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A Microlocal F. and M. Riesz Theorem with Applications

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Introduction

Consider, by way of example, the following F. and M. Riesz theorem for $\mathbb{R}^n$: Let μ be a finite measure on $\mathbb{R}^n$ whose Fourier transform $\hat{\mu}$ is supported in a closed convex cone which is *proper*, that is, which contains no entire line. Then μ is absolutely continuous (*cf.* Stein and Weiss [SW]). Here, as in the sequel, «absolutely continuous» means with respect to Lebesque measure. In this theorem one can replace the condition on the support of $\hat{\mu}$ by a similar condition on the wave front set $WF(\mu)$ of μ, while keeping the same conclusion. The resulting «microlocal F. and M. Riesz theorem» can be applied with great flexibility to derive F. and M. Riesz theorems for measures on Lie groups, measures satisfying partial differential equations, etc. This is, essentially, the program of this paper.

Actually, the microlocal F. and M. Riesz theorem which we are going to use is much stronger than the one indicated above: it states that μ is absolutely continuous if $WF(\mu) \cap (-WF(\mu)) = \emptyset$; and, in fact, such a μ will be in the local H^1-space of Goldberg [G]. An important tool for the proof of this result is Uchiyama's characterization of the real Hardy space $H^1(\mathbb{R}^n)$, *cf.* [U]. This will be done in Section 1. In the remainder of this paper we give two applications, which we now describe.

In [B1] the author proved an F. and M. Riesz theorem for the unit sphere $S_{2n-1} \subseteq \mathbb{C}^n$ by completely different (group theoretic and functional analytic) methods. (The result in [B1] was actually for homogeneous spaces of compact groups whose center contains a circle group.) In Section 2 we prove a new theorem of this type for S_{2n-1}, which greatly extends some important special cases of the result of [B1]. An interesting question is whether one can regain the full F. and M. Riesz theorem of [B1] by the methods of the present paper.

It should be noted that the reasoning used in Section 2 can also be applied in more general situations. However, it seemed preferable first to treat a typical example rather than trying to formulate the most general result, *e.g.*, for compact Lie groups. (*Cf.* also [B2], where the main result of [B1] is extended to compact Lie groups.)

To motivate the second application, treated in Section 3, we consider the following formulation of the classical F. and M. Riesz theorem for $\mathbb{R}^n$. Let μ be a finite measure on $\mathbb{R}$ which is boundary value (in the weak-$*$ sense, say) of a holomorphic function $F(x+iy)$ defined in the upper half plane $\{x+iy : y>0\}$. Then μ is absolutely continuous. Holomorphic functions are solutions of the Cauchy-Riemann equations

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)F=0$$

and it is natural to ask whether one can replace the Cauchy-Riemann operator here by other vector fields

$$X=a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}.$$

It turns out that the answer is «yes» if $b(x,0)\neq 0$ for all $x\in\mathbb{R}$, that is, if $\mathbb{R}\times\{0\}$ is not characteristic for X. If a/b is real, this is quite easy to see; for the case that $\operatorname{Im}(a/b)\neq 0$, we use the microlocal F. and M. Riesz theorem together with estimates on $WF(\mu)$; *cf.* Section 3 below for details.

More generally, let $P_1,\ldots,P_N$ be N vector fields (with complex-valued coefficients) on $\mathbb{R}^{n+1}$, and let μ be a measure on $\mathbb{R}^n$ which is the boundary value (in distributional sense) of a function f on $\mathbb{R}^{n+1}_+ = \{(x,t): x\in\mathbb{R}^n, t>0\}$ satisfying $P_j f=0$, $1\leqslant j\leqslant N$. For which P_j is such a μ necessarily absolutely continuous? In Section 3 we give a sufficient condition whose proof uses the microlocal F. and M. Riesz theorem. As a corollary we show that a measure μ on a hypersurface S in $\mathbb{C}^n$ which is the boundary value of a holomorphic function defined on one side of S is absolutely continuous.

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1. A Microlocal F. and M. Riesz Theorem

We use Uchiyama's powerful characterization of $H^1(\mathbb{R}^n)$ to derive an F. and M. Riesz theorem for $\mathbb{R}^n$, which we then microlocalize. We first recall the definitions of $H^1(\mathbb{R}^n)$ and of Goldberg's local Hardy space h^1.

Definition. *A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is in the real Hardy space $H^1(\mathbb{R}^n)$ if for some $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi}(0) \neq 0$,*

$$\|f\|_{H^1} \equiv \left\| \sup_{t>0} |\psi_t * f(\cdot)| \right\|_{L^1(\mathbb{R}^n)} < \infty \tag{1.1}$$

(*where, as usual, $\psi_t(x) = t^{-n}\psi(x/t)$*).
f is in $h^1(\mathbb{R}^n)$, Goldberg's local Hardy space, if

$$\|f\|_{h^1} \equiv \left\| \sup_{0<t<1} |\psi_t * f(\cdot)| \right\|_{L^1(\mathbb{R}^n)} < \infty. \tag{1.2}$$

For equivalent definitions and further properties of these spaces, *cf.* Fefferman and Stein [FS], Goldberg [G]. Note that both H^1 and h^1 are contained in $L^1(\mathbb{R}^n)$. In (1.1), one may replace $\psi_t(x)$ by the Poisson-Kernel for the upper half space. The interest of h^1 is that it can also be defined on manifolds (in the usual way, using coordinate charts), *cf.* [G, Proposition 3]. We will also use the following two properties of $h^1(\mathbb{R}^n)$, *cf.* [G]:

$$\mathcal{S} \subseteq h^1(\mathbb{R}^n) \tag{1.3}$$

$$\mathcal{S}(\mathbb{R}^n) \cdot h^1(\mathbb{R}^n) \subseteq h^1(\mathbb{R}^n) \tag{1.4}$$

(and more generally, $h^1(\mathbb{R}^n)$ is closed under 0-th order pseudo-differential operators).

