Revista Matemática Iberoamericana
Vol. 5, n.os 3 y 4, 1989

Quasiconformal Mappings Onto John Domains

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1. Introduction

In this paper we study quasiconformal homeomorphisms of the unit ball $\mathbb{B} = \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$ onto John domains. We recall that John domains were introduced by F. John in his study of rigidity of local quasi-isometries [J]; the term John domain was coined by O. Martio and J. Sarvas seventeen years later [MS]. From the various equivalent characterizations we shall adapt the following definition based on diameter carrots, *cf.* [V4], [V5], [NV].

Let E be an arc in $\mathbb{R}^n$ with end points x_0 and x_1, and let $E[x_1, x]$ denote the subarc of E between $x_1, x \in E$. For $b \geqslant 1$ the open set

$$\operatorname{car}(E, b) = \bigcup \{B(x, b^{-1} \operatorname{diam} E[x_1, x]) : x \in E\}$$

is called a *b-carrot* (or *b-cone* [GHM]) with vertex x_1 joining x_1 to x_0. Here $B(x, r)$ denotes the open n-ball centered at x with radius r. A domain D in $\mathbb{R}^n$ is said to be a *b-John domain* with center x_0 if there is $x_0 \in D$ such that each $x_1 \in D$ can be joined to x_0 by a b-carrot in D. It follows that if $D \neq \mathbb{R}^n$ is b-John, then it is bounded; indeed, $D \subset B(x_0, b \operatorname{dist}(x_0, \partial D))$.

Among simply connected planar domains John domains can be recognized from a number of different geometric properties as well as from the properties of the Riemann mapping [P2], [NV], [GHM]. It is our purpose in this paper to show that certain analogues of those results can be found also in higher dimensions. In fact, if D is a bounded domain in $\mathbb{R}^n$ and quasiconformally equivalent to the unit ball, then our main theorem provides nine equivalent conditions for D to be John. Two of those conditions were previously known [V5], [NV]. It is interesting to note that in our main theorem, Theorem 3.1, the requirement «D is quasiconformally equivalent to the unit ball» cannot be replaced *e.g.* by «D is homeomorphic to the unit ball» or «D is a Jordan domain». Thus, among all John domains those which can be quasiconformally mapped to a ball lend themselves to more clear pictures.

The main theorem is stated in Section 3 after some preliminary discussion. Our proofs are mainly based on the modulus method but, at least implicitly, also the analytic aspects of the higher dimensional quasiconformal theory are present. We also feel that J. Väisälä's theory of quasisymmetric mappings has come to be an indispensable guide to the geometry of John domains.

The proof of the main theorem leads us to consider more general subinvariance properties of certain domains under quasiconformal mappings. These phenomena were previously studied in [FHM] and [V5]. In Section 6 we present a quite general theorem which describes the internal distortion of quasiconformal mappings and extends a recent result of J. Väisälä [V5, Theorem 2.20]. A few corollaries will be discussed in Section 7; we demonstrate, for instance, that broad domains are subinvariant under quasiconformal mappings.

There is a beautiful theorem due to F. W. Gehring and W. K. Hayman [GH] which states that in simply connected planar domains the hyperbolic geodesic essentially has the least length (or diameter, see [P1, pp. 136]) among all paths with same endpoints. In proving our main theorem we shall require a similar result which can be viewed as a quasiconformal analogue of the Gehring-Hayman Theorem and which as such may have some independent interest. This result, Theorem 4.1, is stated and proved in Section 4. Having seen the first draft of this paper, R. Näkki informed the author that Theorem 4.1 also follows from [HN, Theorem 2] after a simple limiting procedure.

I wish to thank J. Väisälä for generously showing me his unpublished work and P. Koskela whose question about the equivalence of I and VIII in Theorem 3.1 partly led me to investigate the problems in this paper. I also thank the referee for a meticulous reading of the paper and for many useful comments. Indeed, the proofs for the implications III $\Rightarrow$ IV and III $\Rightarrow$ VI in Theorem 3.1 are due to the referee; this route of reasoning substantially shortened my original arguments.

2. Some Definitions and Lemmas

Before stating and proving our main theorem we shall record in this section some definitions and results needed later on.

2.1. Notation

Our basic notation is fairly standard and generally as in [V1]. For example, D and D' will denote proper subdomains of $\mathbb{R}^n$, $n \geqslant 2$, and $f\colon D \to D'$ includes the assumption that f is a homeomorphism onto D'. Open balls and spheres in a metric space (X, e) are denoted by $B_e(x, r)$ and $S_e(x, r)$, respectively; whenever X is a subset of $\mathbb{R}^n$ with the euclidean metric in it, the subscript e is omitted. We abbreviate $B(r) = B(0, r)$, $S(r) = S(0, r)$ and $\mathbb{S} = \partial\mathbb{B}$, where $\mathbb{B} = B(0, 1)$ is the unit ball. By a boundary cap $I \subset \mathbb{S}$ we mean a set of the form $\bar{B}(x, r) \cap \mathbb{S}$ for some $x \in \bar{\mathbb{B}}$. The (euclidean) diameter of a set A is $d(A)$ and the (euclidean) distance between sets A and B is $d(A, B)$. For brevity, $d(\{x\}, A) = d(x, A)$. If E is an arc in $\mathbb{R}^n$ and $x, y \in E$, then $E[x, y]$ will denote the closed subarc of E between x and y. The closed line segment between points $x, y \in \mathbb{R}^n$ is denoted by $[x, y]$.

2.2. John domains and cigars

In addition to the definition given in the introduction we will need the following cigar property of John domains. Recall that if E is an arc in $\mathbb{R}^n$ with endpoints x_1 and x_2, then for $b \geqslant 1$ the open set

$$\operatorname{cig}(E, b) = \bigcup \Big\{ B\Big(x, b^{-1} \min_{i=1,2} d(E[x_i, x])\Big) : x \in E \Big\}$$

is a *b-cigar* (or *double cone* [GHM]) joining x_1 and x_2.

Then a bounded domain D is a b-John domain if and only if each pair of points in D can be joined by a b'-cigar in D; the constants b and b' depend only on each other [NV, Theorem 2.16].

This equivalence allows us to define John domains in the compactified space $\bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$: a domain D in $\bar{\mathbb{R}}^n$ is a b-John domain if each pair of points in $D \cap \mathbb{R}^n$ can be joined by a b-cigar in D, see [NV]. John domains in $\bar{\mathbb{R}}^n$ are briefly discussed in Section 6.

2.3. The internal metric

The internal metric δ_D in D is defined by

$$\delta_D(x, y) = \inf d(E)$$

where the infimum is taken over all arcs joining x and y in D. We shall often abbreviate $\delta_D = \delta$. The internal distance between two sets $A, B \subset D$ is written as $\delta_D(A, B)$, and the internal diameter of $A \subset D$ is $\delta_D(A)$.

2.4. Broad domains

Let $\varphi: (0, \infty) \to (0, \infty)$ be a decreasing homeomorphism. We say that D is *φ-broad* if for each $t > 0$ and each pair (C_0, C_1) of continua in D the condition $\delta_D(C_0, C_1) \leqslant t \min\{d(C_0), d(C_1)\}$ implies $M(\Delta(C_0, C_1; D)) \geqslant \varphi(t)$. Recall that $\Gamma = \Delta(C_0, C_1; D)$ is the family of all paths joining C_0 and C_1 in D, and $M(\Gamma)$ denotes the modulus of Γ.

Broad domains were introduced in [V5] and it was later proved in [NV] that a simply connected planar domain is broad if and only if it is John. Broad domains also provide some new insight to internal distortion properties of quasiconformal mappings, *cf.* [V5, Theorem 2.20] and Theorem 6.1 below.

The definition for broad domains in $\bar{\mathbb{R}}^n$ is similar.

2.5. Linearly locally connected sets

Suppose that A is a subset of D and $b \geqslant 1$. We say that A is *b-LLC_2* (with respect to δ_D) in D if for all $x \in A$ and $r > 0$ the points in $A \backslash \bar{B}(x, br)$ (in $A \backslash \bar{B}_{\delta_D}(x, br)$) can be joined in $D \backslash \bar{B}(x, r)$ (in $D \backslash \bar{B}_{\delta_D}(x, r)$). If $A = D$, we say D is *b-LLC_2* or *b-LLC_2* with respect to δ_D.

The expression LLC_2 is used because the condition above is but the second of the two requirements placed on linearly locally connected domains (then $A = D$), *cf.* [G], [GM1], [V3].

