathbb{C})$ and

$$K(X) = \{f \in CMO(\mathbb{C}) : f = 0 \text{ a.e. on } X\}.$$

Since $CMO(\mathbb{C})^* = H^1(\mathbb{C})$, it is clear that

$$VMO(X)^* = \{h \in H^1(\mathbb{C}) : h = 0 \text{ a.e. on } X^c\}.$$

Let $h \in VMO(X)^*$ and assume that h annihillates $H(X)$. We must show that h annihillates $h(X)$. Set $f = \log|z| * h$, so that f vanishes on X^c. Since f is continuous, it vanishes also on $\overset{\circ}{X^c}$. Because of Theorem 1.6 we are just left with the task of showing that $\langle h, (1/z) * \mu\rangle = 0$, whenever μ is a complex Borel measure supported on $(\overset{\circ}{X})^c$ satisfying the growth condition

$$|\mu|(\Delta(z, r)) \leqslant \epsilon(r)r, z \in \mathbb{C}, r > 0, \tag{16}$$

for some $\epsilon(r) \to 0$ as $r \to 0$.

We have

$$2\pi\left\langle h, \frac{1}{z} * \mu\right\rangle = 4\left\langle \bar\partial\partial f, \frac{1}{z} * \mu\right\rangle = -4\pi\langle \partial f, \mu\rangle = -4\pi \int \partial f \, d\mu.$$

Let D be the set of points at which f has an ordinary differential. Now, μ vanishes on sets of σ-finite length because of the growth condition (16) and the complement of D has zero length because of Lemma 2.1. Hence

$$\int \partial f \, d\mu = \int_{D \cap (\mathring{X})^c} \partial f \, d\mu.$$

On the other hand f vanishes on $(\mathring{X})^c$, and thus

$$\int \partial f \, d\mu = \int_E \partial f \, d\mu,$$

where E is the set of points at which f vanishes and f has a non-zero ordinary differential. But E has σ-finite length because of Lemma 2.3 and consequently $\mu(E) = 0$. Then $\int \partial f \, d\mu = 0$, as desired. □

3. Spectral Synthesis

We begin by observing that the mapping

$$I_2(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$$
$$f \to \Delta f$$

is an onto isomorphism. Hence the dual of $I_2(\mathbb{R}^2)$ is $BMO(\mathbb{R}^2)$ and the action of a $b \in BMO(\mathbb{R}^2)$ on a $f \in I_2(\mathbb{R}^2)$ is given by

$$b(f) = \langle b, \Delta f \rangle$$

where $\langle \, , \rangle$ is the $BMO - H^1$ duality.

Let $b \in BMO(\mathbb{R}^2)$ and assume

$$(17) \qquad b(\varphi) = 0, \qquad \varphi \in C_0^\infty(U), \qquad U = E^c.$$

We must show that $b(f) = 0$, f being the function in the statement of Theorem 2. Obviously (17) is equivalent to the harmonicity of b on U. The strategy of the proof consists in using Vitushkin's constructive scheme to approximate b in the weak* topology of $BMO(\mathbb{R}^2)$ by simpler functions β for which we know that $\beta(f)$ is either zero or small.

It will be first shown that, without loss of generality, one can suppose b to be harmonic outside a compact subset of E. Let $\varphi \in C_0^\infty(\mathbb{C})$, supp$(\varphi) \subset \{|z| < 2\}$ and $\varphi = 1$ on $|z| < 1$. Set $\varphi_n(z) = \varphi(z/n)$, $n = 1, 2, \ldots$ $b_n = V_{\varphi_n} b$ and $R_n = b - b_n$. Then b_n is harmonic on $U \cup \{|z| > 2n\}$ and $\|b_n\|_* \leqslant C\|b\|_*$. Passing to a subsequence we get $R_{n_j} \to R$ weak* in $BMO(\mathbb{R}^2)$ as $j \to \infty$, and it turns out that R is harmonic on $\mathbb{R}^2$ because R_n is harmonic on $\{|z| < n\}$. Hence R is constant, or, in other words, $b_n \to b$ weak* in $BMO(\mathbb{R}^2)$. Replacing b by b_n it becomes clear that we can assume that b is harmonic on the complement V of a compact subset K of E.