The following notation will be useful:

$$h^1_{\text{loc}}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) \text{: for every } \phi \in C_c^\infty(\mathbb{R}^n)\ \phi \cdot f \in h^1(\mathbb{R}^n) \}.$$

We now recall Uchiyama's characterization of $H^1(\mathbb{R}^n)$ (*cf.* [U]):

Theorem 1.1. *Let $\phi_1, \ldots, \phi_k \in C^\infty(\mathbb{R}^n \setminus 0)$ be homogeneous of degree 0 such that*

$$\operatorname{rank}\begin{pmatrix} \phi_1(\xi) & \cdots & \phi_k(\xi) \\ \phi_1(-\xi) & \cdots & \phi_k(-\xi) \end{pmatrix} = 2, \quad \textit{for every} \quad \xi \in S. \tag{1.5}$$

Let K_j be the multiplier operator associated to ϕ_j: $(K_j f)^\wedge(\xi) = \phi_j(\xi)\hat{f}(\xi)$. Then, for $f \in L^1(\mathbb{R}^n)$

$$C_1 \|f\|_{H_1} \leqslant \sum_{j=1}^{k} \|K_j f\|_1 \leqslant C_2 \|f\|_{H_1}$$

(*with constants* C_1, C_2 *only depending on the* ϕ_j *and on* n).

If X is a manifold we let $M(X)$ and $M_{\text{loc}}(X)$ denote the spaces of finite and of locally finite measures on X. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, $\hat{u}$ denotes the Fourier transform of u.

From Theorem 1.1 one can derive the following F. and M. Riesz theorem.

Theorem 1.2. *Let* $F \subseteq \mathbb{R}^n$ *be a closed conic subset such that* $F \cap (-F) = \{0\}$. *Let* $\mu \in M(\mathbb{R}^n)$ *be such that* $\operatorname{supp} \hat{\mu} \subseteq F$. *Then* μ *is in* $H^1(\mathbb{R}^n)$ (*and in particular,* μ *is absolutely continuous*).

PROOF. Let $F' = F \cap S$, $S = S_{n-1}$ the unit sphere. Since $F' \cap (-F') = \emptyset$, there exists an open set $U \supseteq F'$, $U \subseteq S$, such that $U \cap (-U) = \emptyset$. Let

$$W = S \setminus (F' \cup -F')$$

and let Q_j denote the j-th «quadrant» in $\mathbb{R}^n$:

$$Q_1 = \{\xi = (\xi_1, \ldots, \xi_n) : \xi_1 \geqslant 0, \xi_2 \geqslant 0, \ldots, \xi_n \geqslant 0\},$$
$$Q_2 = \{\xi_1 \leqslant 0, \xi_2 \geqslant 0, \ldots, \xi_n \geqslant 0\},$$

etc. Let $U_j = W \cap (\epsilon\text{-neighborhood of } Q_j)$, where ϵ is so small that $U_j \cap (-U_j) = \emptyset$. Then $\{U, -U, U_1, \ldots, U_{2^n}\}$ is an open cover of S. Relabel the elements of this cover as $\{V_1, \ldots, V_k\}$, with $V_1 = U$ (and $k = 2^n + 2$).

Let $\{\phi_1, \ldots, \phi_k\}$ be a partition of unity subordinate to this cover, such that $\phi_1 \equiv 1$ on $F' \subseteq V_1 = U$. Then these ϕ_j's satisfy (1.5).

Now consider a $\mu \in M(\mathbb{R}^n)$ satisfying $\operatorname{supp} \hat{\mu} \subseteq F$. Let

$$P_y(x) = c_n \frac{y}{(|x|^2 + y^2)^{(n+1)/2}} \qquad (x \in \mathbb{R}^n, \quad y > 0)$$

denote the Poisson-kernel and define $f_\epsilon = P_\epsilon * \mu$. Then, with K_j the singular integral operator associated to ϕ_j as in Theorem 1.1,

$$K_1 f_\epsilon = f_\epsilon, \qquad K_j f_\epsilon = 0 \quad \text{for} \quad j \neq 1.$$

By Theorem 1.1, $\|f_\epsilon\|_{H^1} \leqslant C\|f_\epsilon\|_1 \leqslant C\|\mu\|$. By taking $\psi(x) = P_1(x)$ in (1.1) and letting $\epsilon \downarrow 0$ it now follows that

$$\left\| \sup_{y>0} |P_y * \mu(\cdot)| \right\|_1 \leqslant C\|\mu\|$$

(we use that $P_{t_1} * P_{t_2} = P_{t_1+t_2}$), which implies that $\mu \in H^1(\mathbb{R}^n)$. □

The following corollary will be needed below

Corollary 1.3. *Let $\nu \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered measure on $\mathbb{R}^n$ such that* supp $\hat{\nu} \subseteq F$, *F as in Theorem* 1.2. *Then $\nu \in h^1_{\text{loc}}(\mathbb{R}^n)$.*

PROOF. We may suppose, without loss of generality, that supp $\hat{\nu} \cap \{|\xi| \leqslant 1\} = \varnothing$. For if $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ for $|\xi| \leqslant 2$, $\chi(\xi) = 1$ for $|\xi| \geqslant 3$, then $(1 - \chi)\hat{\nu}$ is the Fourier transform of a tempered C^∞-function g, and $\nu - g$ is a tempered measure such that supp $(\nu - g)^\wedge \cap \{|\xi| \leqslant 1\} = \varnothing$. By (1.3), $g \in h^1_{\text{loc}}(\mathbb{R}^n)$.

