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Estimates of Kernels on Three-Dimensional *CR* Manifolds

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Introduction

(i) Description of results.

Let $\mathcal{M} = b\Omega$ be the three-dimensional smooth pseudo-convex boundary of a bounded domain $\Omega \subset \mathbb{C}^2$. On $\mathcal{M}$ there is a natural Cauchy-Riemann operator $\bar\partial_b$ mapping functions to sections of a suitable line bundle $\mathcal{B}^{1,0}$ of 1-forms. (See below.) The restriction to $\mathcal{M}$ of an analytic function on Ω satisfies the tangential Cauchy-Riemann equations $\bar\partial_b F = 0$. We denote by $\mathcal{H}_b$ the closed subspace of $L^2(\mathcal{M})$ annihilated by $\bar\partial_b$, and define the Szegö projection S_b as the orthogonal projection from $L^2(\mathcal{M})$ onto $\mathcal{H}_b$. The Szegö projection is given in terms of an integral kernel as

$$S_b f(x) = \int_{\mathcal{M}} K(x,y) f(y)\, dy.$$

One of our goals is to describe the singularities of the Szegö kernel $K(x,y)$.

Another goal is to solve the inhomogeneous Cauchy-Riemann equations $\bar\partial_b u = f$. To obtain a well-posed problem, we must take into account that both $\bar\partial_b$ and its adjoint have large nullspaces. Thus we pick a Hermitean metric on the bundle $\mathcal{B}^{1,0}$ and let $S_b^{1,0}$ be the orthogonal projection from L^2-sections of $\mathcal{B}^{1,0}$ to the subspace $\mathcal{H}_b^{1,0}$ annihilated by $\bar\partial_b^*$. The correct statement of the inhomogeneous Cauchy-Riemann equations is

$$\bar\partial_b u = f - S_b^{1,0} f \quad \text{with} \quad u \perp \mathcal{H}_b.$$

Again, the solution u is given by an integral kernel,

$$u(x) = \int_{\mathfrak{M}} H(x, y) f(y)\, dy$$

and another of our goals is to describe the singularities of $H(x, y)$. Finally we will also consider the equation

$$\Box_b u = \bar{\partial}_b^* \bar{\partial}_b u = f - S_b f$$

with $u \perp \mathcal{H}_b$. Then, u is given by

$$u(x) = \int_{\mathfrak{M}} R(x, y) g(y)\, dy$$

and we will describe the singularities of $R(x, y)$.

Let r denote a defining function of $b\Omega$, such that r is C^∞ in a neighborhood of $b\Omega$, $dr \neq 0$, and $r(P) > 0$ if $P \notin \bar{\Omega}$, $r(P) = 0$ if $P \in b\Omega$, $r(P) < 0$ if $P \in \Omega$. Let L, $\bar{L}$ and T be the vector fields on $b\Omega$ defined by

(1)
$$L = r_{z_2} \frac{\partial}{\partial z_1} - r_{z_1} \frac{\partial}{\partial z_2},$$

$$\bar{L} = r_{\bar{z}_2} \frac{\partial}{\partial \bar{z}_1} - r_{\bar{z}_1} \frac{\partial}{\partial \bar{z}_2},$$

$$T = \sqrt{-1} \left(r_{\bar{z}_1} \frac{\partial}{\partial z_1} + r_{\bar{z}_2} \frac{\partial}{\partial z_2} - r_{z_1} \frac{\partial}{\partial \bar{z}_1} - r_{z_2} \frac{\partial}{\partial \bar{z}_2} \right).$$

These vector fields are linearly independent at each $P \in b\Omega$ and the vector field $[L, \bar{L}]$ is tangent to $b\Omega$ hence we may write

(2)
$$[L, \bar{L}] = \frac{1}{\sqrt{-1}} \theta T + aL + b\bar{L}.$$

The condition that Ω is pseudo-convex is then expressed by $\theta \geqslant 0$. Let $\mathcal{H}_b \subset L_2(b\Omega)$ denote the space of square integrable fuctions which are annihilated by $\bar{L}$ in the distribution sense (*i.e.* $h \in \mathcal{H}_b$ means $h \in L_2(b\Omega)$ and $(h, \bar{L}^* v) = 0$ for all $v \in C^\infty(b\Omega)$). It is proved in [K] and in [BS] that the range of $\bar{L}$ in $L_2(b\Omega)$ is closed. Thus given $f \in L_2(b\Omega)$ with f orthogonal to the null space of $\bar{L}^*$ there exists a unique solution u such that

(3)
$$\bar{L} u = f \quad \text{and} \quad u \perp \mathcal{H}_b.$$

Also, given $g \perp \mathcal{H}_b$ there exists a unique u such that

(4)
$$\bar{L}^* \bar{L} u = g \quad \text{and} \quad u \perp \mathcal{H}_b.$$

Denote by S_b: $L_2(b\Omega) \to \mathcal{H}_b$ the orthogonal projection. Note that if $f, \bar{L}f \in$ $\in L_2(b\Omega)$ and if u satisfies

(5) $$\bar{L}u = \bar{L}f \quad \text{with} \quad u \perp \mathcal{H}_b$$

then, by linear algebra we have

(6) $$S_b f = f - u.$$

Thus there exist distributions K, H and R on $b\Omega \times b\Omega$ such that

(7) $$S_b f(x) = \int_{b\Omega} K(x, y) f(y)\, dy,$$

(8) $$u(x) = \int_{b\Omega} H(x, y) f(y)\, dy$$

implies that u satisfies (3) whenever f is orthogonal to the null space of $\bar{L}^*$, and

(9) $$u(x) = \int_{b\Omega} R(x, y) g(y)\, dy$$

implies that u satisfies (4) whenever $g \perp \mathcal{H}_b$.

The purpose of this paper is to prove the optimal non-isotropic estimates for K, H and R in a neighborhood of (P, P) where $P \in b\Omega$, is of finite type. Such estimates have been obtained by M. Christ (see [Ch2]) and the estimates of the Szegö kernel are contained in Nagel, Rosay, Stein, and Wainger (see [NRSW]). We will describe our results more precisely below but first we formulate the above in a slightly more general setting.

Let $\mathfrak{X}$ be a two-dimensional complex manifold and let $\Omega \subset \mathfrak{X}$ be an open set such that $\bar{\Omega}$ is compact and its boundary $b\Omega$ is smooth and pseudo-convex. Now the formulas given in (1) make sense only locally and, in fact, there may not be a global non-vanishing vector field tangent to $b\Omega$ which is of type $(1, 0)$, *i.e.* a combination of the $\partial/\partial z$. To overcome this difficulty let $T^{1,0}(b\Omega)$ be the subbundle of the complexified tangent bundle $\mathbb{C}T(b\Omega)$ consisting of vectors of type $(1, 0)$. Let $T^{0,1}(b\Omega) = \overline{T^{1,0}(b\Omega)}$ and denote by $B^{0,1}(b\Omega)$ the dual space of $T^{0,1}(b\Omega)$. We define the operator $\bar{\partial}_b$: $C^\infty(b\Omega) \to$ sections of $B^{0,1}(b\Omega)$, by

(10) $$\langle (\bar{\partial}_b u)_P, \bar{Z} \rangle = \bar{Z}(u),$$

where $P \in b\Omega$ and $\bar{Z} \in T_P^{0,1}(b\Omega)$.

On each fiber $\mathbb{C}T_P(b\Omega)$ we construct a hermitean inner product, depending smoothly on P, so that $T_P^{1,0}(b\Omega)$ is orthogonal to $T_P^{0,1}(b\Omega)$. We choose a volume element on $b\Omega$ which is compatible with this inner product and we define the spaces of square-integrable functions $L_2(b\Omega)$ and square-integrable $(0, 1)$-forms $L_2^{0,1}(b\Omega)$; this is the completion of the space of sections of

$B^{0,1}(b\Omega)$. Now, in case the range of $\bar\partial_b$ is closed, there exists a unique u such that

$$\bar\partial_b u = f \quad \text{and} \quad u \perp \mathcal{H}_b, \tag{3'}$$

whenever $f \in L_2^{0,1}(b\Omega)$ and $f \perp$ null space of $\bar\partial_b^*$, here $\mathcal{H}_b$ denotes the null space of $\bar\partial_b$ in $L_2(b\Omega)$. We also have for each $g \in L_2(b\Omega)$, $g \perp \mathcal{H}_b$ a unique u such that

$$\Box_b u = \bar\partial_b^* \bar\partial_b u = g \quad \text{and} \quad u \perp \mathcal{H}_b. \tag{4'}$$

The condition that the range of $\bar\partial_b$ is closed is not always satisfied. In [K] it is proved that the range of $\bar\partial_b$ is closed whenever there exists a strictly plurisubharmonic function defined in a neighborhood of $b\Omega$. Thus, in particular, if Ω is a pseudo-convex domain in a Stein manifold then the range of $\bar\partial_b$ is closed. The kernels defined by (7), (8), and (9) can thus be interpreted in this more general situation and our results also apply to them. Our results apply to the case when $b\Omega$ is replaced by an abstract three-dimensional, pseudo-convex, compact *CR* manifold $\mathcal{M}$. The *CR* structure on a three-dimensional manifold $\mathcal{M}$ is given by a one-dimensional subbundle $T^{1,0}(\mathcal{M})$ of the complexified tangent bundle $\mathbb{C}T(\mathcal{M})$ with the property that

$$T^{1,0}(\mathcal{M}) \cap \overline{T^{1,0}(\mathcal{M})} = \{0\}.$$

All of the above notions can then be defined on $\mathcal{M}$ and if the range of $\bar\partial_b$ is closed then the kernels given by (7), (8), and (9) make sense and our results apply to them.

Consider $P_0 \in \mathcal{M}$, with $\mathcal{M}$ a three-dimensional, pseudo-convex *CR* manifold. Then there exist a neighborhood U of P and linearly independent vector fields L, $\bar L$ and T on U such that L has values in $T^{1,0}(\mathcal{M})$, T is real and (2) is satisfied with $\theta \geqslant 0$. We say that P_0 is of *finite type* if for some m the vector fields L, $\bar L$ and their commutators of order less than or equal to m span the tangent space of $b\Omega$ at P. We say that P_0 is of *type m* if m is the least number satisfying this condition. If P_0 is of type m, then there is a small neighborhood U of P_0 in which every point is of type less than or equal to m. For the statements of our results we fix such a neighborhood U.

The results in this paper are stated in terms of a family of non-Euclidean balls on «cylinders». That such cylinders are crucial is clear to anyone who has studied the work of Stein *e.g.* [NSW1], [NSW]. We define these cylinders in Section 2 and recall their basic properties. For each $P \in U$ and each $\delta \in (0, 1]$ the «cylinder» is a neighborhood of P denoted by $B(P, \delta)$. If $\delta_1 < \delta_2$ then $B(P, \delta_1) \subset B(P, \delta_2)$ and $\bigcap_\delta B(P, \delta) = \{P\}$. The $B(P, \delta)$ have height proportional to δ and a base whose radius is proportional to $\gamma(P, \delta)$ with $C\delta^{1/2} \leqslant \gamma(P, \delta) \leqslant C'\delta^{1/m}$. Let $X_1 = \mathrm{Re}\,(L)$ and $X_2 = \mathrm{Im}\,(L)$. Our main theorem is formulated in terms of the following definition.

Definition. *If Q is a distribution on $\mathfrak{M} \times \mathfrak{M}$ and if $P_0 \in \mathfrak{M}$ is of type m with a neighborhood U as above then Q is of non-isotropic order q, with $q = 0, 1$, and 2 if whenever p_k is a homogeneous polynomial in four non-commuting variables then there exists C_k such that*

$$|p_k(X_x, X_y)Q(x, y)| \leqslant C_k \frac{\gamma(P, \delta)^{q-k}}{\gamma(P, \delta)^2 \delta} \tag{11}$$

whenever $x \in B(P, \delta/3)$ and $y \in B(P, \delta) \backslash B(P, \delta/3)$. Here X_x and X_y denote X_1, X_2 acting on x and y respectively.

The purpose of this paper is to give a self-contained proof of the following theorem:

Main theorem. *Let $\mathfrak{M}$ be a three-dimensional compact pseudo-convex CR manifold. Suppose that the operator $\bar{\partial}_b$ has closed range in $L_2(\mathfrak{M})$ (as noted above this assumption is satisfied whenever $\mathfrak{M}$ is the boundary of a domain in a Stein manifold). Suppose further that $P_0 \in \mathfrak{M}$ is of type m. Then the distributions defined by* (7), (8), *and* (9) *satisfy the following*:

(*A*) *K is of non-isotropic order zero.*
(*B*) *H is of non-isotropic order one.*
(*C*) *R is of non-isotropic order two.*

These results may be found in Christ [Ch. 1, 2] and in Nagel, Rosay, Stein, and Wainger [NRSW]. The main application of this theorem is to the understanding of the mapping properties of the fundamental solutions on function spaces. This is carried out in [NRSW] by proving also certain cancellation properties of K.

(ii) Description of the proof.

In the equation $\bar{\partial}_b u = f$ the right hand side is of course a section of a line bundle (*i.e.* $B^{0,1}(\mathfrak{M})$). However, virtually all our analysis is local, and we use a local trivialization to write the $\bar{\partial}_b$ equation on U in the form $\bar{L}u = f$. Here, by abuse of notation, f is written as a function, not a section. Similarly the operator $\bar{\partial}_b^*$ when applied to sections over U can be identified with $\bar{L}^*$. By $\bar{L}^*$ we mean the formal adjoint of $\bar{L}$ which can be expressed as

$$\bar{L}^* v = -Lv + av \tag{12}$$

where $a \in C^\infty(\mathfrak{M})$.

The condition that the range of $\bar{\partial}_b$ is closed is equivalent to the estimate

$$\|u\| \leqslant C \|\bar{\partial}_b u\|, \tag{13}$$

for all $u \perp \mathcal{H}_b$. It is also equivalent to the condition that the range of $\bar{\partial}_b^*$ is closed which, in turn, is equivalent to the estimate

$$\|v\| \leqslant c\|\bar{\partial}_b^* v\|, \tag{14}$$

for all v orthogonal to the null space of $\bar{\partial}_b^*$. We will make essential use of the fact that if the range of $\bar{\partial}_b$ is closed then $\mathcal{H}_b$ is the orthogonal complement of the range of $\bar{\partial}_b^*$ and hence whenever $u \perp \mathcal{H}_b$ there exists v so that $u = \bar{\partial}_b^* v$.

The main theorem is a consequence of the following three estimates. First, there exists $C > 0$ such that

$$\|u\|_{B(P,\delta)} \leqslant C\gamma(P,\delta)\left(\sum_1^2 \|X_j u\| + \|u\|\right), \tag{15}$$

for $u \in C^\infty(\mathfrak{M})$, $P \in U$, and $\delta \in (0, 1]$, where $\|u\|_{B(P,\delta)}$ denotes the L_2-norm over $B(P, \delta)$.

Second, there exists $\epsilon > 0$ such that if $\zeta, \zeta' \in C_0^\infty(U)$ with $\zeta' = 1$ in a neighborhood of the support of ζ then for every $s \in \mathbb{R}$ there exists $C_s > 0$ such that

$$\|\zeta u\|_{s+\epsilon} \leqslant C(\|\zeta' \bar{L} u\|_s + \|u\| + \|v\|), \tag{16}$$

whenever $u = \bar{L}^* v$, more precisely we should have $u = \bar{\partial}_b^* G$ with $G = v\bar{\omega}$ in U and v should really be replaced by G. Throughout this paper we will continue to treat sections of $B^{0,1}(\mathfrak{M})$ as if they were functions. This will not lead us to any difficulties since any operations we perform on these sections will be confined to U and on U the sections can be identified with functions as noted above.

Third, suppose that $\bar{\partial}_b u = f$. Then, if $u \perp \mathcal{H}_b$ we have

$$\|L(\zeta u)\| \leqslant C\|f\|. \tag{17}$$

These estimates imply the theorem, as in Christ [Ch. 1], [Ch. 2] and in Nagel, Rosay, Stein and Wainger [NRSW]. We are grateful to E.M. Stein for conversations directing our attention to the search for a simple, direct proof of (17). It follows from (16) that the kernels K, H and R are C^∞ off the diagonal. This can be expressed by the following inequalities: given closed sets $B_1, B_2 \subset U$ with $B_1 \cap B_2 = \emptyset$ and $k > 0$ then there exists $C > 0$ such that

$$\sup |D_x^\alpha D_y^\beta K(x, y)| \leqslant C, \tag{18}$$

where the supremum is taken over $x \in B_1$, $y \in B_2$, and $|\alpha| + |\beta| \leqslant k$. The same inequalities hold for H and R. To go from (18) to the desired estimates, given by (11), we «rescale». Roughly speaking this means that for each $P \in U$ and $\delta \in (0, 1]$ we construct a one-to-one map of $B(P, \delta)$ to a fixed open set $\tilde{U} \subset \mathbb{R}^3$.

This map takes the *CR* structure on U to a *CR* structure on $\tilde{U}$ and the kernels K, H and R to kernels $\tilde{K}$, $\tilde{H}$ and $\tilde{R}$ respectively. Estimates (15), (16) and (17) are then used to show that (18) holds for $\tilde{K}$, $\tilde{H}$ and $\tilde{R}$ with C independent of P and δ. The proof of (18) for the operator $\tilde{H}$ depends on a two step rescaling devised by Christ [Ch. 2], we reproduce his proof in Section 7. The estimates (11) then follow from scaling back to the original kernels.

We will briefly discuss the proofs of the estimates (15), (16) and (17). Estimate (15) follows from the results of Rothschild and Stein (see [RS]), we give a self contained proof using microlocal techniques. Estimate (16) is proved in [K1] and a more detailed proof is presented in this paper. Here we indicate the idea of this proof since it is used again in proving (17).

Choose a coordinate system y_0, y_1, y_2 in U with origin at P_0 such that $T = g\partial/\partial y_0$ with $g(0) > 0$ we assume that U is small enough so that $g \geqslant \text{const.} > 0$ on U. Let ξ_0, ξ_1, ξ_2 be the dual coordinates and let $\xi' = (\xi_1, \xi_2)$. We will microlocalize by making use of multipliers of the Fourier transform in the following classes.

(19)
$$\begin{aligned}
\mathcal{C} &= \{p \in C^\infty(\mathbb{R}^3) : p \geqslant 0,\ p(t\xi) = p(\xi) \text{ whenever } t \geqslant 1 \text{ and } |\xi| \geqslant 1\} \\
\mathcal{C}^+ &= \{p \in \mathcal{C} : \text{there exists } a > 0 \text{ such that } p(\xi) = 0 \text{ when } \xi_0 < a|\xi'| \text{ and } |\xi| \geqslant 1\}. \\
\mathcal{C}^0 &= \{p \in \mathcal{C} : \text{there exists } a > 0 \text{ such that } p(\xi) = 0 \text{ when } |\xi_0| > a|\xi'|\} \\
\mathcal{C}^- &= \{p \in \mathcal{C} : \text{there exists } a > 0 \text{ such that } p(\xi) = 0 \text{ when } \xi_0 > -a|\xi'|\}.
\end{aligned}$$

For $u \in C_0^\infty(U)$ we define $\mathcal{P}^+u(x)$, $\mathcal{P}^0u(x)$, and $\mathcal{P}^-u(x)$ by

$$\int e^{ix\cdot\xi} p(\xi)\hat{u}(\xi)\,d\xi,$$

where $p \in \mathcal{C}^+$, $p \in \mathcal{C}^0$, and $p \in \mathcal{C}^-$ respectively.

