Revista Matemática Iberoamericana
Vol. 4, n.^{os} 3 y 4, 1988

Infinite Group Actions on Spheres

Gaven J. Martin

Introduction

This paper is mainly intended as a survey of the recent work of a number of authors concerning certain infinite group actions on spheres and to raise some as yet unanswered questions. The main thrust of the current research in this area has been to decide what topological and geometrical properties characterise the infinite conformal or Möbius groups. One should then obtain reasonable topological or geometric restrictions on a subgroup G of the homeomorphism group of a sphere which will imply that it can be made conformal after a change of coordinates. That is G is topologically conjugate to a Möbius group. Many aspects of the theory of Kleinian and Möbius groups can be found in the books [Ah 3], [Bea], [Mas 1], [MB] and [Th].

Another focus of an investigation into the topological nature of Möbius groups is that it helps us to better understand these groups and enables us to decide what features are essential to the development of certain aspects of the general theory. As a relatively simple question one may ask whether the classification into elliptic, parabolic and loxodromic elements for a conformal group is analytic or topological in nature? What properties determine the structure of the limit set? To gain deeper insight one is led to ask are the combination and decomposition theorems for Kleinian groups (a lá Maskit), either topological, geometric or analytic in nature? How does Ahlfors' finiteness theorem fit into the picture? Is it really topological in nature or does it definitely depend on the analytic structure of Kleinian groups (as does the proof). Perhaps it is essentially an algebraic statement about finitely generated Kleinian groups! Also what of Bers' Area Theorem and Sullivan's Finiteness of Cusps. Are these too more general phenomena? Is Selberg's theorem on the existence of torsion free finite index subgroups of finitely generated linear groups a phenomena exhibited by any reasonable topological action on a sphere?

In this introduction we will outline the main ideas and some of their consequences, specific details of statements and some proofs appear later. I hope the careful reader (and especially the relevant authors) will forgive the simplifications I have had to make in the proofs that do appear.

A first natural generalisation of a group of conformal homeomorphisms of a sphere is that of a *quasiconformal group* of homeomorphisms. Such groups can be thought of as acting with a uniformly bounded amount of distortion, whereas conformal groups have no distortion (at least infinitesimally). We use J. Väisälä's book [Vä] as our standard reference for the theory of quasiconformal mappings. The following question was first asked by F. W. Gehring and B. Palka [GP] in connection with their work on quasiconformally homogeneous domains:

Is every uniformly quasiconformal group the quasiconformal conjugate of a conformal group?

The answer to this question is unknown in dimension one (where quasiconformal maps are more commonly called quasisymmetric maps), yes in dimension two and generally false in higher dimensions. These, together with some positive results, are mainly due to D. Sullivan [Su 1] and P. Tukia [Tu 1, 2, 3].

We regard quasiconformal groups as a geometric generalisation of conformal groups. A topological generalisation was proposed by Gehring and the author [GM] in their study of quasiconformal groups. The idea is to retain the normal families properties of conformal and quasiconformal mappings, as exhibited by the fact that any sequence of conformal homeomorphisms contains a subsequence whose limit is either a conformal homeomorphism or a constant mapping. This compactness property is clearly a necessary condition for the existence of a topological conjugacy and we call group of homeomorphisms of the n-sphere with this property a convergence group. We then ask.

When is a convergence group topologically conjugate to a conformal group?

Again the answer to this question is largely unknown although there are good positive results. In dimension one it is thought to be always the case while in higher dimensions some additional hypotheses are clearly necessary. We ask the question with regard to conformal groups since we are primarily interested in lower dimensional phenomena or cases where the natural geometric candidate for a possible conjugacy is a conformal group, for instance as in the case of topological analogues of Schottky groups. As a simple example, the group of complex biholomorphisms of the unit ball in $\mathbb{C}^n$, $n \geqslant 2$, acts as a convergence group on $\mathbb{S}^{2n-1}$ but by Margulis super-rigidity this group or any of its discrete cocompact subgroups cannot even be isomorphic to conformal groups as the enveloping Lie algebras are distinct. Actually these groups cannot even be topologically conjugate to quasiconformal groups. There are however quite reasonable hypotheses to make in a general setting which yield good answers to the above question.

There are many closely related questions to those raised above. For instance:

When does the group action on the n-sphere extend to the $(n + 1)$*-ball?*

This question is motivated by the classical Poincaré extension which extends conformal actions on the n-sphere to the $(n + 1)$-ball.

In dimension one this extension problem is equivalent to the general conjugacy problem, [MT]. And due to results of G. Mess [Me] and P. Scott [Sc], the general conjugacy problem is equivalent to the Seifert conjecture for three manifolds:

Is a compact three manifold with a normal infinite cyclic subgroup in its fundamental group is a Seifert fibered space?

Roughly a Seifert fibered space is a three manifold foliated by circles in a nice way. There are other interesting consequences related to Teichmüller theory. Indeed the one dimension question on the conjugacy of convergence groups to Fuchsian groups is essentially a universal version of the Nielsen Realisation Problem. The usual Nielsen Realisation Problem, solved by S. Kerchoff [Ker], is implied by the statement (see [Tu 3]):

A group of homeomorphisms acting on the circle with a Fuchsian subgroup of finite index is topologically conjugate to a Fuchsian group.

Necessarily a group of homeomorphisms with a Fuchsian subgroup of finite index is a convergence group. P. Tukia has shown that convergence groups of the circle are conjugate to Fuchsian groups unless they contain a semitriangle group of finite index [Tu 3]. Such groups are so special as to be presumed not to exist. In addition he and the author have shown that convergence groups of the circle which can be extended to the entire disk are conjugate to Fuchsian groups [MT].

In higher dimensions the problem is essentially more complicated. It is known that the fundamental group of a strictly negatively curved manifold acts as a convergence group on the sphere at infinity of the universal covering space [MS]. Thus reasonable affirmative solutions to the problem imply results concerning the homotopy type of negatively curved compact manifolds. This is especially interesting in the three dimensional case where it is thought that every negatively curved compact three manifold is hyperbolic. There is enough machinery in place, mainly due to Thurston, so that the answer to this question is essentially implied by the conjecture:

A convergence group acting on the two sphere and whose limit set is the whole two sphere is topologically conjugate to a Kleinian group?

For partial results along this line see [MS] and [MT]. There is a relationship here between these questions and those of Gromov concerning his so called hyperbolic groups [Gr. 2]. Also the fact that the fundamental group of a strictly negatively curved manifold acts as a convergence group on the sphere at infinity explains many of the geometric features of the isometry groups of visibility manifolds discovered by Eberlin and O'Neil in their work [EO].

M. Freedman has shown that in higher dimensions, the question of the existence of a solution to the extension problem for even a rather restrictive class of convergence groups is equivalent to the surgery problem [Fr]. In particular the four dimensional surgery problem, at present unsolved, is equivalent to the following question concerning a topological characterisation of certain Schottky groups acting on the three sphere.

Given a convergence group G acting on the three sphere, isomorphic to the free group on a finite number of generators, with limit set a Cantor set and which is of compact type, does G extend to a convergence group of the four ball?

These questions are all very interesting, however the simplicity of their statements often belies the fact that they are mostly quite difficult. But as we will see even partial solutions are often revealing and unify many approaches to varied problems.

Finally we remark that the theory of finite group actions on spheres is subsumed in trying to understand general convergence groups. Fortunately this theory is well developed but there are still many unanswered questions which are rather difficult and deep, for a good survey (particularly the three dimensional aspects) see Morgan and Bass [MB] and the references therein. We will try to concentrate on the aspects of the theory which specifically relate to the group being infinite.

1. Notation and Definitions

We denote by Hom ($\mathbb{S}^n$) the homeomorphism group of the unit sphere $\mathbb{S}^n$ of euclidean $(n + 1)$-space, $\mathbb{R}^{n+1}$. The topology of Hom ($\mathbb{S}^n$) is the usual compact open topology, which is the same as that of the topology of uniform convergence in the *chordal distance* $\rho(x, y)$ that $\mathbb{S}^n$ inherits as a metric subspace of $\mathbb{R}^{n+1}$. A self homeomorphism f of $\mathbb{S}^n$ is said to be *K-quasiconformal*, $1 \leqslant K < \infty$, if for each $x \in \mathbb{S}^n$

$$\limsup_{r \to 0} \frac{\mathcal{L}(x, r)}{\ell(x, r)} \leqslant K$$

where

$$\mathcal{L} = \max \{\rho(f(x), f(y)) \colon \rho(x, y) = r\}$$

and

$$\ell = \min \{\rho(f(x), f(y)) \colon \rho(x, y) = r\}.$$

By a *conformal* homeomorphism of $\mathbb{S}^n$ we mean a Möbius transformation, that is the restriction to $\mathbb{S}^n$ of a finite composition of reflections in spheres or hyperplanes orthogonal to $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, [Ah 3]. In all dimensions, a self homeomorphism of the sphere is conformal if and only if it is 1-quasiconformal [Ge]. Conformal mappings are of course the boundary values of isometries of the usual Riemannian hyperbolic metric $ds^2 = (1 - |x|^2)^{-2}|dx|^2$ of constant negative curvature on the unit ball B^{n+1}.

A subfamily $\mathfrak{F}$ of Hom ($\mathbb{S}^n$) is said to have the *convergence property* if each infinite subfamily of $\mathfrak{F}$ contains a subsequence $\{f_j\}$ for which one of the following two properties holds

(*a*) There is an $f \in$ Hom ($\mathbb{S}^n$) such that $f_j \to f$ and $f_j^{-1} \to f^{-1}$ uniformly.
(*b*) There are points x and y (possibly $x = y$) such that $f_j \to x$ locally uniformly in $\mathbb{S}^n - \{y\}$ and $f_j^{-1} \to y$ locally uniformly in $\mathbb{S}^n - \{x\}$.

One should note that there is a slight redundancy in the definition with regard to the convergence of the inverse mappings.

Definition 1.1. *A subgroup G of* Hom ($\mathbb{S}^n$) *is called a* ***convergence group*** *if G has the convergence property. A subgroup G of* Hom ($\mathbb{S}^n$) *is called a* ***quasiconformal group*** *if there is a finite K such that every $g \in G$ is K-quasiconformal and G is called a* ***Möbius group*** *(sometimes* ***conformal group****) if every $g \in G$ is a Möbius transformation.*

Our interest in the convergence property is twofold. Firstly it is the characteristic compactness property of conformal and quasiconformal mappings, see [GM Thm. 3.2].

Theorem 1.2. *The family $\mathfrak{F}_K$ of all K-quasiconformal homeomorphisms of $\mathbb{S}^n$ has the convergence property.*

Corollary 1.3. *A quasiconformal group G is a convergence group.*

Secondly the convergence property is topologically invariant.

Theorem 1.4. *If G is a convergence group and h is a self homeomorphism of $\mathbb{S}^n$, then the group $H = hGh^{-1}$ is also a convergence group.*

Thus the topological conjugates of Möbius and quasiconformal groups are convergence groups. Clearly then the convergence property is a necessary condition for a group G to be the topological conjugate of a quasiconformal group.

Definition 1.5. *Let Ω be a region in the sphere and $x \in \Omega$. A subfamily $\mathfrak{F}$ of* Hom $(\mathbb{S}^n)$ *is said to act* ***discontinuously at*** *x if there is a neighbourhood U of x in Ω such that for all but finitely many $f \in \mathfrak{F}$,*

$$f(U) \cap U = \varnothing.$$

We say that $\mathfrak{F}$ acts ***discontinuously in*** *Ω if it acts discontinuously at each point of Ω. We say that $\mathfrak{F}$ acts* ***properly discontinuously*** *in Ω if for each compact subset E of Ω*

$$f(E) \cap E = \varnothing$$

for all but finitely many $f \in \mathfrak{F}$.

Definition 1.6. *Let G be a convergence group acting on $\mathbb{S}^n$. The* ***ordinary set*** *for G is the set*

$$O(G) = \{x \in \mathbb{S}^n : G \text{ acts discontinuously at } x\}.$$

The ***limit set*** *of G is the set*

$$L(G) = \mathbb{S}^n - O(G).$$

We say that is ***discrete*** *if the identity is isolated in G.*

Notice that by definition $O(G)$ is open and $L(G)$ is closed. Recall that even in the conformal case it is possible that G is discrete while $O(G)$ is empty. Whenever G is discrete the convergence property (a) can never occur for if $g_j \to g$, then the sequence $g_j g_{j+1}^{-1}$ converges uniformly to the identity in G. The convergence property (b) guarantees $L(G)$ is not empty as soon as G is infinite and discrete. In fact the condition (b) easily implies.

Theorem 1.7. *A discrete convergence group G acts properly discontinuously in its ordinary set O(G).*

We recall that a *Schottky group* is a Möbius group of the n-sphere whose limit set is a *Cantor set*, that is a totally disconnected perfect set. The classical example of a Schottky group is the group generated by reflections in a finite disjoint collection of codimension one round spheres with the property that no sphere separates the collection. Schottky groups are a good example from which to build a general theory, they being the simplest infinite Möbius groups that exhibit interesting geometric and dynamic behaviour. The following result [GM Thm. 7.8] shows that any reasonable topological generalisation of a Schottky group will necessarily be a convergence group.

