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An Extremal Property of Entire Functions with Positive Zeros

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1. Introduction

Let $f(z)$ be a Weierstrass product of finite genus q with zeros $z_\nu \neq 0$ so that

$$f(z) = \prod_{\nu=1}^{\infty} E_q(z/z_\nu) \tag{1.1}$$

where

$$E_q(u) = \begin{cases} 1 - u & q = 0 \\ (1-u)\exp(u + u^2/2 + \cdots + u^q/q) & q > 0 \end{cases} \tag{1.2}$$

is the usual Weierstrass primary factor and

$$\sum_{\nu=1}^{\infty} |z_\nu|^{-q-1} < \infty.$$

Put

$$\hat{f}(z) = \prod_{\nu=1}^{\infty} E_q(z/|z_\nu|), \tag{1.3}$$

and define

$$u(re^{i\varphi}, f) = \sup_{\theta} \{\log|f(re^{i(\theta+\varphi)})| + \log|f(re^{i(\theta-\varphi)})|\}. \tag{1.4}$$

Let $n(r, 0)$ and $N(r, 0)$ be the counting functions for the zeros of f [H; p. 6].

Theorem 1. *For*

$$\pi/2(q+1) \leqslant \varphi \leqslant \pi/2q \tag{1.5}$$

we have

$$u(re^{i\varphi}, f) \leqslant u(re^{i\varphi}, \hat{f}) = 2\log|\hat{f}(re^{i\varphi})|. \tag{1.6}$$

(*When* $q = 0$ *we interpret* (1.5) *as* $\pi/2 \leqslant \varphi \leqslant \pi$.)

Further, the convolution inequalities

$$u(re^{i\varphi}, f) \leqslant \int_0^\infty n(t, 0) J(r/t, \varphi) t^{-1}\, dt, \tag{1.7}$$

$$u(re^{i\varphi}, f) \leqslant \int_0^\infty N(t, 0) K(r/t, \varphi) t^{-1}\, dt \tag{1.8}$$

both hold for φ *in the range* (1.5), *where*

$$J(s, \varphi) = \frac{2s^{q+1}(s \cos q\varphi - \cos(q+1)\varphi)}{1 + s^2 - 2s\cos\varphi} \tag{1.9}$$

and

$$K(s, \varphi) = s\, \partial J(s, \varphi)/\partial s \tag{1.10}$$

satisfy

$$J(s, \varphi) \geqslant 0, \qquad K(s, \varphi) \geqslant 0 \qquad (0 < s < \infty). \tag{1.11}$$

Recall that, for a nondecreasing function $S(r)$ $(0 < r < \infty)$, a sequence $\{r_m\}$ tending to ∞ is a sequence of Pólya peaks of order λ for S if for every $\epsilon > 0$

$$S(u) \leqslant \left(\frac{u}{r_m}\right)^{\lambda - \epsilon} S(r_m) \qquad (1 < u \leqslant r_m)$$

$$S(u) \leqslant \left(\frac{u}{r_m}\right)^{\lambda + \epsilon} S(r_m) \qquad (r_m < u)$$

whenever $m \geqslant m_0(\epsilon)$, from which it follows that $S(r_m) r_m^{-\lambda+\delta} \to \infty$ as $m \to \infty$ for any $\delta > 0$ (cf. [F; p. 136]).

If g is an entire function of nonintegral order λ, then $g(z) = z^k e^{P(z)} f(z)$ where f has the representation (1.1) with $q = [\lambda]$, P is a polynomial of degree at most q, and, by known existence theorems ([H; p. 103], [DS]), $N(r, 0)$ and $n(r, 0)$ each have Pólya peaks of order λ.

The inequalities (1.7) and (1.8) along with the positivity (1.11) of J and K allow for very precise estimates of $u(re^{i\varphi}, f)$ near the Pólya peaks of the counting functions. These estimates will be carried out in Theorem 2.

There are numerous known results on the distribution of values of entire and meromorphic functions of orders $\lambda < 1$ for which the extremal functions have positive zeros, and whose counterparts for $\lambda > 1$ are unknown (cf. [H; pp. 109-119] and [P]). This is due to the particularly simple behavior of $|E_0(re^{i\theta})|$, which for every $r > 0$ is increasing on $(0, \pi)$ and then decreases symmetrically on $(\pi, 2\pi)$. When $q \geqslant 1$, the intervals on which $|E_q(re^{i\theta})|$ increase and decrease depend upon r.

Our Theorem 1 presents a rare instance when an inequality on primary factors is sharp for a range of θ, independent of r, and hence leads directly to extremal properties of $\hat{f}$.

