Revista Matemática Iberoamericana
Vol. 4, n.$^{\text{os}}$ 3 y 4, 1988

Calderón's Problem for Lipschitz Classes and the Dimension of Quasicircles

Kari Astala

1. Introduction

In last years the mapping properties of the Cauchy integral

$$C_\Gamma f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}\,d\xi$$

have been widely studied. The most important question in this area was Calderón's problem, to determine those rectifiable Jordan curves Γ for which C_Γ defines a bounded operator on $L^2(\Gamma)$. The question was solved by Guy David [Da] who proved that C_Γ is bounded on $L^2(\Gamma)$ (or on $L^p(\Gamma)$, $1 < p < \infty$) if and only if Γ is *regular*, *i.e.*

$$\mathcal{H}^1(\Gamma \cap B(z_0, R)) \leqslant CR \tag{1}$$

for every $z_0 \in \mathbb{C}$, $R > 0$ and for some constant C.

Once the L^p-cases are settled it is natural to ask when C_Γ is bounded on the other classical function spaces. In particular, it has been shown by Salaev [Sa], *cf.* also [Dy], that if Γ is regular, then C_Γ is a bounded operator on the Lipschitz classes

$$\Lambda^\alpha(\Gamma) = \left\{ f \colon \|f\|_{\Lambda^\alpha(\Gamma)} = \sup_{x,y\in\Gamma} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \right\},$$

$0 < \alpha < 1$. Recently Zinsmeister [Z] made the interesting discovery that after a suitable reinterpretation, see Section 3, Calderón's problem makes sense for Lipschitz classes $\Lambda^\alpha(\Gamma)$ even on non-rectifiable curves Γ. His result was a follows.

Theorem 1.1. (Zinsmeister.) *If Γ is a bounded K-quasicircle, there is a constant $a \in [1, 40]$ such that the Cauchy operator*

$$C_\Gamma : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma)$$

is bounded whenever $a(K) < \alpha < b(K)$ and $K^{2a} \leqslant (1 + \sqrt{5})/2$;

$$a(K) = \frac{K^{2a} - 1}{K^{2a} + 1},$$

$$b(K) = (2K^{4a} - 1)^{-1}.$$

Here a curve Γ is called a K-quasicircle if $\Gamma = \varphi(\{|z| = 1\})$ for some K-quasiconformal mapping φ of $\bar{\mathbb{C}}$. Similarly a domain D is called a K-quasidisk if ∂D is K-quasicircle. For the many different characterizations of quasicircles and —disks see, for instance, [L].

In this paper we shall obtain boundedness theorems for the Cauchy operator on all quasicircles and all $K \geqslant 1$. In fact, it turns out that for every quasicircle Γ there is a number $\alpha(\Gamma) < 1$, depending on the dimension rather than the dilatation of Γ, such that C_Γ is bounded on $\Lambda^\alpha(\Gamma)$ if $\alpha \in (\alpha(\Gamma), 1)$ and unbounded if $\alpha \in (0, \alpha(\Gamma)]$.

The best way to describe the behaviour of C_Γ is in terms of A_p weights of Muckenhoupt [M]. Recall that a function $w \geqslant 0$ is said to belong to the class A_p, $1 < p < \infty$, if

$$\left(\frac{1}{|B|}\int_B w(z)\, dm(z)\right)\left(\frac{1}{|B|}\int_B w(z)^{-1/(p-1)}\, dm(z)\right)^{p-1} \leqslant C$$

holds for a constant $C < \infty$ and for all disks $B \subset \mathbb{C}$; here $|B|$ denotes the Lebesgue measure of B. Further, $w \in A_1$ if

$$\frac{1}{|B|}\int_B w(z)\, dm(z) \leqslant Cw(x) \quad \text{a.e.} \quad x \in B$$

and $w \in A_\infty$ if

$$\frac{1}{|B|}\int_B w(z)\, dm(z) \leqslant C \exp\left(\frac{1}{|B|}\int_B \log w(z)\, dm(z)\right)$$

hold for all disks B. Then $A_1 \subset A_p \subset A_\infty$ and A_∞ is the union of all A_p classes.

Theorem 1.2. *Let* Γ *be a bounded quasicircle and* $0 < \alpha < 1$. *Then the following conditions are equivalent*

(*a*) $C_\Gamma : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma)$ *is a bounded operator.*
(*b*) $d(z, \Gamma) \in A_p$, $p = 1 + 1/(1 - \alpha)$.

Here $d(z, \Gamma)$ *denotes the euclidean distance from* Γ. *More precisely, if*

$$\alpha(\Gamma) = \inf\{\alpha : d(z, \Gamma) \in A_{1 + 1/(1-\alpha)}\} \tag{2}$$

then $0 \leqslant \alpha(\Gamma) < 1$ *and* C_Γ *is bounded on* $\Lambda^\alpha(\Gamma)$ *whenever* $\alpha(\Gamma) < \alpha < 1$ *and unbounded whenever* $0 < \alpha \leqslant \alpha(\Gamma)$.

The condition (*b*) can be replaced by

(*b*′) $d(z, \Gamma)^{\alpha - 1} \in A_1$

and also by

(*b*″) $d(z, \Gamma)^{\alpha - 1} \in A_\infty$,

in other words by $d(z, \Gamma)^{\alpha-1} \in A_p$ for any p. As it is well known, the A_p-condition also characterizes the boundedness of the 2-dimensional Hilbert transform or the Beurling-Ahlfors transform

$$Hf(z) = \text{p.v.} \int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2}\, dm(\xi),$$

see [CF]. Hence we have

Corollary 1.3. *If* Γ *is a bounded quasicircle*, $0 < \alpha < 1$ *and* $1 < p < \infty$, *the following conditions are equivalent*

(*a*) $\|C_\Gamma f\|_{\Lambda^\alpha(\Gamma)} \leqslant M_1 \|f\|_{\Lambda^\alpha(\Gamma)}$, $f \in \Lambda^\alpha(\Gamma)$.
(*b*) $\int_{\mathbb{C}} |Hf(z)|^p d(z, \Gamma)^{\alpha-1}\, dm(z) \leqslant M_2 \int_{\mathbb{C}} |f(z)|^p d(z, \Gamma)^{\alpha - 1}\, dm(z)$,
$f \in L^p_w(\mathbb{C})$, $w(z) = d(z, \Gamma)^{\alpha - 1}$.

In Theorem 1.2 the assumption that Γ is a quasicircle is not necessary, the proof works for a number of other curves, too. For example we obtain a proof for Salaev's theorem, *cf.* Corollary 3.9.

To see more clearly the geometric meaning of Theorem 1.2(*b*) we must introduce some notation. If E is a bounded subset of the complex plane and $0 < r \leqslant \operatorname{diam}(E)$, set

$$M^\beta(E; r) = \inf\left\{ nr^\beta : E \subset \bigcup_{i=1}^{n} B(x_i, r), n \in \mathbb{N} \right\}.$$

Then $\lim_{r\to 0} \sup M^\beta(E; r) = M^\beta(E)$ is the β—dimensional *Minkowski content* of E, *cf.* [MV]. Instead of the Minkowski content we need to use the following closely related quantity

$$h^\beta(E) = \sup\{M^\beta(E; r): 0 < r \leqslant \operatorname{diam}(E)\}.$$

If $\mathcal{H}^\beta$ denotes the β-dimensional Hausdorff measure, then clearly $\mathcal{H}^\beta(E) \leqslant M^\beta(E) \leqslant h^\beta(E)$.