It turns out that if D is LLC_2, then it is LLC_2 with respect to δ_D; see Lemma 5.12 below. However, the converse need not be true in general (the examples that we have found are somewhat complicated and irrelevant in this connection).

Note that if A is *b-LLC_2* in D, then it need not be connected.

2.6. Quasisymmetric mappings

Let X_1 and X_2 be metric spaces with distance written as $|x - y|$ and let $\eta: [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f: X_1 \to X_2$ is *η-quasisymmetric* if $|a - x| \leqslant t|a - y|$ implies $|f(a) - f(x)| \leqslant \eta(t)|f(a) - f(y)|$ for all $a, x, y \in X_1$. If there is $H \geqslant 1$ such that $|a - x| \leqslant |a - y|$ implies $|f(a) - f(x)| \leqslant H|f(a) - f(y)|$, then f is said to be *weakly H-quasisymmetric.* Clearly an η-quasisymmetric mapping is weakly quasisymmetric (with $H = \eta(1)$) but the converse is not true in general. For background information about quasisymmetric mappings and their role in Geometric Function Theory see [TV], [V2], [V3], [V5].

A quasisymmetric embedding $f: D \to D'$ is always quasiconformal whilst the converse is true only for certain domains [V3].

We also have the following

Lemma 2.7. [V2, Theorem 2.4]. *Suppose that $f: D \to D'$ is K-quasiconformal, $x \in D$, and $0 < \lambda < 1$. Then $f|B(x, \lambda d(x, \partial D))$ is η-quasisymmetric, where η depends only on n, K, and λ.*

The next lemma follows from Lemma 2.7; see also [V1, Theorem 18.1].

Lemma 2.8. *Suppose that $f: D \to D'$ is K-quasiconformal, $x \in D$, and $0 < \lambda < 1$. Then there are positive constants λ_1 and λ_2, depending only on n, K and λ, such that*

$$B(f(x), \lambda_1 d(f(x), \partial D')) \subset f(B(x, \lambda_2 d(x, \partial D))) \subset B(f(x), \lambda d(f(x), \partial D')).$$

2.9. The function a_f

Let $f: D \to D'$ be K-quasiconformal. For $x \in D$ write

$$B_x = B\left(x, \frac{1}{2} d(x, \partial D)\right)$$

and set

$$a_f(x) = \exp\left(\frac{1}{nm(B_x)} \int_{B_x} \log J_f dm\right), \tag{2.10}$$

where J_f is the Jacobian of f and $m(B_x)$ stands for the n-measure of the ball B_x.

It was observed by Astala and Gehring that for certain distortion properties of quasiconformal mappings the function a_f plays a role analogous to that played by $|f'|$ when f is planar and conformal [AG1], [AG2]. In particular,

Lemma 2.11. [AG2, Theorem 1.8]. *There is a constant $c = c(n, K)$ such that*

$$\frac{1}{c} \frac{d(f(x), \partial D')}{d(x, \partial D)} \leqslant a_f(x) \leqslant c \frac{d(f(x), \partial D')}{d(x, \partial D)}$$

for all $x \in D$.

The careful reader notices that in [AG1], [AG2] the integral in (2.10) is defined with $B_x = B(x, d(x, \partial D))$. However, as seen from (2.12) below, these two definitions prove to be equivalent and for our purposes (2.10) is more convenient.

The next lemma derives from Lemma 2.11 and from the n-dimensional version of [AG1, Lemma 5.10]:

$$\left|\frac{1}{m(B_1)} \int_{B_1} \log J_f dm - \frac{1}{m(B_2)} \int_{B_2} \log J_f dm\right| \leqslant c(n, K)\left(\log \frac{m(B_1)}{m(B_2)} + 1\right); \tag{2.12}$$

here $f: D \to D'$ is K-quasiconformal and $B_2 \subset B_1$ are balls in D.

Lemma 2.13. *Let $f\colon \mathbb{B} \to D$ be K-quasiconformal and $x, y \in \mathbb{B}$. Then there are constants c_1, c_2 which depend only on n, K, and the hyperbolic distance between x and y such that*

$$\frac{1}{c_1} a_f(y) \leqslant a_f(x) \leqslant c_1 a_f(y) \tag{2.14}$$

and

$$\frac{1}{c_2} d(f(y), \partial D) \leqslant d(f(x), \partial D) \leqslant c_2 d(f(y), \partial D). \tag{2.15}$$

Recall that the hyperbolic metric in $\mathbb{B}$ is given by the metric density

$$ds = \frac{2|dx|}{1 - |x|^2};$$

the hyperbolic geodesic joining two points x and y in $\mathbb{B}$ is an arc of a circle orthogonal to $\mathbb{S}$.

The final result we record in this section is the following consequence of a theorem due to M. Zinsmeister; see [Z, Theorem 2].

For $x \in \mathbb{B}$ we define the cap $I(x) = \bar{B}(x, 3(1 - |x|)) \cap \mathbb{S}$.

Lemma 2.16. *Let $f\colon \mathbb{B} \to D$ be K-quasiconformal and let $x, y \in \mathbb{B}$ be such that $I(y) \subset I(x)$. Then there is a hyperbolic geodesic L from x to $I(y)$ such that*

$$d(f(L)) \leqslant c d(f(x), \partial D), \tag{2.17}$$

where the constant c depends only on n, K, and the hyperbolic distance between x and y (or, equivalently, on the ratio $d(I(y))/d(I(x))$).

3. Main Theorem

Let f be a K-quasiconformal mapping from $\mathbb{B}$ onto a bounded domain D. We assume further that f has a continuous extension to $\bar{\mathbb{B}}$, which is true if and only if D is finitely connected on the boundary [V1, pp. 58], in particular if D is John [NV, 2.17].

The following is the main result of the paper.

Theorem 3.1. *The following are equivalent*

I. *D is b-John with center $f(0)$;*

II. *D is φ-broad;*

III. *$f\colon \mathbb{B} \to (D, \delta_D)$ is η-quasisymmetric;*

IV. *$d(f(I(x))) \leqslant b d(f(x), \partial D)$ for all $x \in \mathbb{B}$ and $I(x) = \bar{B}(x, 3(1 - |x|)) \cap \mathbb{S}$;*

V. *$d(f([x, w])) \leqslant b d(f(x), \partial D)$ for all $w \in \mathbb{S}$ and $x \in [0, w]$;*

VI. *$a_f(rw)(1 - r)^{1-\alpha} \leqslant b a_f(\rho w)(1 - \rho)^{1-\alpha}$ for all $w \in \mathbb{S}$ and $0 \leqslant \rho \leqslant r < 1$;*

VII. *$\dfrac{d(f(I))}{d(f(J))} \leqslant b\left(\dfrac{d(I)}{d(J)}\right)^{\alpha}$ for all boundary caps $I \subset J \subset \mathbb{S}$;*

VIII. *D is b-LLC_2;*

IX. *D is b-LLC_2 with respect to δ_D;*

X. *$f\colon \mathbb{B} \to (D, \delta_D)$ is weakly H-quasisymmetric.*

The constants b, α, H (not necessarily the same at each occurrence) and the functions φ, η depend only on each other and the data

$$v = \left(n, K, \frac{d(D)}{d(f(0), \partial D)}\right).$$

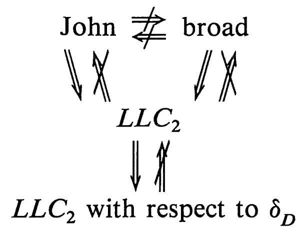
The equivalence of I, II, and III is known. The hard part is to show that I (or II) implies III [V5, Theorem 2.20] whereas it is considerably simpler to demonstrate that John domains and broadness are preserved under quasisymmetric mappings [NV, Theorems 3.6, 3.9]. We shall prove

$$\begin{array}{lllllll} \text{III} \Rightarrow & \text{IV} \Rightarrow & \text{V} \Rightarrow & \text{VIII} \Rightarrow & \text{IX} \Rightarrow & \text{X} \Rightarrow & \text{I} \\ \Downarrow & \Uparrow & & & & & \\ \text{VI} \Rightarrow & \text{VII} & & & & & \end{array}$$

We also provide a new proof for the implications I $\Rightarrow$ III and II $\Rightarrow$ III; see Remark 6.7 (*b*).

In the plane most of the implications are known for conformal mappings. In particular, the equivalence of IV, V, VI, and VII was proved by Ch. Pommerenke [P2].