To prove (16) we use the Hörmander estimate, there exist $\epsilon > 0$, $C > 0$ such that

$$\|u\|_\epsilon^2 \leqslant C\left(\sum \|X_j u\|^2 + \|u\|^2\right) \tag{20}$$

for all $u \in C_0^\infty(U)$, in [H]. Note that

$$\sum \|X_j u\|^2 + \|u\|^2 \sim \|Lu\|^2 + \|\bar{L}u\|^2 + \|u\|^2.$$

Hence to prove (16) for $s = 0$ it is natural to try to estimate $L(\zeta u)$ by $\bar{L}(\zeta u)$. We have

$$\begin{aligned}
\|L(\zeta u)\|^2 &= -(\bar{L}L(\zeta u), \zeta u) + \cdots \\
&= -(L\bar{L}(\zeta u), \zeta u) + ([L, \bar{L}](\zeta u), \zeta u) + \cdots \\
&= \|\bar{L}(\zeta u)\|^2 + (\theta T(\zeta u), \zeta u) + \cdots,
\end{aligned}$$

where the terms denoted by dots are $O((\|L(\zeta u)\| + \|\bar{L}(\zeta u)\|)\|\zeta u\| + \|\zeta u\|^2)$. Thus we have

$$\|L(\zeta u)\|^2 \leqslant (\theta T(\zeta u), \zeta u) + C(\|\bar{L}(\zeta u)\|^2 + \|\zeta u\|^2). \tag{21}$$

If the first term on the right were negative we could delete it and obtain the estimate for $L(\zeta u)$. In (21) we replace ζu by $\zeta'\mathcal{P}^-\zeta u$, with $\zeta' = 1$ on a neighborhood of supp (ξ). Then the first term on the right can be written as $(R\zeta'\mathcal{P}^-\zeta u, \zeta'\mathcal{P}^-\zeta u)$ where R is a first order pseudo-differential operator with principal symbol equal to $\theta(x)g(x)\xi_0$ when $\xi_0 \leqslant 0$ and 0 if $\xi_0 \geqslant 0$. Hence from Garding's inequality, and (20) we obtain

$$\|\mathcal{P}^-\zeta u\|_\epsilon^2 \leqslant C\|\mathcal{P}^-\zeta\bar{L}u\|^2 + \cdots, \tag{22}$$

where the dots represent terms that get absorbed later. Since $\mathcal{P}^0$ microlocalizes to the elliptic region we also have

$$\|\mathcal{P}^0\zeta u\|_1^2 \leqslant C\|\mathcal{P}^0\zeta\bar{L}u\|^2 + \cdots. \tag{23}$$

Let Λ^s be the operator defined by

$$\widehat{\Lambda^s u}(\xi) = (1 + |\xi|^2)^{s/2}\hat{u}(\xi). \tag{24}$$

Replacing ζu by $\Lambda^\epsilon\zeta u$ in (21) and proceeding as in the derivation of (22) we obtain

$$\|\mathcal{P}^-\zeta u\|_{2\epsilon}^2 \leqslant C\|\mathcal{P}^-\zeta\bar{L}u\|_\epsilon^2 + \cdots. \tag{25}$$

Interchanging the role L and $\bar{L}$ we get

$$\|\mathcal{P}^+\zeta v\|_{2\epsilon} \leqslant C\|\mathcal{P}^+\zeta\bar{L}^*v\|_\epsilon + \cdots \tag{26}$$

since $\bar{L}^*v = -Lv + av$.

Since $u = \bar{L}^*v$ we have

$$\begin{aligned}\|\mathcal{P}^+\zeta u\|_\epsilon^2 &= (\mathcal{P}^+\zeta u, \Lambda^{2\epsilon}\mathcal{P}^+\zeta\bar{L}^*v)\\ &= (\mathcal{P}u^+\zeta\bar{L}u, \Lambda^{2\epsilon}\mathcal{P}^+\zeta v) + \cdots\\ &\leqslant \|\mathcal{P}^+\bar{L}u\|\ \|\mathcal{P}^+\zeta v\|_{2\epsilon} + \cdots\\ &\leqslant \|\mathcal{P}^+\mathcal{P}\zeta\bar{L}u\|\ \|\mathcal{P}^+\zeta u\|_\epsilon + \cdots.\end{aligned}$$

We thus obtain

$$\|\mathcal{P}^+\zeta u\|_\epsilon^2 \leqslant C\|\mathcal{P}^+\zeta\bar{L}u\|^2 + \cdots. \tag{27}$$

Now combining (22), (23) and (27) we obtain (16) with $s = 0$. Then replacing ζu by $\Lambda^s(\zeta u)$ with $s \leqslant k\epsilon$ in (21) and proceeding by induction on k we obtain (16).

Our proof of (17) requires a further microlocalization. Let

$$\psi \in C_0^\infty(\{\xi \in \mathbb{R}^3 : 0 < a < |\xi| < b\}).$$

Let M be a fixed large number and $\delta \in (0, 1]$ then for $u \in C_0^\infty(U)$ we define $\Gamma_\delta u$ by

$$\widehat{\Gamma_\delta u(\xi)} = \psi\left(\frac{\delta}{M}\xi\right)\hat{u}(\xi). \tag{28}$$

In Section 5 we introduce for $s \geqslant 0$, the norms $|||u|||_s$, with $u \in C_0^\infty(U)$, defined by

$$|||u|||_s^2 = \int_0^1 \int_{\mathbb{R}^3} \gamma(P, \delta)^{-2s} |\Gamma_\delta u(P)|^2 \, dP \frac{d\delta}{\delta} + \|u\|^2. \tag{29}$$

We prove that

$$M^\epsilon |||u|||_{s+1} \leqslant C_s \sum_1^r |||X_j u|||_s + C\|u\|, \tag{30}$$

with C_s independent of M. The idea is to prove (17) by using the same type of argument as in the proof of (16) but with the $|||\bullet|||_s$-norms instead of the Sobolev norms and with 1 instead of ϵ. In order to carry this out, and to prove (30), we have to also localize in x-space. To do this we construct functions $\sigma_{P\delta}^0$ which essentially localize to $B(P, \delta)$ but which are supported in larger cylindrical neighborhoods denoted by $\tilde{B}(P, \delta)$. The $\tilde{B}(P, \delta)$ have the same size base as $B(P, \delta)$ but their height is proportional to $\delta\gamma(P, \delta)^{-1}$. The basic fact is that $\gamma(Q, \delta)$ is «almost» constant when $Q \in \tilde{B}(P, \delta)$. We then show that

$$|||u|||_s^2 \sim \int_0^1 \int_{\mathbb{R}^3} \gamma(P, \delta)^{-2s} \|\sigma_{P\delta}^0 \Gamma_\sigma u\|^2 \frac{dP}{\operatorname{vol} B(P, \delta)} \frac{d\delta}{\delta} + \|u\|^2. \tag{31}$$

Now the proof of (30) proceeds by applying (20) to u replaced by $\sigma_{P\delta}^0 \Gamma_\delta u$. We derive an estimate of the form

$$M^{2\epsilon}\gamma(P, \delta)^{-2s-2} \|\sigma_{P\delta}^0 u\|^2 \leqslant C\gamma(P, \delta)^{2s} \sum \|\sigma_{P\delta}^0 \Gamma_s X_j u\|^2 + \cdots, \tag{32}$$

where the dots represent terms whose integral with respect to

$$\frac{dP}{\operatorname{vol} B(P, \delta)} \frac{d\delta}{\delta}$$

is bounded by $C|||u|||_{s+1}$, with C independent of M. The terms represented by the dots arise from various commutators and can be controlled because the dual of the ξ-support of Γ_δ is contained in a ball of radius δ and center P, and so the distance of this ball to the complement of $\tilde{B}(P, \delta)$ is greater than $c\delta^{1-1/m}$. To prove (17) the main step is to prove an analogue of (26) with $\|\bullet\|_{2\epsilon}$ and $\|\bullet\|_{\epsilon}$ replaced by $M^{\epsilon}|||\bullet|||_2$ and $|||\bullet|||_1$, respectively. This is done replacing ζu in (21) with $\sigma^0_{P\delta}\Gamma_\sigma u$ and retracing the steps for (21) to (26) finally obtaining

$$(33) \qquad \sum \|\sigma^0_{P\delta}\Gamma_\delta X_j \mathcal{P}^+ \zeta v\|^2 \leqslant C\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ \zeta \bar{L}^* v\|^2 + \cdots .$$

The dots again represent various commutators and are estimated in such a way that after multiplying by

$$\gamma(P, \delta)^{-2} \frac{dP}{\operatorname{vol} B(P, \delta)} \frac{d\delta}{\delta}$$

and integrating we get

$$\sum |||X_j \mathcal{P}^+ \zeta v|||_1^2 \leqslant C|||\mathcal{P}^+ \zeta v|||_1^2 + C|||\mathcal{P}^+ \zeta v|||_2^2 + \cdots .$$

Choosing M sufficiently large in (30) we obtain an estimate for $|||\mathcal{P}^+ \zeta v|||_2$, analogous to (26). We can then estimate $|||\mathcal{P}^+ \zeta u|||_2$. Finally we get

$$\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ \zeta u\|^2 \leqslant \|\sigma^0_{P\delta}\Gamma_\delta \zeta \bar{L} u\|^2 + \cdots$$

and hence, after integration with respect to

$$\frac{dP}{\operatorname{vol} B(P, \delta)} \frac{d\delta}{\delta},$$

the right hand side is bounded by $C\|\bar{L}u\|$ (or more precisely by $C\|\bar{\partial}_b u\|$ since we use the closed range property). Thus we get the desired bound for $\|L\mathcal{P}^+ \zeta u\|$, the bounds for the term $\|L\mathcal{P}^- \zeta u\|$ and $\|L\mathcal{P}^0 \zeta u\|$ follow easily from (21) with ζu replaced by $\mathcal{P}^- \zeta u$ and $\mathcal{P}^0 \zeta u$, respectively. Combining these we get the desired estimates (17). The principal new point is that (17) has a rather simple proof; all else here will be familiar to experts from [Chr], [M], [NRSW].

(iii) Additional remarks.

Our main objective in writing this paper is to give a self-contained proof of the main theorem. The principal point is to prove (17). Here we wish to describe briefly how (17) also follows from our results in [FK]. Recall that in [FK] we use a pseudo-differential operator A with the following properties

$$(34) \qquad A\mathcal{P}^+ \sim -\bar{L}L\mathcal{P}^+ \quad \text{and} \quad A\mathcal{P}^- \sim -L\bar{L}\mathcal{P}^- .$$

First we will show how (17) follows from the inequality

$$(35) \qquad \sum \|X_i X_j w\| \leqslant C(\|Aw\| + \|w\|),$$

for all $w \in C_0^\infty(U)$. It suffices to estimate $\|L\mathcal{P}^+ \zeta u\|$, when $u \perp \mathcal{H}_b$, $\bar\partial_b u = f$. We have $u = \bar{L}^* v \sim -Lv$ so that

$$\bar{L}\zeta u = -\bar{L}L\zeta v + \cdots$$

hence

$$-\bar{L}L\mathcal{P}^+\zeta v = A\mathcal{P}^+\zeta v = \zeta\mathcal{P}^+\zeta'\bar{L}u + \cdots, \tag{36}$$

the dots, as usual represent terms some of which in the end are absorbed by the left hand side and the rest estimated by the right hand side. From (35) and (36) it then follows that

$$\begin{aligned}\|L\mathcal{P}^+\zeta u\| &\leqslant C\sum \|L^2\mathcal{P}^+\zeta v\| + \cdots \\ &\leqslant C'\|f\|.\end{aligned}$$

The estimate (35) is established starting from the estimate

$$\sum \|X_i v\|^2 \leqslant C|(Av, v)| + C\|v\|^2.$$

Substituting $v = X_j w$ we get

$$\begin{aligned}\sum \|X_i X_j w\|^2 &\leqslant C|(AX_j w, X_j w)| + c\|X_j w\|^2 \\ &\leqslant C\{|(Aw, -X_j^2 w)| + |([A, X_j]w, X_j w)|\} + \cdots\end{aligned}$$

so we have

$$\sum \|X_i X_j w\|^2 \leqslant C'\|Aw\|^2 + C|([A, X_j]w, X_j w)| + \cdots$$

in order to estimate the second term on the right side we have to either microlocalize as above or follow a modification of the calculations in sections 5 and 6 of [FK]. The relevant L^2-estimates for A may also be deduced from the general theory in [F].

We refer the reader to [FK] for a discussion of the many related results in the area and for a more complete reference list. The estimates for the kernels obtained here can be used to prove optimal non-isotropic estimates for the corresponding operators as in Christ [Ch. 2] and Nagel, Rosay, Stein and Wainger [NRSW]. These results are also related to estimates for the $\bar\partial$-equation and the Bergman kernel. In fact, the estimates in the isotropic Hölder norms imply the corresponding estimates for $\bar\partial$ and the Bergman kernel. Recently optimal non-isotropic estimates for the Bergman kernel were obtained by McNeal (see [M]) and by Nagel, Rosay, Stein, and Wainger (see [NRSW]).

A very interesting use of these results has been made by Christ, see [Ch. 3]. He proves that a pseudo-convex, compact, three dimensional CR manifold of finite type, on which $\bar\partial_b$ has a closed range, can be imbedded in $\mathbb{C}^n$, for some n. His method shows how these estimates can be used to construct CR functions which were not accessible by the use of estimates in Sobolev norms.

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1. The First Microlocalization

Lemma 1.1. *Let $\mathfrak{M}$ be a compact pseudo-convex three-dimensional manifold with $P_0 \in \mathfrak{M}$. Then there exists a neighborhood U of P_0 and a constant $C > 0$ such that*

$$(1) \qquad \sum_1^2 \|X_j \mathcal{P}^- w\|^2 \leqslant C\|\mathcal{P}^- \bar{L} w\|^2 + C\|\tilde{\mathcal{P}}^- w\|^2 + C\|\mathcal{R} w\|^2$$

for all $w \in C_0^\infty(U)$, where $\mathcal{P}^-$ and $\tilde{\mathcal{P}}^-$ are pseudo-differential operators with symbols $p, \tilde{p} \in \mathcal{C}^-$ such that $\tilde{p}(\xi) = 1$ when $\xi \in \operatorname{supp}(p)$, and $\mathcal{R}$ is a pseudo-differential operator of order $-\infty$.

PROOF. First note that

$$(2) \qquad \|[\mathcal{P}^-, \bar{L}]w\| \leqslant C\|\tilde{\mathcal{P}}^- w\| + C\|\mathcal{R} w\|,$$

where $\mathcal{R} = [\mathcal{P}^-, \bar{L}](\mathcal{P}^- - 1)$.

Since the X_j are combinations of L and $\bar{L}$ it suffices to show that

$$(3) \qquad \|L\mathcal{P}^- w\|^2 \leqslant C\|\bar{L}\mathcal{P}^- w\|^2 + C\|\mathcal{P}^- w\|^2.$$

To prove (3) note that

$$(4) \qquad \|L\mathcal{P}^- w\|^2 = \|\bar{L}\mathcal{P}^- w\|^2 + (\theta T\mathcal{P}^- w, \mathcal{P}^- w) + O(\|\mathcal{P}^- w\|(\|L\mathcal{P}^- w\| + \|\bar{L}\mathcal{P}^- w\| + \|\mathcal{P}^- w\|)).$$

Pseudo-convexity implies that the symbol of $(\mathcal{P}^-)^*\theta T\mathcal{P}^-$ is non-positive on U. Hence by Garding's inequality

$$(5) \qquad (\theta T\mathcal{P}^- w, \mathcal{P}^- w) \leqslant C(\|\tilde{\mathcal{P}}^- w\|^2 + \|\mathcal{R} w\|^2).$$

Thus combining (4) and (5) we obtain (3) which completes the proof of the lemma.

Corollary 1.2. *Under the same assumptions, we have*

$$(6) \qquad \sum_1^2 \|X_j \mathcal{P}^+ w\|^2 \leqslant C\|\mathcal{P}^+ L w\|^2 + C\|\tilde{\mathcal{P}}^+ w\|^2 + C\|w\|^2.$$

for $w \in C_0^\infty(U)$.

Lemma 1.3. *Under the same assumptions as Lemma* 1.1, *we have*

$$\sum_1^2 \|X_j \mathcal{P}^0 w\|^2 \leqslant C\|\mathcal{P}^0 \bar{L} w\|^2 + C\|\tilde{\mathcal{P}}^0 w\|^2 + C\|\mathcal{R} w\|^2 \tag{7}$$

for $w \in C_0^\infty(U)$.

PROOF. Taking U sufficiently small we see that if $\phi', \phi \in C_0^\infty$, and $\phi = 1$ on U, and $\phi' = 1$ on a neighborhood of $\operatorname{supp}(\phi)$ then

$$\begin{aligned}\|\bar{L}\mathcal{P}^0 w\| &\geqslant \|\phi' L \mathcal{P}^0 w\| - \|\mathcal{R} w\| \\ &\geqslant C\sum_1^2 \left\|\phi' \frac{\partial}{\partial y_j} \mathcal{P}^0 w\right\| - \text{small const.}\left\|\phi' \frac{\partial}{\partial y_0} \mathcal{P}^0 w\right\| - C\|\mathcal{R} w\| \\ &\geqslant C\|\mathcal{P}^0 w\|_1 - C\|\mathcal{R} w\|.\end{aligned}$$

This implies (7) proving the Lemma.

Lemma 1.4. *If in addition to the above assumptions we suppose that* $P_0 \in \mathfrak{M}$ *is of finite type then there exists a neighborhood* U *of* P_0 *such that if* $\zeta, \tilde{\zeta} \in C_0^\infty(U)$, *with* $\tilde{\zeta} = 1$ *on the support of* ζ, *then there exists* $\epsilon > 0$ *and for* $s \in \mathbb{R}$ *a constant* C_s *such that*

$$\|\mathcal{P}^- \zeta u\|_{s+\epsilon} \leqslant C_s(\|\mathcal{P}^- \zeta \bar{L} u\|_s + \|\tilde{\mathcal{P}}^- \tilde{\zeta} \bar{L} u\|_{s-1} + \|\mathcal{P}^- \tilde{\zeta} u\| + \|\mathcal{R} \tilde{\zeta} u\|). \tag{8}$$

PROOF. The fact that P_0 is of finite type implies that there exists a neighborhood U and constants $\epsilon > 0$ and $C > 0$ such that

$$\|w\|_\epsilon^2 \leqslant C\left(\sum_1^2 \|X_j w\|^2 + \|w\|^2\right) \tag{9}$$

for all $w \in C_0^\infty(U)$.