We shall see in Lemma 2.2 below that a Jordan curve Γ is regular if and only if $h^1(\Gamma \cap B(z_0, R)) \leqslant CR$ for all $z_0 \in \mathbb{C}$ and $R > 0$. Therefore it is reasonable to say that Γ is *δ-regular* if there is a constant C such that

$$h^\delta(\Gamma \cap B(z_0, R)) \leqslant CR^\delta, \qquad z_0 \in \mathbb{C}, \qquad R > 0. \tag{3}$$

Theorem 1.4. *If Γ is a bounded quasicircle and if Γ is δ-regular, then C_Γ is bounded on $\Lambda^\alpha(\Gamma)$ whenever $\delta - 1 < \alpha < 1$. Conversely, if C_Γ is bounded on $\Lambda^\alpha(\Gamma)$, then Γ is $(1 + \alpha)$-regular. Thus*

$$\delta(\Gamma) = \inf\{\delta: \Gamma \text{ is } \delta\text{-regular}\} = 1 + \alpha(\Gamma).$$

To illustrate how these results describe the behavior of the Cauchy integral we mention that for the snowflake or Koch curve Γ, $\alpha(\Gamma) = \log(4/3)/\log 3$ and that Γ is $\delta(\Gamma)$-regular. In fact, $\delta(\Gamma) = 1 + \alpha(\Gamma) = \log 4/\log 3 = \dim_H(\Gamma)$, the Hausdorff dimension of Γ.

In the case of a general quasicircle Γ the Hausdorff dimension, the Minkowski dimension $\beta(\Gamma) = \inf\{\beta: M^\beta(\Gamma) < \infty\}$ and the degree of regularity $\delta(\Gamma)$ may be very different. However, the differences vanish if we look at the whole class of all K-quasicircles, *i.e.* as in Theorem 1.1 look for the estimate of $\alpha(\Gamma)$ in terms of the dilation K,

$$\alpha(K) = \sup\{\alpha(\Gamma): \Gamma \text{ is } K\text{-quasicircle}\}.$$

Theorem 1.5. *For each $K \geqslant 1$ the following quantities are equal*

(*a*) $d(K) = \sup\{\dim_H(\Gamma): \Gamma \text{ is } K\text{-quasicircle}\}$.
(*b*) $\beta(K) = \sup\{\beta(\Gamma): \Gamma \text{ is } K\text{-quasicircle}\}$.
(*c*) $\delta(K) = \sup\{\delta(\Gamma): \Gamma \text{ is } K\text{-quasicircle}\}$.

Moreover,

$$1 + \alpha(K) = d(K) = \beta(K) = \delta(K).$$

The above characterization yields numerical estimates for $\alpha(K)$: In a recent article Becker and Pommerenke [BP] estimated the Minkowski dimension of quasicircles and proved that $\beta(K) \leqslant 2 - C_1 K^{-3,41}$ and that for K close to 1, $1 + 0{,}09\varkappa^2 \leqslant \beta(K) \leqslant 1 + 37\varkappa^2$, $\varkappa = (K-1)/(K+1)$.

Thus, if Γ is a K-quasicircle, then the Cauchy integral C_Γ is bounded on $\Lambda^\alpha(\Gamma)$ whenever $\alpha(K) < \alpha < 1$, where

$$\alpha(K) = d(K) - 1 \leqslant 1 - C_1 K^{-3,41}$$

and

$$0{,}09(K-1)^2/(K+1)^2 \leqslant \alpha(K) \leqslant 37(K-1)^2/(K+1)^2$$

for K near 1. The bound is best possible; if $\alpha < d(K) - 1$ we can find a K-quasicircle Γ such that C_Γ is not bounded on $\Lambda^\alpha(\Gamma)$.

It is our pleasure to express our gratitude to M. Zinsmeister for pointing out mistakes in the first version of this paper and, especially, for his help in constructing the correct proof for Lemma 3.4, which is now based on a suggestion of him.

2. Preliminaries

Following the terminology of Väisälä [V] we call a set A *porous* if there is a constant $0 < \lambda < 1$ such that every disk $B(z_0, R)$ in $\mathbb{C}$ contains a disk $B(z, \lambda R)$ with $A \cap B(z, \lambda R) = \varnothing$. We show first that for porous curves Γ the conditions (b), (b') and (b'') in Theorem 1.2 are equivalent.

Lemma 2.1. *Let Γ be a Jordan curve and $0 < \alpha < 1$.*

(*a*) *If there is a constant $C < \infty$ such that*

$$\int_{B(z_0,R)} d(z,\Gamma)^{\alpha-1}\, dm(z) \leqslant C R^{1+\alpha} \tag{4}$$

for all $z_0 \in \mathbb{C}$ and $R > 0$, then $d(z,\Gamma)^{\alpha-1} \in A_1$.

(*b*) *If Γ is porous and $d(z,\Gamma)^{\alpha-1} \in A_\infty$, then* (4) *holds for all $z_0 \in \mathbb{C}$, $R > 0$.*

(*c*) *If $p = 1 + 1/(1-\alpha)$, then $d(z,\Gamma) \in A_p$ if and only if $d(z,\Gamma)^{\alpha-1} \in A_{p'}$, $p' = p/(p-1)$.*

PROOF. Fix $z_0 \in \mathbb{C}$ and $R > 0$ and denote $B = B(z_0, R)$. In (*a*) if Γ intersects $B(z_0, 2R)$, then $d(z,\Gamma) \leqslant 3R$ for $z \in B$. Thus

$$\frac{1}{|B|}\int_B d(x,\Gamma)^{\alpha-1}\, dm(x) \leqslant C_1 d(z,\Gamma)^{\alpha-1}, \qquad z \in B.$$

If Γ does not intersect $B(z_0, 2R)$, then $d(z_0,\Gamma)/2 \leqslant d(z,\Gamma) \leqslant 2d(z_0,\Gamma)$ for every $z \in B$ and hence

$$\int_B d(x,\Gamma)^{\alpha-1}\, dm\,(x) \leqslant 4^{1-\alpha}|B|d(z,\Gamma)^{\alpha-1}, \ z \in B.$$

To prove (*b*) choose a disk $B(x, \lambda R) \subset B(z_0, R)$ which does not intersect Γ. As $d(z,\Gamma)^{\alpha-1}$ belongs to A_∞, $d(z,\Gamma)^{\alpha-1} \in A_p$ for some $p < \infty$ and so

$$\frac{1}{|B|}\int_B d(z,\Gamma)^{\alpha-1}\, dm\,(z) \leqslant C\left(\frac{1}{|B|}\int_B d(z,\Gamma)^{(1-\alpha)/(p-1)}\, dm\,(z)\right)^{1-p}$$
$$\leqslant C|B|^{p-1}(\lambda R/2)^{\alpha-1}|B(x,\lambda R/2)|^{1-p}$$

which gives (4). Finally (*c*) follows from the fact that $w \in A_p$ if and only if $w^{-1/(p-1)} \in A_{p'}$. $\square$

Every quasicircle is porous [V]. There are, of course, many other examples. For instance, it is not difficult to see that regular Jordan curves, or even δ-regular curves with $\delta < 2$, are all porous.