Theorem 2. *Let g have nonintegral order* λ *and* $\{r_m\}$ *be a sequence of Pólya peaks for N of order* λ. *Then*

$$\limsup_{m\to\infty} \frac{u(tr_m e^{i\varphi}, g)}{N(r_m, 0)} \leqslant \frac{2\pi\lambda t^{\lambda}}{\sin \pi\lambda} \cos((\pi - \varphi)\lambda) \tag{1.12}$$

for φ *satisfying* (1.5), *uniformly for t in compact subsets of* $0 < t < \infty$.

Similarly,

$$\limsup_{m\to\infty} \frac{u(tR_m e^{i\varphi}, g)}{n(R_m, 0)} \leqslant \frac{2\pi t^{\lambda}}{\sin \pi\lambda} \cos((\pi - \varphi)\lambda) \tag{1.13}$$

for $\{R_m\}$ *a sequence of Pólya peaks of order* λ *for n, with* φ *in the range* (1.5) *and uniformly for t in compact subsets of* $0 < t < \infty$.

Theorem 2 is sharp and extends a theorem of Fuchs [F] who proved (1.12) for $t = 1$, $\lambda > 1/2$, and φ restricted to the range $\pi/2(q + 1) \leqslant \varphi \leqslant \pi/2\lambda$.

Inequality (1.12) still holds for entire g of finite lower order μ, provided λ is replaced in (1.12) by any finite nonintegral $\rho \in [\mu, \lambda]$ and the r_m are chosen to be *strong peaks* of $N(r, 0)$ in the sense of [MS]. The corresponding remark applies also to (1.13). For proof, combine the arguments used here for Theorem 2 with those of [MS].

2. A Preliminary Lemma

Put $k(z, \varphi) = \log|E_q(ze^{i\varphi})E_q(ze^{-i\varphi})|$. For (1.6)-(1.8) we require

Lemma 1. *For* φ *in the range* (1.5) *and* $|z| = r$ *we have*

$$k(z, \varphi) \leqslant k(r, \varphi) \qquad (0 < r < \infty). \tag{2.1}$$

When $r < 1$, (2.1) is equivalent to

$$-\sum_{k=q+1}^{\infty} (r^k/k)(1 - \cos k\theta)\cos k\varphi \geqslant 0,$$

but a direct proof of this seems difficult. Exponentiating (2.1) leads however to an easy proof.

PROOF OF LEMMA 1. Put

$$G(z) = E_q(ze^{-i\varphi})E_q(ze^{i\varphi}) = \sum_{n=0}^{\infty} g_n z^n. \tag{2.2}$$

To prove (2.1) it then suffices to show that

$$g_n \geqslant 0, \qquad n = 0, 1, \ldots \tag{2.3}$$

for φ in the range (1.5).

Let

$$E(z) = E_q(z) = (1 - z)e^{R(z)}, \qquad R(z) = \sum_{j=1}^{q} \frac{z^j}{j} \quad (= 0 \text{ if } q = 0)$$

so that $E'(z) = -z^q e^{R(z)}$. Thus,

$$\begin{aligned} G'(z) &= e^{i\varphi}E'(ze^{i\varphi})E(ze^{-i\varphi}) + e^{-i\varphi}E(ze^{i\varphi})E'(ze^{-i\varphi}) \\ &= -e^{i\varphi}(ze^{i\varphi})^q e^{R(ze^{i\varphi})}E(ze^{-i\varphi}) - e^{-i\varphi}(ze^{-i\varphi})^q e^{R(ze^{-i\varphi})}E(ze^{i\varphi}) \\ &= -z^q e^{R(ze^{i\varphi})}e^{R(ze^{-i\varphi})}(e^{i(q+1)\varphi}(1 - ze^{-i\varphi}) + e^{-i(q+1)\varphi}(1 - ze^{i\varphi})) \\ &= e^{R(ze^{i\varphi}) + R(ze^{-i\varphi})}(\alpha z^q + \beta z^{q+1}) \end{aligned}$$

where $\alpha = -2\cos(q + 1)\varphi$ and $\beta = 2\cos q\varphi$. Since

$$R(ze^{i\varphi}) + R(ze^{-i\varphi}) = 2\sum_{j=1}^{q} \frac{\cos j\varphi}{j} z^j,$$

we have that $R(ze^{i\varphi}) + R(ze^{-i\varphi})$ and thus $\exp[R(ze^{i\varphi}) + R(ze^{-i\varphi})]$ have nonnegative Taylor coefficients for φ in the range (1.5). Now, $\alpha \geqslant 0$ and $\beta \geqslant 0$ for φ satisfying (1.5) so that G' has nonnegative Taylor coefficients. Finally since $G(0) = 1$ it follows that (2.3) and consequently (2.1) hold for the range (1.5). □