Now let $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi} \geqslant 0$, supp $\hat{\psi} \subseteq \{|\xi| \leqslant 1\}$, $\psi(0) = 1$. Write $\psi^\epsilon(x) = \psi(\epsilon x)$. Then $\nu_\epsilon = \psi^\epsilon \cdot \nu$ is a finite measure on $\mathbb{R}^n$ such that supp $\hat{\nu}_\epsilon =$ supp $(\psi^\epsilon * \hat{\nu})$ is contained in a conic ϵ-neighbourhood of F. By Theorem 1.2, $\nu_\epsilon \in H^1(\mathbb{R}^n) \subseteq h^1(\mathbb{R}^n)$ for sufficiently small ϵ. Now take $\phi \in C^\infty_c(\mathbb{R}^n)$ and let $\epsilon > 0$ be so small that $\nu_\epsilon \in h^1(\mathbb{R}^n)$ and $\psi^\epsilon(x) \neq 0$ on supp ϕ. Then $\phi \cdot \nu = (\phi/\psi^\epsilon) \cdot (\psi^\epsilon \nu)$ is in $h^1(\mathbb{R}^n)$ by (1.4). □

We now microlocalize Theorem 1.2. If $X \subseteq \mathbb{R}^n$ is open and $u \in \mathcal{D}'(X)$ is a distribution on X, we let $WF(u) \subseteq X \times \mathbb{R}^n \backslash 0 = T^*(X)\backslash 0$ denote the wave front set of u (*cf.* Hörmander [H] for the definition). For $x \in X$ let $WF_x(u) = \{\xi \in \mathbb{R}^n \backslash 0 : (x, \xi) \in WF(u)\} = WF(u) \cap T^*_x(X)$. All this also makes sense if X is a manifold.

Theorem 1.4. *Let X be a manifold and $\mu \in M_{\text{loc}}(X)$ a locally finite measure such that*

$$WF_x(\mu) \cap -WF_x(\mu) = \varnothing, \quad \textit{for every} \quad x \in X. \tag{1.6}$$

Then μ is in $h^1_{\text{loc}}(X)$. In particular, μ is absolutely continuous with respect to any Lebesque measure on X.

Here $\mu \in h^1_{\text{loc}}(X)$ means that $\phi \cdot \mu \in h^1(\mathbb{R}^n)$ for all $\phi \in C^\infty_c(X)$ supported in a coordinate neighborhood.

PROOF. It suffices to prove the theorem for X an open subset of $\mathbb{R}^n$. We show that

(1.7) Given $x \in X$ there exists a neighborhood U_x of x such that for any $\phi \in C^\infty_c(U_x)$ $\phi \cdot \mu \in h^1(\mathbb{R}^n)$.

If $v \in \mathcal{E}'(\mathbb{R}^n)$ is a compactly supported distribution, let $\Sigma(v) \subseteq \mathbb{R}^n \backslash 0$ be the closed conic subset defined as follows: $\xi \notin \Sigma(v)$ if and only if there exists a conic neighborhood Γ of ξ such that for $\eta \in \Gamma$ and $N \in \mathbb{N}$

$$|\hat{v}(\eta)| \leqslant C_N(1 + |\eta|)^{-N}. \tag{1.8}$$

Then $WF_x(u) = \bigcap \{\Sigma(\phi u): \phi \in C_c^\infty(X),\ \phi(x) \neq 0\}$ and $\Sigma(\phi u) \to WF_x(u)$ as $\operatorname{supp} \phi \to \{x\}$, ϕ ranging over $C_c^\infty(X)$-functions for which $\phi(x) \neq 0$ (*cf.* [H], Section 8.1).

Now suppose that $\mu \in M_{\mathrm{loc}}(X)$ satisfies (1.6) and let $x \in X$. There exists a conic open $\Gamma \subseteq \mathbb{R}^n \backslash 0$ such that $\Gamma \supseteq WF_x(\mu)$ and such that $\Gamma \cap (-\Gamma) = \varnothing$.

Let $\Delta \subseteq \mathbb{R}^n \backslash 0$ be an open conic subset such that $WF_x(\mu) \subseteq \Delta \subseteq \bar{\Delta} \subseteq \Gamma$ and let $U = U_x$ be an open neighborhood of x such that for $\phi \in C_c^\infty(U)$ with $\phi(x) \neq 0$. $\Sigma(\phi\mu) \subseteq \Delta$. Take such a ϕ. Then $\widehat{\phi\mu}$ is rapidly decreasing (in the sense of (1.8)) on $\mathbb{R}^n \backslash \bar{\Delta}$. Let $\chi \in C^\infty(\mathbb{R}^n)$, $0 \leqslant \chi \leqslant 1$, such that $\operatorname{supp} \chi \subset \Gamma \backslash \{|\xi| \leqslant 1\}$ and such that $\chi = 1$ on $\Delta \backslash \{|\xi| \leqslant 2\}$. The inverse Fourier transform g of $(1 - \chi) \cdot \widehat{\phi\mu}$ is then in $C^\infty(\mathbb{R}^n)$ and $\nu = \phi\mu - g\,dx$ is a tempered measure such that $\operatorname{supp} \hat{\nu} \subseteq \Gamma$. By Corollary 1.3, $\nu \in h^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence $\phi \cdot \mu \in h^1_{\mathrm{loc}}(\mathbb{R}^n)$. This proves (1.7) for those ϕ with $\phi(x) \neq 0$, which obviously suffices. $\square$

2. F. and M. Riesz for the Unit Sphere in $\mathbb{C}^n$

We now use Theorem 1.4 to prove an F. and M. Riesz theorem for the unit sphere S in $\mathbb{C}^n$. For the statement we need some notation from the theory of spherical harmonics on S, *cf.* Rudin [R], Chapter 12. Let σ denote the rotation invariant measure on S, normalized by $\sigma(S) = 1$, say. Let $H(p, q)$ be the set of restrictions to S of harmonic functions u on $\mathbb{C}^n$ which are homogeneous of degree p in z and of degree q in $\bar{z}$. Then $L^2(S, \sigma) = \Sigma_{p,q} H(p, q)$ (orthogonal direct sum). Let π_{pq} denote the orthogonal projection onto $H(p, q)$; π_{pq} can be extended to distributions.

For a finite measure μ we let the spectrum of μ be $\operatorname{spec} \mu = \{(p, q): \pi_{pq}\mu \neq 0\}$.

If $F \subseteq \mathbb{R}_+ \cup \{\infty\}$, let $i(F) = \left\{\frac{1}{\alpha}: \alpha \in F\right\}$ $\left(\text{where } \frac{1}{0} = \infty,\ \frac{1}{\infty} = 0, \text{ as usual}\right)$.