Let it be remarked that the equivalence of I, II, VIII, and IX is not true for general domains when the picture is as follows



Proofs for the implications can be found below in Lemmas 6.2, 7.2, and 5.12. As for the counterexamples, it is clear that throwing in a countable set of points may destroy the carrot property of John domains whereas the modulus remains intact; on the other hand, by judiciously removing open intervals $\{I_i\}$ from $[0, 1]$, the John domain $D = (\mathbb{B}^2 \setminus [0, 1]) \cup \{I_i\colon\ i = 1, 2, \dots\}$ is not broad. Further, if $n \geqslant 3$, a Jordan domain with an outward directed wedge is LLC_2 but neither John nor, if the wedge is sharp enough, broad.

The reader is invited to compare Theorem 3.1 to results in [GM1], [V3] and how the concepts John, broad, LLC_2, and quasisymmetry in δ_D are related to their predecessors: uniform, QED, LLC, and quasisymmetry. The analogue is particularly patent in Theorem 6.1 below from which we derive the implications IX $\Rightarrow$ X, I $\Rightarrow$ III, and II $\Rightarrow$ III as a special case.

Astala and Gehring proved in [AG2] that if f is a bounded K-quasiconformal mapping in $\mathbb{B}$, then f is Hölder continuous in $\bar{\mathbb{B}}$ with the exponent $\alpha, 0 < \alpha \leqslant K^{1/(1-n)}$, if and only if $a_f(x) \leqslant b(1 - |x|)^{\alpha - 1}$. It has been proved by several authors [NP], [GM2], [MV] that quasiconformal mappings onto John domains are Hölder continuous. In light of the Astala-Gehring theorem, Theorem 3.1 VI above shows that slightly more is true.

4. A Distortion Theorem for Quasiconformal Mappings

In this section we establish the following theorem (see [HN, Theorem 2] for a similar result).

Theorem 4.1. *Let $f\colon \mathbb{B} \to D$ be K-quasiconformal and bounded. Let L be the line segment from* 0 *to a point $w \in \mathbb{S}$. If γ is any arc joining* 0 *to w in $\mathbb{B}$, then*

$$d(f(L)) \leqslant b\,d(f(\gamma))$$

where the constant b depends only on n and K.

Proof. We denote the images by primes: $f(x) = x'$, $x \in \mathbb{B}$, $f(A) = A'$, $A \subset \mathbb{B}$. By normalizing we may assume that $d(\gamma') = 1$. Fix a point $z' \in \gamma'$ so that $\gamma' \subset \bar{B}(z', 1)$. For $k = 0, 1, 2, \dots$ let $L_k = [(1 - 2^{-k})w, (1 - 2^{-k-1})w]$. We shall first show that for each k there is a point $x_k \in L_k$ such that $f(x_k) \subset \bar{B}(z', c)$ for some $c = c(n, K) < \infty$. Indeed, if $L'_k \subset \mathbb{R}^n \setminus \bar{B}(z', c)$, then

$$M(\Delta(L'_k, \gamma'; D)) \leqslant \omega_{n-1}(\log c)^{1-n}$$

while the modulus estimate [V1, 10.12] implies

$$M(\Delta(L_k, \gamma; \mathbb{R}^n)) \geqslant c(n).$$

Since

$$\frac{1}{2}M(\Delta(L_k, \gamma; \mathbb{R}^n)) \leqslant M(\Delta(L_k, \gamma; \mathbb{B}))$$
$$\leqslant KM(\Delta(L'_k, \gamma'; D)),$$

the desired upper bound for c can be found.

Next, fix $k \geqslant 1$ and $y \in L_k$. Let $x_{k-1} \in L_{k-1}$ and $x_{k+1} \in L_{k+1}$ be points whose images lie in $\bar{B}(z', c)$. Since

$$\frac{1 - |x_{k-1}|}{1 - |x_{k+1}|} \leqslant \frac{2^{-k+1}}{2^{-k-2}} = 8,$$

the points x_{k-1} and x_{k+1} lie in a hyperbolic ball with fixed radius; by Lemma 2.7 f is $\eta = \eta(n, K)$-quasisymmetric in that ball. Since

$$|y - x_{k+1}| \leqslant |x_{k-1} - x_{k+1}|,$$

then

$$|y' - x'_{k+1}| \leqslant \eta(1)|x'_{k-1} - x'_{k+1}| \leqslant 2\eta(1)c = c',$$

and hence

$$|y' - z'| \leqslant |y' - x'_{k+1}| + |x'_{k+1} - z'| \leqslant c' + c = c(n, K).$$

A similar reasoning shows that if $y \in L_0$, then $y' \in \bar{B}(z', c(n, K))$ as well. Consequently, the diameter of $f(L)$ is bounded by a number which depends only on n and K as required.

Remark 4.2. The above proof shows that the conclusion of Theorem 4.1 is retained if L is the hyperbolic geodesic joining two points $w_1, w_2 \in \mathbb{S}$ and γ is any arc joining those points in $\mathbb{B}$.

5. Proof of Theorem 3.1

Throughout the proof we let c, $c_1, \ldots$ denote positive constants, not necessarily the same at each occurrence, which depend only on the numbers b, α, H, the functions φ, η, and the data

$$v = \left(n, K, \frac{d(D)}{d(f(0), \partial D)}\right).$$

As usual, $c(a, \ldots)$ denotes a constant which depends only on $a, \ldots$.

III ⇒ IV. Fix $x \in \mathbb{B}$ and write $I = I(x)$. We denote images by primes, *i.e.* for $z \in \bar{\mathbb{B}}$, $z' = f(z)$. Choose $w_0 \in \mathbb{S}$ such that $|x' - w_0'| = d(x', \partial D)$. Then for any point w' on the half open line segment $[x', w_0')$ we have $\delta_D(x', w') = |x' - w'|$, and hence by quasisymmetry

$$\left|\frac{x' - z'}{x' - w'}\right| \leqslant \frac{\delta_D(x', z')}{\delta_D(x', w')} \leqslant \eta\left(\left|\frac{x - z}{x - w}\right|\right) \tag{5.1}$$

for $z \in \mathbb{B}$. In particular, if $w_j' \in [x', w_0')$ such that $w_j' \to w_0'$ and $z_j \in \mathbb{B}$ such that $z_j \to z_0 \in I$, then (5.1) implies

$$|x' - z_j'| \leqslant \eta\left(\frac{|x - z_j|}{|x - w_j|}\right)|x' - w_j'|.$$

By letting $j \to \infty$, we obtain

$$\begin{aligned} |x' - z_0'| &\leqslant \eta\left(\frac{|x - z_0|}{1 - |x|}\right) d(x', \partial D) \\ &\leqslant \eta(3)\, d(x', \partial D). \end{aligned}$$

Consequently,

$$d(f(I)) \leqslant 2\eta(3)\, d(x', \partial D)$$

as required.

IV ⇒ V. Fix $w \in \mathbb{S}$ and $x \in [0, w]$. Let L be a non euclidean segment from x to $I = I(x)$ such that $d(f(L)) \leqslant cd(f(x), \partial D)$, see (2.17). It then follows from Theorem 4.1 that

$$\begin{aligned} d(f([x, w])) &\leqslant cd(f(L) \cup f(I)) \\ &\leqslant c(d(f(L)) + d(f(I))) \\ &\leqslant c_1 d(f(x), \partial D) \end{aligned}$$

as required.

III ⇒ VI. Fix $w \in \mathbb{S}$ and $0 \leqslant \rho \leqslant r < 1$. By Lemma 2.11 it suffices to show

$$\frac{d(f(rw), \partial D)}{d(f(\rho w), \partial D)} \leqslant b\left(\frac{1 - r}{1 - \rho}\right)^{\alpha}, \tag{5.2}$$

where b and α depend only on η and the data v. It follows from [TV, 3.12] that η may be assumed to be of the form $\eta(t) = c\max(t^{\alpha}, t^{1/\alpha})$, where

$c = c(\eta) > 0$ and $\alpha = \alpha(\eta) \leqslant 1$. Thus for $r < s < 1$ we have

$$\frac{\delta_D(f(rw), f(sw))}{\delta_D(f(\rho w), f(sw))} \leqslant \eta\left(\frac{s-r}{s-\rho}\right) \leqslant c\left(\frac{s-r}{s-\rho}\right)^{\alpha}. \tag{5.3}$$

Since

$$\delta_D(f(rw), f(sw)) \geqslant |f(rw) - f(sw)|$$

and since

$$\delta_D(f(\rho w), f(sw)) \leqslant d(f([\rho w, w])) \leqslant bd(f(\rho w), \partial D)$$

by V, (5.3) implies

$$\frac{|f(rw) - f(sw)|}{bd(f(\rho w), \partial D)} \leqslant c\left(\frac{s-r}{s-\rho}\right)^{\alpha}; \tag{5.4}$$

note that we have already established the implications III $\Rightarrow$ IV $\Rightarrow$ V. Finally, by letting $s \to 1$ in (5.4) establishes (5.2) and the proof of III $\Rightarrow$ VI is complete.

