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Existence of Solutions for some Elliptic Problems with Critical Sobolev Exponents

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geqslant 3$. In this paper we are concerned with the problem of finding $u \in H_0^1(\Omega)$ satisfying the nonlinear elliptic problems

$$\Delta u + |u|^{\frac{n+2}{n-2}} + f(x) = 0 \tag{1.1}$$

in Ω and $u(x) = 0$ on $\partial\Omega$, and

$$\Delta u + u + |u|^{\frac{n+2}{n-2}} + f(x) = 0 \tag{1.2}$$

in Ω and $u(x) = 0$ on $\partial\Omega$, when of $f \in L^\infty(\Omega)$.

The exponent $q = (n + 2)/(n - 2)$ is critical for the Sobolev embedding of $H_0^1(\Omega)$ in $L^{q+1}(\Omega)$. This embedding is not compact and therefore the operator $F: H_0^1(\Omega) \to H_0^1(\Omega)$, for which its fixed points are solutions of (1.1) or (1.2), is not compact. For these reasons, standard fixed-points methods can not be applied to find solutions of (1.1) and (1.2). In this paper we study (1.1) and (1.2) by making use of a fixed point theorem as well as one from approximation methods. Problems of type (1.1) and (1.2) have been studied in [2] and [4]. In these papers the authors find positive solutions. Their methods are variational and their work is related to the Yamabe Problem. For a complete description of the Problem of Yamabe, we refer to [6].

If $f = 0$, it is well known that the equation (1.1) has no positive solutions. (See [2 p. 422].) If $f \neq 0$ we will see in Theorem 3.3 that the equation (1.1) has a nontrivial solution and in the case $f > 0$, by the maximum principle, we have that the equation (1.1) has a positive solution.

Also, if $f = 0$, the equation (1.2) has a positive solution in the case that $1 \in (0, \lambda_1)$, where λ_1 denote the first eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω. (See [2, p. 441].) If $f \neq 0$, we will see in Theorem 3.4 that equation (1.2) has nontrivial solution and that if $f > 0$ then the solution is positive.

2. Preliminaries

Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by $g(x, u) = |u|^s + f$ in the case of problem (1.1), or $g(x, u) = u + |u|^s + f$, in the case of problem (1.2), and $0 < s \leqslant (n + 2)/(n - 2)$. Then the operator of Nemytsky $G: L^{s+1}(\Omega) \to L^{\frac{s+1}{s}}(\Omega)$ defined by $G(u)(x) = g(x, u(x))$ is continuous and bounded, so that for every $\epsilon > 0$ there exists $r = r(\epsilon)$ such that if $\|u\|_{L^{s+1}(\Omega)} \leqslant r$, then $\|g(x, u) - g(x, 0)\|_{L^{(s+1)/s}(\Omega)} \leqslant \epsilon$ and we have the following inequality, (see [5, p. 26]),

$$\|g(x, u)\|_{L^{(s+1)/s}(\Omega)} \leqslant \left[\left(\frac{\|u\|_{L^{s+1}(\Omega)}}{r}\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|f\|_{L^{(s+1)/s}(\Omega)}. \tag{2.1}$$

For short, we will indicate $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$, for all $p > 0$.

It is well-known (see [1, p. 40]) that if $1 \leqslant s + 1 < 2n/(n - 2)$, $n \geqslant 3$, the inclusion $H_0^1(\Omega) \to L^{s+1}(\Omega)$ is completely continuous. If $1 \leqslant s + 1 \leqslant 2n/(n - 2)$ the inclusion is only continuous and we have

$$\|u\|_{s+1} \leqslant \hat{K}(s)\|u\|_{1,2}, \tag{2.2}$$

where $\|\cdot\|_{1,2}$ is the norm of the space $H_0^1(\Omega)$. In the case $s = 1$, $\hat{K}(1) = 1/\sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the operator $-\Delta$.

If

$$s = \frac{n+2}{n-2}$$

then

$$\hat{K}\left(\frac{n+2}{n-2}\right) = \frac{n-1}{n-2}\,\frac{1}{\sqrt{n}}. \tag{2.3}$$

(see [1, p. 41]).