3. Proof of Theorem 1

By Lemma 1 and (1.1) it follows that for any z with $|z| = r$ and φ satisfying (1.5) we have

$$\begin{aligned}
(3.1)\qquad \log|f(ze^{i\varphi})f(ze^{-i\varphi})| &= \sum_{\nu=1}^{\infty} k(z/z_\nu, \varphi) \\
&\leqslant \sum_{\nu=1}^{\infty} k(r/|z_\nu|, \varphi) \\
&= \int_0^\infty k(r/t, \varphi)\, dn(t, 0) \\
&= 2\log|\hat{f}(re^{i\varphi})| \\
&= \int_0^\infty n(t, 0) k_1(r/t, \varphi) r t^{-2}\, dt
\end{aligned}$$

where

$$(3.2)\qquad k_1(s, \varphi) = \frac{\partial k(s, \varphi)}{\partial s}.$$

Thus (1.6) holds and if we put

$$(3.3)\qquad J(s, \varphi) = s k_1(s, \varphi)$$

a direct computation with (1.2) shows that J is also given by (1.9) and that $J(s, \varphi) \geqslant 0$ for φ satisfying (1.5). From (1.4) and (3.1) we then obtain (1.7).

Continuing on from (3.1) with another integration by parts and K as in (1.10), we get

$$\log|f(ze^{i\varphi})f(ze^{-i\varphi})| \leqslant \int_0^\infty N(t, 0) K(r/t, \varphi) t^{-1}\, dt$$

which implies (1.8).

It remains only to verify that $K(s, \varphi) \geqslant 0$. In fact with G again as in (2.2), then $k = \log G$ so that from (3.2), (3.3), and (1.10) we have

$$\begin{aligned}
K(s, \varphi) &= s\frac{\partial}{\partial s}\left(\frac{s\partial \log G}{\partial s}\right) \\
&= G(s)^{-2}[sG'(s)G(s) + s^2G''(s)G(s) - (sG'(s))^2] \\
&= G(s)^{-2} \sum_{n=0}^{\infty} b_n s^n
\end{aligned}$$

where $G(s)^{-2} > 0$ and $\sum b_n s^n$ has infinite radius of convergence. Thus, it suffices to show that $b_n \geqslant 0$, $n = 0, 1, \ldots$.

With the notation of (2.2) we have

$$\begin{aligned} b_n &= \sum_{k=0}^{n} k g_k g_{n-k} + \sum_{k=0}^{n} k(k-1) g_k g_{n-k} - \sum_{k=0}^{n} k g_k (n-k) g_{n-k} \\ &= \sum_{k=0}^{n} (2k^2 - kn) g_k g_{n-k} \\ &= \sum_{j=0}^{n} (2(n-j)^2 - n(n-j)) g_{n-j} g_j. \end{aligned}$$

Thus, for φ satisfying (1.5) it follows from (2.3) that

$$\begin{aligned} 2b_n &= \sum_{k=0}^{n} (2k^2 - kn + 2(n-k)^2 - n(n-k)) g_k g_{n-k} \\ &= \sum_{k=0}^{n} (n-2k)^2 g_k g_{n-k} \geqslant 0. \end{aligned}$$

4. Proof of Theorem 2

Since $\{r_m\}$ is a sequence of Pólya peaks of order λ of N we have from (1.8) that

$$u(tr_m e^{i\varphi}, g) \leqslant N(r_m, 0)\left[\int_0^\infty K(t/\sigma, \varphi)\sigma^{\lambda-1}\,d\sigma + \eta_m(t)\right] \tag{4.1}$$

where

$$\begin{aligned} \eta_m(t) = \frac{1}{N(r_m, 0)}\Bigg[&\int_0^1 N(\sigma, 0) K(tr_m/\sigma, \varphi)\,d\sigma \\ &+ \int_{r_m^{-1}}^1 K(t/\sigma, \varphi)(\sigma^{\lambda-1-\epsilon} - \sigma^{\lambda-1})\,d\sigma \\ &+ \int_1^\infty K(t/\sigma, \varphi)(\sigma^{\lambda-1+\epsilon} - \sigma^{\lambda-1})\,d\sigma \\ &+ k\log(tr_m) + C(tr_m)^q\Bigg]. \end{aligned} \tag{4.2}$$

Using (1.9) and (1.10) we find that for $\varphi \neq 0$,

$$\begin{aligned} |K(s, \varphi)| &\leqslant C_1(q, \varphi)s^{q+1} \qquad s < 1 \\ |K(s, \varphi)| &\leqslant C_2(q, \varphi)s^{q} \qquad s \geqslant 1. \end{aligned}$$

These inequalities along with the fact that $N(r_m, 0)r_m^{-\lambda+\delta} \to \infty (\delta > 0)$ as $m \to \infty$ imply that for φ in the range (1.5) and t in a compact subset of $(0, \infty)$ we may take η_m arbitrarily small, for sufficiently small ϵ and large m in (4.2).