Also, let $\Sigma(F) \subseteq \mathbb{N} \times \mathbb{N}$ be defined by

$$\Sigma(F) = \left\{(p, q) \in \mathbb{N} \times \mathbb{N}: \frac{q}{p} \in F\right\}.$$

Our F. and M. Riesz theorem for S is the following.

Theorem 2.1. *Suppose that $F \subseteq \mathbb{R}_+ \cup \{\infty\}$ is a closed subset such that $F \cap i(F) = \varnothing$. Let μ be a finite measure on S such that* $\operatorname{spec} \mu \subseteq \Sigma(F)$. *Then μ is in $h^1(S)$. In particular, μ is absolutely continuous with respect to σ.*

EXAMPLES.

(i) $F = [0, \alpha]$, $\alpha < 1$, and $\Sigma(F) = \{(p, q): q \leqslant \alpha p\}$. This special case of Theorem 2.1 is also contained in [B1, Theorem 1.1].

(ii) $F = [0, \alpha] \cup [\beta, \gamma]$ with $\alpha < 1 < \beta \leqslant \gamma$ and $\gamma < 1/\alpha$. In this case $\Sigma(F)$ looks like the union of two cones such that the reflection of one with respect to the line $p = q$ has zero intersection with the other. Note that, contrary to one of the conditions of Theorem 1.1 of [B1], $\{p - q: (p, q) \in \Sigma(F)\}$ is not bounded from below or from above anymore.

PROOF OF THEOREM 2.1. We are going to exploit the fact that $H(p, q)$ is the simultaneous eigenspace of two commuting self-adjoint differential operators on S, namely the Laplace-Beltrami operator Δ_S and the tangential vector field T defined by

$$Tf(\zeta) = \frac{1}{i}\frac{d}{d\theta} f(e^{i\theta}\zeta)\Big|_{\theta=0}.$$

In fact, if we write

$$\nu = (-\Delta_S + (n-1)^2)^{1/2} - (n-1),$$

then ν is a first order pseudo-differential operator on S with eigenvalue k on the eigenspace $\mathcal{H}(k) = \{u: \Delta u = 0 \text{ on } \mathbb{C}^n,\ u(rz) = r^k u(z) \text{ for } r \geqslant 0\}$ (*cf.* Taylor [T2, Chapter 4]). Since

$$\mathcal{H}(k) = \sum_{p+q=k} H(p, q),$$

it follows that

$$H(p, q) = \{u \in L^2(S): \nu(u) = (p+q)u,\ T(u) = (p-q)u\}. \tag{2.1}$$

Let $F \subseteq \mathbb{R}_+ \cup \{\infty\}$ be a closed subset satisfying $F \cap i(F) = \varnothing$, and let $a(x, y) \in C^\infty(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree 0 such that

$$F = \{y/x: a(x, y) = 0\}. \tag{2.2}$$

Now if spec $\mu \subseteq \Sigma(F)$, μ is annihilated by the operator

$$\sum_{p,q} a(p, q)\pi_{p,q} \tag{2.3}$$

(because of (2.2)). Writing

$$\tilde{a}(x, y) = a\left(\frac{x+y}{2}, \frac{x-y}{2}\right),$$

we see that (2.3) is equal to the operator $\tilde{a}(\nu, T)$ as defined using the spectral theorem, *cf.* (2.1). By a result of Strichartz [S] (*cf.* also Section 12.1 in Taylor [T1]), $\tilde{a}(\nu, T)$ is a first order pseudo-differential operator on S and, if we denote the principal symbol of a pseudo-differential operator A by $\sigma(A)$,

$$\sigma(\tilde{a}(\nu, T)) = \tilde{a}(\sigma(\nu), \sigma(T)). \tag{2.4}$$

(Strictly speaking, we should have made $\tilde{a}$ smooth in 0, but this would only change $\tilde{a}(\nu, T)$ by a smoothing operator.)

Now $\tilde{a}(\nu, T)\mu = 0(\bmod C^\infty)$ implies that

$$WF(\mu) \subset \operatorname{Char} \tilde{a}(\nu, T) = \{(z, \xi) \in T^*(S): \sigma(\tilde{a}(\nu, T))(z, \xi) = 0\}.$$

To finish the proof we compute $\sigma(\tilde{a}(\nu, T))$. If $z \in S$ we let

$$T_z(S) = T_z^{\mathbb{C}}(S) + \mathbb{R} \cdot iz$$

be the usual splitting of the tangent space in $\mathbb{C}^n$, with

$$T_z^{\mathbb{C}}(S) = \{\xi \in \mathbb{C}^n: \langle z, \xi \rangle = 0\}$$

the maximal complex subspace of $T_z(S)$ ($\langle \bullet, \bullet \rangle$ being the standard Hermitian inner product on $\mathbb{C}^n$). Identify $T_z(S)$ and $T_z^*(S)$, using the Riemannian metric on S induced by $\mathbb{C}^n$. If $\xi \in T_z(S)$, $\xi = \xi' + \theta \cdot iz$ with $\xi' \in T_z^{\mathbb{C}}(S)$, $\theta \in \mathbb{R}$, then

$$\begin{aligned} \sigma(\nu)(z, \xi) &= c \cdot (|\xi'|^2 + \theta^2)^{1/2} \\ \sigma(T)(z, \xi) &= \theta. \end{aligned} \tag{2.5}$$

By (2.4),

$$\sigma(\tilde{a}(\nu, T))(z, \xi) = a\left(\frac{c(|\xi'|^2 + \theta^2)^{1/2} + \theta}{2}, \frac{c(|\xi'|^2 + \theta^2)^{1/2} - \theta}{2}\right).$$

Now suppose that there exists a $(z, \xi) \in \operatorname{Char}(\tilde{a}(\nu, T))$ such that also $(z, -\xi) \in \operatorname{Char}(\tilde{a}(\nu, T))$, $\xi = \xi' + \theta \cdot iz$ as above, $\xi \neq 0$. Then

$$\frac{c(|\xi'|^2 + \theta^2)^{1/2} - \theta}{c(|\xi'|^2 + \theta^2)^{1/2} + \theta} \in F \cap i(F),$$

contradicting the assumption on F. Hence $WF(\mu)$ satisfies the condition of Theorem 1.4 and hence μ is in $h^1(S)$. □

Probably this type of argument can be used in more general situations, *e.g.*, measures on homogeneous spaces of compact Lie groups. However, the formulation of an analogon of Theorem 2.1 is likely to become much more complicated. *Cf.* for example [B2], where an F. and M. Riesz theorem for arbitrary compact Lie groups can be found which generalizes the one of [B1]. The reason for these complications is that one has to refine the notion of spectrum.