Since Measure $(\Omega) = |\Omega|$ is finite then

$$\|u\|_{s+1} \leqslant |\Omega|^{\frac{1}{s+1} - \frac{n-2}{2n}} \cdot \|u\|_{\frac{2n}{n-2}}. \tag{2.4}$$

(2.2) – (2.4) yield, for all $u \in H_0^1(\Omega)$,

$$\|u\|_{s+1} \leqslant K(s)\|u\|_{1,2}, \tag{2.5}$$

where

$$K(s) = \frac{1}{\sqrt{n}} \frac{n-1}{n-2} |\Omega|^{\frac{1}{s+1} - \frac{n-2}{2n}}.$$

SOLUTIONS OF (1.1) AND (1.2). Let

$$g(x, u) = |u|^s + f \quad \text{or} \quad g(x, u) = |u|^s + u + f, \qquad s \leqslant \frac{n+2}{n-2}.$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.1) and (1.2) respectively, if for all $v \in H_0^1(\Omega)$

$$\langle u, v \rangle_{1,2} = \int_\Omega g(x, u(x))v(x)\,dx. \tag{2.6}$$

For $u \in H_0^1(\Omega)$ fixed, the right side of (2.6) defines a linear, continuous functional. Then by Riesz's Theorem there exists $F: H_0^1(\Omega) \to H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\langle F(u), v \rangle_{1,2} = \int_\Omega g(x, u)v \tag{2.7}$$

Then, by virtue of (2.6) and (2.7), $u \in H_0^1(\Omega)$ is a weak solution of (1.1) or (1.2) if and only if u is a fixed point of F. It is well-known that only for $s < (n+2)/(n-2)$, F is completely continuous. Our main tool will be the following Theorem due to Krasnosel'skii.

Theorem 2.1. *Let $F: H \to H$ be a completely continuous operator defined on a Hilbert space H. Let $D \subset H$ be an open and bounded set such that $0 \notin \partial D$. Suppose that for all $u \in \partial D$, $\langle F(u), u \rangle \leqslant \|u\|^2$. Then F has a fixed point in $\bar{D}$.*

PROOF. See [1, p. 271].

3. The Main Results

First, we are concerned with the following general problem

$$\left.\begin{aligned} \Delta u + g(x, u) &= 0 \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where $g: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is measurable in x, continuous in u and satisfies

$$|g(x,u)| \leqslant a(x) + b|u|^s, \tag{3.2}$$

where $a(x) \in L^{\frac{s+1}{s}}(\Omega)$, $b > 0$, $1 < s+1 < 2n/(n-2)$ and $n \geqslant 3$. Then we have

Theorem 3.1. *Suppose that* $g(x,u)$ *satisfies* (3.2) *and* $0 < s < 1$, *then* (3.1) *has a weak solution.*

PROOF. In the same way as (2.1) we have

$$\|g(x,u)\|_{\frac{s+1}{s}} \leqslant \left[\left(\frac{\|u\|_{s+1}}{r}\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|g(x,0)\|_{\frac{s+1}{s}}. \tag{3.3}$$

(2.5), (2.7) and (3.3) give

$$\langle F(u), u\rangle_{1,2} \leqslant \left\{\left[\left(\frac{K}{r}\|u\|_{1,2}\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|g(x,0)\|_{\frac{s+1}{s}}\right\} K\|u\|_{1,2}. \tag{3.4}$$

We claim that there exists $y > 0$ such that for $\|u\|_{1,2} = y$

$$\left\{\left[\left(\frac{K}{r}y\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|g(x,0)\|_{\frac{s+1}{s}}\right\} Ky \leqslant y^2. \tag{3.5}$$

Since $0 < s < 1$ it is easy to see that, for $y > 0$ sufficiently large, we get (3.5). For (3.4) and (3.5) we have that $\langle F(u), u\rangle_{1,2} \leqslant \|u\|_{1,2}^2$, where $\|u\|_{1,2}$ is sufficiently large. Then Theorem 3.1 follows from Theorem 2.1.

Remark. Theorem 3.1 is a consequence of Theorem 2.5 in [3] (see Example 2.9, p. 122). There, Example 2.9 is done from a variational point of view.

Theorem 3.2. *Suppose that* $g(x,u)$ *satisfies* (3.2) *and* $s = 1$. *Then* (3.1) *has a weak solution if* $\lambda_1 > b$.