VI $\Rightarrow$ VII. Let $I \subset J \subset \mathbb{S}$ be two boundary caps. Choose distinct points $x_I, x_J \in \mathbb{B} \setminus \{0\}$ such that $|x_J| \leqslant |x_I|$ and that

$$(1 - |x_I|) \sim d(x_I, I) \sim d(I), \qquad (1 - |x_J|) \sim d(x_J, J) \sim d(J),$$

where $A \sim B$ means that the ratio A/B is bounded from above and below by an absolute constant. It then follows that the hyperbolic distance from x_I (or x_J) to a point $|x_I|x$, $w \in I$ (or $|x_J|w$, $w \in J$) is bounded by an absolute constant. In particular, Lemmas 2.7 and 2.11 yield

$$\begin{aligned} |f(|x_I|w) - f(x_I)| &\leqslant c(n,K)d(f(x_I), \partial D) \\ &\leqslant c(n,K)(1 - |x_I|)a_f(x_I) \end{aligned} \tag{5.5}$$

for all $w \in I$. Likewise, by (2.14)

$$\frac{1}{c} a_f(|x_I|w) \leqslant a_f(x_I) \leqslant c a_f(|x_I|w), \qquad c = c(n,K), \qquad w \in I. \tag{5.6}$$

Similar estimates hold for x_J. In the conformal case the conclusion follows from (5.5) and (5.6) by integrating $|f'|$ along a line, see [P2, pp. 81]. We need to make the following detour.

Fix $w \in I$ and let $x_1, x_2, \ldots$ be the points on $[|x_I|w, w]$ determined by

$$x_1 = |x_I|w, \qquad 1 - |x_j| = \frac{3}{4}(1 - |x_{j-1}|) \quad \text{for} \quad j \geqslant 2.$$

Write

$$B_j = B(x_j, |x_j - x_{j+1}|) = B\left(x_j, \frac{1}{4}(1 - |x_j|)\right).$$

It again follows from Lemmas 2.7 and 2.11 that

$$|f(x_j) - f(x_{j+1})| \leqslant c d(f(x_j), \partial D) \leqslant c(1 - |x_j|) a_f(x_j). \tag{5.7}$$

We obtain from (5.7) and VI that

$$\begin{aligned} |f(x_j) - f(x_{j+1})| &\leqslant c(1 - |x_j|) a_f(x_j) \\ &\leqslant c(1 - |x_j|)^{\alpha}(1 - |x_I|)^{1-\alpha} a_f(|x_I| w) \\ &= c\left(\frac{3}{4}\right)^{\alpha(j-1)} (1 - |x_I|) a_f(|x_I| w) \end{aligned}$$

whence

$$\begin{aligned} |f(|x_I| w) - f(w)| &\leqslant \sum_{j=1}^{\infty} |f(x_j) - f(x_{j+1})| \\ &\leqslant c(1 - |x_I|) a_f(|x_I| w). \end{aligned} \tag{5.8}$$

Thus, for $w_1, w_2 \in I$

$$\begin{aligned} |f(w_1) - f(w_2)| &\leqslant |f(|x_I| w_1) - f(w_1)| + |f(|x_I| w_1) - f(|x_I| w_2)| \\ &\quad + |f(|x_I| w_2) - f(w_2)| \\ &\leqslant c(1 - |x_I|) a_f(x_I); \end{aligned}$$

here (5.5), (5.6) and (5.8) were utilized. We conclude

$$d(f(I)) \leqslant c(1 - |x_I|) a_f(x_I). \tag{5.9}$$

Next, suppose that we have the lower bound

$$d(f(J)) \geqslant c d(f(|x_J|\, |x_I|^{-1} x_I), \partial D). \tag{5.10}$$

Then Lemma 2.11 implies $d(f(J)) \geqslant c(1 - |x_J|) a_f(x_0)$, where $x_0 = |x_J|\, |x_I|^{-1} x_I$. By combining this with (5.9) we arrive at

$$\frac{d(f(I))}{(1 - |x_I|) a_f(x_I)} \leqslant \frac{c d(f(J))}{(1 - |x_J|) a_f(x_0)}$$

which, in view of VI, is the desired inequality because $1 - |x_J| \sim d(J)$, $1 - |x_I| \sim d(I)$ and $a_f(x_0) \sim a_f(x_J)$.

It remains to establish (5.10) or generally

$$d(f(y), \partial D) \leqslant c(n, K) d(f(J)) \tag{5.11}$$

whenever $J \subset \mathbb{S}$ is a boundary cap and $y \in \mathbb{B}$ is such that $1 - |y| \sim d(y, J) \sim d(J)$. To see why this is true, consider the path family $\Gamma = \Delta(\bar{B}_y, J; \mathbb{B})$ where $B_y = B(y, (1 - |y|)/2)$. Then $0 < c(n) \leqslant M(\Gamma)$, and hence $0 < c(n, K) \leqslant M(f(\Gamma))$. It follows again from (2.15) that

$$\frac{1}{c} d(f(x), \partial D) \leqslant d(f(y), \partial D) \leqslant c d(f(x), \partial D), \qquad x \in B_y, \qquad c = c(n, K),$$

and hence, if

$$\frac{d(f(y), \partial D)}{d(f(J))} = R$$

is very large, we have

$$0 < c(n, K) \leqslant M(f(\Gamma)) \leqslant \omega_{n-1} (\log cR)^{1-n}.$$

This establishes (5.11) and, therefore, the implication VI $\Rightarrow$ VII.

VII $\Rightarrow$ IV. Let $I = I(x) \subset \mathbb{S}$ be a cap as in IV. We follow the idea presented in [P2, pp. 81-82].

Indeed, by VII we can choose a constant $c_1 = c_1(\alpha, b)$ such that if $J \subset I$ and $d(J) < c_1 d(I)$, then $d(f(J)) < d(f(I))/4$. Let c be the constant in Lemma 2.16 corresponding to the value c_1 above. That is, if $J \subset I$ and $d(J) \geqslant c_1 d(I)$, then there is an arc L from x to J satisfying (2.17).

Next consider the set

$$A = \{ z \in \operatorname{int}_{\mathbb{S}} I \colon |f(z) - f(x)| > c d(f(x), \partial D) \}.$$

Then A is open in I and can be written as a countable union of boundary caps A_k with the property that $\bar{A}_k \cap (I \setminus A) \neq \varnothing$. Fix one A_k. Necessarily $d(A_k) < c_1 d(I)$, for otherwise one can find a curve L from x to A_k such that

$$d(f(L)) \leqslant c d(f(x), \partial D) < |f(z) - f(x)| \quad \text{for all} \quad z \in A_k,$$

which is absurd. Thus

$$d(f(A_k)) < \frac{1}{4} d(f(I))$$

by the choice of c_1. Let z_1, z_2 be two interior points of I. If $z_1 \in A_{k_1}$ for some k_1, choose a point $z_1' \in \bar{A}_{k_1} \cap (I \setminus A)$. If $z_1 \notin A$, set $z_1' = z_1$. Define z_2' similarly.

Then

$$\begin{aligned}|f(z_1)-f(z_2)| &\leqslant |f(z_1)-f(z_1')| + |f(z_1')-f(x)| \\ &\quad + |f(x)-f(z_2')| + |f(z_2')-f(z_2)| \\ &\leqslant d(f(A_{k_1})) + 2cd(f(x),\partial D) + d(f(A_{k_2})) \\ &\leqslant \frac{1}{2}d(f(I)) + 2cd(f(x),\partial D).\end{aligned}$$

Since $z_1, z_2 \in I$ were arbitrary, we have

$$d(f(I)) \leqslant bd(f(x),\partial D)$$

as required.

V ⇒ VIII. Fix $x \in D$ and $r > 0$. Suppose that two components, D_1 and D_2, of $D\setminus\bar{B}(x,r)$ meet $D\setminus\bar{B}(x,cr)$. We shall show that $c \leqslant 4b$.