Theorem 1.8. *Let G be a subgroup of* Hom $(\mathbb{S}^n)$ *which acts properly discontinuously in* $\mathbb{S}^n - E$, *where E is a closed totally disconnected set. Then G is a discrete convergence group and L(G) lies in E.*

SKETCH OF PROOF. Let $\{g_i\}$ be an infinite sequence of elements of G and $x_0 \in \mathbb{S}^n - E$. Without loss of generality we may assume that $g_i(x_0) \to a$ and $g_i^{-1}(x_0) \to b$. Necessarily $a, b \in E$. We claim $g_i \to a$ locally uniformly in $\mathbb{S}^n - \{b\}$. Let $E_\delta = \{x \in \mathbb{S}^n : \rho(x, E) < \delta\}$. Then $E = \bigcap E_\delta$ and the maximum diameter of a component of E_δ tends to zero with δ. Assume that E_δ has connected complement. For sufficiently large i, $g_i(\mathbb{S}^n - E_\delta) \cap (\mathbb{S}^n - E_\delta) = \emptyset$ as G acts properly discontinuously in the complement of E. Since this set is connected its image must lie in a component of E_δ and as x_0 eventually lies in E_δ and $g_i(x_0) \to a$ we see that eventually $g_i(\mathbb{S}^n - E_\delta)$ lies in the component U_δ of E_δ containing a. The boundary of all the components of E_δ also must lie in this set and so the image of all but one of these components must lie in U_δ. It is not difficult to see that this one component must be that component V_δ of E_δ containing b. Since $U_\delta \to a$ and $V_\delta \to b$ the desired convergence follows. The details are in showing the assumption that the complement of E_δ is connected is not necessary.

Here is another useful characterisation of convergence groups. Let T_n denote the *triple space*

$$T_n = \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n - \Delta$$

where Δ is the big diagonal $\Delta = \{(x, y, z) : x = y \text{ or } y = z \text{ or } x = z\}$. Thus T_n is the space of distinct triples of points on the sphere.

The space T_1 is homeomorphic to $B^2 \times \mathbb{S}^1$. Let $n \geqslant 2$ and $\zeta = (x, y, z) \in T_n$. We think of x, y and z as being three distinct points on the boundary $\mathbb{S}^n$ of hyperbolic $(n+1)$-space B^{n+1}. Let $\mathfrak{L}$ be the hyperbolic line joining x to y and

$p(\zeta)$ as the projection of z onto $\mathfrak{L}$. That is $p(\zeta)$ is the point of $\mathfrak{L}$ such that the hyperbolic ray joining $p(\zeta)$ to z is orthogonal to $\mathfrak{L}$. For each $w \in B^{n+1}$ the fiber $p^{-1}(w)$ is homeomorphic to the space of two frames of the tangent space $T_w B^{n+1}$ the fiber $p^{-1}(w)$ is homeomorphic to the space of two frames of the tangent space $T_w B^{n+1}$ (the inner product comes from the hyperbolic metric). The homeomorphism is given by the directions of x and z at $w = p(\zeta)$. Since $T_w B^{n+1} = B^{n+1}$ we find

$$T_n = B^{n+1} \times V_{n+1,2},$$

where $V_{n+1,2} = \{(u, v) \in \mathbb{S}^n \times \mathbb{S}^n : u \perp v\}$ is the Steiffel manifold of two frames on hyperbolic space. Thus we think of T_n as being hyperbolic space up to a compact factor.

Given a subgroup G of $\mathrm{Hom}(\mathbb{S}^n)$ there is a natural homomorphism ϕ embedding G in $\mathrm{Hom}(T_n)$ given by $\phi(g)(\zeta) = (g(x), g(y), g(z))$ where $\zeta = (x, y, z)$.

The point of this exercise is the following (see [GM 2]).

Theorem 1.9. *The following assertions are equivalent*:

(1) *G is a discrete convergence group acting on $\mathbb{S}^n$.*
(2) *$\phi(G)$ acts properly discontinuously on T_n.*

Furthermore, the action of $\phi(G)$ on T_n is effective if G is torsion free.

Usually we will forget the homomorphism ϕ and say that G acts on the triple space.

P. Scott has conjectured that a finitely generated group G of homeomorphisms of $\mathbb{S}^1$ is isomorphic to a Fuchsian group if and only if $\phi(G)$ acts properly discontinuously on T_1 [Sc. pp. 175]. This is conjecture is evidently implied by the existence of a topological conjugacy of any one dimensional convergence group to a Fuschsian group. It is here that we find the relationship with the Seifert conjecture. If G is convergence group acting on $\mathbb{S}^1$, then $M^3 = T_1/\phi(G)$ is a three manifold for which we have the short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(M^3) \to G \to 1.$$

If the Seifert conjecture is true, then M^3 is Seifert fibered and G is isomorphic to a Fuchsian group. Given this isomorphism one can construct a topological conjugacy [MT]. Conversely, given an orientable three manifold M with $\mathbb{Z}$ as a normal subgroup and if M is not a circle bundle over the sphere then the universal cover of M is $B^2 \times \mathbb{R}^1$. Passing to the appropriate subcover given by the normal $\mathbb{Z}$ one can construct an action of the quotient group on the circle (which is thought of as ∂B^2) which acts properly discontinuously on the triple space $T_1 \approx B^2 \times \mathbb{S}^1$. If this action is topologically conjugate to a Fuchsian group, then the quotient group is isomorphic to a Fuchsian group and consequently M is a Seifert fibered space, [Sc] [Me].

From [St. p. 132] we find a calculation of the homotopy groups of the Steiffel manifolds. In particular $\pi_i(V_{n+1,2}) = 0$, $i < n - 1$ and $\pi_n(V_{n+1,2})$ is $\mathbb{Z}$ or $\mathbb{Z}_2$ depending on whether n is odd or even. Hence T_n is simply connected for $n > 2$ and thus a torsion free discrete convergence group of $\mathbb{S}^n$, $n > 2$, is isomorphic to the fundamental group of a $3n$-manifold.

This also suggests that an approach to the extension problem (from $\mathbb{S}^n$ to B^{n+1}) might be to try and split off the compact manifold factor from the action of the group on the triple space.

Theorem 1.9 has other important consequences. If M is a negatively curved $(n + 1)$ manifold, then it is well known that the universal cover of M is diffeomorphic to the open $(n + 1)$ ball. One can define a compactification of this universal cover: A point at infinity is an equivalence class of asymptotic geodesics (that is geodesics which remain a bounded distance apart). If the sectional curvature $K(M) \leqslant K < 0$, then the points at infinity form a topological n-sphere S_∞ which is the boundary of the universal covering space [EO]. It is not difficult to see that the elements of $\pi_1(M)$ act naturally on S_∞ via their action on the geodesics and that this action is as a group of homeomorphisms. The curvature assumption also implies that there is a unique geodesic line in the universal cover connecting any two points of the sphere at infinity. Using this fact, together with the fact that $\pi_1(M)$ is a discrete group of isometries and is therefore properly discontinuous on the universal cover, it is not too difficult to establish the following consequence of Theorem 1.9, see [MS Thm. 5.6].

Theorem 1.10. *Let M be a strictly negatively curved $(n + 1)$-manifold. Then $\pi_1(M)$ acts naturally as a convergence group on the sphere at infinity of the universal cover.*

This provides yet another reason for studing convergence groups. Results of this type for compact M are implicit in earlier work of Gromov [Gr 3].

2. The Limit Set and the Elementary Convergence Groups

The following lemma is a useful tool in the study of the limit set of a discrete convergence group.

Lemma 2.1. *Let G be a discrete convergence group. Then for each point $x_0 \in L(G)$ there is a point $y_0 \in L(G)$ and a sequence $\{g_i\}$ of elements of G such* *that*

$$g_i \to x_0 \text{ locally uniformly in } \mathbb{S}^n - \{y_0\}$$

and

$$g_i^{-1} \to y_0 \text{ locally uniformly in } \mathbb{S}^n - \{x_0\}$$

It is essentially this property which implies the following classical result on the structure of the limit set.

Theorem 2.2. *Let G be a discrete convergence group. Then*

(1) *$L(G)$ consists of either* 0, 1, 2 *points or else $L(G)$ is a perfect set.*
(2) *$L(G)$ is either nowhere dense or coincides with $\mathbb{S}^n$.*
(3) *If* card $L(G) > 2$, *then $L(G)$ is the smallest closed G invariant set.*

If $L(G) = \mathbb{S}^n$, then we say G is of the *first kind.* Otherwise G is of the *second kind.*

Theorem 2.2 holds in much more generality of course. A more general construction of a limit set was given by R. Kulkarni [Ku]. He points out that there is a close connection between the number of components of the limit set and the number of ends of the group (a purely algebraic concept). Extending ideas of H. Hopf and H. Freudenthal he went on to develop the ends inequality [Ku, Thm. 4] which for (finitely generated) convergence groups is

The number of components of $L(G)$ is no more than the number of ends of G.

This result has a nice consequence (realised by Kulkarni in the conformal case) on the algebraic structure of certain convergence groups which we will see later.

In a discrete Möbius group there are three types of elements; elliptic, parabolic and loxodromic. Here is a topological generalisation of these notions.

Definition 2.3. *Let G be a discrete convergence group. We define*

$$\operatorname{ord}(g) = \inf\{m : m \geqslant 1 \text{ and } g^m = \text{identity}\}$$

and

$$\operatorname{fix}(g) = \{x \in \mathbb{S}^n : g(x) = x\}.$$

*We say g is **elliptic** if* $\operatorname{ord}(g) < \infty$, *$g$ is **parabolic** if* $\operatorname{ord}(g) = \infty$ *and* card $\{\operatorname{fix}(g)\} = 1$ *and finally g is **loxodromic** if* $\operatorname{ord}(g) = \infty$ *and* card $\{\operatorname{fix}(g)\} = 2$.

Definition 2.4. *Let G be a discrete convergence group. We say that G is* ***elementary*** *if card* $L(G) \leqslant 2$.

Then, analogously to the classical case, it is quite natural to classify the elementary discrete convergence groups [GM 1].

Theorem 2.5. *$L(G) = \varnothing$ if and only if G is a finite group of elliptic elements.*

$L(G)$ consists of a single point if and only if G is an infinite group consisting only of elliptic and parabolic elements all of which must fix this point.

$L(G)$ consists of a pair of points if and only if G is an infinite group consisting only of elliptic and loxodromic elements each of which fixes or interchanges these points.

In the last case it is not difficult to show that G does indeed contain a loxodromic element. It is not known whether there must be a parabolic element in the case that $L(G)$ is a point. This is equivalent to the following question. (Recall we do not require the action to be effective!)

Is there an infinite purely torsion convergence group acting properly discontinuously on euclidean n-space? What if we suppose in addition that the group is finitely generated?

The answer is no in both cases when $n = 1$ or 2. This question should be regarded as a first step towards a more general Selberg Lemma [Se] as the existence of a torsion free subgroup of finite index (in the finitely generated case) implies the group is finite. In the conformal case, the answer is always no, [Wat].

The following result must be regarded as the first real interplay between topology and algebra in this theory. We repeat the sketch of proof from [GM 3].

Theorem 2.6. *Let G be a discrete convergence group which is abelian. Then G is elementary.*

SKETCH OF PROOF. Suppose that x_1, x_2, x_3 are three distinct points of $L(G)$. From Lemma 2.1 we obtain sequences and points such that for $k = 1, 2, 3$

$$g_{j,k} \to x_k \quad \text{as} \quad j \to \infty \quad \text{locally uniformly in } \mathbb{S}^n - \{y_k\}.$$

By symmetry we need only consider the two cases

(i) $x_1 \neq y_2$ and $x_2 \neq y_1$.
(ii) $x_1 \neq y_3$, $x_2 \neq y_1$ and $x_3 \neq y_2$.

Suppose (i) holds. Choose disjoint neighbourhoods U_1, U_2 of x_1, x_2 such that $y_1 \notin cl(U_1)$ and $y_2 \notin cl(U_2)$. Fix $y \notin \{y_1, y_2\}$. Then by the convergence property for all sufficiently large j

$$U_1 \supseteq g_{j,1}(\{y\} \cup U_2) \quad \text{and} \quad U_2 \supseteq g_{j,2}(\{y\} \cup U_1).$$

Thus

$$g_{j,2} \circ g_{j,1}(x) \in U_2 \quad \text{and} \quad g_{j,1} \circ g_{j,2}(x) \in U_1$$

and this is impossible as G is abelian. The argument is similar but a little more complicated when (ii) holds.

If we now consider the cyclic group generated by an element of a discrete convergence group we find that our classification of elements is exhaustive.

Corollary 2.7. *Let G be a discrete convergence group. Then every element of G is elliptic, parabolic, or loxodromic.*

A classical result of Newman [Ne] which says that periodic homeomorphisms of the sphere which fix an open set are the identity together with the above classification now implies a uniqueness result reminiscent to analyticity.

Proposition 2.8. *Let G a discrete convergence group and f, $g \in G$. If $f = g$ on an open set, then $f \equiv g$.*

The convergence property also implies the following dynamical behaviour of the elements of infinite order which is identical to that of the elements of infinite order in a discrete Möbius group.

Theorem 2.9. *If g is parabolic with fixed point x_0, then*

$$g^{\pm j} \to x_0 \quad \textit{as} \quad j \to \infty \quad \textit{locally uniformly in} \quad \mathbb{S}^n - \{x_0\}$$

If g is loxodromic with fixed points x_0 and y_0, then these points can be labeled so that

$$g^{j} \to x_0 \quad \textit{as} \quad j \to \infty \quad \textit{locally uniformly in} \quad \mathbb{S}^n - \{y_0\}$$

and

$$g^{-j} \to y_0 \quad \textit{as} \quad j \to \infty \quad \textit{locally uniformly in} \quad \mathbb{S}^n - \{x_0\}.$$

In light of this result and in view of the general question we have raised concerning topological conjugacy, one is naturally led to ask.

Are the elements in a discrete convergence group topologically conjugate to Möbius transformations?