PROOF. If $\|u\|_2 \leqslant r$ then $\|g(x,u)\|_2 \leqslant \|a(x)\|_2 + br$. Let $\epsilon = \|a(x)\|_2 + br$. (3.5) yields

$$(K^2y^2 + r^2)^{1/2}\frac{\epsilon}{r} + \|g(x,0)\|_2 \leqslant \frac{y}{K}. \tag{3.6}$$

It is clear that there exists $y > 0$ such that y satisfies (3.6) if

$$\frac{1}{K^2} > K^2\frac{\epsilon^2}{r^2}. \tag{3.7}$$

In this case, we know that $K^2 = \dfrac{1}{\lambda_1}$, therefore (3.7) is equivalent to

$$\lambda_1 > \frac{\|a(x)\|_2}{r} + b. \tag{3.8}$$

For r sufficiently large we get (3.8). Theorem 2.1 now implies our result.

Now we will return to our main problems (1.1) and (1.2). We have the following

Theorem 3.3. *Assume that $f \in L^\infty(\Omega)$. Suppose that at least one of the following inequalities holds*

$$\|f\|_\infty < B(n)|\Omega|^{-\frac{n+2}{2n}}, \quad \textit{if} \quad |\Omega| > 1, \tag{3.9}$$

or

$$\|f\|_\infty < B(n), \qquad \textit{if} \quad |\Omega| \leqslant 1, \tag{3.10}$$

where

$$B(n) = \frac{\dfrac{4}{n+2}\left(\dfrac{n-2}{n+2}\right)^{\frac{n-2}{4}}}{\left(\dfrac{n-1}{n-2}\dfrac{1}{\sqrt{n}}\right)^{\frac{n+2}{2}}}, \qquad n \geqslant 3.$$

Then problem (1.1) *has at least a weak solution if we assume that $\partial\Omega$ is sufficiently smooth.*

PROOF. First we will consider the following problem

$$\left.\begin{aligned} \Delta u + |u|^s + f = 0 \quad &\text{in} \quad \Omega, \\ u(x) = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{3.11}$$

and $1 < s < \dfrac{n+2}{n-2}$.

Let $g(x, u) = |u|^s + f$. As in Theorems 3.1 and 3.2, the problem (3.11) has a weak solution $u_s \in H_0^1(\Omega)$ if there exists $y > 0$ such that

$$\left[\left(\frac{K}{r}y\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|f\|_{\frac{s+1}{s}} \leqslant \frac{y}{K}, \tag{3.12}$$

where K is the constant in (2.5). In this case, it is easy to see that $r = r(\epsilon) = \epsilon^{1/s}$.

Therefore (3.12) becomes

$$[(K^2x)^{s+1} + \epsilon^{\frac{s+1}{s}}]^{\frac{s}{s+1}} + \|f\|_{\frac{s+1}{s}} \leqslant x, \tag{3.13}$$

where $x = y/K$.

Since we can take ϵ sufficiently small, (3.13) has a solution $x > 0$ if

$$(K^2x)^s + \|f\|_{\frac{s+1}{s}} < x \tag{3.14}$$

has a solution $x > 0$.

Now, it is easy to see that (3.14) has the solution

$$x_0 = \left(\frac{1}{sK^{2s}}\right)^{\frac{1}{s-1}}$$

if

$$\|f\|_{\frac{s+1}{s}} < \frac{\left(\frac{1}{s}\right)^{\frac{1}{s-1}} - \left(\frac{1}{s}\right)^{\frac{s}{s-1}}}{K^{\frac{2s}{s-1}}} = \lambda(s). \tag{3.15}$$

Finally,

$$\lim_{s \to \frac{n-2}{n-2}} \lambda(s) = B(n). \tag{3.16}$$

Also, for $s \in R$ such that $1 < s < \dfrac{n+2}{n-2}$ we have

$$\|f\|_{\frac{s+1}{s}} \leqslant \|f\|_\infty |\Omega|^{\frac{s}{s+1}}.$$

Therefore, if $|\Omega| > 1$, (3.9) yields

$$\|f\|_{\frac{s+1}{s}} \leqslant \|f\|_\infty |\Omega|^{\frac{s}{s+1}} < \|f\|_\infty |\Omega|^{\frac{n+2}{2n}} < B(n). \tag{3.17}$$

Or, if $|\Omega| \leqslant 1$, by (3.10) we have

$$\|f\|_{\frac{s+1}{s}} \leqslant \|f\|_\infty < B(n). \tag{3.18}$$

Using (3.16), (3.17) and (3.18) we get that there exists s_0 such that $1 < s_0 < \dfrac{n+2}{n-2}$ and, if $s \in \left(s_0, \dfrac{n+2}{n-2}\right)$, then (3.14) has a solution $x > 0$ and therefore (3.11) has a weak solution.