3. Absolute Continuity of Measures Arising as Boundary Values of Solutions of Partial Differential Equations

Let $X \subseteq \mathbb{R}^n$ be open and let U be an open neighborhood of $X \times \{0\}$ in $\mathbb{R}^{n+1}$; let $U_+ = U \cap \{(x,t): x \in \mathbb{R}^n, t > 0\}$. Let $P_1, \ldots, P_N$ be a set of first order linear partial differential operators with C^∞-coefficients defined on the closure of U_+. Consider measures μ on X which arise in the following way: there is an $f \in C^1(U_+)$ satisfying $P_j f = 0$, $1 \leqslant j \leqslant N$, such that μ is the limit, in $\mathfrak{D}'(X)$, of $f(x,t)$ as $t \downarrow 0$. The question with which we concern ourselves here is for which P_j such a μ necessarily is absolutely continuous. We will give a sufficient condition for P's which are vectorfields:

$$P_j = c_j \partial_t + \langle a_j, \partial_x \rangle = c_j(x,t)\frac{\partial}{\partial t} + \sum_\mu a_{j\mu}(x,t)\frac{\partial}{\partial x_\mu}. \tag{3.1}$$

For x in X let $J(x) = \{j: 1 \leqslant j \leqslant N, c_j(x,0) \neq 0\}$. Then the main result of this section is the following:

Theorem 3.1. *All notation as above. Let P_j be given by* (3.1). *Suppose that for all $x \in X$ the following closed convex cone is proper* (*i.e. contains no straight lines*):

$$\bigcap_{j \in J(x)} \{\xi \in \mathbb{R}^n: \operatorname{Im}(c_j(x,0)^{-1}\langle a_j(x,0), \xi\rangle) \leqslant 0\} \tag{3.2}$$

$$\cap \bigcap_{j,k} \{\xi \in \mathbb{R}^n: c_j(x,0)\langle a_k(x,0), \xi\rangle = c_k(x,0)\langle a_j(x,0), \xi\rangle\}.$$

Let μ be a locally finite measure on X which is the distributional boundary value $\lim_{t\downarrow 0} f(\cdot, t) = \mu$ of an $f \in C^1(U_+)$ that satisfies $P_j f = 0$, $1 \leqslant j \leqslant N$, and for which there exists an $M \in \mathbb{N}$ such that

$$|f(x,t)|, |\partial_x f(x,t)| = O(t^{-M}), \tag{3.3}$$

uniformly on compacta of X. Then μ is in $h^1_{\mathrm{loc}}(X)$.

If $a_j(x,0) = \partial_t a_j(x,0) = \cdots = \partial_t^{k-1} a_j(x,0) \equiv 0$ on X one may replace $a_j(x,0)$ in the first line of (3.2) *by $\partial_t^k a_j(x,0)$.*

Before giving the proof of Theorem 3.1 let us make some remarks. If one takes $n = N = 1$ and P_1 to be the Cauchy-Riemann operator on $\mathbb{C} = \mathbb{R}^2$, one obviously obtains (a local version of) the classical F. and M. Riesz theorem for $\mathbb{R}$. More generally, one can show using Theorem 3.1 that a measure on a hypersurface S in $\mathbb{C}^n$ which is boundary value (in distribution sense) of a holomorphic function defined on one side of S is absolutely continuous with respect to surface measure (just straighten out S locally and then apply Theorem 3.1).

It is clear that Theorem 3.1 is meaningless if the hypersurface $\{t = 0\}$ is characteristic for all P_j. The following easy example shows that Theorem 3.1 is false in this case: Take $n = N = 1$ and let

$$f(x, t) = \pi^{-1/2} t^{-1} e^{-x^2/2t^2}, \qquad x \in \mathbb{R}, \quad t > 0.$$

Then $f(x, t) \to \delta(x)$ as $t \downarrow 0$ and f satisfies the partial differential equation

$$(x^2 - t^2)\frac{\partial f}{\partial x} + xt\frac{\partial f}{\partial t} = 0.$$

Also, the cone on the right hand side of (3.1) can not be proper if $c_j^{-1} a_j$ is real on $\{t = 0\}$ for all j. But if one of the P's, say P_1, has real coefficients and $\{t = 0\}$ is not characteristic for P_1, μ is absolutely continuous for trivial reasons: $P_1 f = 0$ means that f is constant along the characteristics of P_1, which intersect $\{t = 0\}$ transversally. Hence one can extend f in a C^1-way to a neighborhood of $X \times \{0\}$ in $\mathbb{R}^{n+1}$ in such a way that this extension of f is still annihilated by P. If follows that Theorem 3.1 is only interesting in case all non-characteristic P's have complex coefficients.

Finally note that if $n > 1$, f has to satisfy an *overdetermined* system for the cone (3.1) to be proper.

The proof of Theorem 3.1 consists of showing that $WF(\mu)$ is contained in the cone (3.1). The conclusion then follows by Theorem 1.4. To estimate $WF(\mu)$ we first estimate the wave front set of a distribution which is the boundary value of a function annihilated by a single vector field. The arguments we will use have been inspired by Hörmander's treatment of this problem for the Cauchy-Riemann operator (*cf.* [H]) but are more involved since we are dealing with variable coefficient operators.

Let P be a vector field of the form $P = \partial_t + \langle a, \partial_x \rangle$. (This could be relaxed at times.) Let P^* denote the formal (real) adjoint of P: $P^* = -\partial_t - \langle \partial_x, a \rangle$.