We now turn to answer this fundamental question. Notice before we proceed that the classification necessarily applies to the elements of a quasiconformal group and so one can raise the above question in the quasiconformal context. We say g is a K-quasiconformal parabolic (respectively loxodromic) if the cyclic group $\langle g \rangle$ generated by g is a K-quasiconformal group and g is a parabolic (respectively loxodromic) element. We consider the three cases separately.

(1) Suppose g is parabolic.

Then in dimension one the answer is easily seen to be yes. In higher dimensions the convergence property for parabolics is known as Sperner's condition and homeomorphisms of $\mathbb{S}^n$ with this property are called *quasitranslations.* Results due to Sperner and Kérékjárto [Ke 1] show that the answer is again yes in dimension two (basically the action of $\langle g \rangle = \{g^m : m \in \mathbb{Z}\}$ on $\mathbb{R}^2$ is effective and so must cover an annulus). However it is not true in general that quasitranslations are topologically conjugate to translations, for examples see Kinoshita [Ki] in dimension 3 and Husch [Hu 1] for all higher dimensions. Here is an outline of a construction of a counterexample in dimension four. Let W^3 be a Whitehead three manifold. That is W^3 is a contractible three manifold not homeomorphic to $\mathbb{R}^3$. It is known that $W^3 \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. Then $W^3 \times \mathbb{S}^1$ and $\mathbb{R}^3 \times \mathbb{S}^1$ have the same universal covering space $\mathbb{R}^4$. The covering action extends to a quasitranslation of the sphere and these two actions are topologically distinct as the quotients are not homeomorphic.

If in addition g is a K-quasiconformal parabolic, the existence of a K'-quasiconformal conjugacy to a translation, where K' depends only on K and not g, is due to Hinkkanen [Hi] in dimension one (where it is surprisingly difficult) and the aforementioned results of Sullivan and Tukia in dimension two (which we outline later). The general question in the quasiconformal case is unknown in higher dimensions and we ask

Is every quasiconformal parabolic transformation topologically (quasiconformally) conjugate to a Möbius translation?

A generalization of Sullivan and Tukia's method yields an affirmtive answer for parabolic quasiconformal diffeomorphisms of the sphere, see [Ma 2]. Here the conjugacy is quasiconformal with dilatation depending only on that of the group and the dimension. There is an example however of a purely parabolic abelian quasiconformal group of $\mathbb{S}^n$, $n \geqslant 3$, which is not quasiconformally conjugate to a group of translations (it is topologically conjugate, see Section 5).

(2) Suppose g is loxodromic.

Then in all dimensions loxodromic transformations are topologically conjugate to Möbius transformations. Homeomorphisms satisfying the conclusion of Theorem 2.9 for loxodromic mappings are called *topological dilations*. Results due to Kérékjárto [Ke 2], Homma and Kinoshita [HK] and Husch [Hu 2] imply that orientation preserving topological dilations or their inverses are always conjugate to the dilation $x \to 2x$. Here is an outline of the proof of this fact:

We may assume that the fixed points of g are 0 and ∞ and that $g^m \to \infty$ as $m \to \infty$. Assume that there is a topological sphere S of codimension one separating 0 and ∞ and such that $g(S) \cap S = \varnothing$. Set $S_m = g^m(S)$. The convergence properties imply that S_m is a disjoint collection of spheres with 0 and ∞ as their only accumulation point. The Annulus Theorem implies that the region between S_m and S_{m+1} is homeomorphic to an annulus. Now it is easy to see how to use this annular structure to construct the desired conjugacy. We point out however that the use of the Annulus Theorem here is an essential use of this deep theorem which has only recently been established in all dimensions. We are left with the problem of constructing the sphere S. The convergence properties imply that eventually every sphere S separating 0 from ∞ moves of itself as it is compactly supported away from the limit set. In low dimensions one can use a finite cut and paste process to modify such an S to find one that moves off itself. In higher dimensions one must use more sophisticated methods. When $n \geqslant 3$, $\mathbb{S}^n - \{0, \infty\}$ is simply connected. The action of $\langle g \rangle$ on this space is effective and must cover a compact manifold with fundamental group $\mathbb{Z}$. This manifold must fiber over the circle and using some homotopy theory one can show that there is a sphere transverse to the fibration which lifts. This is the sphere we seek.

The existence of a topological conjugacy implies of course that

$$(\mathbb{S}^n - \{0, \infty\})/\langle g \rangle = \mathbb{S}^{n-1} \times \mathbb{S}^1.$$

In the quasiconformal case the existence of a topological conjugacy together with the fact that the quotient is compact implies (via the quasiconformal Hauptvermutung) that there is a quasiconformal conjugacy. It is not known if the dilatation of this conjugating map can be made to depend only on the dilatation of the group and the dimension. However this is the case if g is a diffeomorphism (where a more direct proof can be found, see Section 5).

(3) Suppose g is elliptic.

Then by a Theorem in part due to Brouwer, Kérékjárto and Eilenberg [Ei] every periodic homeomorphism of the two sphere is topologically conjugate to a rotation. By suspension this also implies the result in dimension one.

However this result is not true in higher dimensions. In dimension three Montgomery and Zippin [MZ] constructed an involution of $\mathbb{S}^3$ whose fixed point set is a wild knot. Such a map cannot of course be conjugate to an orthogonal rotation. However the affirmative solution to the Smith conjecture [MB] implies that periodic diffeomorphisms of $\mathbb{S}^3$ with a nonempty fixed point set are conjugate. The result is unknown if the fixed point set is empty. Examples due to Giffen and others [Gi] of smooth diffeomorphisms of all orders whose fixed points sets are smoothly knotted codimension two spheres show that there is no hope of topological conjugacy. Note that a finite group of diffeomorphisms is a quasiconformal group. For the general quasiconformal case in dimension one the existence of a quasiconformal conjugacy whose dilatation depends only on that of the group and not the order of the group is established in [Hi], while in dimension two the result follows form [Tu].

This summarises what is known about the elements in a discrete convergence group. There is the remaining question about the elements in a convergence group which is not discrete. It is clear we need only consider the cyclic case and we ask what can be said about a self homeomorphism of $\mathbb{S}^n$ with the property that for some sequence of integers $m(j)$ we have

$$g^{m(j)} \to \text{Identity uniformly in } \mathbb{S}^n.$$

In dimension one it is simple consequence of Denjoy's Theorem that such maps are conjugate to irrational rotations. The quasiconformal case can be found in [Hi]. In dimension two again it is always true that such maps are conjugate to irrational rotations of the two sphere [HM]. I do not know of any results in higher dimensions. Presumably it is not the case that such topological irrational rotations are conjugate to orthogonal irrational rotations in any higher dimension, even in the quasiconformal case. I do not know the answer to the analogous identity theorem for such maps;

Suppose that g is a topological irrational rotation and that g is the identity on an open set. Is g necessarily the identity?

It is worthwhile to recall a result of A. Hinkkanen, [Hi 2], who shows that every non-discrete convergence group of $\mathbb{S}^1$ is topologically conjugate to a Möbius group.

Returning to the structure of the limit set we observe from the classification of the elementary groups that every nonelementary discrete convergence group contains at least one loxodromic element. The following theorem shows that actually the loxodromic fixed points are pairwise dense in $L(G) \times L(G)$, see [GM, Thm. 6.7].

Theorem 2.10. *Let G be a nonelementary discrete convergence group and suppose that U and V are open sets which both meet the limit set. Then there is a loxodromic element in G with one fixed point in U and the other in V.*

Another result in this direction analogous to a classical result is

Theorem 2.11. *Let G be a discrete convergence group and $f, g \in G$. If f and g have a common fixed point and if g is loxodromic, then f and g have two fixed points in common and there is a integer k such that*

$$f \circ g^k = g^k \circ f.$$

A simple consequence of the plethora of loxodromic elements in a nonelementary discrete convergence group is

Theorem 2.12. *Let G be a discrete nonelementary convergence group. Then for every m, G contains a subgroup isomorphic to the free group of rank m.*

Finally here is another nice interplay between algebra and topology for convergence groups (for a conformal version see [Ku]). Stalling's theorem [Sta] states that a finitely generated group G with infinitely many ends splits as a free product with amalgamation or as an HNN extension. If G is a convergence group whose limit set is a Cantor set, then the ends inequality (see the note following Theorem 2.2) and Stalling's theorem imply that G splits. Each of these spliting subgroups has fewer than three ends or has infinitely many and the limit set correspondingly must consist of fewer than three points or is again a Cantor set. One can then inductively proceed to split those splitting subgroups with infinitely many ends. In the end (if this splitting process terminates) we are left with a complete splitting of G into subgroups all of whose limit set consists of fewer than three points. These are the elementary groups.

Theorem 2.13. *Let G be a finitely generated discrete convergence group whose limit set is a Cantor set. Then G splits as* HNN *extensions and free products with amalgamation (possibly infinite) and each factor is an elementary convergence group.*

In the special case that G is torsion free the splitting process must stop by Grushko's theorem. As we have seen the elementary convergence groups with a single limit point are hard to classify (they could be isomorphic to Fuchsian groups for instance). However Theorem 2.11 easily implies those torsion free elementary convergence groups with two limit points are just the infinite cyclic groups generated by a loxodromic. There is an easy way to assert that in a convergence group with Cantor limit set there are no elementary subgroups with a single limit point and that is to assume that every element is loxodromic. Thus we have the following nice corollary to the above [GM 2].

Corollary 2.14. *Let G be a finitely generated convergence group of* $\mathbb{S}^n$ *whose limit set is a Cantor set and for which every element is loxodromic. Then G is isomorphic to a free group.*

B. Maskit has a stronger version of this result, including a partial converse, in the Kleinian case [Mas 2].

3. Groups Acting on the Circle

Here is the best known result concerning discrete convergence groups acting on the circle. It is due to P. Tukia [Tu, Thm. 6 B]. We will sketch a proof from notes of Tukia's talk at the Nevanlinna Colloquium. Tukia's proof is motivated by the partial solutions to the Nielsen realisation problem obtained by Nielsen and Zieschang. Central to the proof is the existence of a simple axis.

Theorem 3.1. *Let G be a discrete convergence group of the circle. Then either G is topologically conjugate to a Fuchsian group or G has a semitriangle group of finite index.*

A *semitriangle group* is a discrete nonelementary convergence group generated by two elements f and g such that for some $p, q, r > 1$

$$f^p = g^q = (f \circ g)^{-r}.$$

Such groups are factors of triangle groups and are of such a special nature it might be supposed that they do not exist (as subgroups of Hom $(\mathbb{S}^1)$). Here are some conditions which imply that G does not contain a semitriangle group of finite index and so G is topologically conjugate to a Fuchsian group.

(1) G is torsion free.
(2) G is isomorphic to a Fuchsian group.
(3) $L(G) \neq \mathbb{S}^1$.
(4) G contains a parabolic element.
(5) G is infinitely generated.
(6) $T_1/\phi(G_0)$ is noncompact or virtually Haken (G_0 is the orientation preserving subgroup).

It should be noted that if there are semitriangle convergence groups. Then it is not true (as is conjectured) that every 3-manifold is virtually Haken (that is finitely covered by a Haken manifold). The 3-manifold $T_1/\phi(G)$ is the counterexample.

SKETCH OF PROOF. We assume that G is orientation preserving. From the above every element of G is topologically conjugate to a Möbius transformation. We want to extend the action of G to the unit disk and apply the following result of Martin and Tukia [MT, Thm. 4.4] whose proof we also outline.

Theorem 3.2. *Let G be a discrete convergence group acting on the closed unit disk $\mathfrak{D}$. Then G is topologically conjugate to a Fuchsian group.*

PROOF. $(\mathfrak{D} - L(G))/G$ is a (possibly bordered) surface with isolated branch points. Using this surface construct a G-invariant triangulation of $\mathfrak{D} - L(G)$. Redefine the conformal structure of $\mathfrak{D} - L(G)$ so that G now acts conformally, this can be done as in Ahlfors and Sario [AS, p. 127]. This conformal structure on $\operatorname{int}(\mathfrak{D})$ is conformally that of the standard unit disk or the entire complex plane. This last case is impossible as it implies G is elementary. If $L(G)$ is totally disconnected the conjugacy is easily extended to the whole disk $\mathfrak{D}$ while if $L(G) = \mathbb{S}^1$ one can extend the conjugacy using the fact that every limit point has a neighbourhood basis bounded by loxodromic axes by Theorem 2.10.

Returning to our sketch of Tukia's proof, we define an *axis* to be a pair (a, b) of distinct points of $\mathbb{S}^1$. We identify and axis with the hyperbolic line joining the points and we say two axes intersect if the corresponding hyperbolic lines meet. If $A = (a, b)$ is an axis and $g \in G$ we set $g(A) = (g(a), g(b))$. We say an axis is *simple* with respect to G if

(i) A and $g(A)$ do not intersect for any $g \in G - \{\text{identity}\}$, and
(ii) for any $\epsilon > 0$, $|g(a) - g(b)| < \epsilon$ for all but finitely many $g \in G$.

Suppose that there is a simple axis A. We form the one complex

$$X_A = \mathbb{S}^1 \cup \{g(A) : g \in G\}.$$

The simplicity of A easily enables us to extend the action of G to this complex retaining the convergence property. A component C of $\mathfrak{D} - X_A$ has boundary homeomorphic to a circle and the stabilizer G_C of this boundary acts as a convergence group of this circle. The limit set of G_C lies in $\mathbb{S}^1$ and so the group is of the second kind. It is not difficult to show that there is a simple axis for groups of the second kind (the endpoints of an interval of discontinuity). We repeat the construction of the one complex in this component C and then transfer it to all components conjugate to C in G. In this way we obtain a convergence group of an extended complex $Y_A \supseteq X_A$.