Now we have

$$\begin{aligned}\|\mathcal{P}^- \zeta u\|_{s+\epsilon}^2 &= \|\Lambda^s \mathcal{P}^- \zeta u\|_\epsilon^2 \\ &\leqslant \|\tilde{\zeta}\Lambda^s \mathcal{P}^- \zeta u\|_\epsilon^2 + \|\mathcal{R}\zeta u\|^2 \\ &\leqslant C\left(\sum_1^2 \|X_j \tilde{\zeta}\Lambda^s \mathcal{P}^- \zeta u\|^2 + \|\mathcal{P}^- \tilde{\zeta} u\|_s^2 + \|\mathcal{R}\zeta u\|^2\right) \\ &\leqslant C\left(\sum_1^2 \|X_j \mathcal{P}^- \tilde{\zeta}\Lambda^s \zeta u\|^2 + \|\mathcal{P}^- \tilde{\zeta} u\|_s^2 + \|\mathcal{R}\zeta u\|^2\right) \\ &\leqslant C(\|\mathcal{P}^- \bar{L}\tilde{\zeta}\Lambda^s \zeta u\|^2 + \|\tilde{\mathcal{P}}^- \tilde{\zeta} u\|_s^2 + \|\mathcal{R}\zeta u\|^2) \\ &\leqslant C(\|\mathcal{P}^- \zeta \bar{L} u\|_s^2 + \|\mathcal{P}^- \tilde{\zeta} u\|_s^2 + \|\mathcal{R}\zeta u\|^2).\end{aligned} \tag{10}$$

The desired estimate (8) then follows by induction on k, if we choose $s \leqslant k\epsilon$.

Analogously, using (6) and (7) instead of (3) we obtain

$$(11)\quad \|\mathcal{P}^0\zeta u\|_{s+\epsilon} \leqslant C_s(\|\mathcal{P}^0\zeta\bar{L}u\|_s + \|\tilde{\mathcal{P}}^0\tilde{\zeta}\bar{L}u\|_{s-1} + \|\tilde{\mathcal{P}}^0\tilde{\zeta}u\| + \|\mathcal{R}\tilde{\zeta}u\|)$$

$$(12)\quad \|\mathcal{P}^+\zeta v\|_{s+\epsilon} \leqslant C_s(\|\mathcal{P}^+\zeta\bar{L}^*v\|_s + \|\tilde{\mathcal{P}}^+\tilde{\zeta}\bar{L}^*v\|_{s-1} + \|\tilde{\mathcal{P}}^+\tilde{\zeta}v\| + \|\mathcal{R}\tilde{\zeta}v\|).$$

Proposition 1.5. *Under the same assumptions as Lemma* 1.4, *if* $u = \bar{L}^*v$ *then*

$$(13)\qquad \|\zeta u\|_{s+\epsilon} \leqslant C_s(\|\tilde{\zeta}\bar{L}u\|_s + \|\tilde{\zeta}u\| + \|\tilde{\zeta}v\|).$$

PROOF. From (8) and (11) we obtain

$$(14)\qquad \|\mathcal{P}^-\zeta u\|_{s+\epsilon} + \|\mathcal{P}^0\zeta u\|_{s+\epsilon} \leqslant C_s(\|\tilde{\zeta}\bar{L}u\|_s + \|\tilde{\zeta}u\|).$$

Hence it will suffice to prove that

$$(15)\qquad \|\mathcal{P}^+\zeta u\|_{s+\epsilon} \leqslant C_s(\|\tilde{\zeta}\bar{L}u\|_s + \|\tilde{\zeta}u\| + \|\tilde{\zeta}v\|).$$

To prove (15) we write

$$(16)\qquad \begin{aligned} \|\mathcal{P}^+\zeta u\|^2_{s+\epsilon} &= (\mathcal{P}^+\zeta u, \Lambda^{2s+2\epsilon}\mathcal{P}^+\zeta\bar{L}^*v) \\ &= (\mathcal{P}^+\zeta\bar{L}u, \Lambda^{2s+2\epsilon}\mathcal{P}^+\zeta v) \\ &\quad + O(\|\mathcal{P}^+\zeta u\|_{s+\epsilon}(\|\tilde{\mathcal{P}}^+\tilde{\zeta}v\|_{s+\epsilon} + \|\tilde{\zeta}v\|)) \\ &\quad + ([\bar{L}, \mathcal{P}^+\zeta]\tilde{\zeta}u, \Lambda^{2s+2\epsilon}\mathcal{P}^+\zeta v). \end{aligned}$$

Setting $\mathcal{E} = [\bar{L}, \mathcal{P}^+\zeta]$ we can write the last term above as

$$(17)\quad \begin{aligned} (\mathcal{E}\tilde{\zeta}u, \Lambda^{2s+2\epsilon}\mathcal{P}^+\zeta v) &= (\mathcal{P}^+\zeta u, \Lambda^{2s+2\epsilon}\mathcal{E}\tilde{\zeta}v) \\ &\quad + O(\|\tilde{\mathcal{P}}^+\tilde{\zeta}u\|^2_{s+\epsilon-1} + \|\tilde{\mathcal{P}}^+\tilde{\zeta}v\|^2_{s+\epsilon} + \|\tilde{\zeta}u\|^2 + \|\tilde{\zeta}v\|^2). \end{aligned}$$

Combining (16) and (17) we get

$$(18)\quad \begin{aligned} \|\mathcal{P}^+\zeta u\|^2_{s+\epsilon} &\leqslant \text{large. const. } \|\mathcal{P}^+\zeta\bar{L}u\|^2_s + \text{small const. } \|\mathcal{P}^+\zeta v\|^2_{s+2\epsilon} \\ &\quad + \text{small const. } \|\mathcal{P}^+\zeta u\|^2_{s+\epsilon} + \text{large const. } \|\tilde{\mathcal{P}}^+\tilde{\zeta}v\|^2_{s+\epsilon} \\ &\quad + C\|\tilde{\mathcal{P}}^+\tilde{\zeta}u\|^2_{s+\epsilon-1} + C\|\tilde{\zeta}u\|^2 + C\|\tilde{\zeta}v\|^2. \end{aligned}$$

Now by (12) we obtain

$$(19)\qquad \|\mathcal{P}^+\zeta v\|^2_{s+2\epsilon} \leqslant C_s(\|\mathcal{P}^+\zeta u\|^2_{s+\epsilon} + \|\tilde{\mathcal{P}}^+\tilde{\zeta}u\|^2_{s+\epsilon-1} + \|\tilde{\zeta}v\|^2)$$

and

$$(20)\qquad \|\tilde{\mathcal{P}}^+\tilde{\zeta}v\|^2_{s+\epsilon} \leqslant C_s(\|\tilde{\tilde{\mathcal{P}}}^+\tilde{\tilde{\zeta}}u\|^2_s + \|\tilde{\tilde{\zeta}}v\|^2).$$

Combining the above and changing notation we have

$$\|\mathcal{P}^+\zeta u\|_{s+\epsilon}^2 \leqslant C\|\mathcal{P}^+\zeta\bar{L}u\|_s^2 + C(\|\tilde{\mathcal{P}}^+\tilde{\zeta}u\|_s^2 + \|\tilde{\zeta}u\|^2 + \|\tilde{\zeta}v\|^2). \tag{21}$$

Hence, by induction on k with $s \leqslant k\epsilon$, we obtain the desired estimate (15) and conclude the proof.

If $\mathfrak{M}$ has the property that the range of $\bar{\partial}_b$ is closed then $u \perp \mathcal{H}_b$ if and only if there exists v so that $u = \bar{L}^*v$ and furthermore there exists a constant $C > 0$ so that

$$\|u\| \leqslant C\|\bar{L}u\| \quad \text{whenever} \quad u \perp \mathcal{H}_b, \tag{22}$$

and

$$\|v\| \leqslant C\|\bar{L}^*v\| \tag{23}$$

whenever v is orthogonal to the null space of $\bar{L}^*$.

Theorem 1.6. *Suppose that $\mathfrak{M}$ is a compact, three dimensional, pseudo-convex CR manifold on which the range of $\bar{\partial}_b$ is closed. Suppose further that $P_0 \in \mathfrak{M}$ is a point of finite type. Then there exists a neighborhood U of P_0 and an $\epsilon > 0$ such that if u is in the domain of $\bar{L}$, if $u \perp \mathcal{H}_b$, and if $\zeta\bar{L}u \in H^s$ for all $\zeta \in C_0^\infty(U)$ then $\zeta u \in H^{s+\epsilon}$ for all $\zeta \in C_0^\infty(U)$.*

PROOF. Applying (22) and (23) to (13) we obtain

$$\|\zeta u\|_{s+\epsilon} \leqslant C_s(\|\tilde{\zeta}\bar{L}u\|_s + \|\bar{L}u\|). \tag{24}$$

The theorem is then obtained using standard smoothing operators.

Theorem 1.7. *Under the same assumptions as in the theorem above denote by $S_b\colon L_2(\mathfrak{M}) \to \mathcal{H}_b$ the orthogonal projection. Then there exists a neighborhood U of P_0 such that if f has the property that $\zeta f \in H^s$ for all $\zeta \in C_0^\infty(U)$ then $\zeta S_b(f) \in H^s$ for all $\zeta \in C_0^\infty(U)$.*

PROOF. Let $g = f - S_b(f)$ then $\bar{L}g = \bar{L}f$, with $g \perp \mathcal{H}_b$ and $g = \bar{L}^*v$. Then, applying the previous estimates

$$\begin{aligned}
\|\zeta g\|_{s+\epsilon}^2 &= (\zeta g, \Lambda^{2s+2\epsilon}\zeta\bar{L}^*v)\\
&= (\zeta\bar{L}g, \Lambda^{2s+2\epsilon}\zeta v) + O(\|\tilde{\zeta}g\|_{s+\epsilon/2}\|\tilde{\zeta}v\|_{s+3\epsilon/2})\\
&= (\zeta\bar{L}f, \Lambda^{2s+2\epsilon}\zeta v) + O(\bullet)\\
&= (\zeta f, \Lambda^{2s+2\epsilon}\zeta\bar{L}^*v) + O(\|\tilde{\zeta}f\|_{s+\epsilon}\|\tilde{\zeta}v\|_{s+\epsilon} + \|\tilde{\zeta}g\|_{s+\epsilon/2}\|\tilde{\zeta}v\|_{s+3\epsilon/2})\\
&= O(\|\tilde{\zeta}f\|_{s+\epsilon}(\|\zeta g\|_{s+\epsilon} + \|\tilde{\zeta}v\|_{s+\epsilon}) + (\|\tilde{\zeta}g\|_{s+\epsilon/2}\|\tilde{\zeta}v\|_{s+3\epsilon/2}))\\
&\leqslant C\|\tilde{\zeta}f\|_{s+\epsilon}^2 + \text{small const. } \|\zeta g\|_{s+\epsilon}^2 + C\|\tilde{\zeta}g\|_{s+\epsilon/2}^2.
\end{aligned}$$

Thus we obtain

$$\|\zeta g\|_{s+\epsilon} \leqslant C(\|\tilde{\zeta} f\|_{s+\epsilon} + \|f\|)$$

hence, replacing $s+\epsilon$ by s, we obtain

$$\begin{aligned} \|\zeta S_b(f)\|_s &\leqslant C\|\zeta f\|_s + C\|\zeta g\|_s \\ &\leqslant C\|\tilde{\zeta} f\|_s + C\|f\|. \end{aligned}$$

Again the proof is concluded by applying smoothing operators.

2. Localization

In this section we recall the localization studied in Sections 2 and 3 of [FK]. Our treatment here is self-contained.

Let X_1 and X_2 be vector fields in a neighborhood U of the origin in $\mathbb{R}^3$. Let T be any vector field on U such that X_1, X_2, and T are linearly independent.

Definition 2.1. *For each k-tuple* $(i_1, \ldots, i_k)$, *with* $i_j = 1$ *or* 2, *we define the functions* $\theta_{i_1 \cdots i_k} \in C^\infty(U)$ *by*

$$[X_{i_k}[X_{i_{k-1}}, \ldots, [X_{i_2}, X_{i_1}]] \ldots] = \theta_{i_1 \cdots i_k} T \bmod (X_1, X_2).$$

We say that the origin is of type m if there exists an m-tuple $(j_1, \ldots, j_m)$ *such that* $\theta_{j_1 \cdots j_m}(0) \neq 0$ *and if for each k-tuple* $(i_1, \ldots, i_k)$ *with* $k < m$ *we have* $\theta_{i_1 \cdots i_k}(0) = 0$.

Then we have

$$(1) \quad X_{j_1} \cdots X_{j_s}(\theta_{i_1 \cdots i_k}) = \theta_{i_1 \cdots i_k j_1 \cdots j_s} + \sum_\alpha a_{i_1 \cdots i_k \alpha} \theta_{i_1 \cdots i_k \alpha} + \sum_{|\beta| \leq s+1} a_\beta \theta_\beta$$

when $a_{i_1 \cdots i_k \alpha} \in C^\infty(U)$ and the α run over all proper subsets of $\{j_1, \ldots, j_s\}$ and $a_\beta \in C^\infty(U)$.

Note that the $\theta_{i_1 \cdots i_k}$ depend on the choice of T, if $\theta'_{i_1 \cdots i_k}$ corresponds to T' then there is a constant $C > 0$ such that

$$C^{-1}|\theta_{i_1 \cdots i_k}(P)| \leqslant |\theta'_{i_1 \cdots i_k}(P)| \leqslant C|\theta_{i_1 \cdots i_k}(P)| \quad \text{for all} \quad P \in U.$$

Let y_0, y_1, y_2 be coordinates in $\mathbb{R}^3$ chosen so that, for suitably small U, we have X_1, X_2, and $\partial/\partial y_0$ are linearly independent on U. For the remainder of this paper we will set

$$T = \frac{\partial}{\partial y_0}.$$

Definition 2.2. *For each $P \in U$ we define λ_k^P by*

$$\lambda_k^P = \sum_{j \leq k} |\theta_{i_1 \dots i_j}(P)|.$$

From now on we will assume that the origin is of type m, then we have $\lambda_m^P \geqslant$ const. > 0 for all $P \in U$.

As in Section 2 of [FK] we define for each $P \in U$ a coordinate system x_0^P, x_1^P, x_2^P by

$$\begin{aligned} x_0^P &= y_0 - y_0(P) - F^P(y_1 - y_1(P), y_2 - y_2(P)) \\ x_0^P &= y_j - y_j(P), \qquad j = 1, 2. \end{aligned} \tag{2}$$

The F^P are polynomials in two variables defined as follows. First we choose vector fields X_1', X_2' such that

$$X_i' = \sum_{j=1}^{2} h_i^j X_j, \tag{3}$$

the h_i^j are smooth with $\det(h_i^j)$ bounded away from zero in a neighborhood U of the origin and

$$X_i' = a_i' \frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_i} \qquad i = 1, 2. \tag{4}$$

Lemma 2.3. *For each $P \in U$ there exists a unique polynomial F^P in two variables, defining x^P by (2), such that $F^P(0,0) = 0$, and the functions b_i^P, defined by*

$$X_i' = b_i^P \frac{\partial}{\partial x_0^P} + \frac{\partial}{\partial x_i^P}, \tag{5}$$

satisfy

$$\left[\left(\frac{\partial}{\partial x_1^P} \right)^j b_1^P \right]_P = 0 \quad \textit{for} \quad j = 0, 1, \dots, m-1 \tag{6}$$

and

$$\left[\left(\frac{\partial}{\partial x_1^P} \right)^i \left(\frac{\partial}{\partial x_2^P} \right)^j b_2^P \right]_P = 0 \quad \textit{for} \quad i + j = 0, 1, \dots, m-1. \tag{7}$$

PROOF. From (2), (4), and (5) we obtain

$$\begin{aligned} b_i^P(x^P) = {} & a_i'(x_0^P + y_0(P) + F^P(x_1^P, x_2^P), x_1^P + y_1(P), x_2^P(P) + y_2(P)) \\ & - F^P_{x_i^P}(x_1^P, x_2^P). \end{aligned} \tag{8}$$

Then (6) together with (8) determine

$$\left[\left(\frac{\partial}{\partial x_1^P}\right)^i F^P\right]_0 \quad \text{for} \quad i = 1, \ldots, m$$

and together with (7) the

$$\left[\left(\frac{\partial}{\partial x_1^P}\right)^i \left(\frac{\partial}{\partial x_2^P}\right)^j F^P\right]_0$$

are determined for $i + j \leqslant m$. □

In terms of the x^P-coordinates we have

(9) $$X_i = a_i^P \frac{\partial}{\partial x_0^P} + \sum_{j=1}^{2} a_{ij}^P \frac{\partial}{\partial x_j^P}.$$

Lemma 2.4. *There exists $C > 0$ such that for all $P \in U$ we have*

(10) $$\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^s \left(\frac{\partial}{\partial x_2^P}\right)^t a_i^P\right]_0\right| \leqslant C\lambda_{s+t+1}^P,$$

and

(11) $$\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^s \left(\frac{\partial}{\partial x_2^P}\right)^t \theta_{i_1 \cdots i_k}\right]_0\right| \leqslant C\lambda_{k+s+t}^P.$$

PROOF. First observe that

$$a_i^P = \sum_{j=1}^{2} g_i^j b_j^P,$$

where (g_i^j) is the inverse of (h_i^j) defined by (3). Hence it suffices to prove (10) with a_i^P replaced by b_i^P, in fact by b_1^P because of (7). Furthermore, defining $\theta'_{i_1 \cdots i_k}$ by

(12) $$[X'_{i_k}, [X'_{i_{k-1}}, \ldots [X'_{i_2}, X'_{i_1}]], \ldots] = \theta'_{i_1 \cdots i_k} T \operatorname{mod}(X'_1, X'_2),$$

we have

(13) $$\theta_{i_1 \cdots i_k} = \sum_{s \leq k} g^{j_1 \cdots j_s} \theta'_{j_1 \cdots j_s},$$

where the $g^{i_1 \cdots i_s}$ are polynomials in $X_{r_1} \cdots X_{r_t}(g_i^j)$; thus it suffices to prove (11) with $\theta_{i_1 \cdots i_k}$ replaced by $\theta'_{i_1 \cdots i_k}$. From (12) we see that

(14) $$\theta'_{12} = b_{1x_2}^P - b_{2x_1}^P + b_2^P b_{1x_0}^P - b_1^P b_{2x_0}^P$$

so (10) for $s + t = 1$ and (11) for $s + t = 0$, $k = 2$ follow.