Lemma 2.2. *A Jordan curve* Γ *is regular if and only if it is* 1*-regular in the sense of* (3).

PROOF. Since $\mathcal{H}^1(E) \leqslant h^1(E)$, 1-regularity implies the usual regularity. Conversely, if Γ is regular and if Γ intersects $B(z_0, R)$, let $\rho \leqslant \operatorname{diam}(\Gamma \cap B(z_0, R))$. Then there exist points $x_j \in \Gamma \cap B(z_0, R)$, $1 \leqslant j \leqslant m$, such that $\Gamma \cap B(z_0, R) \subset \bigcup_1^m B(z_j, \rho)$ and each point of $\Gamma \cap B(z_0, R)$ is contained in at most M of the balls $B(x_j, \rho)$ (see, for example, [St]). Here M is an absolute constant. As $x_j \in \Gamma$,

$$\begin{aligned} m\rho &\leqslant m\mathcal{H}^1(\Gamma \cap B(x_j, \rho)) \\ &\leqslant M\mathcal{H}^1(\Gamma \cap \textstyle\bigcup_1^m B(x_j, \rho)) \\ &\leqslant M\mathcal{H}^1(\Gamma \cap B(z_0, 3R)) \leqslant CR. \end{aligned}$$

Consequently, $h^1(\Gamma \cap B(z_0, R)) \leqslant CR$ and Γ is 1-regular. $\square$

Lemma 2.3. *Let* Γ *be a porous Jordan curve and* $0 < \alpha < 1$. *If* $d(z,\Gamma)^{\alpha-1} \in A_1$, *then* Γ *is* $(1+\alpha)$*-regular, and if* Γ *is* δ*-regular* $d(z,\Gamma)^{\alpha-1} \in A_1$ *whenever* $1 + \alpha > \delta$. *In particular,* $\delta(\Gamma) = 1 + \alpha(\Gamma)$.

PROOF. Assume first that Γ is δ-regular and denote $B(t) = B(0, t)$. If $t \leqslant R$, it follows from basic covering theorems, *cf.* [MV, Lemma 3.1], that

$$\begin{aligned} t^{\delta-2}|\{z \in B(z_0, R): d(z,\Gamma) < t\}| &\leqslant |\Gamma \cap B(z_0, 2R) + B(t)|/t^{2-\delta} \\ &\leqslant C_1 h^\delta(\Gamma \cap B(z_0, 2R)) \\ &\leqslant C_2 R^\delta. \end{aligned}$$

Integrating this we have for $\alpha \in (\delta - 1, 1)$

$$\begin{aligned}\int_{B(z_0,R)} d(z,\Gamma)^{\alpha-1}\,dm(z) &= (1-\alpha)\int_0^\infty |\{z \in B(z_0,R): d(z,\Gamma) < t\}|\,t^{\alpha-2}\,dt \\ &\leqslant C_2 R^\delta \int_0^R t^{\alpha-\delta}\,dt + (1-\alpha)\pi R^2 \int_R^\infty t^{\alpha-2}\,dt \\ &\leqslant C_3 R^{1+\alpha},\end{aligned}$$

$C_3 = \pi + C_2/(1 + \alpha - \delta)$. According to Lemma 2.1 $d(z,\Gamma)^{\alpha-1} \in A_1$.

On the other hand, in case $d(z,\Gamma)^{\alpha-1} \in A_1$, we may apply as above [MV, Lemma 3.1] and Lemma 2.1 to obtain

$$\begin{aligned}h^{1+\alpha}(\Gamma \cap B(z_0,R)) &\leqslant C \sup_{0<\rho<2R} |\Gamma \cap B(z_0,R) + B(\rho)|/\rho^{1-\alpha} \\ &\leqslant C \sup_{0<\rho<2R} \rho^{\alpha-1}|\{x \in B(z_0,3R): d(x,\Gamma)^{\alpha-1} > \rho^{\alpha-1}\}| \\ &\leqslant C \int_{B(z_0,3R)} d(z,\Gamma)^{\alpha-1}\,dm(z) \\ &\leqslant C_4 R^{1+\alpha}. \quad \square\end{aligned}$$

3. The Cauchy Integral

Let Γ be first a rectifiable Jordan curve and let D be the bounded component of $\mathbb{C}\backslash\Gamma$. If F is a C^∞-function with compact support and if $f = F|_\Gamma$, it then follows from Stokes' theorem that

$$C_\Gamma f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}\,d\xi = -\frac{1}{\pi}\int_{\mathbb{C}\backslash D} \frac{\bar\partial F(\xi)}{\xi - z}\,dm(\xi), \qquad z \in D.$$

Here the latter expression is well defined even if Γ is not rectifiable. Hence we can take it as the definition of $C_\Gamma f$ in case of a general or non-rectifiable Γ:

$$(5)\quad C_\Gamma f(z) = -\frac{1}{\pi}\int_{\mathbb{C}\backslash D} \frac{\bar\partial F(\xi)}{\xi - z}\,dm(\xi), \qquad z \in D, f = F|_\Gamma \ \text{ and } \ F \in C_0^\infty(\mathbb{C}).$$

Note that by applying the generalized Cauchy integral formula we see immediately that $C_\Gamma f$ as defined in (5), does, indeed, depend only on f and not on the specific extension F. Furthermore, it is easily seen that $C_\Gamma f$ extends continuously to $\bar D$ (in fact, formula (5) defines a function continuous at each $z \in \mathbb{C}$ when $F \in C_0^\infty(\mathbb{C})$).

Definition 3.1. *Let Γ be a Jordan curve bounding the domain $D \subset \mathbb{C}$ and let $0 < \alpha < 1$. We say that the Cauchy operator C_Γ is bounded on $\Lambda^\alpha(\Gamma)$, if*

(6) $\qquad \|C_\Gamma f\|_{\Lambda^\alpha(\Gamma)} \leqslant M\|f\|_{\Lambda^\alpha(\Gamma)}$ *for every* $f = F|_\Gamma, \qquad F \in C_0^\infty(\mathbb{C})$,

for some constant M independent of f.

Remark 3.2. In Definition 3.1 the continuity requirement is minimal or the weakest possible and so the definition is the most general one. If C_Γ is bounded in the above sense, then C_Γ extends a priori only to $\Lambda_0^\alpha(\Gamma)$, the closure of C^∞ in $\Lambda^\alpha(\Gamma)$,

$$\Lambda_0^\alpha(\Gamma) = \{f \in \Lambda^\alpha(\Gamma) : |f(x) - f(y)|/|x-y|^\alpha = o(|x-y|)\}.$$

However, the next two lemmas show that if Γ is a quasicircle and C_Γ is bounded on $\Lambda^\alpha(\Gamma)$ in the sense of Definition 3.1, then necessarily *every* $f \in \Lambda^\alpha(\Gamma)$ has an extension F to $\mathbb{C}$ with $\bar\partial F \in L^1(\mathbb{C}\backslash D) \cap C^\infty(\mathbb{C}\backslash D)$. In addition, then $C_\Gamma f(z)$ is well defined via formula (5) for each $f \in \Lambda^\alpha(\Gamma)$ and $z \in D$, $C_\Gamma f(z)$ is analytic in D and it has a continuous extension to $\partial D = \Gamma$ such that (6) holds.