We repeat this process, perhaps infnitely often, to find a one complex X on which G acts as a convergence group and such that if C is a component of $\mathfrak{D} - X$, then the stabilizer of C is an elementary convergence group. It is easy to conjugate the elementary groups to Fuchsian groups and so we are able to extend the action of G on X to the whole disk.

This then leaves us with the problem of constructing a simple axis or showing that there is a semitriangle group of finite index. It is important to note that not all Fuchsian groups have simple axes, for instance cocompact triangle groups do not. The basic starting point is the existence of a *regular axis.* This is an axis A satisfying (ii) above and such that $\{g \in G - G_A: g(A) \cap A = \varnothing\}$ is a finite set, here $G_A = \{g \in G: g(A) = A\}$. The cardinality of this set is the intersection number of A. It can be shown that the fixed points of a loxodromic element of G form a regular axis, this follows more or less from the convergence property. Thus we start with a loxodromic axis A which is regular but not simple and try to produce another axis with smaller intersection number. This is done by carefully studying the intersection patterns of those axes which are conjugate in G to A and meet A. One can show that there is an axis with a smaller intersection number unless a specific pattern occurs which then leads to the construction of a semitriangle group. An argument based on the action of the group on the triple space shows this semitriangle group to be of finite index.

The remaining problem of dealing with the semitriangle groups seems to be quite difficult. Tukia has suggested a way of conjugating every discrete convergence group of the circle to a quasisymmetric group (the one dimensional analogue of quasiconformal groups). The hope is that the problem will be easier given a more geometric constraint on the group. Here is what is known at present about quasisymmetric groups (that does not follows from Tukia's result above), the results are due to Hinkkanen [Hi].

Theorem 3.3. *Let G be a K-quasisymmetric group of the circle such that either G is elementary or G is not discrete. Then G is conjugate to a Fuchsian group by a K'-quasisymmetric mapping, where K' depends only on K and not on G.*

P. Tukia has kindly pointed out the following

Theorem 3.4. *Let G be a discrete convergence group acting on $\mathbb{S}'$. Then G is topologically conjugate to a uniformly quasisymmetric group.*

PROOF. If G does not contain a semitriangle group of finite index, then G is topologically conjugate to a Möbius group, Theorem 3.1. Otherwise the action of G on the triple space is cocompact and by a result of G. Mess [Me] the group G is coarse quasi-isometric to the hyperbolic plane. That is, after a change of coordinates, G acts as a quasisymmetric group of $\mathbb{S}^1$.

Presumably some version of the following question has an affirmative answer and so one can apply Tukia's results to the quasisymmetric case as well.

Suppose that G is a discrete K-quasisymmetric group which is topologically conjugate to a Fuchsian group. Is G conjugate to a Fuchsian group by a K'-quasisymmetric mapping, where K' depends only on K and not on G?

It is not difficult to see a possible argument in the case that G is of the first kind.

We point out here that if one could extend a quasisymmetric group of the circle to a quasiconformal group of the unit disk, then this extension is necessarily quasiconformally conjugate to a Fuchsian group and so too therefore is the boundary group. There are many ways to extend a quasisymmetric map of the circle to the disk and some of these are canonical in many respects, see especially the extension found by Douady and Earle [DE]. However no extension can respect the composition of general quasisymmetric maps and so the extension of a group to the disk may no longer be a group acting on the disk. It might be however, that there is a canonical extension respecting composition for quasisymmetric groups.

We present here an interesting connection between the problem of conjugacy of quasisymmetric groups to Fuchsian groups and Teichmüller theory. The reader who is familiar with Teichmüller theory will recall that the Nielsen realisation problem (the problem of realising a finite group of homotopies of a given surface F^2 as a group of isometries of F^2 in some conformal structure) is implied by showing that a finite group of isometries of the Teichmüller space of F^2 has a common fixed point.

The *Schwarzian derivative* $S(\varphi)$ of a holomorphic mapping φ of the unit disk $\mathfrak{D}$ is defined as

$$S(\varphi) = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2.$$

The Schwarzian derivative measures the deviation of a holomorphic map from a Möbius transformation as seen by the fact that it precisely annihilates the latter. We define

$$\mathfrak{U} = \{S(\varphi) \colon \varphi \text{ is univalent in } \mathfrak{D}\}.$$

Universal Teichmüller space is $\mathfrak{T} = \text{int}(\mathcal{U})$. A quasidisk is the image of the unit disk under a quasiconformal self homeomorphism of $\mathbb{R}^2$. Here is the connection between quasisymmetric mappings and Schwarzian derivatives, relevant details and proofs can be found in [Le].

Let g be a quasisymmetric homeomorphism of the circle. Then there is a quasidisk Ω and univalent mappings $\varphi: \mathfrak{D} \to \Omega$ and $\psi: \mathbb{C} - \mathfrak{D} \to \mathbb{C} - \Omega$, such that

$$g = \psi^{-1} \circ \varphi \mid \mathbb{S}^1.$$

This construction is essentially unique so that the map $g \to S(\varphi) = S_g$ is well defined and, given some normalisation of g, is bijective (it is really the Bers' embedding).

Universal Teichmüller space has a few natural metrics. The Teichmüller metric is defined by

$$d(S_g, S_f) = \inf\{\log K: g \circ f^{-1} \text{ has a } K\text{-quasiconformal extension to } \mathfrak{D}\},$$

other natural metrics and discussions of their various properties can be found in the recent books [Ga], [Le] and [Na]. We point out that this metric is sufficiently regular to enable one to construct geodesics between any two points. Our interest in all of this lies in the following representation of a quasisymmetric group G as a subgroup of a compact subgroup of the isometry group of Universal Teichmüller space.

Let g be a quasisymmetric map of the circle. Define the map $g^*: \mathfrak{T} \to \mathfrak{T}$ by the rule

$$g^*(S_f) = S_h,$$

where $h = f \circ g^{-1}$.

It is not difficult to see that g^* is an isometry of $\mathfrak{T}$ with the Teichmüller metric and that the set $G^* = \{g^*: g \in G\}$ is a group of isometries. Since G is a quasisymmetric group, the set

$$S(G) = \{S_g: g \in G\}$$

is a bounded subset of $\mathfrak{T}$. It is also G^* invariant. This enables us to prove that the closure of G^* is a compact group of isometries. If the elements of G^* have a common fixed point S_h, then

$$g^*(S_h) = S_h \Leftrightarrow h \circ g^{-1} = \gamma \circ h$$

for some Möbius transformation γ. Thus h conjugates G to a Möbius group. There are many natural ways to try to find a fixed point for a compact group action on a contractible space (as $\mathfrak{T}$ is) using the metric structures available.

For instance if the set $S(G)$ has a well defined barycenter. Or if there is a metric with enough «convexity» in which G acts as isometries. The fact that geodesics are not always unique for the Teichmüller metric causes problems for constructions based on this metric. In the finite dimensional case, where this construction works perfectly well, G^* acts as isometries in the Weil-Petersson metric. S. Wolpert [Wo] has shown that this metric is negatively curved and so by an old result of Cartan G^* has a common fixed point. This is another proof of the Nielsen problem. We therefore finish this section by posing the following problem:

Find a metric ρ on universal Teichmüller space in which the action g^* of a quasisymmetric map g is as an isometry and such that a compact group of isometries has a common fixed point.

It is quite possible that the Teichmüller metric is such a metric.

4. Groups Acting on the Two Sphere

As we have mentioned, every quasiconformal group acting on $\mathbb{S}^2$ (or more generally any subdomain) is quasiconformally conjugate to a Möbius group. This result was first proved by Sullivan [Su 1] and his proof was formalised and generalised by Tukia [Tu 2]. The methods developed for the two dimensional quasiconformal case have useful applications in higher dimensions. Thus we will give here a brief outline of the idea of the proof and then present most of the details in a more general setting in Section 5.

Theorem 4.1. *Let G be a K-quasiconformal group acting on $\mathbb{S}^2$. Then there is a K'-quasiconformal self homeomorphism f of the sphere such that the group fGf^{-1} is a Möbius group. Here $K' \leqslant \min\{K^{1/\sqrt{2}}, \sqrt{2K}\}$.*

IDEA OF PROOF. Suppose that G is a group of diffeomorphisms and $x \in \mathbb{S}^2$. What it means for G to have bounded distortion is that for each $g \in G$ the linear mappings $g'(x)^t g'(x)$: $T_x\mathbb{S}^2 \to T_{g(x)}\mathbb{S}^2$ send circles to ellipses of uniformly bounded eccentricity. Since we are only interested in the associated conformal structures we normalise these ellipses so that the product of the length of their axes is one (that is $\det g'(x) = 1$). For each $g \in G$ we construct the ellipse coming from $g^{-1}(x)$ under the map induced by g. This family of ellipses based at x is G-invariant. That is, the family of ellipses based at $g^{-1}(x)$ is sent to the family based at x under the map induced by g. So too then is the average ellipse which we can construct because of the uniform bound on eccentricity and diameter. This constructs a G-invariant ellipse field; a field of ellipses in the tangent bundle which is invariant under the map induced by any element of G and has uniformly bounded eccentricity. The smoothness assumption on the elements of G can be relaxed (quasiconformal mappings are differentiable almost everywhere). In this way we construct a measurable G-invariant ellipse field. It is this much of the construction that can be done in all dimensions. Next in dimension two the existence theorem for quasiconformal mappings (or as it has lately become known, the measurable Riemann mapping theorem) implies that any measurable ellipse field is almost everywhere the ellipse field induced by some quasiconformal mapping. Conjugating by the quasiconformal homeomorphism h corresponding to the G-invariant ellipse field we see that the group $H = hGh^{-1}$ leaves the standard field of round circles invariant. Thus every element of H is conformal and so H is a Möbius group.

As a first application of this result we recall from Theorem 1.10 that the action of the fundamental group Γ of a negatively curved three manifold on the universal cover is a convergence group of the two sphere at infinity S_∞. Evidently this sphere carries a quasiconformal structure in which Γ acts quasiconformally if and only if M has the homotopy type of a hyperbolic space form.

If M has quarter pinched sectional curvatures $-4 < K(M) \leqslant -1$, then one knows that the sphere at infinity has a compatible C^1-structure in which the fundamental group acts as diffeomorphisms, [Gre] and [HP]. D. Sullivan has made the following conjecture in this case [Su 2].

If M is a quarter pinched three manifold, then $\pi_1(M)$ acts as a quasiconformal group on the two sphere at infinity.

As a consequence of course we find that quarter pinched three manifolds have the homotopy type of hyperbolic manifolds. Sullivan has related this conjecture to other questions concerning the expanding properties of the geodesic flow on the tangent spaces to the horospheres. Examples of Gromov and Thurston [GT] of n-manifolds, $n \geqslant 4$, with sectional curvatures close to minus one that do not have the homotopy type of hyperbolic manifolds show that this quasiconformality must be a low dimensional phenomena.

We return to the question of whether or not convergence groups are also conjugate to Möbius groups in dimension two. Here is a simple example that is easily generalised to all dimensions to show that this is not the case (although recall that in dimension two every element is individually conjugate).

Example 4.2. Let G be a nonelementary Fuchsian group acting on $\mathbb{S}^2$. Then G leaves the disk $\mathfrak{D}^2$ invariant. Identify $\mathfrak{D}^2$ to a point x_0 and extend the action of G over this point by agreeing that every element of G fixes x_0. This produces a group of homeomorphisms H of $\mathbb{S}^2/\mathfrak{D} \approx \mathbb{S}^2$ acting properly discontinuously in the complement of the point x_0. Thus H is a convergence group and is not conjugate to any Möbius group since the stabilizer of a point in a Möbius group is virtually abelian and the stabilizer of x_0 is $H \cong G$.

One can see how to generalise this process of taking an equivariant decomposition of the sphere to produce more complicated examples of convergence groups. The point to this example is that in dimension two it seems quite likely that this is the only way to construct convergence groups which are not the topological conjugates of Möbius groups. We now outline the results of R. Skora and the author [MS]. To begin with we need the following terminology

Definition 4.3. *A subset X of $\mathbb{S}^n$ is* ***cellular*** *if $X = \cap B_j$, where $\{B_j\}$ is a collection of closed topological n-balls (cells) such that B_{i+1} lies inside* Int (B_j). *A map $\eta: \mathbb{S}^n \to \mathbb{Y}$ is* ***cellular*** **if** *$\eta^{-1}(y)$ is cellular for every $y \in \mathbb{Y}$. A* ***decomposition*** *$\mathcal{R}$ of $\mathbb{S}^n$ is a disjoint collection of subsets of $\mathbb{S}^n$ such that every point of $\mathbb{S}^n$ lies in some element of $\mathcal{R}$. We say a decomposition $\mathcal{R}$ is* ***cellular*** *if the quotient map $\mathbb{S}^n \to \mathbb{S}^n/\mathcal{R}$ is cellular.*

A subset X of the two sphere is cellular if and only if X is a closed connected nonseparating set (an easy application of the Riemann mapping Theorem).

A decomposition $\mathcal{R}$ of $\mathbb{S}^n$ is *upper semi-continuous* if the quotient map is a closed map and it is *G-invariant* or simply *equivariant* if its elements are permuted by G. The applications we have in mind rest on the following important result of R. L. Moore [Mo].