Now, by Theorem 2.1 we have that for $s \in \left(s_0, \dfrac{n+2}{n-2}\right)$ the weak solution $u_s \in H_0^1(\Omega)$ of (3.11) satisfies,

$$\|u_s\|_{1,2} \leqslant y_0 = Kx_0 = \left(\frac{1}{s}\right)^{\frac{1}{s-1}} K^{\frac{1+s}{1-s}}. \tag{3.19}$$

By (3.19) we obtain that for $s \in \left(s_0, \dfrac{n+2}{n-2}\right)$ the set $\{u_s\}$, such that u_s is a weak solution of (3.11), is bounded. Then there exists $\{u_k\} \subset \{u_s\}$ such that

$$W \lim_{k \to \frac{n+2}{n-2}} u_k = u_{\frac{n+2}{n-2}}, \tag{3.20}$$

for some $u_{\frac{n+2}{n-2}} \in H_0^1(\Omega)$. ($W\lim$ indicates weak limit.)

For simplicity we will use $\dfrac{n+2}{n-2} = N$.

Our next step is to show that u_N is a weak solution of (1.1).

Since $f \in L^\infty(\Omega)$, by an iterative argument called a bootstrapping procedure, (see [1, p. 50], we can see that $u_s \in H_0^1(\Omega)$, the weak solution of (3.11), satisfies that $u_s \in C^{0,\alpha}(\bar{\Omega})$, and since $\partial\Omega$ is sufficiently smooth, u_s is, in particular, continuous on $\bar{\Omega}$.

As in (2.7) let $F_s: H_0^1(\Omega) \to H_0^1(\Omega)$ be defined as

$$\langle F_s(u), v\rangle_{1,2} = \langle |u|^s + f, v\rangle_2, \tag{3.21}$$

for all $v \in H_0^1(\Omega)$.

For $s < N$, F_s is completely continuous. As we saw, there exists $s_0 < N$ such that for $s \in (s_0, N)$, $F_s(u_s) = u_s$, where u_s is a weak solution of (3.11).

Let $\{u_k\}$ a subsequence of $\{u_s\}$, $s \in (s_0, N)$, such that $u_k \rightharpoonup u_N$ if $k \to N$. ($\rightharpoonup$ denotes weak convergence). By (3.21) we have

$$\langle F_k(u_k), v\rangle_{1,2} = \langle u_k, v\rangle_{1,2} = \langle |u_k|^k + f, v\rangle_2, \tag{3.22}$$

for all $v \in H_0^1(\Omega)$. And therefore, for all $v \in H_0^1(\Omega)$

$$\lim_{k \to N} \langle |u_k|^k + f, v\rangle_2 = \langle u_N, v\rangle_{1,2}. \tag{3.23}$$

On the other hand, u_k, $k \in (s_0, N)$, is continuous. Hence

$$\lim_{r \to N} \| |u_k|^r - |u_k|^N \|_{\frac{2n}{n+2}} = 0,$$

and by Lebesgue's dominated convergence Theorem we have that $|u_k|^r \to |u_k|^N$ in $L^{\frac{2n}{n+2}}(\Omega)$ if $r \to N$. Now, for each $v \in H_0^1(\Omega)$ fixed, $\langle v, \cdot\rangle_2$ defines a linear and bounded functional on $L^{\frac{2n}{n+2}}(\Omega)$. Hence, for all $v \in H_0^1(\Omega)$

$$\lim_{r \to N} \langle |u_k|^r + f, v\rangle_2 = \langle |u_k|^N + f, v\rangle_2. \tag{3.24}$$

Also, $\{u_k\} \subset H_0^1(\Omega)$ is bounded, and since $H_0^1(\Omega)$ is embedded in $L^{\frac{2n}{n+2}}(\Omega)$, $\{u_k\}$ is bounded in $L^{\frac{2n}{n+2}}(\Omega)$ as well. Furthermore the Nemytsky operator defined by $|u|^N$ is bounded, so that $\{|u_k|^N\}$ is bounded in $L^{\frac{2n}{n+2}}(\Omega)$. Therefore, there exists

$h \in_{L^{2n/(n+2)}}(\Omega)$ and a subsequence of $\{u_k\}$, labeled in the same form, such that for all $v \in H_0^1(\Omega)$

$$\lim_{k \to N} \langle |u_k|^N + f, v \rangle_2 = \langle h + f, v \rangle_2. \tag{3.25}$$

By (3.24) and (3.25) we get

$$\lim_{r, k \to N} \langle |u_k|^r + f, v \rangle_2 = \langle h + f, v \rangle_2, \tag{3.26}$$

for each $v \in H_0^1(\Omega)$. (3.23) and (3.26) yield

$$\langle h + f, v \rangle_2 = \langle u_N, v \rangle_{1,2}, \tag{3.27}$$

for each $v \in H_0^1(\Omega)$.