Assume $f(0) = 0$. First observe that $M = d(D) \leqslant 2bd(0,\partial D)$ and therefore $B(0, 2c_1M) \subset D$ where $c_1 = 1/4b$. If $|x| < c_1M$, then $r > c_1M$, and therefore $D\setminus\bar{B}(x,cr) = \varnothing$ as soon as $c > 4b = 1/c_1$; a similar conclusion holds if $r \geqslant |x| \geqslant c_1M$. We may therefore assume that $0 \notin \bar{B}(x,r)$. Then 0 is not in one of the components D_1 and D_2, say $0 \notin D_1$. There is a boundary point $w \in \mathbb{S}$ such that $f(w) \in \partial D_1 \cap (\mathbb{R}^n\setminus\bar{B}(x,cr))$ and that the arc $\gamma = f([0,w])$ approaches $f(w)$ from D_1. In particular, since $0 \notin D_1$, there is $z \in \gamma \cap \bar{B}(x,r)$. Since $d(\gamma[f(w),z]) \geqslant (c-1)r$, V implies $(c-1)r \leqslant bd(z,\partial D) \leqslant 2br$. This shows that $c \leqslant 2b+1 \leqslant 4b$ as required.

The implication VIII ⇒ IX is an immediate consequence of the following lemma.

Lemma 5.12. *Let A be an arcwise connected subset of a domain D. If A is b-LLC_2 in D, then A is b_1-LLC_2 with respect to δ_D in D with $b_1 = b_1(b)$.*

Proof. Fix $x \in A$ and $r > 0$. Pick $z, y \in A\setminus\bar{B}_{\delta_D}(x,cr)$ and suppose they cannot be joined in $D\setminus\bar{B}_{\delta_D}(x,r)$. We shall show that $c \leqslant 2b+1$.

Let α be an arc joining z and y in A. Choose $z_0 \in \alpha$ and $y_0 \in \alpha$ such that

(i) $$\min\{|x-z_0|, |x-y_0|\} > \frac{1}{2}(c-1)r$$

and

(ii) the subarcs $\alpha[z,z_0]$ and $\alpha[y,y_0]$ lie in $D\setminus\bar{B}_{\delta_D}(x,r)$.

This choice is possible since $\alpha \cap \bar{B}_{\delta_D}(x,r) \neq \varnothing$. Now (i) and the LLC_2-property imply that the points z_0 and y_0 can be joined in $D\setminus\bar{B}(x,(c-1)r/2b)$ by an arc

β. Then $\gamma = \alpha[z, z_0] \cup \beta \cup \alpha[y, y_0]$ joins z and y in D. Necessarily, because of (ii), β meets $\bar{B}_{\delta_D}(x, r)$. Therefore, for some $w \in \beta$, $(c-1)r/2b \leqslant |w - x| \leqslant \delta_D(w, x) \leqslant r$ which establishes the desired inequality $c \leqslant 2b + 1$.

We have two more implications to work out. The proof of IX $\Rightarrow$ X is somewhat long and, by the same token, we shall establish a more general result: Theorem 6.1 in Section 6. Assuming Theorem 6.1, we next show how to finish the proof of Theorem 3.1. First, the implication IX $\Rightarrow$ X follows directly from Theorem 6.1 by choosing $A = \mathbb{B}$. The final implication X $\Rightarrow$ I is essentially done in [NV, 3.5] but for convenience we incude a proof:

Fix $x'_0 \in D$, $x_0 = f^{-1}(x'_0)$. Let E be the line segment from 0 to x_0 and set $E' = f(E)$. We may assume that E is nondegenerate, for if $x_0 = 0$, there is nothing to prove. Then fix $x' \in E'$ and write $\rho = d(E'[x'_0, x'])$. We need to show

$$B\left(x', \frac{\rho}{c}\right) \subset D, \qquad c = c(H). \tag{5.13}$$

For this, let $x = f^{-1}(x')$ and let $y \in S(x, |x - x_0|)$ be such that

$$|f(y) - x'| = \min_{|z - x| = |x - x_0|} |f(z) - x'|.$$

Then for all $z \in [x_0, x]$ we have $|x - z| \leqslant |x - y|$, and whence

$$\delta_D(f(z), x') \leqslant H\delta_D(f(y), x') = H|f(y) - x'| \leqslant Hd(x', \partial D).$$

It follows that

$$d(E'[x'_0, x']) \leqslant 2Hd(x', \partial D),$$

which proves (5.13) with $c = 2H$.

Save Theorem 6.1, the proof of Theorem 3.1 is now complete.

6. Remarks on Internal Distortion and Subinvariance

The general subinvariance problem can be described as follows. Suppose that $\mathfrak{D}$ is a class of domains in $\mathbb{R}^n$ and $f: D \to D'$ is quasiconformal. When can one conclude that $A \in \mathfrak{D}$ implies $f(A) \in \mathfrak{D}$ for all subdomains A of D? It was shown in [FHM, pp. 120-121] that the conclusion holds if $\mathfrak{D}$ comprises all QED domains in $\mathbb{R}^n$ and $D' \in \mathfrak{D}$; moreover, in conjunction with [V3, Theorem 5.6] this implies that if D' is uniform, then so is $f(A)$ whenever A is a uniform subdomain of D. (The definitions for QED and uniform domains are recalled in Section 7 below.) Subsequently, J. Väisälä [V5, Theorem 2.20] proved that if

D' is broad, then every John subdomain of D is mapped onto a John subdomain of D'. This interesting phenomenon reflects certain internal distortion properties of quasiconformal mappings which, we believe, are worth deeper study.

In this section we first prove the following theorem which in the case of bounded domains generalizes [V5, Theorem 2.20]. Theorem 6.1 also establishes the missing link in the proof of Theorem 3.1. Some applications of Theorem 6.1 to subinvariance problems are discussed in Section 7.

Theorem 6.1. *Suppose that D, D' are bounded, that $f: D \to D'$ is K-quasiconformal, and that D is φ-broad. If $A \subset D$ is such that $f(A)$ is b-LLC_2 with respect to $\delta_{D'}$ in D', then $f|A: A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$ with H depending only on the data*

$$v = \left(n, K, b, \varphi, \frac{\delta_D(A)}{d(x_0, \partial D)}, \frac{\delta_{D'}(f(A))}{d(f(x_0), \partial D')} \right),$$

where x_0 is some fixed point in A.

Before we turn to the proof, let us indicate why Theorem 6.1 can be regarded as an extension of Väisälä's theorem [V5, 2.20]; the only drawback is that in Theorem 6.1 we require the domains to be bounded.

Theorem 2.20 in [V5] follows from Theorem 6.1 above as soon as the following two facts are established:

(i) if $f(A)$ has a b-carrot property in D', then $f(A)$ is b_1-LLC_2 with respect to $\delta_{D'}$ in D';
(ii) in the situation of [V5, 2.20] the weak quasisymmetry implies quasisymmetry.

The condition (ii) derives from [V5, Theorem 2.9] since A is pathwise connected and both A and $f(A)$ are HTB metric spaces by [V5, 2.14 and 2.18]. (The definition for HTB spaces is recalled below before Theorem 6.6.) The condition (i) is established in the following lemma.

Lemma 6.2. *Let D be a domain in $\mathbb{R}^n$ and let $A \subset D$ be such that each $x \in A$ can be joined to a fixed point $x_0 \in D$ by a b-carrot in D. Then A is both b_1-LLC_2 and b_1-LLC_2 with respect to δ_D in D, where $b_1 = b_1(b)$.*

PROOF. The proof is the same for both assertions. Fix $x \in A$ and $r > 0$. Suppose that there are two points $x_1, x_2 \in A \setminus \bar{B}(x, b_1 r)$ which cannot be joined in $D \setminus \bar{B}(x, r)$. We shall show that $b_1 \leqslant 2b + 1$.

Let E_1 and E_2 be the cores of two b-carrots joining x_1 and x_2, respectively, to x_0. Then $E = E_1 \cup E_2$ joins x_1 and x_2 in D. Necessarily E meets $\bar{B}(x, r)$. Pick

$z \in E \cap \bar{B}(x, r)$ and suppose $z \in E_1$. Since $d(E[x_1, z]) > (b_1 - 1)r$ and since E_1 is the core of a b-carrot, we have $B(z, b^{-1}(b_1 - 1)r) \subset D$. On the other hand,

$$d(z, \partial D) \leqslant |z - x| + d(x, \partial D) \leqslant 2r,$$

and hence $b^{-1}(b_1 - 1)r \leqslant 2r$ or $b_1 \leqslant 2b + 1$ as required.