Theorem 4.4. *If $\mathcal{R}$ is a cellular, upper semi-continuous decomposition of $\mathbb{S}^2$, then $\mathbb{S}^2/\mathcal{R}$ is homeomorphic to $\mathbb{S}^2$.*

An immediate consequence is

Corollary 4.5. *Let G be a discrete convergence group of $\mathbb{S}^2$ and $\mathcal{R}$ an equivariant upper semi continuous cellular decomposition. Then the induced action of G on $\mathbb{S}^2/\mathcal{R}$ ($\approx \mathbb{S}^2$) is a convergence group of $\mathbb{S}^2$.*

We need to make the following technical definition.

Definition 4.6. *A finitely generated group Γ is* ***accessible*** *if every sequence of nontrivial algebraic splittings of Γ as free products with amalgamation along finite subgroups or as* HNN *extensions along finite subgroups is finite.*

It is conjectured by C.T.C. Wall that every finitely generated group is accessible [Wa]. This conjecture has been established in the case that Γ has uniformly bounded torsion or is finitely presented by P. Linnell [Li] and M. Dunwoody [Du] respectively. It is a simple consequence of Grushko's theorem that a finitely generated torsion free group is accessible and since Möbius groups are virtually torsion free (Selberg's Lemma) they too are always accessible.

The main result of [MS] is really the Decomposition Theorem. It implies that an accessible convergence group G with $O(G) \neq \emptyset$ is the fundamental group of a finite graph of groups. The edge groups are finite cyclic and the vertex groups are convergence groups for which every component of the ordinary set is simply connected. A consequence of this result will be that under certain circumstances $O(G)/G$ is of finite topological type. This is a version of Ahlfors' Finiteness Theorem. The proof of the Decomposition Theorem is analogous to Abikoff and Maskit's decomposition of finitely generated Kleinian groups [AM]. If some component of the ordinary set is not simply connected we use a topological version of Maskit's Planarity Theorem [Mas 1] to find a «nice» invariant union $\mathcal{C}$ of simple loops in $O(G)$. From this we construct a tree on which G acts without inversions as follows. Define an equivalence relation on $O(G) - \mathcal{C}$ as by saying that two points are equivalent in no element of $\mathcal{C}$ separates them. If $[x]$ and $[y]$ are equivalence classes then at most finitely many elements of $\mathcal{C}$ separate them. Thus $\mathcal{C}$ determines a tree where every vertex is an equivalence class of points of $O(G) - \mathcal{C}$ and two vertices are joined by an edge if exactly one element of $\mathcal{C}$ separates them. G acts on this tree without inversions if it is orientation preserving. For a vertex v we set $G_v = \{g \in G: g(v) = v\}$ and similarly define the stabilizer of an edge. Then by the Bass-Serre theory of groups acting on graphs G admits a spliting as $G \cong G_1 *_{\mathbb{Z}/n\mathbb{Z}} G_2$ or $G \cong G_1 *_{\mathbb{Z}/n\mathbb{Z}}$ and each factor G_i is a vertex group and the amalgamating groups are the edge groups. One then proceeds inductively to find splitings of G_1 and G_2. If this process terminates we have a complete understanding of G up to these factors, this is why we need the hypothesis of accessibility. In the case that G is a Kleinian group this splitting process terminates by Ahlfors' Finiteness Theorem which says that $O(G)/G$ is of finite type [Ah 2].

Decomposition Theorem 4.7. *Let G be an orientation preserving convergence group which is accessible. Then there exists an invariant union $\mathcal{C}$ of disjoint simple loops in $O(G)$ such that $\mathcal{C}/G$ is a disjoint collection of simple loops and each component of $O(G_v)$ is simply connected for every vertex v in the tree determined by $\mathcal{C}$.*

Actually one gets somewhat more from this result. Namely a picture of how the limit set is built up from the limit sets of the vertex groups [MS, pp 4-5].

The decomposition theorem reduces the problem of classifying those accessible convergence groups with $O(G) \neq \emptyset$ to classifying those for which every component of the ordinary set is simply connected. Unfortunately even in the case of Kleinian groups this classification is far from complete, containing Web groups and all groups of the first kind! In any case we have achieved one of our objectives. It is a weak form of Ahlfors' Finiteness Theorem.

Theorem 4.8. *Let G be an accessible convergence group such that every component of $L(G)$ separates the two sphere into finitely many (and therefore at most two* [GM]*) pieces. Then $O(G)/G$ has finite type. That is it has finitely many components, each component has finite genus, a finite number of ends and a finite number of orbifold branch points.*

There have been other results along this line. R. Kulkarni and P. Shalen [KS] showed that if G is a finitely generated torsion free discrete convergence group acting on the closed unit 3-ball, then $(O(G) \cap \mathbb{S}^2)/G$ is of finite type, except for the possibility that there are infinitely many disks or annuli. It is unknown whether this last possibility can occur. Recently Kulkarni has found a version of Theorem 4.8 in the case that the limit set is a Cantor set. And M. Feighn and D. McCullogh refined the techiques of Kulkarni and Shalen [FM] to obtain topological analogues of Bers' Area Theorem [Be] and Sullivan's Finiteness of Cusps [Su 3] and [Kr]. Our methods too provide weak (but topological) forms of these estimates. One can directly check that under the hypotheses of the Decomposition Theorem and Theorem 4.8, that $\chi(O(G)/G) \geqslant 2(1 - N)$, where χ is the orbifold characteristic and N is the fewest number of generators of G. This is Bers' Area Theorem. Similarly it implies that there are a finite number of conjugacy classes of parabolic fixed points of G and under the additional hypothesis of G being torsion free it implies this number is no more than $3N/2$. This is Sullivan's finiteness of cusps.

We now proceed by making an assumption on the structure of the limit set of the convergence group G. We want to identify those convergence groups G_j in the splitting of G that comes from the Decomposition Theorem as topological conjugates of Fuchsian groups. We wish to eliminate from our considerations, those convergence groups G_j whose ordinary set is simply connected and whose limit set is a nonseparating continuum and those convergence groups G_j whose limit set is connected and has infinitely many components in its complement (principally because we cannot show that these groups are conjugates of Kleinian groups in any reasonable sense). In the first case we would like to say that every such convergence group is obtained from a Fuchsian group by some identification on the limit set $\mathbb{S}^1$. Unfortunately it is not known whether this is even true for Kleinian groups although it is conjectured to be the case. This is the question of the local connectedness of the limit set of a finitely generated function group. The second case concerns Web groups and seems even more difficult. This leaves us with essentially two types of convergence groups to deal with. Firstly the elementary groups. We have already classified these: The finite groups are topologically conjugate to finite Möbius groups, the groups with a single limit point are either conjugate to a euclidean group or are constructed exactly as in Example 4.2. The former are identified by the fact that the stabiliser of the limit point is virtually abelian. Secondly there are the convergence groups whose limit set is a topological circle. These groups are topologically conjugate to Fuchsian groups, see [MT]. This follows more or less from Theorem 3.2.

Given a convergence group G such that every component of the limit set is a point or a circle we decompose G into the groups G_j as given by the decomposition Theorem. We conjugate the G_j to Fuchsian groups by homeomorphisms f_j where this is possible. For the groups arising as in Example 4.2 we need a semi-conjugacy by a cellular mapping η_j. One then recombines these Kleinian conjugate groups using the Klein-Maskit Combination Theorem to obtain a Kleinian group Γ. The map which will «conjugate» G to Γ is locally defined by the f_j and the η_j and will correspond to some equivariant (with respect to Γ) cellular decomposition of the sphere. The elements of the decomposition which are not points will be disks whose boundary lies in the limit set of Γ. Thus the convergence group G is obtained from Γ via this cellular decomposition. We capture this notion in the following definition

Definition 4.9. *Let G and H be convergence groups of $\mathbb{S}^n$. We say that G is covered by H if there is an isomorphism $\varphi: H \to G$ and a cellular map $\eta: \mathbb{S}^n \to \mathbb{S}^n$ such that $\eta/\eta^{-1}(O(G))$ is a homeomorphism and the following diagram commutes*

$$\begin{array}{ccc} H \times \mathbb{S}^n & \longrightarrow & \mathbb{S}^n \\ \downarrow \varphi \quad \downarrow \eta & & \downarrow \eta \\ G \times \mathbb{S}^n & \longrightarrow & \mathbb{S}^n. \end{array}$$

Our discussion has established the first part of the next theorem. The second part follows since if the fixed point data of G is correct, then all the conjugacies are topological.

Theorem 4.10. *Let G be a finitely generated accessible convergence group such that every component of the limit set $L(G)$ is either a point or a topological circle. Then G is covered by a Kleinian group. If in addition G has the correct fixed point data, namely the stabilizer of any point in $L(G)$ is virtually abelian, then G is topologically conjugate to a Kleinian group.*

As a corollary to this and Theorem 1.8 we obtain the following topological characterisation of Schottky groups in the plane, once we observe that as soon as there is an invariant component of the ordinary set of a finitely generated convergence group, then it is accessible [MS, Prop. 5.1 and Cor. 5.5].

Corollary 4.11. *Let G be a finitely generated group of homeomorphisms acting properly discontinuously in the complement of a totally disconnected set. If the stabilizer G_x of any point $x \in \mathbb{S}^2$ is virtually abelian, then G is topologically conjugate to a conformal Schottky group.*

Of course in Corollary 4.11, the assumption on the fixed point data is necessary.

We conclude by conjecturing that Theorem 4.10 is true in much more generality and is really the only way of constructing convergence groups on $\mathbb{S}^2$.

Conjecture. *Every convergence group G acting on $\mathbb{S}^2$ is covered by a Kleinian group.*

This conjecture implies that every negatively curved three manifold has the homotopy type of a hyperbolic space form. It implies the following (weaker?) conjecture which is also true if G is as in Theorem 4.10.

Conjecture. *Every convergence group G acting on $\mathbb{S}^2$ extends to a convergence group G' of B^3.*

This would be a topological Poincaré extension theorem.

5. Higher Dimensions

The examples of C. Giffen [Gi] of smooth involutions of the n-sphere, $n \geqslant 4$, which we mentioned above provide simple examples of smooth uniformly quasiconformal groups which are not the topological conjugates of Möbius groups. Presumably it is possible to construct a quasiconformal version of the example of Montgomery and Zippin [MZ] to provide a three dimensional example (the affirmative solution to the Smith conjecture implies that there is no smooth version).

To obtain more complicated examples, one can use the quasiconformal analogue of the Klein-Maskit combination theorems [Ma, Thm. 4.3] to combine one of the examples above with a various Möbius groups. Indeed such a construction is possible for any Möbius group Γ with nonempty ordinary set. We can easily arrange that the smooth involution acts conformally on some open set and (after an appropriate conjugacy of Γ by a suitable Möbius transformation) that this set contains the complement of a fundamental domain for the action of Γ on $O(\Gamma)$. Using the combination theorem we obtain a uniformly quasiconformal group G which is isomorphic as group to $\Gamma * \mathbb{Z}_p$ and which is not topologically conjugate to any Möbius group.

There are many natural questions that these examples leave unanswered. For instance, is the existence of a fake involution the only way to construct such quasiconformal examples? If we were to assume that the group was torsion free or that every element of the group has the correct fixed point data could there still be such examples? Example 4.2 generalises to give examples in all dimensions of convergence groups that are not topologically conjugate to Möbius or even quasiconformal groups. Is this essentially the only way to construct such examples as it seems in dimension two? We will discuss these questions and then turn to consider some affirmative results regarding the existence of a conjugacy.

The first relevant example is due to P. Tukia [Tu 4].

Theorem 5.1. *For every $n \geqslant 3$ there is a uniformly quasiconformal action of a solvable Lie group on $\mathbb{S}^n$ which is not isomorphic as a topological group to any Möbius group.*

This example was an extension of another example, due also to him, of a uniformly Lipschitz (in the euclidean topology of $\mathbb{R}^n$) action which was topologically conjugate but not quasiconformally conjugate to a group of translations. This latter example was constructed by arranging that the orbit of a point was a hyperplane that was not locally quasiconformally flat (in fact this hyperplane is the product of the Von Koch snowflake and $\mathbb{R}^{n-2}$). This group was not discrete. Tukia's methods were subsequently modified and extended by the author in [Ma 1] to prove the following theorem.

Theorem 5.2. *For each $n \geqslant 3$ there is a discrete uniformly quasiconformal group of parabolic transformations acting on $\mathbb{S}^n$, isomorphic as a group to $\mathbb{Z}^{n-1}$ and topologically but not quasiconformally conjugate to a Möbius group.*

This theorem is especially interesting since as we shall see all smooth abelian examples are quasiconformally conjugate to Möbius groups. But this does raise the question.

Suppose G is a uniformly quasiconformal group of $\mathbb{R}^n$ containing only parabolic transformations (fixing ∞). Is G topologically conjugate to a group of translations?

The next interesting examples are due to M. Freedman and R. Skora [F.S.].

Firstly a convergence group which has all the correct fixed point data and a simple algebraic structure but cannot be made quasiconformal by a change of coordinates. We recall that a Cantor set is *tame* in $\mathbb{S}^n$ if there is a self homeomorphism of $\mathbb{S}^n$ taking the Cantor set to a subset of a smoothly embedded arc in $\mathbb{S}^n$. Otherwise the Cantor set is *wild*.

Theorem 5.3. *For every $n \geqslant 3$ there is a convergence group G with the following properties.*

(i) *G is algebraically isomorphic to the free group on two generators and contains only loxodromic transformations,*
(ii) *$O(G)/G$ is compact and $L(G)$ is a wild Cantor set,*
(iii) *G is not topologically conjugate to any uniformly quasiconformal group.*

The example of Theorem 5.3 provides a nice counterpoint to M. Freedman's topological characterisation of certain Schottky groups [Fr].