Also, since $u_k \rightharpoonup u_N$ in $H_0^1(\Omega)$, we have

$$\lim_{k \to N} \langle |u_k|^r + f, v \rangle_2 = \langle |u_N|^r + f, v \rangle_2, \tag{3.28}$$

for all $v \in H_0^1(\Omega)$, and $r < N$. (3.23), (3.26), (3.27) and (3.28) yield

$$\lim_{r \to N} \langle |u_N|^r + f, v \rangle_2 = \langle h + f, v \rangle_2, \tag{3.29}$$

for each $v \in H_0^1(\Omega)$. Since $|u_N|^r v \leqslant (|u_N|^N + 1)|v|$, by Lebesgue's dominated convergence Theorem we have

$$\lim_{r \to N} \langle |u_N|^r + f, v \rangle_2 = \langle |u_N|^N + f, v \rangle_2, \tag{3.30}$$

for each $v \in H_0^1(\Omega)$. (3.30) and (3.29) yield

$$\langle |u_N|^N + f, v \rangle_2 = \langle u_N, v \rangle_{1,2}, \tag{3.31}$$

for each $v \in H_0^1(\Omega)$. By (3.31) we have that $u_N \in H_0^1(\Omega)$ is a weak solution of (1.1).

Theorem 3.4. *For $f \in L^\infty(\Omega)$ and for $n \geqslant 5$, suppose that*

$$|\Omega|^{2/n} < L(n), \tag{3.32}$$

and that at least one of following inequalities holds

$$\|f\|_\infty < \frac{A(n)}{|\Omega|^{\frac{n+2}{2n}}}, \quad \text{if} \quad |\Omega| > 1, \tag{3.33}$$

or

$$\|f\|_\infty < A(n), \quad \text{if} \quad |\Omega| \leqslant 1, \tag{3.34}$$

where

$$L(n) = \frac{\left[\frac{4}{n+2}\left(\frac{n-2}{n+2}\right)^{\frac{n-2}{4}}\right]^{\frac{2}{n}} - \left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{2\frac{n+2}{n-2}}}{\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{2}} \tag{3.35}$$

and

$$A(n) = \frac{\left(\frac{n-2}{n+2}\right)^{\frac{n^2-4}{8n}} - \left(\frac{n-2}{n+2}\right)^{\frac{(n+2)^2}{8n}}}{\left[\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{2}|\Omega|^{\frac{2}{n}} + \left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{\frac{2(n+2)}{n-2}}\right]^{\frac{n-2}{4}}}. \tag{3.36}$$

Then problem (1.2) *has at least a weak solution if we suppose that* $\partial\Omega$ *is sufficiently smooth.*

PROOF. As in Theorem 3.3 we will consider here the problem

$$\left.\begin{aligned} \Delta u + u + |u|^s + f = 0 \quad &\text{in} \quad \Omega, \\ u(x) = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{3.37}$$

with $1 < s < \dfrac{n+2}{n-2}$.

Let $g(x, u) = u + |u|^s + f$. Then if $\|u\|_{s+1} \leqslant r$, $\|u + |u|^s\|_{\frac{s+1}{s}} \leqslant \epsilon$ and the relationship betwen r and ϵ can be taken to be

$$r|\Omega|^{\frac{s-1}{s+1}} + r^s = \epsilon. \tag{3.38}$$

As in the Theorem 3.3, problem (3.37) has a weak solution whenever there exists $x > 0$ such that

$$[(K^2x)^{s+1} + r^{s+1}]^{\frac{s}{s+1}}\frac{\epsilon}{r^s} + \|f\|_{\frac{s+1}{s}} \leqslant x. \tag{3.39}$$