PROOF OF THEOREM 6.1. It is no loss of generality to assume that $x_0 = 0 = f(0)$ and that $1 = d(0, \partial D) = d(0, \partial D')$. We shall denote images by primes: $f(x) = x'$, $x \in D$, $f(E) = E'$, $E \subset D$; and also for brevity $\delta = \delta_D$, $\delta' = \delta_{D'}$, $M = \delta(A)$, $M' = \delta'(A')$.

Thus, let a, x, y be three distinct points in A with $\delta(a, x) \leqslant \delta(a, y)$. (Note that the claim is vacuous if the cardinality of A is less than three.) We need to find an upper bound for

$$H = \frac{\delta'(a', x')}{\delta'(a', y')}.$$

For this we consider two cases.

Case 1.

$$\delta'(a', 0) > c_0 \delta'(a', y')$$

where $c_0 = c_0(v)$ is a constant to be determined later on. We shall show that with an appropriate choice of c_0 too large H generates a contradiction.

In Case 1 we separate to subcases.

Subcase 1*a*.

$$x \in B(0, c_1/2)$$

where

$$c_1 = c_1(n, K), \qquad c_2 = c_2(n, K) < \frac{1}{2}$$

are chosen to satisfy

$$\begin{cases} f^{-1}(B(0, 2c_2)) \subset B\left(0, \dfrac{1}{2}\right); \\ f(B(0, c_1)) \subset B\left(0, \dfrac{c_2}{2}\right) \quad \text{and} \quad f(S(0, c_1)) \cap S\left(0, \dfrac{c_2}{2}\right) \neq \varnothing, \end{cases}$$

see Lemma 2.8.

In proving the claim in the first subcase we again distinguish two possibilities:
(i) $a' \in B(0, c_2)$ or (ii) $a' \notin B(0, c_2)$. If (i) occurs, then

$$H = \frac{\delta'(a', x')}{\delta'(a', y')} = \frac{|a' - x'|}{\delta'(a', y')} \leqslant \left| \frac{a' - x'}{a' - y'} \right|,$$

and because f is $\eta = \eta(n, K)$-quasisymmetric in $B(0, 1/2)$, we obtain $H \leqslant \eta(1)$ provided that $y' \in B(0, 2c_2)$. On the other hand, if $y' \notin B(0, 2c_2)$, then $|a' - y'| \geqslant c_2$, and hence $H \leqslant M'/c_2$.

We may therefore suppose that $a' \notin B(0, c_2)$. Choose a point $z \in S(0, c_1)$ such that $z' \in S(0, c_2/2)$ and let α be the line segment from x to z. Then $\alpha' \subset B(0, c_2/2)$. Let next β' be an arc joining a' and y' in D' with $d(\beta') < 2\delta'(a', y')$, and let $\beta = f^{-1}(\beta')$ be its preimage in D. Then

$$\delta(\alpha, \beta) \leqslant \delta(a, x) \leqslant \delta(a, y) \leqslant d(\beta)$$

and

$$\delta(\alpha, \beta) \leqslant \frac{\delta(a, x) d(\alpha)}{d(\alpha)} \leqslant \frac{2M}{c_1} d(\alpha).$$

Because D is φ-broad, we thus obtain

$$M(\Delta(\alpha, \beta; D)) \geqslant c_3 = c_3(v) > 0,$$

and the quasiconformality of f yields

$$M(\Delta(\alpha', \beta'; D')) \geqslant c_4 = c_4(v) > 0. \tag{6.3}$$

Observe further, that if $\alpha' \cap \bar{B}_{\delta'}(a', c_0 \delta'(a', y')) \neq \varnothing$, then for some $w' \in \alpha' \subset B(0, c_2/2)$

$$\frac{c_2}{2} \leqslant \delta'(a', w') \leqslant c_0 \delta'(a', y')$$

whence

$$H = \frac{\delta'(a', x')}{\delta'(a', y')} \leqslant \frac{2c_0 M'}{c_2} = c(v) < \infty$$

and the proof is complete. We may therefore assume that

$$\alpha' \subset D' \setminus \bar{B}_{\delta'}(a', c_0 \delta'(a', y')).$$

Let us leave the subcase $1a$ for a moment and consider

Subcase 1*b*.

$$x \notin B(0, c_1/2).$$

We still have $\delta'(a',0) > c_0\delta'(a',y')$ and $\delta'(a',x') > c_0\delta'(a',y')$ for otherwise H is less than $c_0 = c_0(v)$, completing the proof. Because of the LLC_2-property, we can join x' to 0 by an arc α' in $D'\backslash\bar{B}_{\delta'}(a', c_5\delta'(a',y'))$ where $c_5 = c_0/b \to \infty$ as $c_0 \to \infty$. Let β' be an arc joining a' and y' as in the subcase 1a, and let again $\alpha = f^{-1}(\alpha')$, $\beta = f^{-1}(\beta')$. Then

$$\delta(\alpha, \beta) \leqslant \delta(a, x) \leqslant \delta(a, y) \leqslant d(\beta)$$

and

$$\delta(\alpha, \beta) \leqslant \frac{\delta(a, x)d(\alpha)}{d(\alpha)} \leqslant \frac{2M}{c_1}d(\alpha).$$

Consequently, as in the subcase 1a the broadness and quasiconformality imply the estimate (6.3) for the present continua α' and β' as well.

Thus in both subcases we have arrived at the situation where

(i) x' is joined to a point in D' by an arc α' which lies entirely outside the ball $\bar{B}_{\delta'}(a', c_6\delta'(a',y'))$, where the constant c_6 depends only on b and c_0, and $c_6 \to \infty$ as $c_0 \to \infty$,

(ii) a' is joined to y' by an arc β' which lies entirely inside the ball $B_{\delta'}(a', 2\delta'(a',y'))$, and

(iii) the modulus estimate (6.3) holds.

It is desirable that this leads to a contradiction if $c_0 = c_0(v)$ is large enough. But that is indeed the case, for if γ is a path joining α' and β' in D', then $d(\gamma) \geqslant (c_6 - 2)\delta'(a',y')$ which means that γ joins the spheres $S(a', 2\delta'(a',y'))$ and $S(a', c_7\delta'(a',y'))$, where $c_7 \to \infty$ as $c_0 \to \infty$. In conclusion,

$$M(\Delta(\alpha', \beta'; D')) \leqslant \omega_{n-1}\left(\log\frac{c_7}{2}\right)^{1-n}$$

contradicting (6.3) for too large c_0. We have thus shown that $H \leqslant c(v) < \infty$, and the proof is complete in Case 1.

Case 2.

$$\delta'(a',0) \leqslant c_0\delta'(a',y')$$

where $c_0 = c_0(v)$ is the constant in Case 1.

We start by observing that because

$$\delta'(a',0) \leqslant c_0\delta'(a',y') \leqslant \frac{c_0M'}{H}, \tag{6.4}$$

we are allowed to assume that $\delta'(a', 0) = |a'|$. In the rest of the proof we let ϵ denote any function which depends only on the data v and satisfies $\epsilon(H) \to 0$ as $H \to \infty$. In particular, we have by (6.4)

$$|y'| \leqslant |a'| + |a' - y'| \leqslant \delta'(a', 0) + \delta'(a', y') \leqslant \epsilon(H).$$

Assuming that $\epsilon(H) < 1$, let α' be the line segment joining y' and a' in $B(0, \epsilon(H))$. Then

$$M(\Delta(\alpha', \partial D'; D')) \leqslant \omega_{n-1}\left(\log \frac{1}{\epsilon(H)}\right)^{1-n}$$

whence

$$M(\Delta(\alpha, \partial D; D)) \leqslant \epsilon(H),$$

where $\alpha = f^{-1}(\alpha')$. Since α joins a and y in D, it follows from the standard Teichmüller estimate [V1, Theorem 11.9] that

$$M(\Delta(\alpha, \partial D; D)) \geqslant \varkappa_n\left(\frac{d(a, \partial D)}{|a - y|}\right)$$

where $\varkappa_n$ is positive decreasing and $\varkappa_n(t) \to 0$ as $t \to \infty$. Therefore, by choosing H large enough, we infer that $f^{-1}(B(a', d(a', \partial D')))$ contains the ball $B(a, |a - y|)$ which in turn is contained in $B(a, d(a, \partial D)/2)$, see Lemma 2.8. In particular, $x \in B(a, |a - y|)$ in view of our initial assumption $\delta(a, x) \leqslant \delta(a, y) = |a - y|$. Hence $\delta'(a', x') = |a' - x'|$, and this finishes the proof because f is $\eta(n, K)$-quasisymmetric in $B(a, d(a, \partial D)/2)$.