Lemma 3.2. *Let $T > 0$ and let $u(x, t)$, $v(x, t) \in C^1(\mathbb{R}^n \times [0, T])$ be such that* supp $v(\cdot, t)$ *is contained in a fixed compactum $K \subset \mathbb{R}^n$ for all t, $0 \leqslant t \leqslant T$. Then*

$$\int_{\mathbb{R}^n} u(x, T) v(x, T)\, dx - \int_{\mathbb{R}^n} u(x, 0) v(x, 0)\, dx = \int_0^T \int_{\mathbb{R}^n} ((Pu)v - u(P^* v))\, dx\, dt.$$

PROOF. Integration by parts. $\square$

For simplicity we first consider the case where $P = \partial_t + \langle a, \partial_x \rangle$ with a not depending on t.

Theorem 3.3. *Let $X \subseteq \mathbb{R}^n$ be open, U an open neighborhood of $X \times \{0\}$ in $\mathbb{R}^{n+1}$, $U_+ = U \cap \mathbb{R}^{n+1}_+$. Let $P = \partial_t + \langle a, \partial_x \rangle, a = a(x)$ C^∞ on X. Let $f \in C^1(U_+)$ be such that $Pf \in L^\infty(U_+)$ while for some $N \in \mathbb{N}$,*

$$|f(x,t)| = O(t^{-N}) \quad as \quad t \downarrow 0,$$

uniformly on compacta of X. Then $\lim_{t \downarrow 0} f(x,t) = f(x, 0+)$ exists in $\mathfrak{D}'(X)$.

PROOF. Let $\phi \in C_c^\infty(X)$. Let $T > 0$ be such that $\operatorname{supp} \phi \times [0, 2T] \subseteq X \cup U_+$. Let $k \in \mathbb{N}$. Determine $\phi_0, \phi_1, \ldots, \phi_k \in C^\infty(\bar{U}_+)$ such that

$$\Phi(x,t) \equiv \Phi^{(k)}(x,t) = \sum_{j=0}^{k} \phi_j(x,t) \frac{t^j}{j!}$$

satisfies the conditions

(i) $\Phi(x, 0) = \phi(x)$.
(ii) $|P^*\Phi(x,t)| \leqslant Ct^k$, $x \in \operatorname{supp} \phi$, $0 \leqslant t \leqslant T$.

The constant C here depends on the derivatives of ϕ up till order $N + 1$.

To prove the existence of Φ, write $P = \partial_t + Q(x, \partial_x)$. Then

$$P^*[\Phi] = \sum_{j=0}^{k-1} \left(-\frac{\partial \phi_j}{\partial t} + Q^*[\phi_j] - \phi_{j+1} \right) \frac{t^j}{j!} + \left(-\frac{\partial \phi_k}{\partial t} + Q^*[\phi_k] \right) \frac{t^k}{k!}$$

and one need only take $\phi_0(x,t) = \phi(x)$, $\phi_j = -\partial_t \phi_{j-1} + Q^*[\phi_{j-1}]$, $1 \leqslant j \leqslant k$. In the present case the ϕ_j do not depend on the variable t, but they will do it in the proof of the next theorem, when Q depends on t. Note, that $\operatorname{supp} \Phi(\cdot, t) \subseteq \operatorname{supp} \phi$, $0 \leqslant t \leqslant T$.

Let $0 < \epsilon < T$ and write $f_\epsilon(x,t) = f(x, t + \epsilon)$ $(t < T)$. Apply Lemma 3.2 with $u = f_\epsilon$, $v = \Phi^{(k)} = \Phi$. Then

$$\int_X f(x,\epsilon)\phi(x)\,dx = \int_X f(x, T+\epsilon)\Phi(x,T)\,dx - \int_{X \times (0,T)} (Pf_\epsilon)\Phi\,dx\,dt + \int_{X \times (0,T)} f_\epsilon(P^*\Phi)\,dx\,dt.$$

Now

$$|f_\epsilon(x,t) \cdot P^*\Phi(x,t)| \leqslant Ct^{k-N}, \qquad (x,t) \in \operatorname{supp} \phi \times (0,T),$$

C independent of ϵ, and

$$\sup |P[f_\epsilon](x,t)| \leqslant \|Pf\|_{L^\infty(U_+)},$$

since $P[f_\epsilon](x,t) = P[f](x, t+\epsilon)$.

Take $k \geqslant N$ and let $\epsilon \to 0$. Then, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}\langle f(\bullet, 0+), \phi\rangle &= \lim_{\epsilon \downarrow 0} \langle f(\bullet, \epsilon), \phi\rangle \\ &= \int_X f(x, T)\Phi^{(k)}(x, T)\,dx - \int_{X \times (0, T)} Pf \cdot \Phi^{(k)}\,dx\,dt \\ &\quad + \int_{X \times (0, T)} fP^*\Phi^{(k)}\,dx\,dt. \qquad \square\end{aligned} \tag{3.4}$$

If a is allowed to depend on t in Theorem 3.3 the proof may fail: the main problem is that $\sup_{\epsilon > 0} |Pf_\epsilon|$ (where $f_\epsilon(x, t) = f(x, t + \epsilon)$) need not be in L^1. We now show how to modify the proof in this case in order to arrive at the following result.

Theorem 3.4. *Let X, U_+ be as in Theorem* 3.3, $P = \partial_t + \langle a, \partial_x\rangle$, $a = a(x, t)$ *be of class C^∞ on $X \cup U_+$. Let $f \in C^1(U_+)$ be such that $Pf \in L^\infty(U_+)$ while for some $N \in \mathbb{N}$,*

$$|f(x, t)|, |\partial_x f(x, t)| = O(t^{-N}) \quad \text{as} \quad t \to 0, \tag{3.5}$$

uniformly on compacta of X. Then $\lim_{t \to 0} f(x, t) = f(x, 0+)$ exists in $\mathfrak{D}'(X)$ and formula (3.4) *for $f(\bullet, 0+)$ remains valid.*

PROOF. The idea is to replace $f(x, t + \epsilon)$ in the proof of Theorem 3.3 by

$$f_\epsilon(x, t) = f(x + \Psi_\epsilon(x, t), t + \epsilon),$$

where Ψ_ϵ is a C^∞-function on $X \cup U_+$ which is to be determined such that

$$P[f_\epsilon](x, t) = P[f](x + \Psi_\epsilon, t + \epsilon) + O(1), \tag{3.6}$$

uniformly in ϵ and x, t. Furthermore, Ψ_ϵ has to satisfy

$$\Psi_\epsilon(x, 0) = 0, \tag{3.7a}$$

$$\Psi_\epsilon(x, t), \partial_t\Psi_\epsilon(x, t), \partial_x\Psi_\epsilon(x, t) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.7b}$$

Retracing the steps of the previous proof with this f_ϵ one sees that Theorem 3.4 is true: Lebesgue's dominated convergence theorem can be applied as at the end of the proof of Theorem 3.3 because of (3.6). Finally, because of (3.7b), $P[f_\epsilon](x, t) \to P[f](x, t)$ as $\epsilon \to 0$ so that formula (3.4) remains valid also.