Rewriting (1), we have with $s + t < m$

$$(X'_1)^s(X'_2)^t(\theta'_{i_1\cdots i_k}) = \theta'_{i_1\cdots i_k\underbrace{1\cdots1}_{s\text{-times}}\underbrace{2\cdots2}_{t\text{-times}}} + \sum_{|\alpha|<s+t} a_{i_1\cdots i_k\alpha}\theta_{i_1\cdots i_k\alpha} + \sum_{|\beta|<s+t+1} a_\beta\theta'_\beta.$$

We also have

$$\left(\frac{\partial}{\partial x_1^P}\right)^s\left(\frac{\partial}{\partial x_2^P}\right)^t\theta'_{i_1\cdots i_k} = (X'_1)^s(X'_2)^t(\theta'_{i_1\cdots i_k})$$

$$+ \sum_{r_1+r_2+q_1+q_2<s+t} a_{r_1r_2q_1q_2}\left[\left(\frac{\partial}{\partial x_1^P}\right)^{r_1}\left(\frac{\partial}{\partial x_2^P}\right)^{r_2} b_j^P\right]\left[\left(\frac{\partial}{\partial x_1^P}\right)^{q_1}\left(\frac{\partial}{\partial x_2^P}\right)^{q_2}\theta'_{i_1\cdots i_k}\right].$$

Differentiating (14) we get

$$\left(\frac{\partial}{\partial x_1^P}\right)^{r_1}\left(\frac{\partial}{\partial x_2^P}\right)^{r_2} b_1 = \left(\frac{\partial}{\partial x_1^P}\right)^{r_1}\left(\frac{\partial}{\partial x_2^P}\right)^{r_2}\theta'_{12}$$

$$+ \sum_{j_1+j_2<r_1+r_2} c_{j_1j_2}\left(\frac{\partial}{\partial x_1^P}\right)^{j_1}\left(\frac{\partial}{\partial x_2^P}\right)^{j_2} b_1$$

$$+ \sum_{j_1+j_2\leq r_1+r_2} d_{j_1j_2}\left(\frac{\partial}{\partial x_1^P}\right)^{j_1}\left(\frac{\partial}{\partial x_2^P}\right)^{j_2} b_2.$$

Since b_2 vanishes to order m at 0 we get

$$\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^{r_1}\left(\frac{\partial}{\partial x_2^P}\right)^{r_2} b_1\right]_0\right| \leqslant C\sum_{j_1+j_2\leq r_1+r_2}\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^{j_1}\left(\frac{\partial}{\partial x_2^P}\right)^{j_2}\theta'_{12}\right]_0\right|.$$

Thus we obtain

$$\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^{s}\left(\frac{\partial}{\partial x_2^P}\right)^{t}\theta'_{i_1\cdots i_k}\right]_0\right| \leqslant C\lambda^P_{s+t+k}$$

$$+ C\sum_{j_1+j_2+q_1+q_2<s+t}\left|\left(\frac{\partial}{\partial x_1^P}\right)^{j_1}\left(\frac{\partial}{\partial x_2^P}\right)^{j_2}\theta'_{12}\right]_0\left|\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^{q_1}\left(\frac{\partial}{\partial x_2^P}\right)^{q_2}\theta'_{i_1\cdots i_p}\right]\right|.$$

Hence (10) and (11) follow by induction.

Definition 2.5. *For $P \in U$ and $2 \leqslant k \leqslant m$ we define $U^{P,k}$ to be the open set given by*

$$U^{P,k} = \left\{Q \in U\colon \sum_{j=1}^{m-k}\lambda^P_{j+k}|(x^P)^j(Q)|^j < \frac{\lambda_k^P}{C_0} \text{ and } |x_0^P(Q)| < \frac{\lambda_k^P}{C_0}\right\}, \tag{15}$$

where

$$|(x^P)'(Q)| = (x_1^P(Q)^2 + x_2^P(Q)^2)^{1/2}$$

and C_0 is suitably large.

Lemma 2.6. *For all $Q \in U^{P,k}$ we have*

$$\frac{1}{2}\lambda_k^P \leqslant \lambda_k^Q \leqslant 2\lambda_k^P. \tag{16}$$

PROOF. Expanding $\theta_{i_1 \dots i_r}$ in Taylor series in the x^P-coordinates and applying (11) we obtain

$$\begin{aligned}(17)\ |\theta_{i_1\dots i_r}(P) - \theta_{i_1\dots i_r}(Q)| &\leqslant C \sum \left| \left[\left(\frac{\partial}{\partial x_1^P}\right)^s \left(\frac{\partial}{\partial x_2^P}\right)^t \theta_{i_1\dots i_r} \right]_0 \right| |(x^P)'(Q)|^{s+t} \\ &\quad + C|(x_0^P)(Q)| + C|(x^P)'(Q)|^m \\ &\leqslant C \sum \lambda_{r+j}^P |(x^P)'(Q)|^j + C|(x_0^P)(Q)|.\end{aligned}$$

Taking $r \leqslant k$ we have $\lambda_{r+j}^P \leqslant \lambda_{k+j}^P$ and since $Q \in U^{P,k}$ we have

$$|\theta_{i_1\dots i_r}(P) - \theta_{i_1\dots i_r}(Q)| \leqslant \frac{2C}{C_0}\lambda_k^P. \tag{18}$$

Thus (16) is established since

$$\lambda_k = \sum_{r \leq k} |\theta_{i_1\dots i_k}|$$

and C_0 is large.

Definition 2.7. *For each $P \in U$ and $\delta > 0$ we define $\gamma(P, \delta)$ to be the positive number satisfying*

$$\delta = \sum_{j=2}^{m} \lambda_j^P \gamma(P, \delta)^j. \tag{19}$$

We define the δ-order at P, denoted by $k(P, \delta)$, to be the least integer such that

$$\lambda_{k(P,\delta)}^P \gamma(P, \delta)^{k(P,\delta)} = \max_j \lambda_j^P \gamma(P, \delta)^j. \tag{20}$$

Then

$$\lambda_{k(P,\delta)}^P \gamma(P, \delta)^{k(P,\delta)} \leqslant \delta \leqslant (m-1)\lambda_{k(P,\delta)}^P \gamma(P, \delta)^{k(P,\delta)}. \tag{21}$$

Hence

$$\gamma(P,\delta) \sim \left(\frac{\delta}{\lambda^P_{k(P,\delta)}}\right)^{1/k(P,\delta)} = \min_j \left(\frac{\delta}{\lambda^P_j}\right)^{1/j}. \tag{22}$$

Lemma 2.8. *There exists* $C > 0$ *such that*

$$\gamma(Q,\delta) \leqslant C\gamma(P,\delta) \quad \textit{for all} \quad Q \in U^{P,k(P,\delta)}. \tag{23}$$

PROOF. By (22) and (16), we have

$$\gamma(Q,\delta) \leqslant C\left(\frac{\delta}{\lambda^Q_{k(P,\delta)}}\right)^{1/k(P,\delta)} \leqslant C'\left(\frac{\delta}{\lambda^P_{k(P,\delta)}}\right)^{1/k(P,\delta)} \leqslant C''\gamma(P,\delta).$$

Lemma 2.9. *There exists* $\eta > 0$, *independent of* P *and* δ *such that the set*

$$\tilde{B}(P,\delta) = \left\{ Q \in U \colon |(x^P)'(Q)| \leqslant \eta\gamma(P,\delta),\ |x_0^P(Q)| \leqslant \eta \frac{\delta}{\gamma(P,\delta)} \right\} \tag{24}$$

is contained in $U^{P,k(P,\delta)}$.

PROOF. From (21) we have

$$\begin{aligned} |x_0^P(Q)| &\leqslant \eta \frac{\delta}{\gamma(P,\delta)} \\ &\leqslant \eta(m-1)\lambda^P_{k(P,\delta)}\gamma(P,\delta)^{k(P,\delta)-1} \\ &\leqslant \frac{\lambda^P_{k(P,\delta)}}{C_0} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{m-k} \lambda^P_{j+k(P,\delta)}|(x^P)'(Q)|^j &\leqslant \eta \sum \lambda^P_{j+k(P,\delta)}\gamma(P,\delta)^j \\ &\leqslant \eta\gamma(P,\delta)^{-k(P,\delta)} \sum \lambda^P_{j+k(P,\delta)}\gamma(P,\delta)^{j+k(P,\delta)} \\ &\leqslant \eta\gamma(P,\delta)^{-k(P,\delta)}\delta \leqslant C\eta\lambda^P_{k(P,\delta)} \leqslant \frac{\lambda^P_{k(P,\delta)}}{C_0}. \end{aligned}$$

Proposition 2.10. *There exists* $C > 0$, *independent of* P *and* δ, *such that*

$$\frac{1}{C}\gamma(P,\delta) \leqslant \gamma(Q,\delta) \leqslant C\gamma(P,\delta) \quad \textit{for all} \quad Q \in \tilde{B}(P,\delta). \tag{25}$$

PROOF. The second inequality follows from Lemmas 1.8 and 1.9. To show that $\gamma(P,\delta) \leqslant C\gamma(Q,\delta)$ first note that (22) can be rewritten as

$$\lambda_j^P \leqslant \delta\gamma(P,\delta)^{-j}.$$

Next, from (17), we have

$$\begin{aligned}|\theta_{i_1,\dots,i_r}(P) - \theta_{i_1,\dots,i_r}(Q)| &\leqslant C\sum_{j=1}^{m-r}\lambda_{r+j}^P\gamma(P,\delta)^j + C\delta\gamma(P,\delta)^{-2}\\ &\leqslant C\delta\gamma(P,\delta)^{-r}.\end{aligned}$$

Now

$$\begin{aligned}\lambda_{k(Q,\delta)}^Q &= \sum_{r\leq k(Q,\delta)}|\theta_{i_1,\dots,i_r}(Q)|\\ &\leqslant \lambda_{k(Q,\delta)}^P + \sum_{r\leq k(Q,\delta)}|\theta_{i_q,\dots,i_r}(P) - \theta_{i_1,\dots,i_r}(Q)|\\ &\leqslant C'\delta\gamma(P,\delta)^{-k(Q,\delta)}\end{aligned}$$

hence

$$\gamma(P,\delta) \leqslant \left(\frac{C'\delta}{\lambda_{k(Q,\delta)}^Q}\right)^{1/k(Q,\delta)} \leqslant \text{const.}\,\gamma(Q,\delta).$$

Lemma 2.11. *There exists $C>0$ such that*

(26) $$\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^s\left(\frac{\partial}{\partial x_2^P}\right)^t a_i^P\right]_Q\right| \leqslant C\delta\gamma(P,\delta)^{-s-t-1}$$

and

(27) $$\left|\left[\left(\frac{\partial}{\partial x_1^P}\right)^s\left(\frac{\partial}{\partial x_2^P}\right)^t \theta_{i_1,\dots,i_k}\right]_Q\right| \leqslant C\delta\gamma(P,\delta)^{-k-s-t}$$

for all $Q\in\tilde{B}(P,\delta)$.

PROOF. Setting

$$D_1 = \frac{\partial}{\partial x_1^P},$$

$$D_2 = \frac{\partial}{\partial x_2^P}$$

and dropping the P's and the δ's we have, using (10)

$$
\begin{aligned}
|D_1^t D_2^s a_i(Q)| &\leqslant C \sum_{l,r} |D_1^{t+l} D_2^{s+r} a_i(0)| \, \Big| \, |x'(Q)|^{l+r} + O(|x_0(Q)| + |x'(Q)|^m) \\
&\leqslant C\Big(\sum_{l,r} \lambda_{t+l+s+r+1} \gamma^{l+r} + \delta\gamma^{-1}\Big) \\
&\leqslant C'\delta\gamma^{-s-t-1}
\end{aligned}
$$

thus proving (26). To prove (27) we use (11) and obtain

$$
\begin{aligned}
|D_1^t D_2^s \theta_{i_1 \cdots i_k}(Q)| &\leqslant C \sum_{l,r} |D_1^{t+l} D_2^{s+r} \theta_{i_1 \cdots i_k}(0)| \, \Big| \, |x'(Q)|^{l+r} \\
&\quad + O(|x_0(Q)| + |x'(Q)|^m) \\
&\leqslant C\Big(\sum_{l,r} \lambda_{t+l+s+r+k} \gamma^{l+r} + \delta\gamma^{-1}\Big) \\
&\leqslant C'\delta\gamma^{-k-s-t}.
\end{aligned}
$$

Definition 2.12. *Let $B(P,\delta)$ denote the subset of $\tilde{B}(P,\delta)$, defined by*

$$B(P,\delta) = \{Q \in \tilde{B}(P,\delta) : |x_0^P(Q)| < \eta\delta\}.$$

Proposition 2.13. *There exists a constant C_1 such that whenever $Q \in B(P,\delta)$ then $B(Q, C_1\delta) \supset B(P,\delta)$ and whenever $Q \in \tilde{B}(P,\delta)$ then $\tilde{B}(Q, C_1\delta) \supset \tilde{B}(P,\delta)$.*

PROOF. Note that, for $C_1 \geqslant 1$ we have

$$
\begin{aligned}
C_1^{1/m}\gamma(Q,\delta) &\leqslant \text{const.}\, \gamma(Q, C_1\delta) \\
&\leqslant C_1^{1/2}\gamma(Q,\delta).
\end{aligned} \tag{28}
$$

Thus it suffices to show that if $Q \in B(P,\delta)$ then there exists a $C_2 > 0$ such that whenever $\bar{Q} \in B(P,\delta)$ we have

$$
\begin{cases}
|x_0^Q(\bar{Q})| \leqslant C_2\eta\delta \\
|(x^Q)'(\bar{Q})| \leqslant C_2\eta\gamma(Q,\delta)
\end{cases} \tag{29}
$$

and whenever $Q, \bar{Q} \in \tilde{B}(P,\delta)$ then

$$
\begin{cases}
|x_0^Q(\bar{Q})| \leqslant C_2\eta\delta\gamma^{-1}(Q,\delta) \\
|(x^Q)'(\bar{Q})| \leqslant C_2\eta\gamma(Q,\delta).
\end{cases} \tag{29$'$}
$$

To prove (29) and (29)$'$ we express the x^Q-coordinates in terms of the x^P-coordinates, we have

$$
\begin{cases}
x_0^Q = x_0^P - x_0^P(Q) - F^{Q,P}(x_1^P - x_1^P(Q), x_2^P - x_2^P(Q)) \\
(x^Q)' = (x^P)' - (x^P)'(Q)
\end{cases} \tag{30}
$$

where $F^{Q,P}$ is a polynomial of order m of the form

$$F^{Q,P}(z_1, z_2) = \sum_{i+j=1}^{m} C_{ij}^{Q,P} z_1^i z_2^j. \tag{31}$$

As in (8) the polynomial $F^{Q,P}$ satisfies

$$F_{x_i^Q}^{Q,P}(x_1^Q, x_2^Q) = b_i^P(x_0^Q + x_0^P(Q) + F^{Q,P}(x_1^Q, x_2^Q), x_1^Q + x_1^P(Q), x_2^Q + x_2^P(Q)) - b_i^Q(x_1^Q, x_2^Q). \tag{32}$$

Hence

$$\sum_{i+j=1} |c_{ij}^{QP}| \leqslant \sum_{1}^{2} |b_i^P(Q)| \leqslant C\delta\gamma(P, \delta)^{-1}.$$

Differentiating (32) with respect to x_1^Q and x_2^Q and evaluating at Q we obtain

$$\sum_{i+j=r} |c_{ij}^{QP}| \leqslant C\delta\gamma(P, \delta)^{-r}.$$

Hence from (30) we get

$$|x_0^Q(\bar{Q})| \leqslant \text{const. } \eta\delta$$

and

$$|(x^Q)'(\bar{Q})| \leqslant \text{const. } \eta\gamma(P, \delta),$$

when $Q, \bar{Q} \in B(P, \delta)$, and

$$|x_0^Q(\bar{Q})| \leqslant \text{const. } \eta\delta\gamma^{-1}(P, \delta)$$
$$|(x^Q)'(\bar{Q})| \leqslant \text{const. } \eta\gamma(P, \delta),$$

when $Q, \bar{Q} \in \tilde{B}(P, \delta)$.

The desired inequalities (29) and (29)′ then follow from Proposition 2.10.

To conclude this section we define $\sigma_{P\delta}^0 \in C_0^\infty(B(P, \delta))$ and $\sigma_{P\delta} \in C_0^\infty(\tilde{B}(P, \delta))$. Let $\tau^0, \tau^1 \in C_0^\infty([0, \infty))$ such that $\tau^0 = 1$ in a neighborhood of 0, $\tau^1 = 1$ in a neighborhood of the support of τ^0, and such that the support of τ^1 is sufficiently small so that

$$\operatorname{supp}\{\tau^1(|x_0^P|\gamma(P, \delta)\delta^{-1})\tau^1(|(x^P)'|\gamma(P, \delta)^{-1})\} \subset \tilde{B}(P, \delta).$$

Setting

$$\phi_{P\delta}(x^P) = \tau^1(|x_0^P|\gamma(P, \delta)\delta^{-1})\tau^1(|(x^P)'|\gamma(P, \delta)^{-1}) \tag{33}$$

we define

$$\sigma_{P\delta}^0(x^P) = \tau^0(|x_0^P|\delta^{-1})\tau^0(|(x^P)'|\gamma(P, \delta)^{-1}), \tag{34}$$

and

$$\sigma_{P\delta}(x^P) = \left(1 + \frac{|x_0^P|}{\delta}\right)^{-N} \phi_{P\delta}(x^P), \tag{35}$$

where N is a large number that will be fixed later.

From (26) we easily deduce

$$|X_{i_1} \cdots X_{i_k} \sigma^0_{P\delta}| \leqslant C\gamma(P, \delta)^{-k} \sigma_{P\delta}. \tag{36}$$

3. Second Microlocalization

In this section we study the operators Γ_δ which localize the Fourier transform of a function to a region where $|\xi| \approx M/\delta$. We prove a number of fairly standard properties of these operators, several of these results (or slight variants thereof) were already established in [FK] and we include them here for completeness.

Definition 3.1. *Let* $\psi \in C_0^\infty(\{\xi \in \mathbb{R}^3 : 0 < a < |\xi| < b\})$. *For* $u \in H^{-s_0}$ *and* $\delta > 0$ *we define* $\Gamma_\delta u$ *by*

$$\widehat{\Gamma_\delta u}(\xi) = \psi\left(\frac{\delta}{M} \xi\right) \hat{u}(\xi).$$

The following is part A of Lemma 1 in Section 1 of [FK].

Lemma 3.2. *Let* $R_1, R_2, \ldots, R_n$ *be pseudo-differential operators such that the supports of their symbols have no point in common. If one of the* $R_1, \ldots, R_n$ *has symbol supported in* $|\xi| \approx M/\delta$ *(that is, in the region* $aM/\delta < |\xi| < bM/\delta$*), then*

$$\|R_1 \cdots R_n u\|_{s_0} \leqslant C\left(\frac{\delta}{M}\right)^{\text{power}} \|u\|_{-s_0}. \tag{1}$$

The «power» and s_0 *may be taken arbitrarily large.*

PROOF. Say R_j has symbol supported in $\{|\xi| \approx M/\delta\}$. Let Q be a pseudo-differential operator of order K, with symbol $q(\xi)$ such that

$$q(\xi) = \left(\frac{M}{\delta}\right)^K$$

on the support of R_j and

$$|\partial^\alpha q(\xi)| \leqslant C_\alpha(1 + |\xi|)^{K - |\alpha|}$$

with C_α independent of δ and M. Then, by the standard theory of pseudo-differential operators, we have

$$\| R_1 \cdots R_j Q R_{j+1} \cdots R_n u \|_{s_0} \leqslant \text{const.}\ \| u \|_{-s_0}$$

where the constant is independent of δ and M. The desired inequality (1) then follows since

$$R_j = \left(\frac{\delta}{M}\right)^K R_j Q.$$

The following result is again a slight variant of a well-known property of standard pseudo-differential operators, it is essential in our analysis and appears as Lemma 2 of Section 1 in [FK], we reproduce it here for completeness.

Lemma 3.3. *Let $T = a(x, D)$ with symbol $a(x, \xi)$ of order zero supported in $|\xi| \approx M/\delta$. Let ϕ be a smooth change of coordinates defined in a fixed neighborhood of 0. If u is supported in a small neighborhood of 0, then*

$$Su(x) = [T(u \circ \phi^{-1})] \circ \phi(x)$$

is well-defined for x in a small neighborhood of 0. There exists a symbol $\tilde{a}(x, \xi)$ of order 0, supported in $|\xi| \approx M/\delta$, so that

$$\| Su - \tilde{a}(x, D) u \|_{H^s(\text{small } nbd \text{ of } 0)} \leqslant c\delta^{\text{power}} \| u \|_{-s}.$$

Here, «power» and s are as large as we please.