The integral in (4.1) can now be explicitly evaluated as follows, using the notations of (3.1)-(3.3).

$$\begin{aligned}\int_0^\infty K(t/\sigma, \varphi)\sigma^{\lambda-1}\,d\sigma &= t^\lambda \int_0^\infty K(r, \varphi)r^{-\lambda-1}\,dr\\ &= t^\lambda \int_0^\infty \frac{\partial J(r,\varphi)}{\partial r} r^{-\lambda}\,dr = \lambda t^\lambda \int_0^\infty J(r,\varphi) r^{-\lambda-1}\,dr\\ &= \lambda t^\lambda \int_0^\infty k_1(r,\varphi) r^{-\lambda}\,dr = \lambda^2 t^\lambda \int_0^\infty k(r,\varphi) r^{-\lambda-1}\,dr\\ &= \lambda^2 t^\lambda \int_0^\infty (\log|E_q(re^{i\varphi})| + \log|E_q(re^{-i\varphi})|) r^{-\lambda-1}\,dr\\ &= 2\lambda^2 t^\lambda \int_0^\infty \log|E_q(-re^{i(\pi-\varphi)})| r^{-\lambda-1}\,dr\\ &= \frac{2\lambda t^\lambda \pi \cos((\pi-\varphi)\lambda)}{\sin \pi\lambda}.\end{aligned}$$

The computation of this last integral is done in [HS; p. 222]. This completes the proof of (1.12). The proof of (1.13) is similar and is thus omitted.

5. Estimates of $\log M(r)/N(r, 0)$

Let g be as in Theorem 2. Then $u(re^{i\varphi}, g)$ is easily seen to be subharmonic in $\mathbb{C}$. We may therefore form a *local indicator* $h(\theta)$ as in [E] where the details are carried out for the case $u = \log|g|$, but they go through without essential change for $u = u(re^{i\varphi}, g)$. The functions

$$V(r) = N(r_m, 0)(r/r_m)^\lambda \qquad (\sigma^{-1} r_m < r < \sigma r_m, \quad \sigma > 1)$$

serve as valid comparison functions in the sense of [E], since for each fixed $t > 0$

$$\limsup_{m\to\infty} \frac{\log M(tr_m, g)}{t^\lambda N(r_m, 0)} \leqslant B = B(\lambda) < \infty. \tag{5.1}$$

To verify (5.1) we observe that the argument of [H; *p.* 102] proves

$$\begin{aligned}\log M(r, g) &\leqslant c_1(q)\left(q \int_0^r (r/s)^q N(s, 0)\,ds/s + (q+1) \int_r^\infty (r/s)^{q+1} N(s, 0)\,ds/s \right)\\ &\quad + O(r^q) + O(\log r)\\ &\leqslant 2(q+1)c_1(q) \int_0^\infty \frac{(r/s)^{q+1}}{1 + r/s} N(s, 0)\frac{ds}{s} + O(r^q) + O(\log r)\end{aligned}$$

where $c_1(q) = 2(q+1)\{2 + \log(q+1)\}$. Since the r_m are Pólya peaks, we have for fixed $t > 0$ and $0 < \epsilon < \min(\lambda - q, q + 1 - \lambda)$ that

$$\log M(tr_m, g) \leqslant c_2(q) \int_0^\infty \frac{(tr_m/s)^{q+1}}{1 + tr_m/s} N(s,0)\frac{ds}{s} + O((tr_m)^q) + O(\log r_m)$$

$$\leqslant c_2(q)N(r_m,0)\left\{t^{\lambda-\epsilon}\int_t^\infty \frac{u^{q-\lambda+\epsilon}}{1+u}du + t^{\lambda+\epsilon}\int_0^t \frac{u^{q-\lambda-\epsilon}}{1+u}du\right\}$$

$$+ O((tr_m)^q) + O(\log r_m)$$

where $c_2(q) = 2(q+1)c_1(q)$. Letting $m \to \infty$ and then $\epsilon \to 0$ yields

$$\limsup_{m\to\infty} \frac{\log M(tr_m, g)}{t^\lambda N(r_m, 0)} \leqslant c_2(q)\int_0^\infty \frac{u^{q-\lambda}\,du}{1+u} = c_2(q)\frac{\pi}{|\sin \pi\lambda|}$$

as claimed in (5.1).