A straightforward calculation shows that, with the notations $\tilde{x} = x + \Psi_\epsilon(x, t)$, $\tilde{t} = t + \epsilon$, $\Psi = \Psi_\epsilon$, Ψ'_x = Jacobian of Ψ with respect to x,

$$P[f_\epsilon](x, t) - P[f](\tilde{x}, \tilde{t}) = \langle \partial_t\Psi + (Id + \Psi'_x)a(x, t) - a(\tilde{x}, \tilde{t}), \partial_x f(\tilde{x}, \tilde{t})\rangle.$$

Beacuse of (3.5) it suffices to determine $\Psi = \Psi_\epsilon$ in such a way that

$$\partial_t \Psi + a(x, t) + \Psi'_x \cdot a(x, t) - a(\tilde{x}, \tilde{t}) = O(t^N). \tag{3.8}$$

We try a solution of the form

$$\Psi_\epsilon = \sum_{j=1}^{N} \Psi_{j,\epsilon}(x) \frac{t^j}{j!}.$$

With such a Ψ_ϵ equation (3.7*a*) is automatically satisfied. Expand the left hand side of (3.8) in a Taylor series in t, up till order N, while treating $a(x, t)$ in the following way:

$$\begin{aligned} a(x + \Psi_\epsilon, t + \epsilon) &\approx \sum_{\substack{\alpha \in \mathbb{N}^n \\ k \in \mathbb{N}}} \frac{1}{\alpha!\, k!} (\partial_x^\alpha \partial_t^k a)(x, \epsilon)(\Psi_\epsilon)^\alpha t^k \\ &= a(x, \epsilon) + (\partial_t a(x, \epsilon) + \partial_x a(x, \epsilon) \cdot \Psi_{1,\epsilon}) t \\ &\quad + \Bigg(\partial_t^2 a(x, \epsilon) + \partial_x a(x, \epsilon) \cdot \Psi_{2,\epsilon} + \sum_{|\alpha| = 2} \frac{2!}{\alpha!} \partial_x^\alpha a(x, \epsilon)(\Psi_{1,\epsilon})^\alpha \\ &\quad + \sum_{|\alpha| = 1} 2!\, \partial_x^\alpha \partial_t a(x, \epsilon)(\Psi_{1,\epsilon})^\alpha \Bigg) \frac{t^2}{2!} + \cdots \end{aligned}$$

If one sets the coefficient of t^j in (3.8) equal to 0 for $j < N$ one obtains a system of equations for $\Psi_{j,\epsilon}(x)$ of the form

$$\Psi_{j,\epsilon}(x) = \{\text{expression in } \Psi_{1,\epsilon}, \ldots, \Psi_{j-1,\epsilon} \text{ and their derivatives}\}$$

which can be solved recursively in a unique way.

The first equation yields

$$\Psi_{1,\epsilon}(x) = a(x, \epsilon) - a(x, 0).$$

It is clear that $\Psi_{j,\epsilon}(x)$ is a C^∞-function of $\epsilon \geqslant 0$ and x and that $\Psi_{j,0}(x) \equiv 0$. Hence equations (3.7*b*) are satisfied. □

We now estimate $WF(f(\cdot, 0+))$ for solutions f of $Pf = 0$.

Theorem 3.5. *All notations as in Theorem* 3.4. *Suppose that* $f \in C^1(U_+)$ *satisfies* (3.5) *and*

$$|Pf(x, t)| = O(t^k), \qquad k = 1, 2, 3, \ldots, \tag{3.9}$$

uniformly on compacta of X. Then

$$WF(f(\cdot, 0+)) \subseteq \{(x, \xi) \in X \times \mathbb{R}^n \backslash 0 \colon \operatorname{Im} \langle a(x, 0), \xi \rangle \leqslant 0\}.$$

Proof. We have to show that if $\operatorname{Im}\langle a(x_0,0),\xi_0\rangle > 0$, there exists a $\phi \in C_c^\infty(X)$, $\phi(x_0) \neq 0$, such that

$$|\langle f(\bullet, 0+), e^{-i\langle \bullet, \xi\rangle}\phi\rangle| \leqslant C_k(1+|\xi|)^{-k}, \qquad k = 1, 2, \ldots$$

for ξ in a conic neighborhood of ξ_0.

Fix $k \in \mathbb{N}$. We are going to determine functions $b_1, \ldots, b_k$ of (x, ξ) such that

$$u_{\xi,k}(x,t) = \exp\left(-i\langle x, \xi\rangle + \sum_{j=1}^{k} b_j(x,\xi)\frac{t^j}{j!}\right)$$

is an approximate solution of $Pu = 0$ in the sense that on compacta of X and for small t,

$$|Pu_{\xi,k}(x,t)| \leqslant C|\xi|t^k.$$

Note that $u_{\xi,k}(x,0) = e^{-i\langle x,\xi\rangle}$.