Theorem 5.4. *Let G be a discrete convergence group of $\mathbb{S}^n$, $n \neq 4$, with the following properties.*

(i) *G is algebraically isomorphic to a free group and contains only loxodromic elements.*
(ii) *$O(G)/G$ is compact.*
(iii) *$L(G)$ is a tame Cantor set.*

Then G is topologically conjugate to a Schottky group.

Freedman goes on in that paper to equate the four dimensional surgery problem and the five dimensional *s*-cobordism «theorem» to the problem of extending to the four ball every convergence group G of $\mathbb{S}^3$ satisfying (i) and (ii) of Theorem 5.4 and whose limit set is a Cantor set. This reduction is quite complicated and related to the reduction of the surgery problem to the so-called atomic surgery problems [CF] (whether it is possible to solve the surgery problem in four manifolds with fundamental group isomorphic to a free group).

The hypothesis that $O(G)/G$ is compact effectively forbids the existence of parabolics and hence the sort of elementary convergence groups that arose as in Example 4.2. We say that G is of *compact type* if $O(G)/G$ is compact. A relevant question is

Suppose that G is a (finitely generated?) discrete convergence group with $O(G) \neq \varnothing$. Show G is of compact type if and only if G contains no parabolics.

The second result of Freedman and Skora is a quasiconformal example similar to Theorem 5.3.

Theorem 5.5. *For every $n \geqslant 3$ there is an $r > 1$ and a smooth uniformly quasiconformal group G of compact type, acting on $\mathbb{S}^n$ and having the following properties:*

(i) *G is algebraically isomorphic to the semidirect product of a free group of rank r and a cyclic group of order $2r$.*
(ii) *$L(G)$ is a wild Cantor set.*
(iii) *G contains only loxodromic and elliptic elements and every element of G is quasiconformally conjugate to a Möbius transformation.*
(iv) *G is not the topological conjugate of any Möbius group.*

This example is good because it does not rely on the existence of a fake elliptic and it has simple algebraic structure. Unfortunately there is still the problem of an elliptic. Also we know that r is necessarily large in this example (is this really necessary in general?). The fact that in both of the examples of Theorems 5.3 and 5.5 the limit set is a wild Cantor set (as evidenced by the fact that its complement is not simply connected) led Freedman and Skora to raise the hope that this was enough to guarantee that a convergence (or quasiconformal) group was not conjugate to a Möbius group. Unfortunately we now know this is not the case as D. Cooper and M. Bestvina [BC] have constructed a Schottky group satisfying (i) and (ii) of Theorem 5.3. However Freedman and Skora's results do raise the problem

Find invariants of Cantor sets which imply they cannot be the limit of Schottky groups. Then if possible construct convergence (or quasiconformal) groups with these sets as limit sets.

As Freedman and Skora point out, the example of [BC] shows such invariants must be subtle. In particular it is not known if the existence of the elliptic in the example of Theorem 5.5 is necessary. That is whether the free part of the action alone is not conjugate. Hopefully subtle invariants may be used to decide that in a very real sense the limit set of the example is wilder than the limit sets of the so far constructed Schottky groups. In any case this still leaves the problem

Construct a discrete uniformly quasiconformal group of $\mathbb{S}^n$, $n \geqslant 3$, which is torsion free and not topologically conjugate to a Möbius group.

It was pointed out to me by R. Skora that there is still the possibility that every convergence group of $\mathbb{S}^n$ whose limit set is a Cantor set arises from some Schottky group via an equivariant cellular decomposition (not necessarily supported on the limit set). Is this possible?

To give a rough idea of what is going on in these examples we outline the constructions. Recall first that the natural way to construct a classical Schottky group is to take a collection of pairs of round hyperspheres, none of which separates the collection. For each pair choose the (essentially) unique orientation preserving Möbius transformations that takes one of the paired spheres to the other. The group generated by these pairings is a Schottky group with limit set a Cantor set. This construction works more or less because the inside and outside of a sphere are homeomorphic. It was Freedman and Skora's observation that one could play the same game with tori (the inside and outside of a torus in $\mathbb{S}^3$ are homeomorphic). One could take a collection of paired tori (no one separating the collection) and look at the group generated by a suitable choice of homeomorphisms of $\mathbb{S}^3$ identifying elements of the pairing. This is not so easy to do conformally as attached to a torus is the conformal-invariant of its modulus. This construction of identifying tori provides nothing new unless one chooses the tori so that some linking occurs. Then one finds that the homeomorphism which maps the inside of one of the paired tori A to the outside of the other B causes the remaining tori to form a nontrivial link inside B. This process repeats itself as we look at other elements of the group. We find a construction going on like that of Antoine's necklace [Ru]. In general one must be a little careful to arrange that the limit set is indeed a Cantor set. If the linking is very tight (few tori all linked), then there is so much stretching (to ensure that the image of the link wraps around the core of the torus under the identifying homeomorphism) and squeezing (to ensure that everyting fits in) that the group cannot be quasiconformal. It cannot be made quasiconformal because this stretching and squeezing pattern is topological in nature (the linking pattern of tori is preserved by any homeomorphism). This is the key to the first example. If one makes the linking less severe by using a large number of tori one can arrange that the amount of distortion stays bounded. The analogy is with a coarse necklace with few pieces, each piece must bend a lot for the whole thing to bend a lot, and a fine necklace, each pieces suffers little even when one bends the whole. At this stage it would be nice to conclude that the limit set built out of this linking pattern (in the quasiconformal case) is not the limit set of a conformal group. Unfortunately this has not been done. However if one is careful in the construction and symmetrically places the linked tori one can extend the group by a finite rotation group preserving this symmetry. We then use the linking pattern to show the fixed point circles of certain conjugates of the elements of the group preserving the symmetry also form a link and that this link cannot be realised by the linking of round circles (which must be the fixed points of Möbius elliptics). This shows the group is not topologically conjugate to a Möbius group and explains the presence of the elliptic in Freedman and Skora's second example.

Having provided a number of counterexamples to the general question we shall now go on to look at what can be said in specific circumstances for a uniformly quasiconformal group G. The results are actually quite interesting and useful. The most powerful tool is the construction of a G-invariant ellipse field (as it is in dimension two). One obtains conjugacy results by showing this invariant field is the pullback of the standard field (of round spheres in the tangent space) under a quasiconformal mapping (an existence theorem) or by somehow linearising this field to find a new invariant field which is the pullback of the standard field by an affine transformation. Since this idea is central to essentially all that is known in higher dimensions we will provide a reasonably detailed account.

We define the space $S(n) = SL(n, R)/SO(n)$. Thus $S(n)$ is the space of real, symmetric, positive definite $n \times n$ matrices with determinant equal to one. The general linear group $GL(n, R)$ acts transitively on the right of $S(n)$ via the rule

$$X[A] = |\det X|^{-2/n} X^t A X, \qquad X \in GL(n, R), \qquad A \in S(n).$$

The Riemannian metric,

$$ds^2 = (1/2)\sqrt{n}\,\mathrm{tr}\,(Y - 1dY)^2$$

on $S(n)$ gives rise to a metric distance which we denote by $d(A, B)$ for $A, B \in S(n)$. This metric is invariant under the right action of $GL(n, R)$ and makes $S(n)$ a globally symmetric Riemannian manifold, which is complete, simply connected, and of nonpositive sectional curvature, see [He] (we include the factor $\sqrt{n}/2$ so that when $n = 2$ we get hyperbolic space of constant curvature -1). Then

$$d(A) = d(\text{Identity}, A) = (1/2)\sqrt{n}\,((\ln \lambda_1)^2 + (\ln \lambda_2)^2 + \cdots + (\ln \lambda_n)^2)^{1/2},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, see [Maa]. Other distances can now be calculated from the transitivity of the $GL(n, R)$ action.

The *matrix dilatation* of a homeomorphism f of $\mathbb{S}^n$, which is differentiable with nonzero Jacobian matrix almost everywhere, is the measurable map $\mu_f \colon \mathbb{S}^n \to S(n)$ defined by

$$\mu_f(x) = |\det f'(x)|^{-2/n} f'(x)^t f'(x) = f'(x)[\mathrm{Id}]$$

where $f'(x)$ is the Jacobian matrix of f and Id denotes the identity.

A *conformal structure* on $\mathbb{S}^n$ is a pair $(\mathbb{S}^n, \mu)$, where $\mu \colon \mathbb{S}^n \to S(n)$ is a measurable mapping for which

$$\operatorname{ess\,sup}\,\{d(\mu(x)) \colon x \in \mathbb{S}^n\} = d(\mu) < \infty.$$

We define $D(\mu) = \exp(d(\mu))$ and call this the dilatation of μ. The standard conformal structure is $\beta(x) \equiv \mathrm{Id}$. As we have said, one should think of a conformal structure as a measurable ellipse field on the tangent space to the sphere, such that the eccentricity of every ellipse is uniformly bounded. A homeomorphism of $\mathbb{S}^n$ viewed as a mapping between two conformal structures μ_1 and μ_2

$$f: (\mathbb{S}^n, \mu_1) \to (\mathbb{S}^n, \mu_2)$$

is called $D(\mu_1, \mu_2)$-quasiconformal if

(1) $f \in W^{1,n}_{\mathrm{loc}}$, *i.e.* f has locally L^n integrable first derivatives, and
(2) $D(f, \mu_1, \mu_2) = \operatorname{ess\,sup}\{\exp(d(\mu_1(x), f'(x)[\mu_2(f(x))])) : x \in \mathbb{S}^n\} < \infty$.

If $\mu_1 = \mu_2 = \beta$, the standard structure, we obtain the usual notion of quasiconformality and if no conformal structures are present we mean quasiconformal in this usual sense. In this case the quantity $\exp(d(f'(x)[\mathrm{Id}]))$ is often referred to as the Ahlfors-Earle dilatation of a quasiconformal mapping at a point x, see [Ah 1]. We call the essential supremum of $D(f, \mu_1, \mu_2)$ over $\mathbb{S}^n$ the D-dilatation of f and we will denote this quantity $D(f)$. Notice that by definition $D(f) = D(\mu_f)$. We say that f is *conformal* if $D(f, \mu_1, \mu_2) = 1$, that is

$$\mu_1(x) = f'(x)[\mu_2(f(x))]$$

almost everywhere.

For the basic facts concerning quasiconformal structures, see [Tu 1, Sec. D]. One should notice that a mapping which is quasiconformal in one structure is automatically quasiconformal in all other structures, however the dilation will vary from structure to structure. Recall too the Louiville theorem that a one-quasiconformal in the usual sense is conformal in the usual sense and so a Möbius transformation.

The notion of D-dilatation is especially useful in our situation. The usual notion (that is of our reference [Vä]) of K-quasiconformality of a homeomorphism f is

$$K(f) = \operatorname{ess\,sup}\{\log(\lambda_{\max}(x)), \log(1/\lambda_{\min}(x))\}^{n/2}$$

where f satisfies (1), (2) and $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ are the largest and smallest eigenvalues of the matrix $f'(x)[\mathrm{Id}]$. Here is the sharp relation between the D-dilatation and the K-dilatation, see [Tu 1, (D5)].

$$(3) \qquad \frac{K(f)}{\sqrt{n-1}} \leqslant D(f) \leqslant K(f) \quad (n \text{ even}) \quad \text{or} \quad \frac{K(f)}{\sqrt{\dfrac{n-1}{n}}} \quad (n \text{ odd}).$$

Notice the implication that

$$K(f) = D(f) \quad \text{when} \quad n = 2.$$

For a quasiconformal group G we set

$$D(G) = \sup \{D(g): g \in G\}.$$

Similarly we define $K(G)$. For a quasiconformal group G, a *G-invariant conformal structure* μ on $\mathbb{S}^n$ is a conformal structure $(\mathbb{S}^n, \mu)$ such that each g in G is conformal as a map

$$g: (\mathbb{S}^n, \mu) \to (\mathbb{S}^n, \mu).$$

We observe that if g is quasiconformal, then

$$h: (\mathbb{S}^n, \mu_h) \to (\mathbb{S}^n, \beta) \quad \text{and} \quad h^{-1}: (\mathbb{S}^n, \beta) \to (\mathbb{S}^n, \mu_h)$$

are conformal. Hence if μ is a G-invariant conformal structure and h is a homeomorphism satisfying (1) and (2) and $\mu_h(x) = \mu(x)$ almost everywhere, then the group $h \circ G \circ h^{-1}$ is a conformal group.

One can always construct a G-invariant quasiconformal structure for a quasiconformal group g. This is due to the $GL(n, R)$ invariance of the metric distance d in $S(n)$ and to the fact that $S(n)$ is nonpositively curved. In fact if $P(E)$ denotes the center of the smallest ball containing a bounded subset E of $S(n)$ (which is well defined and unique since $S(n)$ is nonpositively curved and simply connected), then

$$\mu(x) = P(\{\mu_g(x): g \in G\})$$

will be the desired G-invariant conformal structure, see [Tu. 1, Sections D and E]. A simple geometric argument based on the law of cosines in a negatively curved manifold implies that $D(\mu) \leqslant D(G)^{1/\sqrt{2}}$. In higher dimensions ($n \geqslant 3$) there is no measurable Riemann mapping theorem and we will have to use different techniques to produce a conjugating mapping other than appealing to an existence theorem.