Fix $\delta > 0$ and let $(\Delta_j, \varphi_j, b_j)$ a δ-Vitushkin scheme for the approximation of b. Then $b_j = (1/2\pi)\log|z| * \varphi_j \Delta b$ and $b = \sum_j b_j$, where the sum is over those j such that Δ_j intersects K. We proceed, as in the proof of Theorem 1.6, to localize again each b_j, this time using Lemma 1.4 with $h(t) = t$. We get functions

$$P_j(z) = \sum_k c_k \log|z - w_k| + 2\,\mathrm{Re}\left(\frac{1}{z} * \mu_j\right),$$

where $c_k \in \mathbb{C}$, $w_k \in K$ and μ_j is a complex Borel measure with compact support contained in K, satisfying $|\mu_j|(\Delta(z, r)) \leqslant Cr$, $z \in \mathbb{C}$, $r > 0$. Moreover

$$b_j = P_j + O(|z|^{-2}) \quad \text{as} \quad z \to \infty,$$

and

$$\|P_j\|_* \leqslant C\|b_j\|_* \leqslant C\|b\|_*.$$

We need a triple zero at ∞, so we set

$$b_j - P_j = \frac{c_j}{(z - z_j)^2} + \frac{\bar{c}_j}{(\bar{z} - \bar{z}_j)^2} + O(|z|^{-3}) \quad \text{as} \quad z \to \infty,$$

z_j being the center of Δ_j. We recall that

$$|c_j| \leqslant C\delta^2\|b_j - P_j\|_* \leqslant C\delta^2\|b\|_*.$$

Let $\psi_j \in C_0^\infty(\Delta_j)$, $\int \psi_j = 1$ and $\|\psi_j\|_\infty \leqslant C\delta^{-2}$. Define $B_j = 2\,\mathrm{Re}\,B(c_j\psi_j)$ (B is the Beurling transform).

Thus

$$b_j = P_j + B_j + O(|z|^{-3}) \quad \text{as} \quad z \to \infty$$

and

$$\|B_j\|_* \leqslant C\|b\|_*.$$

Set

$$\beta_\delta = \sum_j P_j + B_j \quad \text{and} \quad D_\delta = b - \beta_\delta.$$

We claim that $\beta_\delta \to b$ as $\delta \to 0$, uniformly on compact subsets of V. To prove the claim consider a compact $H \subset V$ and choose δ so that $3\delta < d$, d being the distance from H to K. We have the decay estimate

$$|b_j(z) - P_j(z) - B_j(z)| \leqslant C\delta^3|z - z_j|^{-3}\|b\|_*, \qquad |z - z_j| > 2\delta,$$

and therefore, for $z \in H$ and any positive integer N,

$$\sum_j |b_j(z) - P_j(z) - B_j(z)| \leqslant \Big(\sum_{|z-z_j|\le N\delta} + \sum_{|z-z_j|>N\delta}\Big) C\delta^3 |z-z_j|^{-3}\|b\|_*$$
$$\leqslant CN^2\delta^3 d^{-3}\|b\|_* + C\|b\|_* + C\|b\|_* \sum_{n=N}^{\infty} n^{-2}.$$

One concludes the proof of the claim by taking N big enough and then δ small enough.

Because of Lemma 1.1 $\|D_\delta\|_* \leqslant C\|b\|_*$. Take now a sequence $\delta_n \to 0$ and set $\beta_n = \beta_{\delta_n}$, $D_n = D_{\delta_n}$. Passing to a subsequence we can assume $\beta_n \to \beta$ and $D_n \to D$ weak* in $BMO(\mathbb{R}^2)$. Since $D_n \to 0$ uniformly on compact subsets of V, $D = 0$ on V. Hence $\langle D, \Delta f\rangle = 0$ because D lives on V^c and Δf on $U \subset V$. Thus

$$b(f) = \langle \beta, \Delta f\rangle.$$

Now, $\langle \beta, \Delta f\rangle = 0$ will follow from

$$\langle P_j, \Delta f\rangle = 0 \tag{18}$$

and

$$\Big\langle \sum_j B_j, \Delta f\Big\rangle \to 0 \quad \text{as} \quad \delta \to 0. \tag{19}$$

To show (18) write

$$\langle \log|z - w_k|, \Delta f\rangle = 2\pi f(w_k),$$

which vanishes because $w_k \in F$, and

$$-\Big\langle \frac{1}{z} * \mu_j, \Delta f\Big\rangle = 4\pi\langle \mu_j, \partial f\rangle = 4\int \partial f\, d\mu_j,$$

which is zero because μ_j lives on F, μ_j vanishes on sets of zero length and ∇f vanishes M^1 – a.e. on F.