Lemma 3.3. *Let D be a bounded Jordan domain and let* $0 < \alpha < 1$. *Suppose further that* $v \in C_0^\infty(\mathbb{C})$, $\operatorname{supp} v \subset B(z_0, R)$, $z_0 \in \Gamma = \partial D$, $\bar\partial v(z) \geqslant 0$ *for* $z \notin D$ *and that* $\|v\|_{\Lambda^\alpha(\Gamma)} = 1$. *Then, if the Cauchy operator* C_Γ *is bounded on* $\Lambda^\alpha(\Gamma)$,

$$\int_{\mathbb{C}\backslash D} \bar\partial v(z)\, dm(z) \leqslant C_1 R^{1+\alpha}.$$

PROOF. Choosing points $w, w' \in \Gamma$ such that $|w - z_0| = 5R$ and $|w' - z_0| = 10R$ we can estimate

$$\int_{\mathbb{C}\backslash D} \bar\partial v(z)\, dm(z) \leqslant 20R \left| \int_{\mathbb{C}\backslash D} \frac{\bar\partial v(z)}{z-w}\, dm(z) - \int_{\mathbb{C}\backslash D} \frac{\bar\partial v(z)}{z-w'}\, dm(z) \right| \cdot$$

Here the latter expression is equal to $20R|C_\Gamma v(w) - C_\Gamma v(w')|$ and since the Cauchy operator is bounded on $\Lambda^\alpha(\Gamma)$,

$$|C_\Gamma v(w) - C_\Gamma v(w')| \leqslant C_0 |w - w'|^\alpha \leqslant 15^\alpha C_0 R^\alpha.$$

Consequently

$$\int_{\mathbb{C}\backslash D} \bar\partial v(z)\, dm(z) \leqslant C_1 R^{1+\alpha},$$

$C_1 = 20 \cdot 15^\alpha C_0$. $\square$

Lemma 3.4. *Let D be a bounded K-quasidisk and let* $0 < \alpha < 1$. *If the Cauchy operator is bounded on* $\Lambda^\alpha(\Gamma)$, $\Gamma = \partial D$, *then*

(7) $$\int_B d(z, \Gamma)^{\alpha-1}\, dm(z) \leqslant CR^{1+\alpha}$$

for each disk B of radius R.

PROOF. Let $B = B(z_0, R)$ and $2B = B(z_0, 2R)$. If B does not intersect Γ, (7) follows trivially since then $d(z, \Gamma)^{\alpha-1} \leqslant d(z, \partial B)^{\alpha-1}$ for each $z \in B$. Thus we may assume that $z_0 \in \Gamma$. Moreover, by a similar reasoning it suffices to study only the case $R \leqslant \operatorname{diam}(\Gamma)$.

To prove (7) we shall find a $v \in C_0^\infty(\mathbb{C})$, satisfying the assumptions of Lemma 3.3 with $\operatorname{supp} v \subset B(z_0, 4R)$, such that

$$(8) \qquad \int_B d(z, \Gamma)^{\alpha-1}\, dm(z) \leqslant C_2(K) \int_{\mathbb{C}\setminus D} \bar{\partial} v(z)\, dm(z).$$

Indeed, it is enough to find such a v in the special case $R = 1$ since otherwise we may change variables and set $v_R(z) = R^\alpha v(z_0 + (z - z_0)/R)$ for $R \neq 1$.

Assuming that $R = 1$ choose for each $n \in \mathbb{N}$ a maximal set of points $x_i = x_i^n \in \Gamma \cap 2B$, $1 \leqslant i \leqslant k_n$, such that

$$|x_i^n - x_j^n| \geqslant 2^{2-n}, \qquad i \neq j.$$

From the basic distortion properties of quasiconformal mappings we see that Γ is porous in the following slightly stronger sense: There exists a constant $\lambda = \lambda(K)$ such that $B(x_j^n, 2^{-n-1})$ contains a point $w_j^n \in D$ with $d(w_j^n, \Gamma) \geqslant \lambda 2^{-n}$.

We can now construct the function v. Define first $g(x) = x - 1$ if $\lambda \leqslant x \leqslant 1$, $g(x) = (\lambda - 1)(x/\lambda)^2$ if $0 \leqslant x \leqslant \lambda$ and $g(x) = 0$ if $x \in \mathbb{R}\setminus[0, 1]$. Next set $\varphi(z) = g(|z|)/z$, $z \in \mathbb{C}$. Clearly $\operatorname{supp} \varphi \subset B(1)$, $|\varphi(z) - \varphi(w)| \leqslant \lambda^{-2}|z - w|$ and

$$(9) \qquad \bar{\partial}\varphi(z) = 1/2|z| \quad \text{if} \quad \lambda < |z| < 1.$$

After these preparations let

$$v(z) = \sum_{n=0}^{m} u_n(z), \qquad u_n(z) = \sum_{j=1}^{k_n} 2^{-n\alpha} \varphi(2^n(z - w_j^n)),$$

where $m \in \mathbb{N}$ will be chosen later.

It follows easily that the support of v is contained in $B(z_0, 4)$ and, by (9), that $\bar{\partial} v(z) \geqslant 0$ if $z \notin D$. Since a standard smoothening gives a function $v \in C_0^\infty(\mathbb{C})$ with the same properties, it remains to show that, for all m, $\|v\|_{\Lambda^\alpha(\Gamma)} \leqslant C_3(K) < \infty$ and that (8) holds when m is large enough.

We start with the Lipschitz estimate. Suppose $z, w \in \Gamma$ and $2^{-p} \leqslant |z - w| < 2^{-p+1}$. Since for a fixed n the disks $B(w_j^n, 2^{-n})$ are disjoint, $|u_n(z) - u_n(w)| \leqslant |u_n(z)| + |u_n(w)| \leqslant 2 \cdot 2^{-n\alpha}$ for $n \geqslant p$ and $|u_n(z) - u_n(w)| \leqslant \lambda^{-2} 2^{n - n\alpha}|z - w|$ for $n < p$. Hence

$$\begin{aligned} |v(z) - v(w)| &\leqslant \sum_{n=0}^{p-1} \lambda^{-2} 2^{n(1-\alpha)} |z - w| + 2 \sum_{n=p}^{m} 2^{-n\alpha} \\ &\leqslant C_3(|z - w| 2^{p(1-\alpha)} + 2^{-p\alpha}) \\ &\leqslant C_4 |z - w|^\alpha. \end{aligned}$$

Here C_4 depends only on α and K (or λ).