The first idea is due to Gromov [Gr 1] and Tukia [Tu 1]. It is this: suppose that μ is a G-invariant conformal structure and that there is a limit point x_0 such that μ is continuous at x_0. We may assume that $x_0 = 0$ and that G acts on $\overline{\mathbb{R}}^n$. Let $\{g_i\}$ be a sequence in G such that $g_i \to 0$ locally uniformly in $\overline{\mathbb{R}}^n - \{y_0\}$ (recall G has the convergence property!). Suppose for the moment that $0 \neq y_0$ and so we can assume (without loss of generality) that $y_0 = \infty$. Let $\{\lambda_i\}$ be a sequence of real numbers such that $|g_i(\lambda_i e_1)| = 1$. Then $\{g_i\}$ forms a normal family and there is a quasiconformal homeomorphism h such that (for some subsequence) $g_i(\lambda_i x) \to h(x)$. The group $\lambda_i^{-1} G \lambda_i$ has the invariant conformal structure

$$\mu_i(x) = \mu\left(\frac{x}{\lambda_i}\right) \to A,$$

where A is some constant matrix. But

$$\lambda_i^{-1} G \lambda_i = \lambda_i^{-1} g_i^{-1} G g_i \lambda_i \to h^{-1} G h$$

and this last group is conjugate to a Möbius group since it has the constant matrix A as an invariant conformal structure. Consequently G is conjugate to a Möbius group.

Now of course one must tidy up all the details. A key result is Tukia's generalization of the good approximation theorem, see [Tu 1, Cor. D].

Theorem 5.6. (Good approximation theorem.) *Let $f_i: U \to \mathbb{S}^n$ be a sequence of K-quasiconformal embeddings. Suppose that $f_i \to f$ for some embedding $f: U \to \mathbb{S}^n$ and that $\mu_{f_i} \to \mu$ in measure for some measurable map $\mu: U \to S(n)$. Then f is K-quasiconformal and $\mu_f = \mu$ a.e. in U.*

If the invariant conformal structure is continuous at a limit point then the above outline works (the assumption that $x_0 \neq y_0$ is unnecessary). Of course we have produced the invariant conformal structure from general methods and it is quite unlikely that it will be continuous anywhere on the limit set (where the action is highly mixing). However the argument is sufficiently robust that all that is necessary is that μ be continuous in measure at a limit point x_0 where we can assert that there is a sequence $g_i \to x_0$ and $g_i^{-1} \to y_0$ with $x_0 \neq y_0$. Such a limit point is called a point of approximation or a *conical limit point*, see [MT]. This last term is because if the action extended to the ball then the orbit of the origin would meet some Stoltz cone based at x_0 infitely often. Since a measurable map is continuous almost everywhre we are more or less done if we assume that the limit set has positive measure (for instance if it is the whole sphere) and that there are plenty of conical limit points. If the action extended to the ball then the hypothesis that $\text{Int}(B^n)/G$ is compact easily implies every point is a conical limit point. Alternatively, if G acts on the triple space T_n so that T_n/G is compact then every limit point is a conical limit point [Tu 1, Cor. G]. We can put all this together to get the following theorem and its corollaries.

Theorem 5.7. *Let G be a quasiconformal group of $\mathbb{S}^n$ and μ a G-invariant conformal structure. If μ is continuous at a limit point of G or continuous in measure at a conical limit point of G, then g is quasiconformally conjugate to a Möbius group by a quasiconformal homeomorphism h satisfying $D(h) \leqslant D(G)^{1/\sqrt{2}}$.*

Corollary 5.8. *Let G be a quasiconformal group of $\mathbb{S}^n$ such that either*

(1) *G extends to B^{n+1} and $\operatorname{int}(B^{n+1})/G$ is compact, or*
(2) *G acts on the triple space T_n so that T_n/G is compact.*

Then G is quasiconformally conjugate to a Möbius group.

As an application of these ideas together with some results on the sort of domains in $\mathbb{R}^n$ that admit quasiconformal actions we find a generalisation of a theorem of Gromov [Gr 1, p. 209], see [Ma 3].

Corollary 5.9. *Let M be a compact n-dimensional manifold (or orbifold) whose universal cover quasiconformally embeds in $\mathbb{S}^n$ and has at least one locally quasiconformally flat boundary point. Then the universal cover of M is quasiconformally the ball and M has the homotopy type of a hyperbolic manifold (respectively orbifold).*

The example $\mathbb{S}^n \times \mathbb{S}^1$ is relevant to Corollary 5.9. The universal cover is $\mathbb{S}^n \times \mathbb{R}^1$ which embedds in $\mathbb{S}^{n+1}$ but $\mathbb{S}^n \times \mathbb{S}^1$ does not admit a hyperbolic structure. The point is that there are two natural ways to embedd $\mathbb{S}^n \times \mathbb{S}^1$ in $\mathbb{S}^{n+1}$. The first is as $\mathbb{S}^n \times (0, 1)$ which has smooth (and therefore locally quasiconformally flat boundary). This embedding cannot be quasiconformal. The only quasiconformal embedding has image $\mathbb{S}^{n+1} - \{0, \infty\}$ which does not have flat boundary.

Theorem 5.7 essentially says that as soon as the limit set of a quasiconformal group G is large enough, then G is quasiconformally conjugate to a Möbius group. We now turn to consider the case where the limit set is small, namely the elementary groups. We will see that smoothness (which played no role in the above) and algebraic simplicity imply quasiconformal conjugacy (as long as there is an element of infinite order).

The following results are from [Ma 2]. It is quite important to recall that although G may be a group of diffeomorphisms the G-invariant conformal structure is almost certainly not very nice. It is easy to construct a loxodromic quasiconformal transformation which is infinitely smooth at both its limit points and yet the invariant ellipse field above is not even approximately continuous there.

Definition 5.10. *A self homeomorphism f of $\mathbb{S}^n$ is said to be **affine** at x_0 if f is differentiable almost everywhere and there is a matrix $A \in S(n)$ such that*

$$d(\mu_f(x), A) \to 0 \quad as \quad x \to x_0.$$

Notice that a diffeomorphism will always be affine at every point.

Theorem 5.11. *Let f be a parabolic quasiconformal homeomorphism of* $\mathbb{S}^n$ *which is affine at its fixed point. Then f is conjugate to a parabolic Möbius transformation by a K-quasiconformal mapping for which*

$$K(\langle f\rangle)^{1/2} \leqslant K \leqslant \sqrt{(n-1)}\, K(\langle f\rangle)^{1+1/\sqrt{2}}.$$

PROOF. Work in $\overline{\mathbb{R}}^n$ and assume the fixed point is infinity. Choose similarity maps $\beta_m = a_m x + b_m$, where a_m is real and b_m lies in $\mathbb{R}^n$, such that

$$f^n \circ \beta_m(0) = 0 \quad \text{and} \quad |f^m \circ \beta_m(e_1)| = 1.$$

Now since the mapping f^m is $K(\langle f\rangle)$-quasiconformal for all m and since every β_m is conformal, the sequence $\{f^m \circ \beta_m\}$ is normal and so contains a uniformly convergent subsequence converging to a $K(\langle f\rangle)$-quasiconformal group of affine transformations. To see this note that for all integers k

$$g^k = h^{-1} \circ f^k \circ h = \lim(\beta_m^{-1} \circ f^{-m} \circ f^k \circ f^m \circ \beta_m) = \lim(\beta_m^{-1} \circ f^k \circ \beta_m)$$

as $m \to \infty$, by the uniform convergence. Consequently g generates a $K(\langle f\rangle)$-quasiconformal group. We now show that g is affine. To do this we will compute its matrix dilatation and show that it is constant (note $g(\infty) = \infty$). Let $g_j = \beta_m^{-1} \circ f \circ \beta_m$ so

$$\mu_{g_j}(x) = \mu_{f \circ \beta_m}(x) = \beta_m'(x)[\mu_f(\beta_m(x))] = \mu_f(\beta_m(x)).$$

The assumption that f is affine at infinity now implies that there is an $A \in S(n)$ such that

$$\mu_{g_j}(x) = \mu_f(\beta_m(x)) \to A, \quad \text{as} \quad j \to \infty.$$

From the good approximation theorem we conclude that $\mu_g \equiv A$. That is, g has constant matrix dilatation. Finally the affine, uniformly quasiconformal group $\langle g\rangle$ is necessarily quasiconformally conjugate to a Möbius group by a linear mapping B whose dilatation is no more than $\sqrt{(n-1)}\, K(\langle g\rangle)^{1/\sqrt{2}}$. Thus $h \circ B$ is a $\sqrt{(n-1)}\, K(\langle f\rangle)^{1+\sqrt{2}}$-quasiconformal mapping conjugating f to a parabolic Möbius transformation.

A similar argument gives us the loxodromic case

Theorem 5.12. *Let f be a loxodromic quasiconformal transformation of* $\mathbb{S}^n$ *whose matrix dilatation is approximately continuous at a fixed point. Then f is conjugate to a loxodromic Möbius transformation by a K-quasiconformal mapping for which*

$$K(\langle f\rangle)^{1/2} \leqslant K \leqslant \sqrt{(n-1)}\, K(\langle f\rangle)^{1+1/\sqrt{2}}.$$

The algebraic idea we need in order to generalise this line of argument is the concept of admissibility

Definition 5.13. *Let G be an abstract group. If g and h are elements of G we define the* **commutator** *of g and h as* $[g, h] = g^{-1}h^{-1}gh$. *If H is a subset of G we define the commutator of G and H as* $[G, H] = \{[g, h]: g \in G$ *and* $h \in H\}$. *Realize that* $[G, H]$ *is just a subset of G and not a subgroup even if H is a subgroup. The* **center** *of G,* $Z(G)$, *is the largest subset of G such that* $[G, Z(G)] = \{\text{Identity}\}$. *We say that a subgroup H of G is* **virtually central** *if the set* [G, H] *is finite.*

Definition 5.14. *An abstract group G is called* **admissible** *if there is an infinite cyclic subgroup which is virtually central.*

Here is the sort of result we have been seeking [Ma 2, Thm. 4.5].

Theorem 5.15. *Let G be a discrete, admissible group of quasiconformal diffeomorphisms of* $\mathbb{S}^n$. *Then G is conjugate to a Möbius group by a K-quasiconformal homeomorphism for which*

$$K(G)^{1/2} \leqslant K \leqslant \sqrt{(n-1)}\, K(G)^{1+1/\sqrt{2}}.$$

PROOF. Since G is admissible there is $f \in G$ such that f is of infinite order and $\langle f \rangle$ is virtually central. Thus $F = [G, \langle f \rangle]$ is a finite set. Because G is a discrete quasiconformal group f is either parabolic or loxodromic and we may assume that f fixes infinity and that if f is loxodromic then infinity is the repulsive fixed point. We proceed as above to find conformal maps β_m such that (for some subsequence) $\beta_m \circ f^m$ converging uniformly to a $K(G)$-quasiconformal mapping h. As before $h^{-1} \circ f \circ h$ will be an affine mapping. Let $g \in G$. We show $h^{-1} \circ g \circ h$ is affine. Again, by the uniform convergence we see

$$g' = h^{-1} \circ g \circ h = \lim (\beta_{m(j)}^{-1} \circ f^{-m(j)} \circ g \circ f^{m(j)} \circ \beta_{m(j)}).$$

Now as $\langle f \rangle$ is virtually central we see that for all $m(j)$ there is $h_{m(j)} \in F$ such that the mapping

$$f^{-m(j)} \circ g \circ f^{m(j)} = g \circ h_{m(j)}.$$

Since F is a finite collection of mappings, we may pass to another subsequence so that

$$f^{-m(j)} \circ g \circ f^{m(j)} = g \circ h$$

for some fixed h in the set F. Then

$$g' = h^{-1} \circ g \circ h = \lim (\beta_{m(j)} \circ g \circ h \circ \beta_{m(j)}^{-1})$$

is a $K(G)$-quasiconformal homeomorphism of $\mathbb{R}^n$. Calculating the matrix dilatations of elements of this sequence and using the fact that the matrix dilatation of $g \circ h$ is continuous everywhere, and so in particular at infinity, then applying the good approximation theorem as we did in Theorem 5.11 we see as before that g' is a $K(G)$-quasiconformal affine mapping. Since g was arbitrary, the group $h^{-1} \circ G \circ h$ is affine and quasiconformal and thus conjugate to a conformal group via a linear mapping.

Corollary 5.16. *Let G be a finitely generated infinite discrete quasiconformal group of diffeomorphisms of $\mathbb{S}^n$. If G is abelian, then G is quasiconformally conjugate to a discrete Möbius group. Consequently the rank of G is at most n.*

We now turn to consider the hypotheses of Theorem 5.15 and to give examples to show that they are essentially all necessary. As we saw earlier the assumption that there is an element of infinite order in the group is necessary, when $n \geqslant 4$, due to the existence of smooth counterexamples to the generalized Smith conjecture. The examples of Freedman and Skora show that some geometric or algebraic restriction is necessary. We raise the following question though

Can the hypothesis of admissibility in Theorem 5.15 be replaced by the hypothesis that G is virtually abelian?

Notice that these two hypotheses are independent. It may be that this is not the case, it seems possible that there is a smooth quasiconformal group containing an exotic involution and isomorphic to the Dihedral group.

We recall Theorem 5.2 which was an example of a uniformly quasiconformal group of $\mathbb{S}^n$ isomorphic to $\mathbb{Z}^{n-1}$ (and so of course admissible) which is not quasiconformally conjugate to a Möbius group. As remarked in that paper [Ma 1], the group can be made smooth except at one point. It cannot be made smooth over this last point by our results above. By J. McKemie's results [McK] the dilatation of this group can be assumed arbitrarily close to one.

It is worth observing the following consequence of Theorem 5.15. Recall we saw earlier that fake elliptics can lie in some infinite discrete quasiconformal groups. However we see.

Proposition 5.17. *Let σ be a periodic diffeomorphism of $\mathbb{S}^n$ of period p. If σ is not topologically conjugate to an orthogonal transformation, then there is no discrete smooth $\mathbb{Z} \times \mathbb{Z}_p$ action on $\mathbb{S}^n$ which contains σ and is of bounded distortion.*

Actually of course, a periodic diffeomorphism which is not conjugate to an orthogonal transformation, can lie inside no finitely generated infinite abelian group of bounded distortion acting smoothly on $\mathbb{S}^n$.