PROOF. For x near 0 and u supported near 0 we have

$$Su(x) = \int e^{i\xi(\phi(x) - \phi(y))} a(\phi(x), \xi) \frac{1}{\det \phi'(y)} u(y)\, dy\, d\xi. \tag{2}$$

Now

$$\xi(\phi(x) - \phi(y)) = \xi\left[\int_0^1 \phi'(tx + (1 - t)y)\, dt\right](x - y) = \eta(x - y)$$

with

$$\eta = G(x, y)\xi, \qquad G(x, y) = \left[\int_0^1 \phi'(tx + (1 - t)y)\, dt\right]^t$$

a smooth matrix-valued function. We have $G(x, x) = (\phi'(x))^t$, so G is invertible for x near y. In our integral x and y are both near 0, so we may introduce the smooth matrix-valued function $\tilde{G}(x, y) = (G(x, y))^{-1}$. Changing variables from y, ξ to y, η in (2), we have

$$\text{(3)} \quad Su(x) = \int e^{i\eta(x-y)}[\det \tilde{G}(x, y)a(\phi(x), \tilde{G}(x, y)\eta)] \frac{1}{\det \phi'(y)} u(y)\, dy\, d\eta$$
$$= \int e^{i\eta(x-y)} b(x, \eta, y) u(y)\, dy\, d\eta$$

with b satisfying

$$\text{(4)} \quad \begin{cases} |\partial^\alpha_{x,y}\partial^\beta_\eta b| \leqslant C_{\alpha\beta}\left(\dfrac{M}{\delta}\right)^{-|\beta|} \\ \operatorname{supp} b(x, \eta, y) \subset \left\{|\eta| \approx \dfrac{M}{\delta}\right\} \end{cases}$$

Taylor-expanding b in y about $y = x$, we have

$$\text{(5)} \quad Su(x) = \sum_{|\alpha| \leq 10l} \frac{1}{\alpha!} \int e^{-\eta(x-y)}(y - x)^\alpha [\partial^\alpha_y b(x, \eta, y)|_{y=x}] u(y)\, dy\, d\eta$$
$$+ \int e^{i\eta(x-y)} b^{\#}(x, \eta, y) u(y)\, dy\, d\eta$$

where $b^{\#}$ satisfies (4) and also vanishes to order $10l$ at $x = y$.

Integrating by parts in η in (5), we get

$$Su(x) = \sum_{|\alpha| \leq 10l} C_\alpha \int e^{i\eta(x-y)}[\partial^\alpha_\eta \partial^\alpha_y b(x, \eta, y)|_{y=x}] u(y)\, dy\, d\eta$$
$$+ C \int \{e^{i\eta(x-y)}[|x - y|^{-2l} \Delta^l_\eta b^{\#}(x, \eta, y)]\}\, dy\, d\eta.$$

On the right side, the first term is of the form $\tilde{a}(x, D)u(x)$ with $\tilde{a}(x, \xi)$ a symbol of order zero supported in $|\xi| \approx M/\delta$.

In the second term, the integrand and its derivatives up to order l in x, y are $O(M/\delta)^{-2l}$ and supported in $|\eta| \approx M/\delta$. Hence, taking l large, we see that

$$Su(x) = \tilde{a}(x, D)u(x) + \int K(x, y)u(y)\, dy,$$
$$|\partial^\alpha_x \partial^\beta_y K| \leqslant C_l \delta^l, \qquad |\alpha| + |\beta| \leqslant l;$$

l as large as we please. The conclusion of the lemma is now obvious.

Corollary. *Let Φ be a smooth coordinate change defined on $\{|y| < 1\} \subset \mathbb{R}^3$, and let $\theta \in C^\infty_0(|y| < 1)$. Thus $\theta[(\Gamma_\delta u) \circ \phi]$ is well-defined on all $\mathbb{R}^3$. Let P be* *a pseudo-differential operator of order zero with symbol $\sigma(\xi)$ supported in $\{|\xi| < c_1 M/\delta\}$. If $c_1 << 1$, then*

$$\|P\{\theta[(\Gamma_\delta u) \circ \phi]\}\|_s \leqslant C\delta^{\text{power}}\|u\|_{-s}.$$

PROOF. Set $\tilde{\theta} = \theta \circ \Phi^{-1}$, and take $\psi \in C_0^\infty$ equal to 1 in a neighborhood of $\operatorname{supp}(\tilde{\theta})$. Lemma 3.3 shows that

$$\theta[(\Gamma_\delta \psi u) \circ \Phi] = \theta\tilde{a}(x, D)[(\psi u) \circ \Phi] + \mathcal{E}[(\psi u) \circ \Phi]$$

with $\operatorname{supp}\tilde{a}(x, \xi) \subset \{|\xi| \approx M/\delta\}$ and $\mathcal{E}: H^{-s} \to H^s$ with norm $O(\delta^{\text{power}})$. Hence

$$P\{\theta[(\Gamma_\delta u) \circ \Phi)]\} = \{P\theta\tilde{a}(x, D)\}[(\psi u)\Phi] + P\mathcal{E}[(\psi u)\Phi] + P[\{\tilde{\theta}\Gamma_\delta(1 - \psi)u\} \circ \Phi].$$

Lemma 3.2 applies to $P\theta\tilde{a}(x, D)$ and to $\tilde{\theta}\Gamma_\delta(1 - \psi)$, so the corollary follows at once.

Lemma 3.4. *Let R be a pseudo-differential operator of order zero with symbol $R(x, \xi)$ which, when expressed in terms of x^P-coordinates has support in the support of Γ_δ. Then*

$$\|\sigma_\delta^0 Ru\| \leqslant C\|\sigma_\delta\tilde{\Gamma}_\delta u\| + C'\delta^{\text{power}}\|u\|_{-s_0}, \tag{6}$$

here $\sigma_\delta^0 = \sigma_{P\delta}^0$ and $\sigma_\delta = \sigma_{P\delta}$. The constant C is independent of M and δ, and C' depends on M but not on δ.

PROOF.

$$\begin{aligned} Ru(x) &= \int R(x, \xi)e^{ix\xi}\hat{u}(\xi)\, d\xi \\ &= \int R(x, \xi)e^{i(x-y)\xi}u(y)\, dy\, d\xi \\ &= \int H(x, y)u(y)\, dy \end{aligned} \tag{7}$$

with

$$H(x, y) = \int R(x, \xi)e^{i(x-y)\xi}d\xi. \tag{8}$$

Using the identity

$$\left(1 + \frac{M}{\delta^2}|x - y|^2\right)^{-K}\left(1 - \frac{M^2}{\delta^2}\Delta_\xi\right)^K e^{i\xi(x-y)} = e^{i\xi(x-y)} \tag{9}$$

and integrating by parts repeatedly, we may rewrite H as

$$H(x, y) = \left(1 + \frac{M}{\delta^2}|x - y|^2\right)^{-K}\int_{\mathbb{R}^3} e^{i\xi(x-y)}\left[\left(I - \frac{M^2}{\delta^2}\Delta_\xi\right)^K R(x, \xi)\right] d\xi. \tag{10}$$

The quantity in brackets is $O(1)$ and supported in $|\xi| \approx M/\delta$ in view of our hypothesis on $R(x, \xi)$. Consequently,

$$|H(x,y)| \leqslant C_K\left(\frac{M}{\delta}\right)^3\left(1+\frac{M^2}{\delta^2}|x-y|^2\right)^{-K}.$$

Substituting this into (7), we get

$$(12)\qquad |\sigma_\delta^0(x)Ru(x)| \leqslant C'_K\left(\frac{M}{\delta}\right)^3 \int \frac{|\sigma_\delta^0(x)|}{\left(1+\frac{M^2}{\delta^2}|x-y|^2\right)^K}|u(y)|\,dy.$$

Now it follows from the definitions of σ_δ^0 and σ_δ that

$$(13)\qquad \frac{|\sigma_\delta^0(x)|}{\left(1+\frac{M}{\delta}|x-y|\right)^{2K}} \leqslant C|\sigma_\delta(y)| + \delta^{\text{power}},$$

where «power» can be taken as large as we wish if K and N are sufficiently large (recall that N occurs in the definition of σ_δ).

Now from (12) we get

$$(14)\quad |\sigma_\delta^0(x)Ru(x)|^2$$

$$\leqslant C_K\left(\frac{M}{\delta}\right)^6 \int \frac{dy}{\left(1+\frac{M}{\delta}|x-y|\right)^K} \int \frac{\sigma_\delta(x)^2|u(y)|^2}{\left(1+\frac{M}{\delta}|x-y|\right)^{2K}}dy$$

$$\leqslant C\left(\frac{M}{\delta}\right)^3 \int \frac{\sigma_\delta(x)^2|u(y)|^2}{\left(1+\frac{M}{\delta}|x-y|\right)^K}dy$$

$$\leqslant C\left(\frac{M}{\delta}\right)^3 \int \frac{\sigma_\delta(y)^2u(y)^2}{\left(1+\frac{M}{\delta}|x-y|\right)^K}dy + C\delta^{\text{power}}\|u\|^2.$$

Integrating with respect to x we get

$$(15)\qquad \|\sigma_\delta^0 Ru\| \leqslant C\|\sigma_\delta u\| + \delta^{\text{power}}\|u\|.$$

Substitute now $\tilde{\Gamma}_\sigma u$ for u.

We may use Lemma 3.3 to view R as a pseudo-differential operator in terms of the y-coordinates in which $\tilde{\Gamma}_\delta$ was originally defined. Since the symbol of $\tilde{\Gamma}_\delta$ equals 1 on the support of the symbol of R, we obtain from Lemma 3.3 that $\|R\tilde{\Gamma}_\delta u - Ru\|_{L^2(\operatorname{supp}\sigma_\delta^0)} \leqslant \delta^{\text{power}}\|u\|_{-s_0}$, provided that u is supported in a small neighborhood of 0. We may remove the restriction on supp (u) by putting ϕu in place of u for a suitable cutoff function ϕ and invoking Lemma 3.2 to control the resulting error terms.

Since also

$$\|\tilde{\Gamma}_\delta u\| \leqslant C\left(\frac{M}{\delta}\right)^{s_0}\|u\|_{-s_0},$$

we obtain

$$\|\sigma_\delta^0 Ru\| \leqslant C\|\sigma_\delta \tilde{\Gamma}_\delta u\| + C'\delta^{\text{power}}\|u\|_{-s_0}. \tag{16}$$

Here C is independent of M and δ, while C' depends on M but not on δ. The proof of Lemma 3.4 is now complete.

Lemma 3.5. *Suppose that R is expressed in x^P-coordinates as a pseudo-differential operator of order zero with symbol $R(x, \xi)$ supported in* supp $\tilde{\Gamma}_\delta$. *Then, again setting $\sigma_\delta^0 = \sigma_{P\delta}^0$ and $\sigma_\delta = \sigma_{P\delta}$, we have*

$$\|[\sigma_\delta^0, R]\Gamma_\delta u\| \leqslant \frac{C}{M}\|\sigma_\delta\Gamma_\delta u\| + C'\delta^{\text{power}}\|u\|_{-s_0}, \tag{17}$$

here C is independent on δ and M; and C' is independent of δ.

PROOF. We have

$$\sigma_\delta^0(x)Ru(x) = \int \sigma_\delta^0(x)R(x, \xi)e^{i\xi(x-y)}u(y)\,dy\,d\xi. \tag{18}$$

So

$$[\sigma_\delta^0, R]u(x) = \int H(x, y)u(y)\,dy \tag{19}$$

with

$$H(x, y) = \int (\sigma_\delta^0(x) - \sigma_\delta^0(y))R(x, \xi)e^{i\xi(x-y)}\,d\xi. \tag{20}$$

Again using (9) and integrating by parts, we obtain

$$H(x, y) = \left(1 + \frac{M}{\delta^2}|x - y|^2\right)^{-K}(\sigma_\delta^0(x) - \sigma_\delta^0(y)) \int e^{i\xi(x-y)}\left\{\left(I - \frac{M^2}{\delta^2}\Delta_\xi\right)^{-K} R(x, \xi)\right\}d\xi \tag{21}$$

and, just like in Lemma 3.4, that the absolute value of the integral is bounded by $C_K M^3 \delta^{-3}$, hence

$$(22)\qquad |H(x,y)| \leqslant \frac{C_K |(\sigma_\delta^0(x) - \sigma_\delta^0(y))| \left(\dfrac{M}{\delta}\right)^3}{\left(1 + \dfrac{M^2}{\delta^2}|x-y|^2\right)^K}$$

$$\leqslant C_K \left(\frac{M}{K}\right)^3 \left\{\frac{|\sigma_\delta^0(x) - \sigma_\delta^0(y)|}{\left(1 + \dfrac{|x-y|}{\delta}\right)^K}\right\} \frac{1}{\left(1 + \dfrac{M}{\delta}|x-y|\right)^K}.$$

We will prove that if K is sufficiently large then

$$(23)\qquad \frac{|\sigma_\delta^0(x) - \sigma_\delta^0(y)|}{\left(1 + \dfrac{|x-y|}{\delta}\right)^K} \leqslant C_K \sigma_\delta(y) \frac{|x-y|}{\delta} + C_K \delta^{\text{power}}.$$

To establish (23) we ask when does the following estimate hold

$$(24)\qquad (\delta^{\text{power}} + \sigma_\delta(y))\left(1 + \frac{|x-y|}{\delta}\right)^K \geqslant \text{const}.$$

Clearly (24) holds if $y \in \operatorname{supp}(\sigma_\delta^0)$, for then $\sigma_\delta(y) \sim 1$. Also, (24) holds if $x \in \operatorname{supp}(\sigma_\delta^0)$, by definition of $\sigma_{P,\delta}^0 = \sigma_\delta^0$. If (24) holds, then

$$(25)\qquad \frac{|\sigma_\delta^0(x) - \sigma_\delta^0(y)|}{\dfrac{|x-y|}{\delta}} \leqslant C\left(1 + \frac{|x-y|}{\delta}\right)^K (\delta^{\text{power}} + \sigma_\delta(y)),$$

which amounts to (23).

The only case in which (24) fails is when neither x nor y is in $\operatorname{supp}(\sigma_\delta^0)$, in which case (23) is trivial.

Putting (23) into (22), we get

$$(26)\qquad |H(x,y)| \leqslant C_K \left(\frac{M}{\delta}\right)^3 \frac{|x-y|}{\delta}\left(1 + \frac{M}{\delta}|x-y|\right)^{-K} \sigma_\delta(y)$$

$$+ C_K \delta^{\text{power}} \left(\frac{M}{\delta}\right)^3 \left(1 + \frac{M}{\delta}|x-y|\right)^{-K}$$

$$= H_1(x-y)\sigma_\delta(y) + \delta^{\text{power}} H_2(x-y)$$

with

$$(27)\qquad \|H_1\|_{L_1} \leqslant \frac{C_K}{M} \quad \text{and} \quad \|H_2\|_{L_1} \leqslant C_K.$$

Substituting this in (19) we have

(28) $$|[\sigma_\delta^0, R]u| \leqslant H_1 * |\sigma_\delta u| + \delta^{\text{power}} H_2 * |u|$$

pointwise, so that

(29) $$\|[\sigma_\delta^0, R]u\| \leqslant \frac{C}{M}\|\sigma_\delta u\| + c\delta^{\text{power}}\|u\|.$$

Taking $\Gamma_\delta u$ in place of u, and noting that

(30) $$\|\Gamma_\delta u\| \leqslant C\left(\frac{M}{\delta}\right)^{s_0}\|u\|_{-s_0},$$

we obtain the desired estimate (17) thus completing the proof of Lemma 3.5.

4. Rescaled Subelliptic Estimates

The purpose of this section is to prove the following two estimates

(1) $$M^{2\epsilon}\|\sigma_{P\delta}^0\Gamma_\delta u\|^2 \leqslant C_1\gamma(P,\delta)^2\sum_j\|\sigma_{P\delta}^0\Gamma_\delta X_j u\|^2 + C_1\|\sigma_{P\delta}\tilde{\Gamma}_\delta u\|^2 + C\delta^{\text{power}}\|u\|^2,$$

where C_1 is independent of δ and of M and where C depends on M but not on δ, and

(2) $$\|u\|_{B(P,\delta)} \leqslant C\gamma(P,\delta)\left\{\sum\|X_j u\| + \|u\|\right\},$$

where $\|u\|_{B(P,\delta)}$ denotes the L_2-norm of u over $B(P,\delta)$ and the constant C is independent of P and δ.

We begin with the proof of (1). For each P and δ we introduce the rescaled coordinates (y_0, y_1, y_2) by

(3) $$y_0 = \frac{x_0^P}{\delta}, \quad \text{and} \quad y_j = \frac{x_j^P}{\gamma(P,\delta)} \quad \text{for} \quad j = 1, 2.$$

Let $\mathcal{B} = \{y \in \mathbb{R}^3 : (\delta y_0, \gamma(P,\delta)y') \in B(P,\delta) \text{ for all } P \in U \text{ and } \delta \in (0,1)\}$. Note that $\mathcal{B}$ is of the form $\{y \in \mathbb{R}^3 : |y_0| < C,\ |y'| < C'\}$ where C and C' are independent of P and δ. Let $Y_i = \gamma(P,\delta)X_i$, that is

(4) $$Y_i = \frac{\gamma(P,\delta)}{\delta}a_i^P(\delta y_0, \gamma(P,\delta)y')\frac{\partial}{\partial y_0} + \sum_{j=1}^{2} a_{ij}^P(\delta y_0, \gamma(P,\delta)y')\frac{\partial}{\partial y_j},$$

where the a_i^P and a_{ij}^P are given in (9) of Section 2. The Y_i are C^∞ independently of P and δ and their commutators through order m span, uniformly in P, δ

(as in [FK]). It then follows that there exist $C > 0$ and $\epsilon > 0$ independent of P and δ such that

$$\|w\|_\epsilon \leqslant C\Big(\sum \|Y_j w\| + \|w\|\Big) \tag{5}$$

for all $w \in C_0^\infty(\mathcal{B})$.

Lemma 4.1. *If S is a pseudo-differential operator of order ϵ, $0 \leqslant \epsilon \leqslant 1$, if $\zeta \in C_0^\infty(\mathcal{B})$, and if $\rho(x) = (1 + |x|^2)^{-N}$ then there exists $C > 0$ such that*

$$\|\zeta S u\| \leqslant C\|\zeta u\|_\epsilon + C\|\rho u\| \tag{6}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

PROOF. We have

$$\begin{aligned} \|\zeta S u\| &\leqslant \|S(\zeta u)\| + \|[\zeta, S]u\|, \\ \|S(\zeta u)\| &= \|\Lambda^\epsilon \Lambda^{-\epsilon} S(\zeta u)\| \leqslant C\|\zeta u\|_\epsilon, \end{aligned} \tag{7}$$

and

$$\|[\zeta, S]u\| \leqslant C\|\rho u\|_{\epsilon - 1} \leqslant C\|\rho u\|, \tag{8}$$

Hence combining (7) and (8) we obtain (6).