Thus we are left with the proof of (8). Since the disks $B(x_i^n, 2^{2-n})$, $1 \leqslant i \leqslant k_n$, cover $\Gamma \cap 2B$,

$$\begin{aligned} k_n 2^{(1+\alpha)(2-n)} &\geqslant M^{1+\alpha}(\Gamma \cap 2B; 2^{2-n}) \\ &\geqslant C_5 |\Gamma \cap 2B + B(2^{-n})| 2^{(1-\alpha)n}. \end{aligned}$$

In the latter inequality we used again [MV, 3.1]. Beacuse $B(x_i^n, 2^{-n-1}) \subset B(w_i^n, 2^{-n})$ we can find a disk $B(y_i^n, \lambda 2^{-n})$ inside $B(w_i^n, 2^{-n}) \cap (\mathbb{C} \backslash D)$ and as $\bar{\partial} u_n(z) \geqslant 0$ in $\mathbb{C} \backslash D$, (9) implies

$$\int_{\mathbb{C} \backslash D} \bar{\partial} u_n(z)\, dm(z) \geqslant \sum_{i=1}^{k_n} \int_{B(y_i^n, \lambda 2^{-n})} \bar{\partial} u_n(z)\, dm(z) \geqslant \lambda^2 k_n 2^{-n\alpha - n}.$$

Since Lemma 3.3 yields

$$\sum_{n=0}^{m} \int_{\mathbb{C} \backslash D} \bar{\partial} u_n(z)\, dm(z) = \int_{\mathbb{C} \backslash D} \bar{\partial} v(z)\, dm(z) \leqslant C_1 \cdot C_4 \cdot 4^{1+\alpha} < \infty$$

for all $m \in \mathbb{N}$, we may estimate

$$\begin{aligned} \int_B d(z, \Gamma)^{\alpha - 1}\, dm(z) &= \int_0^\infty |\{z \in B \colon d(z, \Gamma)^{\alpha-1} > t\}|\, dt \\ &\leqslant \sum_{n=0}^{\infty} 2^{n(1-\alpha)} |\{z \in B \colon d(z, \Gamma) < 2^{-n}\}| \\ &\leqslant C_6 \sum_{n=0}^{\infty} \int_{\mathbb{C} \backslash D} \bar{\partial} u_n(z)\, dm(z) \\ &\leqslant 2C_6 \int_{\mathbb{C} \backslash D} \bar{\partial} v(z)\, dm(z) \end{aligned}$$

as soon as m is large enough. Here $C_6 = 8 \cdot 2^\alpha \cdot C_5^{-1} \cdot \lambda^{-2}$. The inequality (7) follows now from Lemma 3.3. □

With Lemma 3.4 and the following corollary to the classical Whitney extension theorem, see [St, p. 174], we can fulfil the promise made in Remark 3.2.

Theorem 3.5. *Let Γ be a bounded Jordan curve in the complex plane and $0 < \alpha < 1$. Then every $f \in \Lambda^\alpha(\Gamma)$ has an extension $F \in \Lambda^\alpha(\mathbb{C})$ such that $\|F\|_{\Lambda^\alpha(\mathbb{C})} < M_0 \|f\|_{\Lambda^\alpha(\Gamma)}$ where M_0 is independent of f, $F \in C^\infty(\mathbb{C} \backslash \Gamma)$, F is compactly supported and $|\operatorname{grad} F(z)| \leqslant M_1 \|f\|_{\Lambda^\alpha(\Gamma)} d(z, \Gamma)^{\alpha - 1}$.*

Indeed, if the Cauchy operator is bounded on $\Lambda^\alpha(\Gamma)$ according to Definition 3.1 and Γ is a quasicircle, then $d(z, \Gamma)^{\alpha-1}$ is locally integrable in $\mathbb{C}$ by Lemma

3.4. Thus Whitney's theorem gives for each $f \in \Lambda^\alpha(\Gamma)$ an extension F such that $\bar\partial F(\xi)(\xi - z)^{-1} \in L^1(\mathbb{C}\backslash D)$ whenever $z \in D$. In particular, the expression

$$C_\Gamma f(z) = -\frac{1}{\pi}\int_{\mathbb{C}\backslash D} \frac{\bar\partial F(\xi)}{\xi - z}\, dm(\xi), \qquad z \in D, \tag{10}$$

is well defined for every $f \in \Lambda^\alpha(\Gamma)$. And it is easily seen that this expression does not depend on the particular choice of the admissible extension F and, furthermore, that in case Γ is rectifiable or $f \in C^\infty$, (10) reduces to the standard definition of the Cauchy integral.

It remains to show that in our situation $C_\Gamma f$ has also boundary values in $\Lambda^\alpha(\Gamma)$ with

$$\|C_\Gamma f\|_{\Lambda^\alpha(\Gamma)} \leqslant M\|f\|_{\Lambda^\alpha(\Gamma)} \quad \text{for all} \quad f \in \Lambda^\alpha(\Gamma). \tag{11}$$

This leads us to the sufficiency of the condition (*b*) in Theorem 1.2; the inequality (11) will then be a consequence of Lemmas 3.4, 2.1 and Corollary 3.7.

Lemma 3.6. *Let σ be a complex measure with compact support $K \subset \mathbb{C}$. If $0 < \alpha < 1$ and $|\sigma|(B(z_0, R)) \leqslant MR^{1+\alpha}$ for all $z_0 \in \mathbb{C}$, $R > 0$, then the Cauchy transform*

$$\hat\sigma(z) = \int_{\mathbb{C}} \frac{d\sigma(\xi)}{\xi - z} \tag{12}$$

is holomorphic and α-Hölder continuous with $\|\hat\sigma\|_{\Lambda^\alpha(G)} \leqslant C(\alpha)M$ in each component G of $\mathbb{C}\backslash K$.

Lemma 3.7 is due to Dolzhenko [Do] but under a different formulation. However, the same proof gives the above result; see also [G, Theorem III.4.4] and its proof.

Corallary 3.7. *Let Γ be a bounded and porous Jordan curve and denote by D the bounded component of $\mathbb{C}\backslash\Gamma$. If $0 < \alpha < 1$ and $d(z, \Gamma)^{\alpha - 1} \in A_1$ then every $f \in \Lambda^\alpha(\Gamma)$ has an extension F such that*

$$C_\Gamma f(z) = -\frac{1}{\pi}\int_{\mathbb{C}\backslash D} \frac{\bar\partial F(\xi)}{\xi - z}\, dm(\xi), \qquad z \in D,$$

is well defined and holomorphic with

$$\begin{aligned}\|C_\Gamma f\|_{\Lambda^\alpha(\Gamma)} &= \|C_\Gamma f\|_{\Lambda^\alpha(D)} \\ &\leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}.\end{aligned}$$

PROOF. If $f \in \Lambda^\alpha(\Gamma)$ is given let F be its Whitney extension as in Theorem 3.5. According to Lemma 2.1 the measure

$$\sigma(A) = \int_A \chi_{\mathbb{C}\setminus D}(\xi)\, \bar\partial F(\xi)\, dm(\xi)$$

satisfies the growth condition of 3.6. Hence $\|C_\Gamma f\|_{\Lambda^\alpha(D)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}$. The equality $\|C_\Gamma f\|_{\Lambda^\alpha(D)} = \|C_\Gamma f\|_{\Lambda^\alpha(\Gamma)}$ follows now from [GHH]. □

Finally we collect the above steps to the

PROOF OF THEOREM 1.2. If $0 < \alpha < 1$, if Γ is a quasicircle and if the Cauchy operator C_Γ is bounded on $\Lambda^\alpha(\Gamma)$, then $d(z, \Gamma) \in A_{1+1/(1-\alpha)}$ by Lemmas 3.4 and 2.1, *i.e.*, (*a*) implies (*b*). Conversely, if $d(z, \Gamma) \in A_{1+1/(1-\alpha)}$, then $d(z, \Gamma)^{\alpha-1} \in A_1$, the Cauchy integral $C_\Gamma f(z)$, formula (10), is well defined not only for $f \in C^\infty$ but for all $f \in \Lambda^\alpha(\Gamma)$ and $z \in D$ and by Corollary 3.7 C_Γ is a bounded operator with $\|C_\Gamma f\|_{\Lambda^\alpha(\Gamma)} = \|C_\Gamma f\|_{\Lambda^\alpha(D)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}$. Thus (*b*) implies (*a*).