The next theorem is a version of Theorem 6.1 for domains containing the point at infinity. We omit the proof of Theorem 6.5; it is similar to but simpler than that of Theorem 6.1.

Theorem 6.5. *Suppose that $f: D \to D'$ is K-quasiconformal and that $f(\infty) = \infty \in D \cap D'$. Suppose further that D is φ-broad, that $A \subset D$, $\infty \in A$, and $f(A)$ is a b-LLC_2 with respect to $\delta_{D'}$ in D'. Then $f|A: A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$ with H depending only on n, K, φ, and b.*

It is not clear to us if one always could draw the more desirable conclusion «$f: A \to f(A)$ is quasisymmetric in the metrics δ_D and $\delta_{D'}$» in Theorem 6.1. The amenable HTB-criterion given in [V5, Theorem 2.9] is not automatically satisfied as there are domains D which are LLC_2 with respect to δ_D but which are not HTB.

What is more, one often wants to know when f is quasisymmetric in the euclidean metric in some subset A of D, which is indeed a much stronger conclusion than that in Theorem 6.1. The following theorem partially answers this question.

We say that $A \subset D$ is of *b-bounded turning* or b-BT in D if each pair of points $x, y \in A$ can be joined by an arc E in D such that $d(E) \leqslant b|x-y|$; if $A = D$, we say D is b-BT. A metric space (X, e) is said to be *k-homogeneously totally bounded* or k-HTB if k: $[1/2, \infty) \to \mathbb{N}$ is an increasing function and if, for each $\alpha \geqslant 1/2$, every closed ball $\bar{B}_e(x, r)$ can be covered by $k(\alpha)$ sets of diameter less than r/α; see [TV], [V5].

Theorem 6.6. *Suppose that $D, D' \subset \mathbb{R}^n$ are bounded domains and $f\colon D \to D'$ is K-quasiconformal. Suppose further that*

(i) *$A \subset D$ is pathwise connected, b_1-LLC_2 with respect to δ_D, and b_2-*BT *in D;*
(ii) *D' is φ-broad and b_3-*BT.

Then $f\colon A \to f(A)$ is η-quasisymmetric with η depending only on the data

$$v = \left(n, K, \varphi, b_1, b_2, b_3, \frac{d(A)}{d(x_0, \partial D)}, \frac{d(f(A))}{d(f(x_0), \partial D')}\right),$$

where x_0 is some fixed point in A.

PROOF. Theorem 6.1 implies that $g = f^{-1}|f(A)$ is weakly $H(v)$-quasisymmetric in the metrics $\delta_{D'}$ and δ_D. On the other hand, $f(A)$ as a subset of the broad domain D' is $k(n, \varphi)$-HTB in $\delta_{D'}$ by [V5, 2.18] and it is easy to see that the bounded turning condition in (i) implies that A is $k(n, b_2)$-HTB in δ_D. We may then deduce from [V5, 2.9] that g, and hence $f|A$, is $\eta(v)$-quasisymmetric in the internal metrics. The theorem follows from this since the bounded turning condition implies that both δ_D and $\delta_{D'}$ are bilipschitz equivalent to the euclidean metric in A and $f(A)$, respectively.

Remark. 6.7.

(*a*) Similarly to Theorem 6.5, Theorem 6.6 admits a formulation for domains containing ∞. Then the assumptions include $f(\infty) = \infty \in A$ whilst the dependence of η on $d(A)/d(x_0, \partial D)$, and $d(f(A))/d(f(x_0), \partial D')$ disappears.

(*b*) As discussed above, Theorem 6.1 implies [V5, Theorem 2.20] if D and D' are bounded. We therefore have obtained a somewhat different proof for the implications I $\Rightarrow$ III and II $\Rightarrow$ III in Theorem 3.1.

(*c*) If A is connected, then $\delta_D(A)$ and $\delta_{D'}(f(A))$ in Theorem 6.1 can be replaced by $d(A)$ and $d(f(A))$, respectively. See [V5, 2.13].

7. Applications of Theorem 6.1

Several interesting corollaries can be drawn from Theorem 6.1. In this final section we present three such results which we feel have some specific interest.

In [FHM] the following subinvariance property of QED domains was proved: if f is a quasiconformal mapping of a domain D onto a QED-domain D', then $f(A) \subset D'$ is a QED domain whenever $A \subset D$ is a QED domain. Recall that a domain A is b-QED if $M(\Delta(C_0, C_1; \mathbb{R}^n)) \leqslant bM(\Delta(C_0, C_1; A))$ for each pair of continua C_0, C_1 in A, see [GM1]. Next we establish an analogous subinvariance result for broad domains.

Theorem 7.1. *Suppose that D, D' are bounded, that $f: D \to D'$ is K-quasiconformal and that D' is φ-broad. If $A \subset D$ is φ_1-broad, then $f(A) \subset D'$ is φ_2-broad with φ_2 depending only on the data*

$$v = \left(n, K, \varphi, \varphi_1, \frac{d(A)}{d(x_0, \partial D)}, \frac{d(f(A))}{d(f(x_0), \partial D')}\right),$$

where x_0 is some fixed point in A.

Proof. Since broad domains are preserved under mappings which are quasisymmetric in the internal metrics [NV, Theorem 3.9], it thus suffices to show that $f: A \to f(A)$ is $\eta(v)$-quasisymmetric in the metrics δ_D and $\delta_{D'}$. Further, since A and $f(A)$ are pathwise connected $k(v)$-HTB metric spaces in δ_D and $\delta_{D'}$, we only need to show that $g = f^{-1}: f(A) \to A$ is weakly $H(v)$-quasisymmetric in $\delta_{D'}$ and δ_D, see [V5, 2.18 and 2.9]. This in turn follows from Theorem 6.1, Remark 6.7(c), and from the lemma below.

Lemma 7.2. *If A is a φ-broad subdomain of D, then A is b-LLC$_2$ in D with b depending only on n and φ. In particular, A is b_1-LLC$_2$ with respect to δ_D in D with $b_1 = b_1(n, \varphi)$.*

Proof. In view of Lemma 5.12, only the first assertion needs to be proved. Fix $x \in A$ and $r > 0$ and suppose that $z, y \in A \setminus \bar{B}(x, br)$ cannot be joined in $D \setminus \bar{B}(x, r)$. We shall show that $b \leqslant b_0(n, \varphi) < \infty$.

We may clearly assume that $b > 2$. Let E be an arc of finite length joining z and y in A, and let ζ_1 be the first point in E with $|\zeta_1 - x| = r$ when traveling from z to y. We define w_1 to be the first point in E with $|w_1 - x| = br$ when traveling from ζ_1 to z and w_1' to be the first point in E with $|w_1' - x| = br$ when traveling from ζ_1 to y. Next, let E_1 and E_1' be two disjoint subarcs of $E[w_1, w_1']$ joining the spheres $S(x, br)$, $S(x, br/2)$ in $\bar{B}(x, br) \setminus B(x, br/2)$. Then suppose that $E[w_i, w_i']$ and E_i, E_i' have been chosen for $i \geqslant 1$. If $E[w_i', y]$ does not meet

$\bar{B}(x, r)$, then stop. Otherwise let ζ_{i+1} be the first point in $\bar{B}(x, r)$ when traveling from w_i' to y, and define the points w_{i+1}, w_{i+1}' as above: w_{i+1} is the first point in E with $|w_{i+1} - x| = br$ when traveling from ζ_{i+1} to w_i' and w_{i+1}' is the first point in E with $|w_{i+1}' - x| = br$ when traveling from ζ_{i+1} to y. For E_{i+1} and E_{i+1}' we choose two disjoint subarcs of $E[w_{i+1}, w_{i+1}']$ which join $S(x, br)$ and $S(x, br/2)$ in $\bar{B}(x, br)\backslash B(x, br/2)$. Since E has finite length, the process stops at some integer $p \geqslant 1$.