In a different vein using more topological methods we conclude this section by recalling a result of P. Tukia and the author regarding the conjugacy of certain convergence groups in the three dimensional case. The main idea of the arguments used is that if G is a convergence group acting on $\bar{B}^3$ and which is of compact type, then the quotient three manifold can have no incompressible tori. It follows then (by Thurston's theorem, see [MB]) that the quotient manifold admits a hyperbolic structure if for instance the manifold has boundary or more generally is Haken. This produces the desired conjugacy to a Möbius group on the ordinary set. There is in general some problem in extending the conjugacy over the limit set, but this is automatic if G were for instance uniformly quasiconformal or if $L(G)$ were a Cantor set.

Theorem 5.18. *Let G be a torsion free uniformly quasiconformal group acting on $\bar{B}^3$ which is of compact type and of the second kind (or if G is of the first kind assume B^3/G is Haken). Then g is quasiconformally conjugate to a Möbius group.*

Theorem 5.19. *Let G be a torsion free convergence group acting on $\bar{B}^3$ which is of compact type and whose limit set is a Cantor set. Then G is topologically conjugate to a Möbius group.*

There is a suggested outline of how to extend the above results to the geometrically finite case in [MT]. The hope is to prove the general conjecture.

A uniformly quasiconformal group acting on B^3 is quasiconformally conjugate to a Möbius group.

Note that from the results of Section 4, the action on the boundary may be assumed conformal.

6. Leftovers

There is much that we have left unsaid. As an example one can define a Poincaré series for a discrete quasiconformal group acting on B^n [GM 2]. The convergence and divergence of this series depends on the structure of the limit set. The radial limit set can be defined and was shown by Gehring, Garnett and Jones [GGJ] to satisfy the classical dichotomy of zero or full measure when $n \geqslant 2$. J. Garnett has raised the following question when $n = 1$ (recall that unlike quasiconformal mappings quasisymmetric mappings need not preserve Lebesgue null sets).

Let G be a quasisymmetric group of the circle. Is the measure of the conical limit set zero or full?

One can define what it means for a quasiconformal group to be geometrically finite and correspondingly solve Ahlfors' measure zero problem in this case using essentially no properties of analyticity.

Other questions regarding discreteness criteria (such as Jøgensen's inequality) and universal constraints (as are imposed by discreteness on Fuchsian groups) may be raised for quasiconformal groups in all dimensions.

Another project would be to more closely relate Gromov's theory of hyperbolic groups [Gr 2] to that of convergence groups. What are the consequences for convergence groups (or for hyperbolic groups)?

We hope that the questions and conjectures raised in this paper and others will motivate further investigation into convergence and quasiconformal groups.

References

[AM] Abikoff, W. and Maskit, B. Geometric Decomposition of Kleinian Groups, *Amer. J. Math.*, **99**(1974), 687-697.

[Ah 1] Ahlfors, L. V. A Somewhat New Approach to Quasiconformal Mappings in $\mathbb{R}^n$, *Lecture Notes in Math.* **599**, Springer-Verlag, (1977).

[Ah 2] —, Finitely Generated Kleinian Groups, *Amer. J. Math.*, **86**(1964), 413-429.

[Ah 3] —, Möbius Transformations in Several Dimensions, *Lecture notes. School of Math.*, Univ. of Minnesota (1981).

[AS] Ahlfors, L. V. and Sario, L. Riemann surfaces, Princeton Univ. Press, (1960).

[Bea] Beardon, A. The Geometry of Discrete Groups, Springer-Verlag, (1983).

[Be] Bers, L. On Ahlfors' Finiteness Theorem, *Amer J. Math.*, **89**(1967), 1078-1082.

[BC] Bestvina, M. and Cooper, D. A wild Cantor set as the limit set of a Schottky Group. *Proc. Amer. Math. Soc.* To appear.

[DE] Douady, A. and Earle, C. Conformally Natural Extensions of Homeomorphisms of the Circle. *Acta Math.*, **157**(1986), 23-48.

[Du] Dunwoody, M. J. Accessibility of finitely presented groups, *Invent. Math.*, **81**(1985), 449-457.

[EO] Eberlin, P. and O'Neil, B. Visibility Manifolds, *Pac. J. Math.*, **46**(1973), 45-109.

[Ei] Eilenberg, S. Sur les transformations périodiques de la surface du sphere, *Fund. Math.*, **22**(1934), 28-41.

[FM] Feighn, M. and McCullough, D. Finiteness Conditions for Three Manifolds with Boundary, *Amer. J. Math.*, **109**(1987), 1155-1169.

[Fr] Freedman, M. A Geometric Reformulation of Four Dimensional Surgery, *Top. Appl.*, **24**(1986).

[FS] Freedman, M. and Skora, R. Strange actions of groups on spheres. *J. Diff. Geometry*, **25**(1987), 75-98.

[Ga] Gardiner, F. P. Teichmüller Theory and Quadratic Differentials, *Pure and Appl. Math.*, Wiley Interscience (1987).

[Ge] Gehring, F. W. Rings and Quasiconformal Mappings in space, *Trans. Amer. Math. Soc.*, **103**(1962), 353-393.

[GGJ] Gehring, F. W., Garnett, J. and Jones, P. Quasiconformal Groups and the Conical Limit Set, *Holomorphic Functions and Moduli*, Springer-Verlag (1988), 59-68.

[GM 1] Gehring, F. W. and Martin, G. J. Discrete Quasiconformal Groups I. *Proc. London Math. Soc.*, **55**(1987), 331-358.

[GM 2] Gehring, F. W. and Martin, G. J. Discrete Quasiconformal Groups II, to appear.

[GM 2] —, Discrete Convergence Groups. *Lecture Notes in Math.* **1275**. Springer Verlag (1987), 158-168.

[GP] Gehring, F. W. and Palka, B. Quasiconformally Homogeneous Domains, *J. Analyse Math.*, **30**(1976), 172-199.

[Gi] Giffen, C. H. The Generalized Smith Conjecture. *Amer. J. Math.* **88**(1966), 187-198.

[Gre] Green, L. The Generalised Geodesic Flow, *Duke Math. J.*, **41**(1974), 115-126.

[Gr 1] Gromov, M. Hyperbolic Manifolds, Groups and Actions, *Ann. Math. Stud.* **97**. Princeton Univ. Press. 183-213.

[Gr 2] —, Hyperbolic groups, *Essays in Group Theory*, Springer-Verlag (1987).

[Gr 3] —, Structures Métriques pour les Variétés Riemanniannes. *Textes Mathématiques.* Cedric/Fernand Nathan (1981).

[GT] Gromov, M. and Thurston, W. Variable Negative Curvature and Constant Negative Curvature, *Invent. Math.*, to appear.

[He] Helgason, S. Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, (1978).

[Hi] Hinkkanen, A. Uniformly Quasisymmetric Groups. *Proc. London Math. Soc.*, **51**(1985), 318-338.

[Hi 2] —, Abelian and non-discrete convergence groups of the circle. To appear.

[HM] Hinkkanen, A. and Martin, G. J. Nondiscrete Convergence Groups in the Plane, to appear.

[HK] Homma, T. and Kinoshita, S. On a topological characterisation of the dilation in E^3, *Osaka Math. J.*, **6**(1954), 135-143.

[HP] Hirsch, M. and Pugh, C. Smoothness of Horocycle Foliations, *J. Diff. Geometry*, **10**(1975), 225-238.

[Hu 1] Husch, L. A Homotopy Theoretic Characterisation of the Translations in E^n, *Compositio Math.*, **24**(1972), 55-61.

[Hu 2] Husch, L. A Topological Theoretic Characterisation of the Dilation in E^n, *Proc. Amer. Math. Soc.*, **28**(1971), 234-236.

[Ker] Kerchoff, S. The Nielsen Realisation Problem, *Annals of Math.*, **117**(1983), 235-265.

[Ke 1] B. von Kerékjártó. Über die fixpunktfreien Abbildungen der Ebene, *Acta Litt. Sci. Szeged*, **6**(1934), 226-234.

[Ke 2] —, Topologische Charakterisierung der linearen Abbildungen, *Acta Litt. Sci. Szeged*, **6**(1934), 235-262.

[Kr] Kra, I. On the Cohomology of Kleinian Groups, *J. Analyse Math.*, **43** (1983/84), 51-87.

[Ku] Kulkarni, R. Infinite Regular Coverings and Kleinian Groups, *Proc. Symp. Pure Math.*, **32**(1978), 407-412.

[KS] Kulkarni, R. and Shalen, P. On Ahlfors' Finiteness Theorem, *Advances in Math.*

[Le] Lehto, O. Univalent Functions and Teichmüller Spaces, Springer-Verlag.

[Li] Linnell, P. A. On Accessibility of Groups, *J. Pure Appl. Alg.*, **30**(1983), 39-46.

[Maa] Maass, H. Siegel's Modular Forms and Dirichlet Series. *Lecture Notes in Math.*, **216**. Springer Verlag (1971).

[Ma 1] Martin, G. J. Discrete Quasiconformal Groups that are not the Quasiconformal Conjugates of Möbius Groups, *Ann Acad. Sci. Fenn. Ser.*, A.I., **11**(1986), 179-202.

[Ma 2] Martin, G. J. On Quasiconformal and Affine Groups, *J. Diff. Geometry*, **29**(1989), 427-448.

[Ma 3] Martin, G. J. Quasiconformal Actions on Domains in Space, *Holomorphic Functions and Moduli*, Springer-Verlag, (1988), 87-112.

[MS] Martin, G. J. and Skora, R. Group Actions on the Two Sphere, *Amer. J. Math.*, **111**(1989), 387-402.

[MT] Martin, G. J. and Tukia, P. Convergence and Möbius Groups, *Holomorphic Functions and Moduli*, Springer-Verlag, (1988), 113-140.

[Mas 1] Maskit, B. Kleinian Groups, Springer-Verlag, (1988).

[Mas 2] —, A Characterisation of Schottky Groups, *J. Analyse Math.*, **19**(1967), 227-230.

[McK] McKemie, J. Quasiconformal Groups with Small Dilatation, *Ann. Acad. Sci. Fenn. Ser. AI,* **12**(1987), 95-118.

[Me] Mess, G. Centers of 3-manifold groups and groups which are coarse quasi-isometric to planes. To appear.

[MZ] Montgomery, D. and Zipping, L. Examples of Transformation Groups, *Proc. Amer. Math. Soc.*, **5**(1954), 460-465.

[Mo] Moore, R. L. Concerning Upper-semicontinuous Collections of Continua, *Trans. Amer. Math. Soc.*, **27**(1925), 416-428.

[MB] Morgan, J. W. and Bass, H. The Smith Conjecture, Academic Press, (1984).

[Ne] Newman, M. H. A. A Theorem on Periodic Transformations of Spaces, *Quart. J. Math.*, **2**(1931), 1-8.

[Ru] Rushing, T. B. Topological Embeddings, Academic Press, (1973).

[Na] Nag, S. The Complex Analytic Theory of Teichmüller Spaces, *Canadian Math. Soc. Monographs*, Wiley Interscience (1988).

[Sc] Scott, G. P. The Geometries of Three Manifolds, *Bull. London Math. Soc.*, **15**(1983), 401-487.

[Se] Selberg, A. On Discontinuous Groups in Higher Dimensional Symmetric Spaces, *Int. Colloquium on Function Theory.* Tata Inst. Bombay, (1960).

[Sta] Stallings, J. Group Theory and Three Dimensional Manifolds. Yale Univ. Press., (1971).

[St] Steenrod, N. The Topology of Fiber Bundles, Princeton Univ. Press, (1951).

[Su 1] Sullivan, D. On the Ergodic Theory at Infinity of an Arbitrary Discrete Group of Hyperbolic Motions. *Ann. Math. Stud.* **97**. Princeton Univ. Press, 465-496.
[Su 2] —, Quasiconformal homeomorphisms in Dynamics, Topology and Geometry, *Proc. Int. Cong.*, Berkeley (1986).
[Su 3] —, Finiteness Theorems for Cusps, *Acta Math.*, **147**(1981), 289-299.
[Th] Thurston, W. The Geometry and Topology of Three Manifolds, Notes, Princeton Univ., (1981).
[Tu 1] Tukia, P. On Quasiconformal Groups, *J. Analyse Math.*, **46**(1986), 318-346.
[Tu 2] —, On Two Dimensional Quasiconformal Groups, *Ann. Acad. Sci. Fenn. Ser.*, A.I., **5**(1980), 73-78.
[Tu 3] —, Homeomorphic Conjugates of Fuchsian Groups, *J. Reine Angew. Math.* **391**(1988), 1-54.
[Tu 4] —, A Quasiconformal Group not Isomorphic to a Möbius Group, *Ann. Acad. Sci. Fenn. Ser.*, A.I., **6**(1981), 149-160.
[Vä] Väisälä, J. Lectures on *n*-Dimensional Quasiconformal Mappings, *Lecture Notes in Math.*, **229**, Springer-Verlag, (1971).
[Wa] Wall, C. T. C. Pairs of Relative Cohomological Dimension One, *J. Pure Appl. Alg.*, **1**(1971), 141-154.
[Wat] Waterman, P. L. Purely Elliptic Möbius Groups. *Holomorphic Functions and Moduli*, Springer-Verlag, (1988), 173-178.

Gaven J. Martin*
Yale University and
University of Auckland
U.S.A.

* Reseach partially supported by a U.S. National Science Foundation Grant.