Setting $w = \zeta u$ and combining (5) and (6) we get

$$\begin{aligned} \|\zeta S u\|^2 &\leqslant C \sum \|Y_j(\zeta u)\|^2 + C\|\rho u\|^2 \\ &\leqslant C \sum \|\zeta Y_j u\|^2 + C'\|\rho u\|^2. \end{aligned} \tag{9}$$

If supp $\hat{u} \subset \{\xi \in \mathbb{R}^3 : |\xi| > M\}$ then, letting S be the operator with symbol

$$S(\xi) = \chi\left(\frac{|\xi|^2}{M^2}\right) M^\epsilon,$$

where

$$\chi(t) = \begin{cases} 0 & \text{if } \ t < 1/2 \\ 1 & \text{if } \ t \geqslant 1, \end{cases} \tag{10}$$

we have $Su = M^\epsilon u$ and then

$$M^{2\epsilon}\|\zeta u\|^2 \leqslant C \sum_j \|\zeta Y_j U\|^2 + C\|\rho u\|^2. \tag{11}$$

By rescaling we then obtain

$$M^{2\epsilon}\|\sigma_{P\delta}^0 u\|^2 \leqslant C\gamma(P, \delta)^2 \sum_j \|\sigma_{P\delta}^0 X_j u\|^2 + C\|\rho_\delta u\|^2 \tag{12}$$

for u with

$$\operatorname{supp}(\hat{u}) \subset \left\{ \xi \in \mathbb{R}^n : |\xi| > \text{const.}\frac{M}{\delta} \right\},$$

where

$$\rho_\delta(x^P) = \left(1 + \frac{|x_0^P|^2}{\delta^2} + \frac{|(x^P)'|^2}{\gamma(P,\delta)^2}\right)^{-N}$$

Now for arbitrary u, $\Gamma_\delta u = u^\# + u_{\text{error}}$ with

$$\operatorname{supp}(\hat{u}^\#) \subset \left\{ |\xi| > C\frac{M}{\delta} \right\} \quad \text{and} \quad \|u_{\text{error}}\|_{C^k} \leqslant C\delta^{\text{power}} \|u\|_{-s_0},$$

where k, «power» and s_0 may be chosen arbitrarily large. Applying (12) to $u^\#$ we get

$$(13)\quad M^{2\epsilon}\|\sigma^0_{P\delta}\Gamma_\delta u\|^2 \leqslant C\gamma(P,\delta)^2 \sum_j \|\sigma^0_{P\delta}X_j\Gamma_\delta u\|^2 + C\|\rho_\delta\Gamma_\delta u\|^2 + \delta^{\text{power}}\|u\|^2.$$

Since

$$\sigma^0_{P\delta}X_j\Gamma_\delta u = \sigma^0_{P\delta}\Gamma_\delta X_j u + \sigma^0_{P\delta}[X_j, \Gamma_\delta]u$$

we obtain, by the results of Section 3

$$\|\sigma^0_{P\delta}[X_j, \Gamma_\delta]u\| \leqslant C\|\sigma_{P\delta}\tilde{\Gamma}_\delta u\| + \delta^{\text{power}}\|u\|.$$

Combining this with (13) we get

$$(14)\quad M^{2\epsilon}\|\sigma^0_{P\delta}\Gamma_\delta u\|^2 \leqslant C\gamma(P,\delta)^2 \sum_j \|\sigma^0_{P\delta}\Gamma_\delta X_j u\|^2 + C\|\rho_\delta\tilde{\Gamma}_\delta u\|^2 + \delta^{\text{power}}\|u\|^2,$$

Recalling the definition of $\phi_{P\delta}$, given by (33) in Section 2, we set $\phi_\delta = \phi_{\rho\delta}$ and replace u in (14) by $\phi_\delta u$ obtaining

$$(15) M^{2\epsilon}\|\sigma^0_\delta\Gamma_\delta\phi_\delta u\|^2 \leqslant C\gamma(\delta)^2 \sum_j \|\sigma^0_\delta\Gamma_\delta X_j\phi_\delta u\|^2 + C\|\rho_\delta\tilde{\Gamma}_\delta\phi_\delta u\|^2 + \delta^{\text{power}}\|u\|^2,$$

here we have dropped P, which remains fixed through the rest of this section.

Now observe that $\gamma(P,\delta) < C\delta^{1/m}$. Thus, introducing the coordinates $x^* = x^P\delta^{1/m-1}$ we note that the ϕ_δ are «smooth» in these coordinates, that is all their derivatives are bounded independently of δ. Similarly we define ϕ^1_δ, ϕ^2_δ, ϕ^3_δ etc., so that $\phi^{i+1}_\delta = 1$ on a neighborhood of the support of ϕ^i_δ and we set $\phi^0_\delta = \phi_\delta$. The coordinates ξ^* dual to x^* are given by $\xi^* = \xi\delta^{1-1/m}$ and hence the support of the symbol of Γ_δ lies in $\{|\xi^*| > C\delta^{-1/m}\}$. Since $\sigma^0_\delta = \sigma^0_\delta\phi^1_\delta$ we have

$$\sigma_\delta^0\Gamma_\delta u = \sigma_\delta^0\Gamma_\delta\phi_\delta u + \sigma_\delta^0[\phi_\delta^1\Gamma_\delta((1-\phi_\delta)u)]. \tag{16}$$

The L_2-norm of the second term on the right is $O(\delta^{\text{power}}\|u\|)$ since ϕ_δ^1 and $1-\phi_\delta$ have disjoint supports. Similarly

$$\sigma_\delta^0\Gamma_\delta X_j u = \sigma_\delta^0\Gamma_\delta X_j\phi_\delta u + \sigma_\delta^0[\phi_\delta^1\Gamma_\delta X_j(1-\phi_\delta)u] \tag{17}$$

and again the second term is $O(\delta^{\text{power}}\|u\|)$ since δX_j is a first order operator in the x^*-coordinates.

So now we have

$$M^{2\epsilon}\|\sigma_\delta^0\Gamma_\delta u\|^2 \leqslant C\gamma(\delta)^2\sum_j\|\sigma_\delta^0\Gamma_\delta X_j u\|^2 + C\|\rho_\delta\tilde{\Gamma}_\delta\phi_\delta u\|^2 + \delta^{\text{power}}\|u\|^2. \tag{18}$$

Reasoning as above we have

$$\|(1-\phi_\delta^1)\tilde{\Gamma}_\delta\phi_\delta u\|^2 \leqslant \delta^{\text{power}}\|u\|^2. \tag{19}$$

Next we have

$$\phi_\delta^1\tilde{\Gamma}_\delta\phi_\delta = Q\phi_\delta^2\tilde{\tilde{\Gamma}} + \delta^{\text{power}}Q_{\text{error}}, \tag{20}$$

where Q, Q_{error} are pseudo-differential operators of order zero in the x^*-coordinates and the symbol of Q is supported on $\{|\xi^*| \sim M\delta^{-1/m}\}$. This is due to the fact that $\phi_\delta^2\tilde{\tilde{\Gamma}}_\delta$ is elliptic on places where the symbol of $\phi_\delta^1\tilde{\Gamma}_\delta\phi_\delta$ is essentially supported. Thus

$$\|\rho_\delta\phi_\delta^1\tilde{\Gamma}_\delta\phi_\delta u\| \leqslant \|\rho_\delta Q\phi_\delta^2\tilde{\tilde{\Gamma}}_\delta u\| + \delta^{\text{power}}\|u\|. \tag{21}$$

In the x^*-coordinates, ρ_δ is constant on a scale $\sim\delta^{1/m}$ while Q is order zero with symbol supported in $\{|\xi^*| \sim M\delta^{-1/m}\}$. First we show that

$$\|\rho_\delta Qw\| \leqslant C\|\rho_\delta w\|. \tag{22}$$

The proof of (11) in Section 3 gives

$$Qw(x^*) = \int H(x^*, y^*)w(y^*)\,dy^*$$

with

$$|H(x^*, y^*)| \leqslant C_k(M\delta^{-1/m})^3(1 + |x^* - y^*|\delta^{-1/m}M)^{-K}.$$

From the definitions of ρ_δ and the x^*-coordinate system we read off

$$\frac{\rho_\delta(x^*)}{\rho_\delta(y^*)} \leqslant C(1 + |x^* - y^*|\delta^{-1/m})^N.$$

Hence

$$\rho_\delta Q w(x^*) = \int \frac{\rho_\delta(x^*)}{\rho_\delta(y^*)} H(x^*, y^*)\rho_\delta w(y^*)\, dy^*.$$

Taking K much larger than N we obtain (22) from the above.

Now we apply (22) to $w = \phi_\delta^2 \tilde{\tilde{\Gamma}}_\delta u$ and combining with (21) we obtain

$$\|\rho_\delta \tilde{\Gamma}_\delta \phi_\delta u\| \leqslant C\|\rho_\delta \phi_\delta^2 \tilde{\tilde{\Gamma}}_\delta u\| + \delta^{\text{power}}\|u\|. \tag{23}$$

From the definition of σ_δ we have $|\rho_\delta \phi_\delta^2| \leqslant$ const. $|\sigma_\delta|$. Hence, combining with (18), we obtain the desired estimate (1) with $\tilde{\tilde{\Gamma}}_\delta$ instead of $\tilde{\Gamma}_\delta$ so that, changing notation, (1) is proved.

Now we turn to the proof of (2). The first step is to construct a function $\phi \in C_0^\infty(\{y \in \mathbb{R}^3 : |y| < 1\})$ with the property that $\hat{\phi} = 1$ to high order at 0, *i.e.* $\hat{\phi}(0) = 1$ and $D^\alpha \hat{\phi}(0) = 0$ for $|\alpha| \leqslant N$. Throughout this proof we will keep P fixed. Then for each γ we define $\delta(\gamma)$ by

$$\delta(\gamma) = \sum_{j=2}^{m} \lambda_j^P \gamma^j, \tag{24}$$

which is consistent with Definition 2.7. We define ϕ_γ by

$$\phi_\gamma(x_0, x') = \frac{1}{\delta(\gamma)\gamma^2} \phi\left(\frac{x_0}{\delta(\gamma)}, \frac{x'}{\gamma}\right), \tag{25}$$

here (x_0, x') denotes $(x_0^P, (x^P)')$.

Lemma 4.2. *Setting*

$$B\left(\frac{\gamma}{M}\right) = B\left(P, \delta\left(\frac{\gamma(P)}{M}\right)\right)$$

we have

(26)

$$|u*\phi_{2\gamma/M}(0) - u*\phi_{\gamma/M}(0)| \operatorname{vol} B\left(\frac{\gamma}{M}\right) \leqslant C\gamma \sum \|X_j u\|_{B(\gamma)} + C_s M^{-s}\|u\|_{B(2\gamma)}$$

and

$$\|u - u*\phi_{\gamma/M}(0)\|_{B(\gamma)} \leqslant C\gamma \sum \|X_j u\|_{B(\gamma)} + C_s M^{-s}\|u\|_{B(2\gamma)}. \tag{27}$$

PROOF. Let Y_j be smooth vector fields given by $Y_j = \gamma X_j$. Given λ_0, λ', $\tilde{\lambda}_0$, $\tilde{\lambda}'$ we define

$$\psi(y_0, y') = \frac{1}{\lambda_0|\lambda'|^2}\phi\left(\frac{y_0}{\lambda_0}, \frac{y'}{\lambda'}\right)$$

and

$$\tilde{\psi}(y_0, y') = \frac{1}{\tilde{\lambda}_0|\tilde{\lambda}'|^2}\phi\left(\frac{y_0}{\tilde{\lambda}_0}, \frac{y'}{\lambda'}\right).$$

If $M >> 1$ and if the λ and $\tilde{\lambda}$ lie in $[M^{-K}, M^{-1}]$ with K large then

$$|\psi * u(0) - \tilde{\psi} * u(0)|^2 \leqslant C \sum \| Y_j u\|^2_{\{|y|<1\}} + C_s M^{-s} \|u\|^2_{\{|y|<2\}}. \tag{28}$$

PROOF OF (28). Define Ωu by $\widehat{\Omega u}(\xi) = w(M^{-1/2}\xi)\hat{u}(\xi)$ with $w \in C^\infty(\mathbb{R}^3)$, $w = 1$ if $|\xi| > 1$, $w = 0$ if $|\xi| < 1/2$. We begin by applying (11) to Ωv. (Note that $\widehat{\Omega v}$ is supported on $|\xi| > M/2$ so (11) applies with $M^{1/2}$ in place of M.) Thus

$$M^\epsilon \|\zeta\Omega v\|^2 \leqslant C\sum_j \|\zeta Y_j \Omega v\|^2 + C\|v\|^2 \leqslant C\sum_j \|\Omega\zeta Y_j v\|^2 + C\|v\|^2. \tag{28'}$$

Apply this to $v = \tilde{\zeta}\tilde{\Omega}u$, where $\tilde{\zeta}$ and $\tilde{\Omega}$ are analogous to $\zeta\Omega$ but symbol of $\tilde{\zeta}\tilde{\Omega} = 1$ on the support of the symbol of ζQ. Lemma 3.2 with $\delta = 1$ and M replaced by $M^{1/2}$, gives

$$\zeta\Omega v = \zeta\Omega\tilde{\zeta}\tilde{\Omega}u = \zeta\Omega u + \mathcal{E}u \quad \text{with} \quad \|\mathcal{E}u\| \leqslant C_s M^{-s}\|u\|.$$

Similarly,

$$\Omega\zeta Y_j v = \Omega\zeta Y_j \tilde{\zeta}\tilde{\Omega}u = \Omega\zeta Y_j u + \mathcal{E}'u \quad \text{with} \quad \mathcal{E}'u \leqslant C_s M^{-s}\|u\|.$$

Hence (28)′ becomes

$$\begin{aligned} M^\epsilon \|\zeta\Omega u\|^2 &\leqslant C\sum_j \|\Omega\zeta Y_j u\|^2 + C\|\tilde{\zeta}\tilde{\Omega}u\|^2 + C_s M^{-s}\|u\|^2 \\ &\leqslant C\sum_j \|\zeta Y_j u\|^2 + C\|\tilde{\zeta}\tilde{\Omega}u\|^2 + C_s M^{-s}\|u\|^2. \end{aligned}$$

The term $\|\tilde{\zeta}\tilde{\Omega}u\|^2$ is analogous to the left-hand side, but we have gained a factor M^ϵ. Hence an obvious induction gives

$$M^\epsilon \|\zeta\Omega u\|^2 \leqslant C\sum_j \|Y_j u\|^2_{L^2\{|x|<1\}} + C_s M^{-s}\|u\|^2. \tag{29}$$

Since $\psi * \Omega u(0) - \tilde{\psi} * \Omega u(0) = (\psi - \tilde{\psi}) * \zeta\Omega u(0)$, (29) implies

$$|\psi * \Omega u(0) - \tilde{\psi} * \Omega u(0)|^2 \leqslant C\sum_j \|\zeta Y_j u\|^2 + C_s M^{-s}\|u\|^2. \tag{30}$$

The condition $\hat{\phi} = 1$ to high order at 0 implies that the Fourier transform of $\psi - \tilde{\psi}$ has L^2-norm $O(M^{-s})$ on $G = \operatorname{supp}(1 - w(M^{-1/2}|\xi|))$. Hence

(31) $$|\psi*(I-\Omega)u(0)-\tilde{\psi}*(I-\Omega)u(0)|^2 \leqslant \|\hat{\psi}-\hat{\tilde{\psi}}\|^2_{L^2(G)}\|(I-\Omega)u\|^2 \leqslant C_sM^{-s}\|u\|^2.$$

Combining (30) and (31), we get

$$|\psi*u(0)-\tilde{\psi}*u(0)|^2 \leqslant C\sum_j \|Y_ju\|^2_{L^2\{|x|<1\}} + C_sM^{-s}\|u\|^2.$$

Replacing u by the product of u and the characteristic function of $\{x: |x|<2\})$, we obtain (28).

Setting

$$\lambda_0 = \delta\left(\frac{\gamma}{M}\right)\delta(\gamma)^{-1}, \qquad \lambda' = (M^{-1}, M^{-1}), \qquad \tilde{\lambda}_0 = \delta\left(\frac{2\gamma}{M}\right)\delta(2\gamma)^{-1},$$

and $\tilde{\lambda}' = (2M^{-1}, 2M^{-1})$ we have

(32) $$\psi(y_0, y') = \phi_{\gamma/M}(x_0, x')\operatorname{vol} B\left(\frac{\gamma}{M}\right).$$

$$\tilde{\psi}(y_0, y') = \phi_{2\gamma/M}(x_0, x')\operatorname{vol} B\left(\frac{2\gamma}{M}\right).$$

Thus we obtain (26) by rescaling (28).

The proof of (27) is analogous to the proof of (26). In place of (28) we have

(33) $$\|u-u*\psi(0)\|_{\{|y|<1\}} \leqslant C\sum_j \|Y_ju\|_{\{|y|<1\}} + C_sM^{-s}\|u\|.$$

This is proved by again setting $u = \Omega u + (I-\Omega)u$. We then obtain (27) by rescaling (33) which concludes the proof of the lemma.

Let $A = \sup_{0<\gamma<1}\gamma^{-2}\|u\|^2_{B(\gamma)}$. Then from (26) we get, for $2^{i+1}\gamma M^{-1}<1$, the estimate

(34) $$\begin{aligned}|u*\phi_{2^{i+1}\gamma/M}(0) - u*\phi_{2^i\gamma/M}(0)|^2 &\leqslant C\delta(2^i\gamma)^{-1}\sum\|X_ju\|^2 \\ &\quad + C_sM^{-s}\delta(2^i\gamma)^{-1}(2^i\gamma)^{-2}\|u\|^2_{B(2^i\gamma)} \\ &\leqslant C\delta(2^i\gamma)^{-1}\sum\|X_ju\|^2 + C_sM^{-s}\delta(2^i\gamma)^{-1}A.\end{aligned}$$

Since $\delta(2^i\gamma)^{-1} \leqslant 4^{-i}\delta(\gamma)^{-1}$, we obtain, by summing over i,

(35) $$|u*\phi_{\gamma/M}(0)|^2 = \delta(\gamma)^{-1}\left(C\sum\|X_ju\|^2 + C_sM^{-s}A\right) + C_M\|u\|^2.$$

Then

(36) $$|u*\phi_{\gamma/M}(0)|^2\operatorname{vol} B\left(\frac{\gamma}{M}\right) \leqslant \gamma^2\left(C\sum\|X_ju\|^2 + C_sM^{-s}A\right) + C_M\delta(\gamma)\gamma^2\|u\|^2.$$

Combining this with (27) we obtain

$$A \leqslant C\Big(\sum_j \|X_j u\|^2 + \|u\|^2\Big) + C_s M^{-s} A$$

so that (2) follows by taking M sufficiently large.

5. Scaled Sobolev Norm

Definition 5.1. *For each $s \geqslant 0$ and $u \in C_0^\infty(U)$ we define the scaled Sobolev norm of u, denoted by $|||u|||_s$, by*

$$(1) \qquad |||u|||_s^2 = \int_0^1 \int_{\mathbb{R}^3} \gamma(P,\delta)^{-2s} |\Gamma_\delta u(P)|^2 \, dP \frac{d\delta}{\delta} + \|u\|^2,$$

where Γ_δ is defined in terms of ψ in Definition 3.1 *and $\psi(\xi) = 1$ when $a < a' < |\xi| < b' < b$.*

The purpose of this section is to prove that for $M >> 1$

$$(2) \qquad M^\epsilon |||u|||_{s+1} \leqslant C_1 \sum |||X_j u|||_s + C\|u\|,$$

where C_1 is independent of M.