It follows form the work of Gehring and Väisälä [GV], either via the original proof or via Theorem 1.5, that every quasicircle is δ-regular for some $\delta < 2$. Hence, according to Lemma 2.3, $\alpha(\Gamma) = \inf\{\alpha : d(z, \Gamma)^{\alpha-1} \in A_1\} < 1$. Moreover, if w is a weight in the A_1-class, then by Jensen's inequality $w^\beta \in A_1$ whenever $0 < \beta < 1$. Thus C_Γ is bounded on $\Lambda^\alpha(\Gamma)$ for each α in the open interval $(\alpha(\Gamma), 1)$. If C_Γ were bounded in $\Lambda^\alpha(\Gamma)$ for some positive $\alpha \leqslant \alpha(\Gamma)$, then $w(z) = d(z, \Gamma)^{\alpha(\Gamma)-1} \in A_1$ and by Muckenhoupt's theorem [M, p. 214] $w^{1+\epsilon} \in A_1$ for some $\epsilon > 0$. But that is clearly impossible as $(\alpha(\Gamma) - 1)(1 + \epsilon) < \alpha(\Gamma) - 1$. The proof of Theorem 1.2 is complete. □

PROOF OF THEOREM 1.4. If Γ is δ-regular and $\delta < \alpha + 1 < 2$, then $d(z, \Gamma)^{\alpha-1} \in A_1$ and $C_\Gamma : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma)$ by Lemma 2.3 and Theorem 1.2. Conversely, if C_Γ is bounded on $\Lambda^\alpha(\Gamma)$, $d(z, \Gamma)^{\alpha-1} \in A_1$ and Γ is $(1 + \alpha)$-regular. □

Remark 3.8. The proofs described here for Theorems 1.2 and 1.4 remain valid, in addition to the quasicircles Γ, also to a number of other Jordan curves. In fact, the only property of quasicircles we used was that they were «biporous»: There is a constant λ such that whenever $x_0 \in \Gamma$ and $R < \operatorname{diam}(\Gamma)$, then both $D \cap B(x_0, R)$ and $(\mathbb{C}\setminus D) \cap B(x_0, R)$ contain a disk of radius λR. Consequently, Theorems 1.2 and 1.4 hold for all biporous Jordan curves.

The above approach gives also a proof for Salaev's theorem in a generalized form.

Corollary 3.9. *If Γ is a δ-regular Jordan curve and $\delta < 2$, then $C_\Gamma : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma)$ for each $\delta - 1 < \alpha < 1$.*

PROOF. Since δ-regular curves, $\delta < 2$, are porous $d(z,\Gamma)^{\alpha-1} \in A_1$ by Lemma 2.3. The claim follows therefore from Corollary 3.7. $\square$

There are many other ways to see that $d(z,\Gamma)^{\alpha-1} \in A_1$ for Γ regular and $0 < \alpha < 1$. For example, one can show directly that $d(z,\Gamma) \in A_p$ for $p > 2$ (M. Zinsmeister, private communication) or we can use the Hardy-Littlewood maximal function $M\mu(x)$ of the arclength measure μ on Γ,

$$M\mu(x) = \sup_{x\in B} \frac{1}{|B|}\int_B d\mu = \sup_{x\in B} \frac{l(\Gamma\cap B)}{|B|};$$

here the supremum is taken over all disks B containing x. Indeed, by regularity $C_1 d(z,\Gamma)^{-1} \leqslant M\mu(z) \leqslant C_2 d(z,\Gamma)^{-1}$ and according to a theorem of Coifman and Rochberg [CR] $(M\mu)^\epsilon$ belongs to the class A_1 whenever $0 < \epsilon < 1$.

Similar arguments yield the correct estimates of the boundedness of the Cauchy integral on many other curves, too. For instance, if Γ is the standard Koch curve or the snowflake curve, then by [H] Γ carries a natural measure μ such that $C_1 d(z,\Gamma)^{\beta-2} \leqslant M\mu(z) \leqslant C_2 d(z,\Gamma)^{\beta-2}$ where $\beta = \log 4/\log 3$ is the Hausdorff dimension of Γ. Hence C_Γ is bounded on $\Lambda^\alpha(\Gamma)$ if $\log(4/3)/\log 3 < \alpha < 1$. Conversely, $\beta \leqslant \delta(\Gamma) = 1 + \alpha(\Gamma)$ and thus C_Γ is not bounded on $\Lambda^\alpha(\Gamma)$ if $0 < \alpha \leqslant \log(4/3)/\log 3$. We also note that by combining these estimates with the proof of Lemma 2.3 one can show that the snowflake Γ is $(\log 4)/(\log 3)$-regular. More generally, if Γ is any (porous) Jordan curve which supports a positive measure μ such that

$$C_1 R^d \leqslant \mu(B(z_0,R)) \leqslant C_2 R^d$$

whenever $z_0 \in \Gamma$ and $R < \operatorname{diam}(\Gamma)$, then

$$d = \dim_H(\Gamma) = \beta(\Gamma) = \delta(\Gamma) = 1 + \alpha(\Gamma)$$

and Γ is d-regular. In particular, *cf.* [MV, 4.19], this holds for all selfsimilar fractal curves satisfying the open set condition of [H, p. 735].

4. The Hausdorff Dimension

In this last section we prove Theorem 1.5, the relation between $\alpha(K)$ and the upper bound for the Hausdorff dimension $d(K) = \sup\{\dim_H(\Gamma)\colon \Gamma \text{ is } K\text{-quasicircle}\}$. According to Theorem 1.4 it will be enough to show that $\delta(K) = d(K)$. For this some lemmas are needed.

Lemma 4.1. *For each* $K \geqslant 1$, $d(K) = \lim_{\epsilon\to 0^+} d(K+\epsilon)$.

PROOF. All bounded $(K+\epsilon)$-quasicircles Γ are of the form $\Gamma = \lambda\varphi(\{|z|=1\}) + \mu$ where $\lambda, \mu \in \mathbb{C}$ and φ is $(K+\epsilon)$-quasiconformal on $\bar{\mathbb{C}}$ with $\varphi(0)=0$, $\varphi(1)=1$ and $\varphi(\infty)=\infty$. Moreover φ admits the factorization $\varphi = \varphi_1 \circ \varphi_2$ where φ_1, φ_2 fix $0, 1, \infty$ and have dilatations $K(\varphi_1) = (K+\epsilon)/K$, $K(\varphi_2) = K$, *cf.* [L, p. 29].