We may then define

$$h_\sigma^{(m)}(\varphi) = \sup_{\sigma - 1 \le t \le \sigma} \frac{u(tr_m e^{i\varphi}, g)}{t^\lambda N(r_m, 0)},$$

$$h_\sigma(\varphi) = \limsup_{m\to\infty} h_\sigma^{(m)}(\varphi),$$

and finally

$$h(\varphi) = \lim_{\sigma\to\infty} h_\sigma(\varphi).$$

Then,

(i) $h(\varphi)$ is subtrigonometric (see [E]);
(ii) for $\beta = 0$ or $\beta = \pi$,

(5.2) $$h(\beta) \geqslant \limsup_{m\to\infty} \frac{2\log M(r_m, g)}{N(r_m, 0)}$$

and for $|\beta - \varphi| < \pi/\lambda$,

(5.3) $$h(\beta)\cos((\beta - \varphi)\lambda) \leqslant h(\varphi);$$

(iii) for φ in the range (1.5),

(5.4) $$h(\varphi) \leqslant \frac{2\pi\lambda}{\sin \pi\lambda}\cos((\pi - \varphi)\lambda).$$

Here (5.4 follows from Theorem 2, (5.3) from [L, p. 56], and (5.2) is immediate from the definitions of u and h.

Following Pólya [P], we seek estimates for

$$C(g) = \liminf_{r\to\infty} \frac{\log M(r, g)}{N(r, 0)}.$$

When $\lambda < 1$ we can take $\beta = \pi$ and $\varphi \in [\pi/2, \pi)$ in (5.2)-(5.4) to deduce $C(g) \leqslant \pi\lambda/\sin \pi\lambda$, a classical result due to Valiron [V] and Pólya [P].

For $\lambda > 1$, good bounds on $C(g)$ are not yet known. To see what (5.2)-(5.4) can tell us, we take $\beta = 0$ and $\varphi = \pi/2(q+1)$ in (5.3) to deduce

$$C(g) \leqslant \frac{\pi\lambda}{|\sin \pi\lambda|} A(\lambda)$$

where the estimate

$$A(\lambda) \leqslant (-1)^q \frac{\cos((\pi-\varphi)\lambda)}{\cos \varphi\lambda} = \frac{\sin((2q+1)\gamma)}{\sin \gamma} \qquad \left(\gamma = \frac{\pi}{2} - \varphi\lambda\right) \tag{5.5}$$

is far from sharp for large λ.

When g has order $1 < \lambda < 2$ we have two explicit estimates:

$$\begin{aligned} A(\lambda) &\leqslant 1 + 2|\cos(\pi\lambda/2)|, \\ A(\lambda) &\leqslant 2|\cos(2\pi\lambda/3)|. \end{aligned} \tag{5.6}$$

The first is equivalent to (5.5) when $q = 1$; the second uses

$$\log M(r,f) \leqslant u(re^{i\pi/3}, \hat{f}) \qquad (0 < r < \infty), \tag{5.7}$$

with f, $\hat{f}$ as in (1.1), (1.3), together with an application of Theorem 2.

The inequality (5.7) follows in case $f = E_1$ from the calculation

$$\max_\theta \log|E_1(re^{i\theta})| = \begin{cases} r^2/2 & (0 < r \leqslant 2) \\ r + \log(r-1) & (2 \leqslant r) \end{cases}$$

together with

$$\begin{aligned} u(re^{i\pi/3}, E_1) &= 2\log|E_1(re^{i\pi/3})| \\ &= r + \log(r^2 - r + 1) \quad (0 < r < \infty). \end{aligned}$$

For f of the form (1.1) we deduce

$$\begin{aligned} \log M(r,f) &\leqslant \sum_{\nu=1}^{\infty} \log M(r, E_1(z/z_\nu)) \\ &\leqslant \sum_{\nu=1}^{\infty} u(re^{i\pi/3}, E_1(z/|z_\nu|)) = u(re^{i\pi/3}, \hat{f}). \end{aligned}$$

Thus the second inequality in (5.6) follows from Theorem 2:

$$C(g) \leqslant \limsup_{m\to\infty} \frac{\log M(r_m, f) + O(r)}{N(r_m, 0)}$$

$$\leqslant \limsup_{m\to\infty} \frac{u(r_m e^{i\pi/3}, \hat{f})}{N(r_m, 0)}$$

$$\leqslant \frac{2\pi\lambda}{\sin \pi\lambda} \cos\left(\frac{2\pi\lambda}{3}\right).$$

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