We turn now to the proof of (19). Set $\chi_j = c_j\psi_j$. Then

$$\begin{aligned} -\Big\langle \sum_j B(\chi_j), \Delta f\Big\rangle &= 4\Big\langle \sum_j B(\bar\partial\chi_j), \partial f\Big\rangle \\ &= 4\Big\langle \sum_j C(\bar\partial\chi_j), \partial^2 f\Big\rangle \\ &= 4\pi\Big\langle \sum_j \chi_j, \partial^2 f\Big\rangle, \end{aligned}$$

and so

$$\Big|\Big\langle \sum_J B_j, \Delta f\Big\rangle\Big| \leqslant C\int_{\cup_j\Delta_j} |\partial^2 f| \to C\int_E |\partial^2 f| \quad \text{as} \quad \delta \to 0.$$

But the last integral vanishes because $\partial^2 f = 0$ a.e. on E, which follows from the fact that $\partial f = 0$ a.e. on E.

Acknowledgements. The idea of the three points Lemma 1.3 was suggested four years ago by J. Garnett to the second author in connection with a similar problem. We are grateful also to J. Dorronsoro for supplying a proof of Lemma 2.1 and to J. J. Carmona who brought to our attention the result of Kolmogorov and Vercenko used in Section 2.

References

[1] Adams, D. R. A note on the Choguet integrals with respect to Hausdorff capacity, Proceedings of the US-Swedish Conference of Function Spaces and Interpolation Theory (1986), Lund, Sweden.

[2] Boivin, A. and Verdera, J. Approximation analytique sur les fermés du plan. Preprint (1988).

[3] Deny, J. Systèmes totaux de fonctions harmoniques, *Ann. Inst. Fourier,* **1**(1949), 103-113.

[4] García-Cuerva, J. and Rubio de Francia, J. L. *Weighted norm inequalities and related topics*, North-Holland Mathematical Studies, Amsterdam, 1985.

[5] Harvcy, R. and Polking, J. Removable singularities for solutions of linear partial differential equations, *Acta Math.* **125**(1970), 39-56.

[6] Hedberg, L. I. Two approximation problems in function spaces, *Ark. Mat.* **16**(1978), 51-81.

[7] Hedberg, L. I. and Wolff, T. H. Thin sets in non-linear potential theory. *Ann. Inst. Fourier,* **33**(1983), 161-187.

[8] Holden, P. J. Extension theorems for functions of vanishing mean oscillation, PhD Thesis (1987), California Inst. of Technology.

[9] Kaufman, R. Hausdorff measure, BMO, and analytic functions, *Pacific J. Math.* **102**(1982), 369-371.

[10] Keldysh, M. V. On the solubility and stability of the Dirichlet problem, *Amer. Math. Soc. Transl.* **(2) 51**(1966), 1-73.

[11] Orobitg, J. Spectral synthesis in spaces of functions with derivatives in H^1. Preprint (1988).

[12] Poornima, S. On the Sobolev spaces $W^{k,1}(\mathbb{R}^n)$, *Lecture Notes Math.*, 992, Springer-Verlag, 1983.

[13] Saks, S. *Theory of the Integral*, Dover Publ., New York, 1964.

[14] Spanne, S. Some function spaces defined using the mean oscillation over cubes, *Ann. Scuola Norm. Sup. Pisa* **19**(1965), 593-608.

[15] Stein, E. M. The differentiability of functions in $\mathbb{R}^n$, *Annals of Math.* **113**(1981), 383-385.

[16] Stein, E. M. *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

[17] Verdera, J. BMO rational approximation and one dimensional Hausdorff content, *Trans. Amer. Math. Soc.* **297**(1986), 283-304.

[18] Verdera, J. C^m approximation by solutions of elliptic equations, and Calderón-Zygmund operators, *Duke Math. J.* **55**(1987), 157-187.

Joan Mateu
Departament de Matemàtiques
Universitat Politècnica de Catalunya
Diagonal 647, 08028 Barcelona.
SPAIN

Joan Verdera
Departament de Matemàtiques
Universitat Autónoma de Barcelona
08193 Bellaterra, Barcelona
SPAIN

Both authors were partially supported by the grant CAYCIT PR 85-0374.