Note that $|||u|||_0 \sim \|u\|$ since

$$c' \ln(b'/a) \leqslant \int_0^1 \left|\psi\left(\frac{\delta}{M}\xi\right)\right|^2 \frac{d\delta}{\delta} \leqslant c \ln(b/a).$$

Also, if $\gamma(P,\delta)$ is replaced by δ, the resulting norm is equivalent to $\|u\|_s$.

Lemma 5.2. *The following are equivalent to $|||u|||_s^2$*

$$(3) \qquad |||u|||_s^2 \sim \|u\|^2 + \int_0^1 \int_{\mathbb{R}^3} \gamma(P,\delta)^{-2s} \|\sigma_{P\delta}^0 \Gamma_\delta u\|^2 \frac{dP}{\mathrm{vol}\,(P,\delta)} \frac{d\delta}{\delta}$$

and

$$(4) \qquad |||u|||_s^2 \sim \|u\|^2 + \int_0^1 \int_{\mathbb{R}^3} \gamma(P,\delta)^{-2s} \|\sigma_{P\delta} \Gamma_\delta u\|^2 \frac{dP}{\mathrm{vol}\,(P,\delta)} \frac{d\delta}{\delta},$$

where $\mathrm{vol}\,(P,\delta) = c\gamma(P,\delta)^2\delta$, *which is the volume of $B(P,\delta)$.*

PROOF. Since $\gamma(Q,\delta) \sim \gamma(P,\delta)$ for $Q \in \tilde{B}(P,\delta)$ it suffices to show that

$$(5) \qquad \int |\sigma_{P\delta}^0(Q)|^2 \frac{dP}{\mathrm{vol}\,(P,\delta)} \sim \int |\sigma_{P\delta}(Q)|^2 \frac{dP}{\mathrm{vol}\,(P,\delta)} \sim 1.$$

The fact that the first term is equivalent to 1 is an immediate consequence of Proposition 2.13. To see that the second term is equivalent to 1 we have, from (30) in Section 2, that

$$x_0^P(Q) = x_0^Q(P) - F^{Q,P}(-x_1^P(Q), -x_2^P(Q)),$$

furthermore

$$|F^{P,Q}(-x_1^P(Q), -x_2^P(Q))| \leqslant \text{const.}\,\delta,$$

for $Q \in \tilde{B}(P, \delta)$. Hence

$$\left(1 + \frac{|x_0^P(Q)|^2}{\delta^2}\right)^{-N} \sim \left(1 + \frac{|x_0^Q(P)|^2}{\delta^2}\right)^{-N}$$

for $Q \in \tilde{B}(P, d)$. So that (4) follows from this and the definition of $\sigma_{P\delta}$, which completes the proof.

The following proposition shows that the norms $|||\bullet|||_s$ are independent of the choice of ψ in the definition of Γ_δ. If Γ_δ is defined by the function ψ as in Definition 3.1 and if further there exist a_1, b_1 so that $\psi(\xi) = 1$ whenever $a_1 < |\xi| < b_1$ then we will denote the norm $|||u|||_s$, given by (1), by $|||u|||_s^\psi$. Similarly, if $\bar{\psi} \in C_0^\infty(\{\xi: 0 < \bar{a} < |\xi| < \bar{b}\})$ and if $\bar{\psi}(\xi) = 1$ when $\bar{a}_1 < |\xi| < \bar{b}_1$, we have the following result.

Proposition 5.3. *The norms* $|||u|||_s^\psi$ *and* $|||u|||_s^{\bar{\psi}}$ *are equivalent.*

PROOF. First assume that $\bar{\psi}$ is elliptic on the support of ψ, i.e. $|\bar{\psi}(\xi)| >$ const. > 0 if $\xi \in \operatorname{supp}\psi$. Then $\Gamma_\delta = Q\bar{\Gamma}_\delta$, where

$$q(\xi) = \frac{\psi\left(\dfrac{\delta}{M}\xi\right)}{\bar{\psi}\left(\dfrac{\delta}{M}\xi\right)}$$

is the symbol of Q then, from (3), (4) and Lemma 3.4, we conclude that $|||u|||_s^\psi \leqslant C|||u|||_s^{\bar{\psi}}$. Now we drop the assumption concerning the support of ψ and we take up the general case. Let

$$r = \frac{\bar{a}_1 + \bar{b}_1}{2\bar{a}_1},$$

let k_0 be an integer so that $\bar{a}_1 r^{-k_0} < a$ and $\bar{b}_1 r^{k_0} \geqslant b$, $\bar{\psi}_j(\xi) = \bar{\psi}(r^j\xi)$, and let

$$\phi = \sum_{-k_0}^{k} \psi_j.$$

The function ϕ is bounded away from zero on the support of ψ and hence we have

$$|||u|||_s^\psi \leqslant C|||u|||_s^\phi \leqslant C \sum_{-k_0}^{k} |||u|||_s^{\bar{\psi}_j}.$$

Now, changing variables in (1), we get

$$(|||u|||_s^{\bar{\psi}_j})^2 = \int_0^{r^j} \int_{\mathbb{R}^3} \gamma(P, r^{-j}\delta)^{-2s} |\Gamma_\delta^{\bar{\psi}} u|^2 \, dP \frac{d\delta}{\delta} + \|u\|^2.$$

Since $\gamma(P, \delta) \leqslant$ const. $\gamma(P, r^{-j}\delta)$ and since

$$\left| \int_1^{r^j} \int \gamma(P, r^{-j}\delta)^{-2s} |\Gamma_\sigma^{\psi} u|^2 \, dP \frac{d\delta}{\delta} \right| \leqslant C\|u\|^2,$$

we obtain $|||u|||_s^\psi \leqslant$ const. $|||u|||_s^{\bar{\psi}}$. Reversing the roles of ψ and $\bar{\psi}$ we conclude that $|||u|||_s^\psi \sim |||u|||_s^{\bar{\psi}}$, thus completing the proof.

To prove the desired estimate (2) we proceed as follows. Multiplying (1) in Section 4 by $\gamma(P, \delta)^{-2s-2}$, integrating with respect to $dP/\text{vol}\,(P, \delta) \cdot d\delta/\delta$, and applying the above lemmas we obtain

$$M^{2\epsilon}|||u|||_{s+1}^2 \leqslant C \sum |||X_j u|||_s^2 + C|||u|||_s^2 + C\|u\|^2.$$

Then (2) follows by choosing M large enough.

6. The Basic Estimate

The purpose of this section is to establish the following result.

Theorem 6.1. *Let $\mathfrak{M}$ be a pseudo-convex compact CR manifold of dimension 3 on which $\bar{\partial}_b$ has closed range in L_2 and suppose that $P_0 \in \mathfrak{M}$ is of finite type. Then there exists a neighborhood U of P_0 such that if $\zeta \in C_0^\infty(U)$ then*

(1) $$\sum_1^2 \|X_j(\zeta u)\| \leqslant C\|\bar{L}u\|$$

for all $u \perp \mathcal{H}(\mathfrak{M})$.

PROOF. The estimate (1) is equivalent to

(2) $$\|L(\zeta u)\| \leqslant C\|\bar{L}u\|.$$

The following estimates follow from (1) and (7) in Section 1 and the fact that the range of $\bar{\partial}_b$ is closed.

$$\|L\mathcal{P}^- \zeta u\| + \|L\mathcal{P}^0 \zeta u\| \leqslant C\|\bar{L}u\|. \tag{3}$$

Thus it will suffice to prove

$$\|L\mathcal{P}^+ \zeta u\| \leqslant C\|\bar{L}u\|. \tag{4}$$

We have $u = \bar{L}^* v = -Lv + gv$. The estimates (4) will be proved in three steps. First, we will show that

$$M^\epsilon |||\mathcal{P}^+ \zeta v|||_1 \leqslant C(\|u\| + \|v\|) \tag{5}$$

and

$$M^\epsilon |||\mathcal{P}^+ \zeta v|||_2 \leqslant C(|||\mathcal{P}^+ \zeta u|||_1 + \|u\| + \|v\|). \tag{5$'$}$$

Second, we will prove that (5) implies that

$$|||\mathcal{P}^+ \zeta u|||_1 \leqslant C\|\bar{L}u\|. \tag{6}$$

Finally, we will prove that (6) implies (4).

We start with (6) in Section 1, namely

$$\sum \|X_j \mathcal{P}^+ w\|^2 \leqslant C\|\mathcal{P}^+ L w\|^2 + C\|\mathcal{P}^+ w\|^2 + C\|\mathcal{R} w\|^2. \tag{7}$$

We wish to use this to estimate $\|\sigma^0_{P\delta}\Gamma_\delta X_j \mathcal{P}^+ \zeta v\|$, from Lemmas 3.1 and 3.4 we have

$$\|\sigma^0_{P\delta}\Gamma_\delta X_j \mathcal{P}^+ \zeta v\| \leqslant \|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ X_j \zeta v\| + \|\sigma_{P\delta}\Gamma_\delta \tilde{\mathcal{P}}^+ \zeta v\| + C\delta^{\text{power}}\|v\|. \tag{8}$$

Letting $\tilde{\mathcal{P}}^+_\delta$ be an operator whose symbol is supported in a neighborhood of the symbol of $\Gamma_\delta \mathcal{P}^+$ and which is 1 on the support of $\Gamma_\delta \mathcal{P}^+$, we choose $\tilde{\mathcal{P}}^+_\delta$, Γ_δ and $\mathcal{P}^+$ so that their symbols depend only on ξ hence we have $\Gamma_\delta \mathcal{P}^+ = \tilde{\mathcal{P}}^+_\delta \Gamma_\delta \mathcal{P}^+$ and thus by Lemma 3.5

$$\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ X_j \zeta v\| \leqslant \|\tilde{\mathcal{P}}^+_\delta \sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ X_j \zeta v\| + \frac{C}{M}\|\sigma_{P\delta}\Gamma_\delta X_j \mathcal{P}^+ \zeta v\| + C'\delta^{\text{power}}\|v\|. \tag{9}$$

The first term on the right above can be estimated by

$$\|\tilde{\mathcal{P}}^+_\delta \sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ X_j \zeta v\| \leqslant \|X_j \tilde{\mathcal{P}}^+_\delta \sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+ \zeta v\| + C\|\sigma_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+ \zeta v\| + \frac{C}{\gamma(P,\delta)}\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+ \zeta v\| + C\delta^{\text{power}}\|v\|. \tag{10}$$

Now, applying (7), with $\mathcal{P}^+$ replaced by $\tilde{\mathcal{P}}_\delta^+$, to the first term on the right we obtain

$$(11)\quad \sum \|X_j \tilde{\mathcal{P}}_\delta^+ \sigma_{P\delta}^0 \Gamma_\delta \mathcal{P}^+ \zeta v\| \leqslant C\|\tilde{\mathcal{P}}_\delta^+ L \sigma_{P\delta}^0 \Gamma_\delta \mathcal{P}^+ \zeta v\| + C\|\sigma_{P\delta}^0 \Gamma_\delta \mathcal{P}^+ \zeta v\|.$$

The first term on the right in (11) is estimated by

$$(12)\quad \|\mathcal{P}_\delta^+ L \sigma_{P\delta}^0 \Gamma_\delta \mathcal{P}^+ \zeta v\| \leqslant C\|\sigma_{P\delta}^0 \Gamma_\delta \mathcal{P}^+ \zeta \bar{L}^* v\| + \frac{C}{\gamma(P,\delta)}\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+ \zeta v\| + C\|\sigma_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+ \tilde{\zeta} v\| + C\delta^{\text{power}}\|v\|.$$

Combining all these we obtain

$$(13)\quad \sum \|\sigma_{P\delta}^0 \Gamma_\delta X_j \mathcal{P}^+ \zeta v\|^2 \leqslant C\|\sigma_{P\delta}^0 \Gamma_\delta \mathcal{P}^+ \zeta \bar{L}^* v\|^2 + \frac{C}{M^2}\sum \|\sigma_{P\delta}\tilde{\Gamma}_\delta X_j \mathcal{P}^+ \zeta v\|^2 + \frac{C}{\gamma(P,\delta)^2}\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+ \zeta v\|^2 + C\|\sigma_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+ \tilde{\zeta} v\|^2 + C\delta^{\text{power}}\|v\|^2.$$

Multiplying by $\dfrac{dP}{\operatorname{vol}(P,\delta)}\,\dfrac{d\delta}{\delta}$ and integrating we obtain

$$\sum \|X_j \mathcal{P}^+ \zeta v\|^2 \leqslant C\|\mathcal{P}^+ \zeta \bar{L}^* v\|^2 + \frac{C}{M^2}\sum \|X_j \mathcal{P}^+ \zeta v\|^2 + C|||\mathcal{P}^+ \zeta v|||_1^2 + C\|v\|^2$$

hence

$$\sum \|X_j \mathcal{P}^+ \zeta v\| < C(\|u\| + \|v\|) + C|||\mathcal{P}^+ \zeta v|||_1.$$

Applying (2) in Section 5 with $s = 0$ and M sufficiently large we obtain (5). Similarly, multiplying (13) by

$$\gamma(P,\delta)^{-2}\frac{dP}{\operatorname{vol}(P,\delta)}\,\frac{d\delta}{\delta}$$

and integrating we obtain

$$\sum |||X_j \mathcal{P}^+ \zeta v|||_1 \leqslant C|||\mathcal{P}^+ \zeta u|||_1 + C|||\mathcal{P}^+ \zeta v|||_2 + C|||\tilde{\mathcal{P}}^+ \tilde{\zeta} v|||_1 + C\|v\|$$

now applying (2) in Section 5 with $s = 1$ and (5) with $\tilde{\mathcal{P}}^+ \tilde{\zeta}$ in place of $\mathcal{P}^+ \zeta$ we get (5)′.

Next, we take up the proof of (6), we have

$$\begin{aligned}\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta u\|^2 &= (\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta u, \sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta \bar{L}^* v)\\ &\leqslant |(\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta \bar{L}u, \sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta v)|\\ &\quad + \frac{C}{\gamma(P,\delta)}\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta u\|\,\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta v\|\\ &\quad + C\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta u\|\,\|\sigma^0_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+\tilde{\zeta} v\|\\ &\quad + C\|\sigma_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+\tilde{\zeta} u\|\,\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta v\|\\ &\quad + C\delta^{\text{power}}(\|u\|^2 + \|v\|^2)\end{aligned}$$

hence

$$\begin{aligned}\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta u\|^2 &\leqslant C\gamma(P,\delta)^2\|\sigma^0_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta \bar{L}u\|^2\\ &\quad + \frac{C}{\gamma(P,\delta)^2}\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta v\|^2\\ &\quad + \text{small const. }\|\sigma_{P\delta}\Gamma_\delta \mathcal{P}^+\zeta u\|^2\\ &\quad + C\|\sigma^0_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+\zeta v\|^2\\ &\quad + C\gamma(P,\delta)^2\|\sigma^0_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+\tilde{\zeta} u\|^2\\ &\quad + C\delta^{\text{power}}(\|u\|^2 + \|v\|^2).\end{aligned}$$

Again, multiplying by $\gamma(P,\delta)^{-2}\dfrac{dP}{\operatorname{vol}(P,\delta)}\dfrac{d\delta}{\delta}$ and integrating, we obtain

$$\begin{aligned}|||\mathcal{P}^+\zeta u|||_1^2 \leqslant C\|\mathcal{P}^+\zeta\bar{L}u\|^2 + C|||\mathcal{P}^+\zeta v|||_2^2 + \text{small const. }|||\mathcal{P}^+\zeta u|||_1^2\\ + C|||\tilde{\mathcal{P}}^+\tilde{\zeta} v|||_1^2 + C(\|u\|^2 + \|v\|^2).\end{aligned}$$

Then from (5) and (5)′ we get, after choosing M sufficiently large

$$|||\mathcal{P}^+\zeta u|||_1^2 \leqslant C\|\bar{L}u\|^2 + C(\|u\|^2 + \|v\|^2).$$

The desired inequality (6) then follows from the closed range property.

We are now in a position to finish the proof of the theorem by proving (4). We have

$$\begin{aligned}\text{(11)}\qquad \|\sigma^0_{P\delta}\Gamma_\delta L\mathcal{P}^+\zeta u\|^2 &\leqslant \|\sigma_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+\tilde{\zeta}\bar{L}u\|^2\\ &\quad + \frac{C}{\gamma(P,\delta)^2}\|\sigma_{P\delta}\tilde{\Gamma}_\delta \tilde{\mathcal{P}}^+\tilde{\zeta} u\|^2\\ &\quad + C\gamma(P,\delta)^2\|\sigma^0_{P\delta}\Gamma_\delta \theta T\mathcal{P}^+\zeta u\|^2\\ &\quad + C\delta^{\text{power}}\|u\|^2.\end{aligned}$$

This estimate is obtained by again integrating by parts as follows

$$\begin{aligned}\|ALBu\|^2 &= (L^*ALBu, ABu) + (ALBu, [A, L]Bu)\\ &= (ALBL^*u, ABu) + ([L^*, ALB]u, ABu) + (ALBu, [A, L]Bu)\\ &= \|ABL^*u\|^2 + (A[L^*, L]Bu, ABu) + \cdots\end{aligned}$$

and using the estimate

$$|(A[L^*, L]Bu, ABu)| \leqslant \frac{1}{\gamma^2}\|ABu\|^2 + \gamma^2\|A\theta TBu\|^2 + \cdots$$

Since $\delta\Gamma_\delta T$ is of order zero and since

$$|\theta| \leqslant \frac{C\delta}{\gamma(P,\delta)^2}$$

we get

$$\|\sigma^0_{P\delta}\Gamma_\delta\theta T\mathcal{P}^+\zeta u\| \leqslant \frac{C}{\gamma(P,\delta)^2}\|\sigma_{P\delta}\tilde{\Gamma}_\delta\mathcal{P}^+\zeta u\| + C\delta^{\text{power}}\|u\|.$$

Substituting this into (11), multiplying by $\dfrac{dP}{\operatorname{vol}(P,\delta)}\dfrac{d\delta}{\delta}$ and integrating we obtain

$$\|L\mathcal{P}^+\zeta u\|^2 \leqslant C\|\bar{L}u\|^2 + C|||\tilde{\mathcal{P}}^+\tilde{\zeta}u|||_1^2 + C\|u\|^2.$$

Hence, applying (6) we obtain (4) and conclude the proof of the theorem.

7. Estimates of Kernels

In this section we prove the main theorem stated in the introduction.