According to Mori's classical distortion theorem φ_1 is $1/K(\varphi_1)$-Hölder continuous on compact subsets of $\mathbb{C}$. Hence

$$\dim_H(\Gamma) \leqslant K(\varphi_1)\dim_H(\varphi_2\{|z|=1\}) \leqslant (1+\epsilon/K)d(K).$$

Since Γ was arbitrary, $d(K+\epsilon) \leqslant (1+\epsilon/K)d(K)$. □

The next lemma is a standard deformation argument. No proof, however, seems to appear in the literature and hence we sketch the details.

Lemma 4.2. *Let φ be a K-quasiconformal mapping on $\bar{\mathbb{C}}$ fixing $0, 1$ and ∞. Then for each $\epsilon > 0$ there is a number $\rho = \rho(K, \epsilon) \in (0, 1/2)$ and a $(K+\epsilon)$-quasiconformal mapping ϕ on $\mathbb{C}$ such that*

(*a*) $\phi(z) = \varphi(z)$ *if* $1/2 \leqslant |z|$
(*b*) $\phi(z) = z$ *if* $|z| \leqslant \rho$.

PROOF. Assume first that φ is conformal in the unit disk $B(1)$. If $\lambda = \varphi'(0)$, then $1/M \leqslant |\lambda| \leqslant M$ and $|\varphi(z) - \lambda z| \leqslant M\rho^2$, $|\varphi'(z) - \lambda| \leqslant M\rho$ for $|z| \leqslant \rho < 1/2$ with a constant M depending only on K. Given a C^∞-function v such that $v(z) = 0$ for $|z| \geqslant 2$ and $v(z) = 1$ for $|z| \leqslant 1$, set

$$g(z) = \varphi(z) + (\lambda z - \varphi(z))v(z/\rho).$$

Then g is quasiconformal on $\mathbb{C}$ and $K(g|_{B(1/2)}) \leqslant 1 + C\rho$ for ρ small. Finally, we replace g by $g(z)(|g(z)|/\lambda\rho)^{\pm\epsilon}$ in an annulus $\rho_1 \leqslant |z| \leqslant \rho$ and obtain a mapping $\tilde{g}$ with the properties: $\tilde{g}(z) = \varphi(z)$ if $|z| > 1/2$, $\tilde{g}(z) = z$ if $|z| < \rho_1$ and $K(\tilde{g}|_{B(1/2)}) \leqslant 1 + \epsilon$, $\rho_1 = \rho_1(K, \epsilon)$.

The general case follows from the above. Indeed, we may factorize $\varphi = k^{-1} \circ h$, where h is conformal in $B(\rho_2)$ and k is conformal outside $\varphi B(\rho_2)$ and deform h and k so that $\varphi(z) = h(z)$ for $|z| > 1/2$ and $h(z) = z$ for $|z| < \rho$. □

Lemma 4.3. *If $d(K) < \delta$, there is a constant*

$$C_0 = C_0(K, \delta)$$

such that

$$h^\delta(\varphi[0,1]) \leqslant C_0$$

for each K-quasiconformal mapping φ on $\bar{\mathbb{C}}$ fixing $0, 1$ and ∞.

PROOF. If the claim is not true, we can find a sequence $\{\varphi_n\}_1^\infty$ of K-quasiconformal mappings on $\bar{\mathbb{C}}$, each fixing 0, 1 and ∞, such that

$$h^\delta(\varphi_n[0,1]) > n, \qquad n \in \mathbb{N}. \tag{13}$$

Using Lemma 4.2 we shall then construct for every $\epsilon > 0$ a new mapping Φ_ϵ on $\bar{\mathbb{C}}$ with $K(\Phi_\epsilon) \leqslant K + \epsilon$ and $\dim_H(\Phi_\epsilon[0,1]) > \delta$. By the Möbius invariance of quasiconformal mappings

$$d(K) = \sup\{\dim_H(\varphi[0,1]) \colon \varphi \text{ is } K\text{-quasiconformal on } \bar{\mathbb{C}},\ \varphi(\infty) = \infty\}$$

and hence we obtain $d(K+\epsilon) \geqslant \delta > d(K)$ for all $\epsilon > 0$. This, however, contradicts Lemma 4.1. Therefore to prove Lemma 4.3 it is enough to find the mappings Φ_ϵ.

Now, assuming the existence of the sequence (13), choose for each n a radius $r_n < 1$ such that $M^\delta(\varphi_n[0,1]; r_n) \geqslant n$. Then choose a maximal set of points $z_i = z_i^n \in [0,1]$, $1 \leqslant i \leqslant k_n$ such that

$$|\varphi_n(z_i) - \varphi_n(z_j)| \geqslant r_n, \qquad i \neq j. \tag{14}$$

Clearly, the union of the balls $B(\varphi_n(z_i), r_n)$ covers $\varphi_n[0,1]$ and thus

$$k_n(r_n)^\alpha \geqslant M^\delta(\varphi_n[0,1]; r_n) \geqslant n. \tag{15}$$

By (14) the disks $B(\varphi_n(z_i), r_n/2)$ are disjoint. Let $B(z_i, \lambda_i)$ be the largest disk, with center z_i, contained in $\varphi_n^{-1}B(\varphi_n(z_i), r_n/2)$.

Next, we deform φ_n and create «holes» at the disks $B(z_i, \lambda_i)$. We shall then fill the holes by similarity-copies of φ_n and as a result obtain a selfsimilar set E, $\dim_H(E) > \delta$, contained in a $(K+\epsilon)$-quasicircle.

To be more precise, we fix $\epsilon > 0$ and apply Lemma 4.2 to find a number $\rho = \rho(K,\epsilon) \in (0, 1/2)$ and for each $n \in \mathbb{N}$ a $(K+\epsilon)$-quasiconformal mapping ϕ_n such that the following conditions hold.

(16a) $\phi_n(z) = \varphi_n(z)$ if $|z| < 2$ and $z \notin \bigcup_i B(z_i, \lambda_i)$

(16b) $\phi_n(z) = z$ if $1/\rho < |z|$.

(16c) In $B_i \equiv B(z_i, \rho\lambda_i)$ $\phi_n = \tau_i$, a similarity with $\tau_i(z_i) = \varphi_n(z_i)$ and $\tau_i(z_i + \lambda_i) = \varphi_n(z_i + \lambda_i)$.

Note that here τ_i, λ_i and B_i depend also on n. From the distortion properties of quasiconformal mappings we deduce

$$\frac{r_n}{C_1\lambda_i} \leqslant |\tau_i'| \leqslant \frac{r_n}{2\lambda_i} \tag{17}$$

where $C_1 = C_1(K)$.