If we take

$$r = K^2, \tag{3.40}$$

K as in (2.5), then (3.39) becomes

$$(x^{s+1} + 1)^{\frac{s}{s+1}}\epsilon + \|f\|_{\frac{s+1}{s}} \leqslant x. \tag{3.41}$$

Now, it is easy to see that

$$x_0 = \left(\frac{1}{s}\right)^{\frac{s}{s^2-1}} \epsilon^{-\left(\frac{1}{s-1}\right)}$$

is a solution of (3.41) if the following two inequalities hold:

$$\epsilon^{\frac{s+1}{s-1}} < \left(\frac{1}{s}\right)^{\frac{1}{s-1}} - \left(\frac{1}{s}\right)^{\frac{s}{s-1}} \tag{3.42}$$

and

$$\|f\|_{\frac{s+1}{s}} < \frac{\left(\frac{1}{s}\right)^{\frac{s}{s^2-1}} - \left(\frac{1}{s}\right)^{\frac{s^2}{s^2-1}}}{\epsilon^{\frac{1}{s-1}}}. \tag{3.43}$$

Now, by (3.38), $\epsilon = |\Omega|^{\frac{s-1}{s+1}}K^2 + K^{2s}$, K as in (2.5). If we take the limit when $s \to N$, in both sides of (3.42) we obtain (3.32). Then (3.32) implies that there exists $s_1 \in R$, $1 < s_1 < N$, such that for all $s \in (s_1, N)$ the inequality (3.42) holds. Also, by (3.33) or (3.34) there exists $s_2 < N$ such that, for all $s \in (s_2, N)$ the inequality holds. Let $s_0 = \text{Max}\{s_1, s_2\}$; then for all $s \in (s_0, N)$ problem (3.37) has a weak solution. We may argue as in Theorem 3.3 and repeat all the formulas in (3.23) to (3.30) and obtain that there exists $u_N \in H_0^1(\Omega)$ a weak solution of the problem (1.2).

Remarks. If $|\Omega| < 1$, Theorem 3.4 holds for $n \geqslant 4$ and in that case, for $n \geqslant 5$, (3.32) is superfluous.

I am grateful to Dr. Yu Takeuchi who showed me that

$$\lim_{n\to\infty} A(n) = \lim_{n\to\infty} B(n) = \lim_{n\to\infty} L(n) = \infty.$$

Following the same argument of Theorem 3.4 we have the following

Theorem 3.5. *For $f \in L^\infty(\Omega)$ and $n \geqslant 4$ suppose that*

$$|\lambda|\,|\Omega|^{2/n} < L(n), \tag{3.44}$$

and that at least one of following inequalities holds

$$\|f\|_\infty < \frac{A(|\lambda|, n)}{|\Omega|^{\frac{n+2}{2n}}}, \quad \text{if} \quad |\Omega| > 1 \tag{3.45}$$

or

$$\|f\|_\infty < A(|\lambda|, n), \quad \text{if} \quad |\Omega| \leqslant 1, \tag{3.46}$$

where

$$(3.47)\quad A(|\lambda|, n) = \frac{\left(\dfrac{n-2}{n+2}\right)^{\frac{n^2-4}{8n}} - \left(\dfrac{n-2}{n+2}\right)^{\frac{(n+2)^2}{8n}}}{\left[\left(\dfrac{n-1}{n-2}\dfrac{1}{\sqrt{n}}\right)^2 |\lambda|\,|\Omega|^{\frac{2}{n}} + \left(\dfrac{n-1}{n-2}\dfrac{1}{\sqrt{n}}\right)^{\frac{2(n+2)}{n-2}}\right]^{\frac{n-2}{4}}}.$$

Then the problem

$$(3.48)\quad \left.\begin{aligned} \Delta u + \lambda u + |u|^N + f = 0 \quad &\text{in} \quad \Omega, \\ u(x) = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}\right\}$$

has at least a weak solution.

If we suppose that $f > 0$ on Ω then, by the Maximum principle for the operator Δ, we have that problems (1.1), (1.2) and (3.48) for $\lambda \geqslant 0$, have positive solutions.

Finally, in Theorem 3.4, if $|\Omega|$ is sufficiently small then $\epsilon = |\Omega|^{\frac{s-1}{s+1}} + r^s$ is small too ($r = K^2$). Then inequality (3.41) holds for some $x > \|f\|_\infty$. We conclude by saying that Problem (1.2) has a weak solution if $|\Omega|$ is sufficiently small.

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