PROOF OF (A). Suppose that f is supported in $B(P, C\delta)$, and let $u = f - S_b f$. We know that $\bar{L}u = \bar{L}f$ and that there exist v and w so that $u = \bar{L}^*v$ and $v = \bar{L}w$. Then we have $\|u\| \leqslant \|f\|$ and

(1) $$\|v\|_{B(P,C\delta)} \leqslant C'\gamma(P,\delta)\|u\| \leqslant C'\gamma(P,\delta)\|f\|.$$

Now we introduce the rescaled functions $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{f}$ defined on $\{\tilde{x}: |\tilde{x}| < 10\}$, here we denote by $\tilde{x}$ the coordinates defined by (3) in Section 4 and we are setting $\tilde{x} = y$. We define

$$\Phi(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) = \left(\frac{\tilde{x}_0}{\delta}, \frac{\tilde{x}_1}{\gamma(P,\delta)}, \frac{\tilde{x}_2}{\gamma(P,\delta)}\right),$$

$$\tilde{f} = f \circ \Phi, \qquad \tilde{u} = u \circ \Phi, \qquad \tilde{v} = \gamma^{-1} v \circ \Phi, \qquad \tilde{w} = \gamma^{-1} w \circ \Phi.$$

Here we are dealing with fixed P and δ and set $\gamma = \gamma(P, \delta)$. Then we have $\bar{\tilde{L}}\tilde{u} = \bar{\tilde{L}}\tilde{f}$, $\tilde{u} = \bar{\tilde{L}}^*\tilde{v}$, $\tilde{v} = \bar{\tilde{L}}\tilde{w}$ with $\|\tilde{u}\|_{\{|\tilde{x}| < 10\}} + \|\tilde{v}\|_{\{|\tilde{x}| < 10\}} \leqslant C\|\tilde{f}\|$, where $\tilde{L} = \gamma(P, \delta)L$.

Recall the subelliptic estimates from Section 1, namely

$$\|\phi\tilde{u}\|_{s+\epsilon} \leqslant C\|\phi'\bar{\tilde{L}}\tilde{f}\|_s + C\|\phi'\tilde{u}\| + C\|\phi'\tilde{v}\|. \tag{2}$$

If we pick f so that $\operatorname{supp}\tilde{f} \cap \operatorname{supp}\phi' = \varnothing$, then the first term on the right is zero so we get

$$\|\phi\tilde{u}\|_{s+\epsilon} < C\|\tilde{f}\|. \tag{3}$$

Now $\tilde{u}(\tilde{x}) = \int \tilde{K}(\tilde{x}, \tilde{y})\tilde{f}(\tilde{y})\, d\tilde{y}$ where $\tilde{K}$ is the rescaled Szegö kernel $\tilde{K}(\tilde{x}, \tilde{y}) = \gamma^2 \delta K(\Phi(\tilde{x}), \Phi(\tilde{y}))$. We write, $\tilde{u} = \tilde{S}_b\tilde{f}$. We have shown that for U, U' contained in $\{\tilde{x}\colon |\tilde{x}| < 10\}$ with $\bar{U} \cap \bar{U}' = \varnothing$, we have

$$\|\tilde{S}f\|_{H^s(U)} \leqslant C\|\tilde{f}\|$$

for $\tilde{f}$ supported in U'. Now $\tilde{S}_b$ is self-adjoint, so we obtain

$$\|\tilde{S}\tilde{f}\|_U \leqslant C\|\tilde{f}\|_{H^{-s}(U')}$$

for $\tilde{f}$ supported in U', by taking duals and interchanging U with U'. Interpolating between the last two estimates, we obtain

$$\|\tilde{S}\tilde{f}\|_{H^{s/2}(U)} \leqslant C\|\tilde{f}\|_{H^{-s/2}(U')}.$$

That means that $\tilde{K} \in C^\infty(U \times U')$ uniformly in δ and P. Hence $|p(\tilde{L}, \bar{\tilde{L}})\tilde{K}| \leqslant C$ for any polynomial p and L, $\bar{\tilde{L}}$ can act either in $\tilde{x}$ or $\tilde{y}$. The desired inequality (2) then follows by reinterpreting this in terms of L, $\bar{L}$ and K.

PROOF OF (B). First we have

$$\|u\|_{B(P, C\delta)} \leqslant C'\gamma(P, \delta)\|f\|. \tag{4}$$

Let $\theta \in C_0^\infty(B(P, C\delta))$ be a function such that $\tilde{\theta} = \theta \circ \Phi \in C_0^\infty(\{x\colon |x| < 10\})$ uniformly in δ and P. Let $u^\# = \theta u - S(\theta u)$. Now

$$\bar{L}u^\# = (\bar{L}\theta)u + \theta(\bar{L}u),$$

hence if $U \subset B(P, C\delta)$ is a neighborhood on which $\theta = 1$ then $\bar{L}^*\bar{L}u^\# = \bar{L}^*\bar{L}u = \bar{L}^*f$ in U. Moreover $\|u^\#\| \leqslant \|\theta u\| \leqslant C'\gamma(P, \delta)\|f\|$. Since $u^\# \perp \mathcal{H}_b$ we have $u^\# = \bar{L}^*v^\#$. Choosing $v^\#$ orthogonal to $\mathfrak{N}(\bar{L}^*)$ we have

$$\|v^\#\|_U \leqslant C\gamma(P, \delta)\|u^\#\| \leqslant C\gamma(P, \delta)^2\|f\|. \tag{5}$$

Now we rescale f, u, $u^{\#}$, $v^{\#}$ by setting $\tilde{f} = f \circ \Phi$, $\tilde{u} = \gamma(P,\delta)^{-1}u \circ \Phi$, $\tilde{u}^{\#} = \gamma(P,\delta)^{-1}u^{\#} \circ \Phi$, and $\tilde{v}^{\#} = \gamma(P,\delta)^{-2}v^{\#} \circ \Phi$. Then we have $\|\tilde{u}^{\#}\| \leqslant C\|\tilde{f}\|$, $\|\tilde{v}^{\#}\| \leqslant C\|\tilde{f}\|$, $\tilde{\bar{L}}^*\tilde{\bar{L}}\tilde{u}^{\#} = \tilde{\bar{L}}^*\tilde{f}$, and $\tilde{L}^*\tilde{v}^{\#} = \tilde{u}^{\#}$ in $\tilde{U} = \Phi^{-1}(U)$, which we may take to be $\{x: |\tilde{x}| < 10\}$. Returning to the subelliptic estimate, we have from Proposition 1.5

$$\|\phi\tilde{u}^{\#}\|_{s+\epsilon} \leqslant C\|\phi'\tilde{f}\|_s + C\|\phi'\tilde{u}^{\#}\| + C\|\phi'\tilde{v}^{\#}\|.$$

Assuming that $\operatorname{supp}\tilde{f} \cap \operatorname{supp}\phi' = \varnothing$, the first term on the right vanishes, while the last terms are dominated by $C\|\tilde{f}\|$. Thus we obtain $\|\phi\tilde{u}^{\#}\|_{s+\epsilon} \leqslant C\|\tilde{f}\|$. What we really want is $\|\phi\tilde{u}\|_{s+\epsilon} \leqslant C\|\tilde{f}\|$, so we look at the difference $\tilde{u} - \tilde{u}^{\#}$ in $\operatorname{supp}\phi$.

Recall that $u^{\#} = \theta u - S_b(\theta u)$ and that $S_b u = 0$. Hence in $\operatorname{supp}\theta$ we have $u^{\#} = u + S_b((1-\theta)u)$, that is for $x \in \operatorname{supp}\theta$

$$(u^{\#} - u)(x) = \int K(x,y)(1-\theta(y))u(y)\,dy. \tag{6}$$

If $p_k(L,\bar{L})$ is a monomial of order k in L, $\bar{L}$, then for $x \in \operatorname{supp}\theta$

$$P_k(L,L)(u^{\#} - u)(x) = \int \{p_k(L,\bar{L})K(x,y)\}(1-\theta(y))u(y)\,dy,$$

where the $p_k(L,\bar{L})$ acts on the x-variable. Since $1 - \theta(y) = 0$ in $B(P, C\delta)$ we may break up the above integral as

(7)
$$p_k(L,\bar{L})(u^{\#} - u)(x) = \sum_{i=-10}^{\infty} \int_{y \in B(P,2^i\delta)\setminus B(P,2^{i-1}\delta)} |p_k(L,\bar{L})K(x,y)|\,|u(y)|\,dy.$$

In the region of integration for fixed i we have

$$|p_k(L,\bar{L})K(x,y)| \leqslant \frac{C\gamma(P,2^i\delta)^{-k}}{2^i\delta\gamma(P,2^i\delta)^2}. \tag{8}$$

Also we have

$$\|u\|_{B(P,2^i\delta)} \leqslant C\gamma(P,2^i\delta)\|f\|$$

so by Cauchy-Schwarz

$$\int_{B(P,2^i\delta)} |u(y)|\,dy \leqslant C\gamma(P,2^i\delta)[\gamma(P,2^i\delta)^2 2^i\delta]^{1/2}\|f\|. \tag{9}$$

Inserting (9) and (8) in (7) and recalling that $\gamma(P,2^i\delta) > 2^{i/2}\gamma(P,\delta)$ we obtain

$$|p_k(L,\bar{L})(u^{\#} - u)(x)| \leqslant C\gamma(P,\delta)^{-k}\delta^{-1/2}\|f\|$$

for $x \in \operatorname{supp}\theta$.

Rewriting this estimate in terms of the rescaled quantities we get

$$|p_k(\tilde{L}, \tilde{\bar{L}})\tilde{u}^{\#} - \tilde{u})| \leqslant C' \| f \|$$

for $|\tilde{x}| < 10$.

Since $\partial/\partial\tilde{x}_j$ is a bounded linear combination of the $p_k(\tilde{L}, \tilde{\bar{L}})$ with $k \leqslant m$ in $\{\tilde{x}: |\tilde{x}| < 10\}$ we conclude that for any s

$$\| \phi(\tilde{u} - \tilde{u}^{\#}) \|_{H^s} \leqslant C'' \| \tilde{f} \|.$$

Combining this with our previous estimate $\| \phi\tilde{u}^{\#} \|_{s+\epsilon} \leqslant c \| \tilde{f} \|$, we find that

$$\| \tilde{u} \|_{H^s(V)} \leqslant C'' \| \tilde{f} \| \tag{10}$$

if $V, V' \subset \{\tilde{x}: |\tilde{x}| < 10\}$ are neighborhoods with disjoint closures with $\operatorname{supp} \tilde{f} \subset V'$.

Denoting the operator that takes f to u by Nf, that is $u = Nf$, we observe that there is a completely analogous estimate for N^* by interchanging the roles of $\bar{L}$ and $\bar{L}^*$. Then by duality and interchanging of V and V', we get

$$\| \tilde{u} \|_V \leqslant C'' \| \tilde{f} \|_{H^{-s}(V')} \quad \text{where} \quad \operatorname{supp} \tilde{f} \subset V'. \tag{11}$$

Then from (10) and (11) we conclude that $\tilde{H} \in C^\infty(V \times V')$ where

$$\tilde{u}(\tilde{x}) = \int \tilde{H}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y})\, d\tilde{y}.$$

In particular, for a monomial $p_k(\tilde{L}, \tilde{\bar{L}})$ of degree k with each L acting either on $\tilde{x}$ or $\tilde{y}$, we get

$$|p_k(\tilde{L}, \tilde{\bar{L}})\tilde{H}(\tilde{x}, \tilde{y})| \leqslant C_k$$

for $\tilde{x} \in V$, $\tilde{y} \in V'$ with constants C_k independent of γ and P. Returning to the original coordinates we obtain the desired estimate (11) in the Introduction thus completing the proof.

PROOF OF (C). Define kernels $K(x, y)$, $\tilde{K}(x, y)$ to provide the relative fundamental solutions of $\bar{L}$, $\bar{L}^*$ respectively. Thus

$$u(x) = \int K(x, y) f(y)\, dy$$

solves $\bar{L}u = f$ modulo the nullspace of $\bar{L}^*$, and u is orthogonal to the nullspace of $\bar{L}$; while

$$\tilde{u}(x) = \int \tilde{K}(x, y) f(y)\, dy$$

solves $\bar{L}^*\tilde{u} = f$ modulo the nullspace of $\bar{L}$, and $\tilde{u}$ is orthogonal to the nullspace of $\bar{L}^*$. From K, $\tilde{K}$ we define composed kernels

$$K_2(x,z) = \int \tilde{K}(x,y)K(y,z)\,dy$$
$$K_3(x,z) = \int K(x,y)K_2(y,z)\,dy.$$

In particular, K_2 is the kernel for the relative fundamental solution of $\bar{L}^*\bar{L}$, so (C) in the Main Theorem amounts to proving estimates on K_2. First we estimate the size of K_2, K_3. Recall from (B) the estimates

$$|K(x,y)|,\ |\tilde{K}(x,y)| \leqslant C/\delta(x,y)\gamma(x,y).$$

Here $\gamma(x,y)$ is the least γ for which $y \in B(x,\gamma)$, while $\delta = \delta(x,\gamma)$. From these estimates follows easily

$$\int_{B(x,\gamma)} |\tilde{K}(x,y)|\,dy \leqslant C\gamma, \qquad \int_{B(z,\gamma)} |K(y,z)|\,dy \leqslant C\gamma.$$

To estimate K_2, we write $\gamma = \gamma(x,z)$ and $\delta(\gamma) = \delta(z,\gamma)$, then break up the region of integration in the definition of K_2. We get

$$\begin{aligned}|K_2(x,z)| &\leqslant \int_{y\in B(x,\gamma/10)} |\tilde{K}(x,y)|\,\frac{C}{\gamma\delta(\gamma)}\,dy\\ &\quad+ \int_{y\in B(z,2\gamma)\setminus B(x,\gamma/10)} \frac{C}{\gamma\delta(\gamma)}\,|K(y,z)|\,dy\\ &\quad+ \sum_{i\geq 1}\int_{y\in B(z,2^{i+1}\gamma)\setminus B(z,2^i\gamma)} \left[\frac{C}{(2^i\gamma)\delta(2^i\gamma)}\right]^2 dy\\ &\leqslant \frac{C'}{\delta(\gamma)} + \sum_{i\geq 1}\frac{C}{\delta(2^i\gamma)} \leqslant \frac{C''}{\delta(\gamma)}.\end{aligned}$$

Similarly, using our estimate for the size of K_2 in the definition of K_3,

$$\begin{aligned}|K_3(x,z)| &\leqslant \int_{y\in B(x,\gamma/10)} |K(x,y)|\,\frac{C}{\delta(\gamma)}\,dy\\ &\quad+ \int_{y\in B(z,2\gamma)\setminus B(x,\gamma/10)} \frac{C}{\gamma\delta(\gamma)}\,|K_2(y,z)|\,dy\\ &\quad+ \sum_{i>1}\int_{y\in B(z,2^{i+1}\gamma)\setminus B(z,2^i\gamma)} \frac{C}{(2^i\gamma)\delta(2^i\gamma)}\,\frac{C}{\delta(2^i\gamma)}\,dy\\ &\leqslant \frac{C\gamma}{\delta(\gamma)} + \sum_{i>1} C'\,\frac{2^i\gamma}{\delta(2^i\gamma)} \leqslant \frac{C''\gamma}{\delta(\gamma)},\end{aligned}$$

since (19) in Section 2 implies that $\delta(2^i\gamma) \geqslant 4^i\delta(\gamma)$.

Now suppose f is a function supported in $B(0, \gamma/10)$ and define functions

$$u_1(x) = \int K(x,y)f(y)\,dy,$$
$$u_2(x) = \int K_2(x,y)f(y)\,dy,$$
$$u_3(x) = \int K_3(x,y)f(y)\,dy.$$

On $B(0,\gamma)$ we rescale f, u_1, u_2, u_3 by setting $\tilde{f} = f \circ \Phi$, $\tilde{u}_1 = \gamma^{-1}u_1 \circ \Phi$, $\tilde{u}_2 = \gamma^{-2}u_2 \circ \Phi$, $\tilde{u}_3 = \gamma^{-3}u_3 \circ \Phi$. These functions are defined in $\{\tilde{x}: |\tilde{x}| < 1\}$. The definitions of our kernels show that $\bar{L}^*u_2 = u_1$, $\bar{L}u_3 = u_2$, hence $\tilde{\bar{L}}^*\tilde{u}_2 = \tilde{u}_1$, $\tilde{\bar{L}}\tilde{u}_3 = \tilde{u}_2$ in $\{\tilde{x}: |\tilde{x}| < 1\}$. Therefore, the subelliptic estimates of Section 1 give

$$\|\tilde{\phi}\tilde{u}_2\|_{s+\epsilon} \leqslant C\|\tilde{\phi}\tilde{u}_1\|_s + C\|\tilde{u}_2\|_{L^2\{|\tilde{x}|<1\}} + C\|\tilde{u}_3\|_{L^2\{|\tilde{x}|<1\}}$$

for $\tilde{\phi} \in C_0^\infty(\{|\tilde{x}| < 1\})$. Our estimates for the size of the kernels K_2, K_3 show easily that $\|\tilde{u}_2\|_{L^2\{|\tilde{x}|<1\}}$, $\|\tilde{u}_3\|_{L^2\{|\tilde{x}|<1\}} \leqslant C\|\tilde{f}\|$. Thus,

$$\|\tilde{\phi}\tilde{u}_2\|_{s+\epsilon} \leqslant C\|\tilde{\phi}\tilde{u}_1\|_s + C\|\tilde{f}\|.$$

In the proof of (B), we showed that $\|\tilde{\phi}\tilde{u}_1\|_s \leqslant C\|\tilde{f}\|$, provided $\tilde{\phi}$ is supported away from $\{\tilde{x}: |\tilde{x}| < 1/10\}$, where $\tilde{f}$ is supported.

So we get

$$(*) \qquad \|\tilde{\phi}\tilde{u}_2\|_{s+\epsilon} \leqslant C'\|\tilde{f}\|$$

for $\operatorname{supp}(f) \subset \{x: |x| < 1/10\}$, $\operatorname{supp}(\tilde{\phi})$ disjoint from $\{\tilde{x}: |\tilde{x}| < 1/10\}$. To interpret this, we introduce the rescaled operator $\tilde{G}_2$ and kernel $\tilde{K}_2(x,y)$, defined by

$$\tilde{G}_2\tilde{f}(\tilde{x}) = \tilde{u}_2(\tilde{x}) = \int \tilde{K}_2(\tilde{x},\tilde{y})f(\tilde{y})\,d\tilde{y}.$$

For $x, y \in B(0,\gamma)$, $K_2(x,y)$ is related to $\tilde{K}_2(\tilde{x},\tilde{y})$ by an obvious scaling. Estimate $(*)$ says that

$$(**) \qquad \tilde{G}_2: L^2(\tilde{U}) \to H^{s+\epsilon}(\tilde{V})$$

for U, V disjoint neighborhoods in $\{\tilde{x}: |\tilde{x}| < 1\}$.

Note also that $\tilde{G}_2$ is self-adjoint, since $K_2(x,y)$ provides the relative fundamental solution of the self-adjoint operator $\bar{L}^*\bar{L}$. Hence by duality we deduce from $(**)$: $\tilde{G}_2: H^{-s-\epsilon}(V) \to L^2(U)$. Reversing the roles of $\tilde{U}$ and $\tilde{V}$ and interpolating with $(**)$, we find that $\tilde{G}_2: H^{-(s-\epsilon)/2}(\tilde{U}) \to H^{(s+\epsilon)/2}(\tilde{V})$. This means that the kernel $\tilde{K}_2(\tilde{x},\tilde{y})$ is C^∞ away from the diagonal. After rescaling, we obtain the estimate (11) of the Introduction with $q = 2$ thus proving (C).

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