If B_i is as in (16c) let u_i be the similarity $u_i(z) = z_i \pm \lambda_i \rho^2 z$ with $u_i B(0, 1/\rho) = B_i$ and $u_i[0, 1] \subset [0, 1]$. Then the holes B_i can be filled in by defining new quasiconformal mappings $\phi_n^{(k)}$ as follows: set $\phi_n^{(1)} = \phi_n$,

$$\phi_n^{(k)}(z) = \phi_n \circ u_i \circ \phi_n^{(k-1)} \circ u_i^{-1}(z), \quad \text{if} \quad z \in B_i, \tag{18}$$

and $\phi_n^{(k)}(z) = \phi_n^{(k-1)}(z)$ otherwise. It is easily seen that each $\phi_n^{(k)}$ is $(K + \epsilon)$-quasiconformal on $\bar{\mathbb{C}}$. In fact, ϕ_n is a similarity on B_i and $u_i \circ \phi_n^{(k-1)} \circ u_i^{-1}$ the identity outside B_i. Consequently, as $k \to \infty$ the $\phi_n^{(k)}$ converge uniformly on $\bar{\mathbb{C}}$ to a $(K + \epsilon)$-quasiconformal mapping Φ_n.

Finally, from (16c) and (18) we have

$$\Phi_n \circ u_i(z) = \tau_i \circ u_i \circ \Phi_n(z), \qquad |z| < 1/\rho. \tag{19}$$

The similarities u_i are contractions and by Hutchinson's theorem [H, 3.2] there is a unique compact set E_u such that

$$\Sigma_u(E_u) = E_u, \qquad \Sigma_u(A) = \bigcup_{i=1}^{k_n} u_i(A).$$

Since these similarities map the unit interval into itself, $E_u \subset [0, 1]$. On the other hand, the similarities $\tau_i \circ u_i$ are also contractions, $r_n \rho^2 / C_1 \leqslant |(\tau_i \circ u_i)'| \leqslant r_n \rho^2 / 2$ by (17). Hence we have a unique compact E for which

$$\Sigma_\tau(E) = E, \qquad \Sigma_\tau(A) = \bigcup_{i=1}^{k_n} \tau_i \circ u_i(A)$$

and it is easily seen from (19) that $\Phi_n(E_u) = E$.

Lastly, we have to estimate the Hausdorff dimension of E. Because the disks B_i are disjoint, E_u and hence E satisfy the open set condition of Hutchinson [H, 5.2]. According to [H, 5.3], $\dim_H(E)$ is then the unique number s for which

$$\sum_{i=1}^{k_n} |(\tau_i \circ u_i)'|^s = 1.$$

But $|(\tau_i \circ u_i)'| \geqslant r_n \rho^2 / C_1$ and when n is large, (15) yields $k_n (r_n \rho^2 / C_1)^\delta \geqslant n \rho^{2\delta} / C_1^\delta > 1$. Therefore

$$\dim_H(\Phi_n[0, 1]) \geqslant \dim_H(E) > \delta, \qquad n \geqslant n_0. \quad \Box$$

PROOF OF THEOREM 1.5. We must show that $d(K) = \delta(K)$. Since $d(K) < \delta(K)$ trivially, it is enough to prove that for each $\delta > d(K)$ and each bounded K-quasicircle Γ there is a constant $C < \infty$ with

$$h^\delta(B(z_0, R) \cap \Gamma) \leqslant C R^\delta, \qquad z_0 \in \mathbb{C}, \qquad R > 0.$$

By Lemma 4.3 and the Möbius invariance of quasiconformal mappings, if $\delta > d(K)$ then

$$h^{\delta}(\Gamma) \leqslant C(K, \delta)\,\mathrm{diam}\,(\Gamma)^{\delta}$$

for all bounded K-quasicircles Γ. If z_0, R are given, take a point $w_0 \in B(z_0, R)$ such that $d(w_0, \Gamma) \geqslant C_0(K)R$ and let $\phi(z) = (z - w_0)^{-1}$. As $|\phi(z) - \phi(z')| \geqslant |z - z'|(2R)^{-2}$ whenever $z, z' \in B(z_0, R) \cap \Gamma$,

$$\begin{aligned} h^{\delta}(B(z_0, R) \cap \Gamma) &\leqslant (2R)^{2\delta} h^{\delta}(\phi\Gamma) \\ &\leqslant (2R)^{2\delta} C\,\mathrm{diam}\,(\phi\Gamma)^{\delta}. \end{aligned}$$

However, $\mathrm{diam}\,(\phi\Gamma)^{\delta} \leqslant (2/C_0R)^{\delta} = C_1 R^{-\delta}$ and hence the claim is proved. The equalities $1 + \alpha(K) = d(K) = \beta(K) = \delta(K)$ follow now from Theorem 1.4. □

References

[BP] Becker, J. and Pommerenke, Ch. On the Hausdorff Dimension of Quasicircles. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **12**(1988).

[CF] Coifman, R. R. and Fefferman, C. C. Weighted Norm Inequalities for Maximal Functions and Singular Integrals. *Studia Math.*, **51**(1974), 241-250.

[CR] Coifman, R. R. and Rochberg, R. Another Characterization of BMO. *Proc. Amer. Math. Soc.*, **79**(1980), 249-254.

[Da] David, G. Opérateurs intégraux singuliers sur certaines courbes du plan complexe. *Ann. Sc. E.N.S.*, **17**(1984), 157-189.

[Do] Dolzhenko, E. P. The Removability of Singularities of Analytic Functions. *Uspeki Mat. Nauk.*, **18**(1963), (112), 135-142 (Russian).

[Dy] Dynkin, E. M. On the Smoothness of Cauchy Type Integrals. *Sem. Leningrad. Otdel. Mat. Inst. Steklov.*, **92**(1979), 115-133.

[G] Garnett, J. Analytic Capacity and Measure. Springer-Verlag, 1972.

[GHH] Gehring, F. W., Hayman, W. K. and Hinkkanen, A. Analytic Functions Satisfying Hölder Conditions on the Boundary. *J. Approx. Theory*, **35**(1982), 243-249.

[GV] Gehring, F. W. and Väisälä, J. Hausdorff Dimension and Quasiconformal Mappings. *J. London Math. Soc.*, **6**(1973), 504-512.

[H] Hutchinson, J. E. Fractals and Selfsimilarity. *Indiana Univ. Math. J.*, **30** (1981), 713-747.

[L] Lehto, O. Univalent Functions and Teichmüller Spaces. Springer-Verlag, 1987.

[MV] Martio, O. and Vuorinen, M. Whitney Cubes, p-Capacity and Minkowski Content. *Expo. Math.*, **5**(1987), 17-40.

[M] Muckenhoupt, B. Weighted Norm Inequalities for the Hardy Maximal Function. *Trans. Amer. Math. Soc.*, **165**(1972), 207-226.

[Sa] Salaev, V. V. Direct and Inverse Estimates for a Singular Cauchy Integral Along a Closed Curve. *Math. Zametki*, **19**(1976), 365-380 (Russian).

[St] Stein, E. M. Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, 1970.

[V] Väisälä, J. Porous Sets and Quasisymmetric Maps. *Trans. Amer. Math. Soc.*, **299**(1987), 525-533.

[Z] Zinsmeister, M. Problèmes de Dirichlet, Neumann, Calderón dans les quasi-disques pour les classes hölderiennes. *Revista Mat. Iberoamericana*, **2**(1986), 319-332.

Kari Astala
Department of Mathematics
University of Helsinki
SF-00100 Helsinki 10
FINLAND