A computation shows that

$$Pu_{\xi,k} = \left(\sum_{j=1}^{k} b_j \frac{t^{j-1}}{(j-1)!} - i\langle a, \xi\rangle + \sum_{j=1}^{k}\langle a, \partial_x b_j\rangle \frac{t^j}{j!}\right)u_{\xi,k}. \tag{3.10}$$

Write

$$a(x,t) = \sum_{l=0}^{k-1} a^{(l)}(x)\frac{t^l}{l!} + a^{(k)}(x,t)\frac{t^k}{k!},$$

where $a^{(j)} = \partial^j{}_t a$, and $a^{(j)}(x) = a^{(j)}(x, 0)$. Then

$$\sum_{j=1}^{k}\langle a, \partial_x b_j\rangle\frac{t^j}{j!} = \sum_{j=1}^{2k}\left(\sum_{l=\max(1,j-k)}^{\min(j,k)}\binom{j}{l}\langle a^{(j-l)}, \partial_x b_l\rangle\right)\frac{t^j}{j!}.$$

Substitute this expression in (3.10) and put the coefficient of t^j equal to 0 for $j \leqslant k-1$. Then

$$b_{j+1} = i\langle a^{(j)}, \xi\rangle - \sum_{l=1}^{j}\binom{j}{l}\langle a^{(j-l)}, \partial_x b_l\rangle, \qquad 0 \leqslant j \leqslant k-1. \tag{3.11}$$

In particular, $b_1(x,\xi) = i\langle a(x,0),\xi\rangle$. Note that $b_1, \ldots, b_k$, as defined by (3.11), do not depend on t, since $a^{(0)}, \ldots, a^{(k-1)}$ only depend on x. It follows from (3.10) that the $b_j(x,\xi)$ are all linear in ξ.

Let $(x_0, \xi_0) \in X \times \mathbb{R}^n\backslash 0$ be such that $\operatorname{Re}(i\langle a(x_0,0),\xi_0\rangle) = -\operatorname{Im}\langle a(x_0,0),\xi_0\rangle < 0$. Then there exist a neighborhood $U(x_0)$ of x_0, a conic neighborhood $V(\xi_0)$ of ξ_0 and a $T > 0$ such that for $x \in U(x_0)$, $\xi \in V(\xi_0)$ and $t \leqslant T$,

$$\operatorname{Im}\langle a(x,0),\xi\rangle > 2c|\xi| > 0 \tag{3.12}$$

(c a suitable constant) and

$$\operatorname{Re} \sum_{j=2}^{k} b_j(x, \xi) \frac{t^j}{j!} \leqslant \frac{1}{2} \operatorname{Im} \langle a(x, 0), \xi \rangle t.$$

Hence for $x \in U(x_0)$, $\xi \in V(\xi_0)$, $t \leqslant T$:

$$|u_{\xi,k}(x, t)| \leqslant e^{-(1/2) \operatorname{Im} \langle a(x,0), \xi \rangle t} \tag{3.13}$$

and

$$|Pu_{\xi,k}(x, t)| \leqslant C(k)|\xi| t^k e^{-(1/2) \operatorname{Im} \langle a(x,0), \xi \rangle t} \tag{3.14}$$

Let $\phi \in C_c^\infty(U(x_0))$ be arbitrary. We now apply formula (3.4) to

$$f(x, 0+)e^{-i\langle x, \xi \rangle} = \lim_{t \downarrow 0} f(x, t) u_{\xi,k}(x, t).$$

Since $P(fu_{\xi,k}) = Pf \cdot u_{\xi,k} + f \cdot Pu_{\xi,k}$ and since Pf and $P^*[\Phi]$ are both $O(t^k)$, $\Phi = \Phi^{(k)}$ as in (3.4), the inequalities (3.12), (3.13) and (3.14) lead to the following estimate for $|\xi| \geqslant 1$ and $k \geqslant N$ (we assume that $T < 1$):

$$|\langle f(\cdot, 0+)e^{-i\langle \cdot, \xi \rangle}, \phi \rangle| \leqslant C \cdot \left(e^{-c|\xi|T} + \int_0^T |\xi| t^{k-N} e^{-c|\xi|t} \, dt \right) \leqslant \frac{C(k)}{|\xi|^{k-N}}.$$

where the constants depend on the supremum norms of ϕ and its derivatives up till order $k + 1$. Since k is otherwise arbitrary, this proves the theorem. □

Remark 3.6. The proof also shows that if

$$a(x, 0) = \partial_t a(x, 0) = \cdots = \partial_t^{l-1} a(x, 0) \equiv 0$$

on X Theorem 3.5 holds with $a(x, 0)$ replaced by $\partial_t^l a(x, 0)$.

PROOF OF THEOREM 3.1. After these preparations we can now easily prove Theorem 3.1. Let $f \in C^1(U_+)$ and μ be as in the theorem. Let $x \in X$ and $j \in J(x)$. Then for y in a neighborhood of x and t small,

$$\partial_t f(y, t) + c_j(y, t)^{-1} \langle a_j(y, t), \partial_x f(y, t) \rangle = 0.$$

By Theorems 3.4 and 3.5,

$$WF(\mu) \subseteq \{(x, \xi) \in X \times \mathbb{R}^n \backslash 0 \text{: for every } j \in J(x)\text{: } \operatorname{Im}(c_j(x, 0)^{-1} \langle a_j(x, 0), \xi \rangle) \leqslant 0\}. \tag{3.15}$$

Now fix j and k and eliminate $\partial_t f$ from $P_j f = P_k f = 0$. It follows that $c_j \langle a_k, \partial_x f \rangle - c_k \langle a_j, \partial_x f \rangle = 0$ on U_+. Hence $\mu = f(\cdot, 0+)$ satisfies the *induced* *equations*

$$L_{jk}\mu = \langle c_j(x,0)a_k(x,0) - c_k(x,0)a_j(x,0), \partial_x \mu \rangle = 0.$$

By [H, Theorem 8.3.1],

$$(3.16)\quad WF(\mu) \subseteq \bigcap_{j,k} \operatorname{Char}(L_{jk})$$

$$= \bigcap_{j,k} \{(x,\xi) \in X \times \mathbb{R}^n \backslash 0 : c_j(x,0)\langle a_k(x,0), \xi\rangle = c_k(x,0)\langle a_j(x,0), \xi\rangle\}.$$

By (3.15), (3.16) and the hypothesis of Theorem 3.1, $WF(\mu)$ is proper. Hence $\mu \in h^1_{\mathrm{loc}}(X)$. □

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