Pick a pair of arcs E_i, E_i', $1 \leqslant i \leqslant p$. By the construction, $\delta_A(E_i, E_i') \leqslant 2br \leqslant 4 \min\{d(E_i), d(E_i')\}$, and hence the broadness of A implies

$$M(\Delta(E_i, E_i'; A)) \geqslant \varphi(4).$$

Next write $\Gamma = \Delta(E_i, E_i'; A)$ and suppose that each path $\gamma \in \Gamma$ goes through $\bar{B}(x, r)$. Since $E_i, E_i' \subset \mathbb{R}^n \backslash B(x, br/2)$, we have by [V1, 6.4]

$$0 < \varphi(4) \leqslant M(\Gamma) \leqslant \omega_{n-1}\left(\log \frac{b}{2}\right)^{1-n}.$$

Therefore, by choosing $b = b(n, \varphi)$ large enough we infer that there is a path γ_i joining E_i and E_i' in $A\backslash\bar{B}(x, r)$. This being true for all $i = 1, \ldots, p$, it is evident that by piecing together all γ_i's and parts of E we can construct a continuum which joins z and y in $A\backslash\bar{B}(x, r)$, hence in $D\backslash\bar{B}(x, r)$, contradicting our initial assumption. It follows that b is bounded by a number which depends only on n and φ, as required.

The proof of Lemma 7.2, and hence that of Theorem 7.1, is complete.

In general, if f maps $\mathbb{B}$ quasiconformally onto a John domain, one cannot hope for better distortion than described in Theorem 3.1. Our next application reveals however that the distortion improves when f is restricted to some Stolz cone.

We recall that a domain D is *b-uniform* if each pair of points $x, y \in D$ can be joined in D by a b-cigar cig (E, b) the core of which satisfies the additional turning condition $d(E) \leqslant b|x - y|$. The *Stolz cone* $C_M(w)$ *with vertex at* $w \in \mathbb{S}$ is defined to be the interior of the closed convex hull of w and the hyperbolic ball centered at 0 with radius $M > 0$.

Theorem 7.3. *Let $C_M(w)$ be a Stolz cone in $\mathbb{B}$ and let $f\colon \mathbb{B} \to D$ be a K-quasiconformal mapping onto a b-John domain D with center $f(0)$. Then $f|C_M(w)$ is η-quasisymmetric with η depending only on the data $v = (n, K, b, M)$. In particular, $f(C_M(w))$ is $b_1(v)$-uniform.*

PROOF. It is well known that quasisymmetric mappings preserve uniform domains, see *e.g.* [V3], and therefore only the first assertion needs to be proved.

We shall show that Theorem 6.6 can be applied to f^{-1} with $D' = \mathbb{B}$ and $A = f(C_M(w))$. For this, observe first that the condition (ii) is immediately met, and clearly is A pathwise connected. Further, since D is φ-broad by Theorem 3.1, we obtain from Theorem 7.1 and Lemmas 7.2 and 5.12 that A is $b(v)$-LLC_2 with respect to δ_D. Note that $d(D) \leqslant bd(f(0), \partial D)$. Hence it remains to verify the bounded turning condition in Theorem 6.6(i).

To this end, let x' and y' be two points in A and denote by x and y their respective preimages in $C = C_M(w)$. Let x^* and y^* be the points on $[0, w]$ with $|x| = |x^*|$, $|y| = |y^*|$. Then the hyperbolic distance between x and x^*, or y and y^*, is bounded by a constant $c_0 = c_0(M)$. We may suppose that the hyperbolic balls $D(x, 3c_0)$ and $D(y, 3c_0)$ do not intersect; for if that were the case, f would be $\eta(n, K, M)$-quasisymmetric on the line segment $[x, y]$ by Lemma 2.7, whence $d(f[x, y]) \leqslant c(v)|x - y|$, proving the assertion.

Next suppose that $0 \in D(x, 2c_0)$. Then there are positive numbers $\lambda_1 = \lambda_1(M)$, $\lambda_2 = \lambda_2(M)$ such that $0 < \lambda_1 < \lambda_2 < 1$ and $x \in B(0, \lambda_1)$, $y \notin B(0, \lambda_2)$. Choose points z_1, $|z_1| = \lambda_1$, and z_2, $|z_2| = \lambda_2$ such that

$$|f(z_1) - f(z_2)| \leqslant |f(x) - f(y)| = |x' - y'|. \tag{7.4}$$

Since f is $\eta(n, K, M)$-quasisymmetric in $B(0, \lambda_2)$, we have

$$\frac{|f(0) - f(z_2)|}{|f(z_1) - f(z_2)|} \leqslant \eta\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)$$

whence

$$|f(z_1) - f(z_2)| \geqslant c_1|f(0) - f(z_2)| \geqslant c_2 d(f(0), \partial D), \qquad c_2 = c_2(n, K, M),$$

where the last inequality again is a consequence of Lemma 2.8. This together with (7.4) insures that any arc E joining x' and y' in D satisfies

$$\begin{aligned} d(E) \leqslant d(D) &\leqslant c_3 \frac{d(D)}{d(f(0), \partial D)} |f(z_1) - f(z_2)| \\ &\leqslant c_4|x' - y'|, \end{aligned}$$

where $c_4 = c_4(v)$.

The proof of the theorem is therefore complete if $0 \in D(x, 2c_0)$ or, by symmetry, if $0 \in D(y, 2c_0)$.

Next we suppose that $0 \notin D(x, 2c_0) \cup D(y, 2c_0)$ and invoke Lemma 3.1 in [GP]: there is a $K_1(M)$-quasiconformal self mapping g of $\mathbb{B}$ such that

$$g(x^*) = x, \qquad g(y^*) = y, \quad \text{and} \quad g(z) = z$$

for all z not in $D(x, 2c_0) \cup D(y, 2c_0)$. Then the $K_2(M)$-quasiconformal mapping

$h = f \circ g: \mathbb{B} \to D$ satisfies $h(0) = f(g(0)) = f(0)$, $h(x^*) = x'$ and $h(y^*) = y'$, and it is therefore no loss of generality to assume originally that x and y lie on the same ray $[0, w] \subset C_M(w)$. In fact, it suffices to show that under the conditions of Theorem 7.3

$$d(f[x, y]) \leqslant b_1 |f(x) - f(y)|, \qquad b_1 = b_1(v), \tag{7.5}$$

whenever x and y lies on a line $[0, w]$, $w \in \mathbb{S}$.

To establish (7.5) we may assume that $|x| \leqslant |y|$. Then the ball $B(x, |x - y|)$ is contained in $\mathbb{B}$, and its image $f(B(x, |x - y|))$ is $\varphi(v)$-broad by Theorem 7.1. The desired conclusion now follows from Theorem 3.1 II and V, applied to the ball $B(x, |x - y|)$.

The proof of Theorem 7.3 is complete.

Our final application divulges a property of conformal mappings, generalizing [FHM, Theorem 1]. J. Väisälä had proved the following theorem before this author in an unpublished manuscript.

Theorem 7.6. *Let D be a doubly connected domain in the Riemann sphere and let $f: D \to \mathfrak{A}$ be a K-quasiconformal mapping onto an annulus $\mathfrak{A} = B(0, R) \setminus \bar{B}(0, r)$. If A is a circle in D, then $f(A)$ is a quasicircle in $\mathfrak{A}$ with constant depending only on K and R/r, the modulus of D.*

PROOF. Note that the theorem is trivial if $r = 0$ and $R = \infty$ so that we may assume $0 \leqslant r < R < \infty$; also if $r = 0$, the assertion follows from [FHM, Theorem 1] but we do not need that result.

By performing preliminary Möbius transformations, we may assume that $\infty \in A$ and that f maps D onto D' with $f(\infty) = \infty$, where D' is the image of $\mathfrak{A}$ under a Möbius transformation. We apply Theorem 6.6 for unbounded domains, see Remark 6.7. Indeed, the line A clearly satisfies the assumptions in (i), and it is not difficult to see that D' is φ-broad with φ depending on the modulus R/r only. Thus f is $\eta(v)$-quasisymmetric on A, in particular $f(A)$ satisfies Ahlfors' three point condition whence it is a quasicircle [G]. Theorem 7.6 is proved.

We close the paper by two questions.

Question 1. (Question of J. Väisälä.) Are John domains subinvariant under quasiconformal mappings? In other words, if $f: D \to D'$ is a quasiconformal mapping onto a John domain D', is it then true that every John subdomain of D is mapped onto a John subdomain of D'?

Question 2. Suppose that D is a b-John domain with center x_0. It was shown in [GHM] that if D is planar and simply connected and if E is a quasihyperbolic geodesic joining a point x to x_0 in D, then $\operatorname{car}(E, b') \subset D$ for some $b' = b'(b)$. Is this property of John disks shared by John domains in $\mathbb{R}^n$ which are quasiconformally equivalent to the unit ball $\mathbb{B}$? Note that for general John domains the answer is no, [GHM].

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* This research was supported in part by the National Science Foundation, Grant DMS 89-02749.