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# Para-Accretive Functions, the Weak Boundedness Property and the *Tb* Theorem

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## Abstract

G. David, J.-L. Journé and S. Semmes have shown that if $b_1$ and $b_2$ are para-accretive functions on $\mathbb{R}^n$, then the «*Tb* Theorem» holds: A linear operator $T$ with Calderón-Zygmund kernel is bounded on $L^2$ if and only if $Tb_1 \in \mathrm{BMO}$, $T^* b_2 \in \mathrm{BMO}$ and $M_{b_2} T M_{b_1}$ has the weak boundedness property. Conversely they showed that when $b_1 = b_2 = b$, para-accretivity of $b$ is necessary for the *Tb* Theorem to hold. In this paper we show that para-accretivity of both $b_1$ and $b_2$ is necessary for the *Tb* Theorem to hold in general. In addition, we give a characterization of para-accretivity in terms of the weak boundedness property and use this to give a sharp *Tb* Theorem for Besov and Triebel-Lizorkin spaces.

## 1. Introduction

We begin by recalling the definitions necessary for the statement of the *Tb* Theorem of G. David, J.-L. Journé and S. Semmes. For $0 < \eta < 1$, let $C_0^\eta(\mathbb{R}^n)$ denote the space of continuous functions $f$ with compact support such that

$$\|f\|_{\mathrm{Lip}\,\eta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta}$$

is finite. Suppose $b_1$ and $b_2$ are complex-valued bounded functions on $\mathbb{R}^n$, and that $T$ is a linear operator such that $M_{b_2}TM_{b_1}$ is continuous from $C_0^\eta(\mathbb{R}^n)$ into its dual $C_0^\eta(\mathbb{R}^n)'$ for all $0 < \eta < 1$. Here $M_b$ denotes the operation of multiplication by $b$. Suppose further that there is a continuous function $K(x, y)$ on $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$, called the kernel of $T$, that represents $T$ in the sense that

$$(1.1) \qquad (M_{b_2}TM_{b_1}\varphi)(\psi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b_2(x)K(x, y)b_1(y)\varphi(y)\psi(x)\,dx\,dy$$

for all $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$, $0 < \eta < 1$, with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \varnothing$. We suppose that $K(x, y)$ satisfies the following size and smoothness estimates for some $\epsilon > 0$:

(1.2) (i) $|K(x, y)| \leqslant C|x - y|^{-n}$ for $x, y \in \mathbb{R}^n$,

(ii) $|K(x, y) - K(x', y)| \leqslant C\left(\dfrac{|x - x'|}{|x - y|}\right)^{\epsilon}|x - y|^{-n}$ for $x, x', y \in \mathbb{R}^n$ with $|x - x'| < \dfrac{1}{2}|x - y|$,

(iii) $|K(x, y) - K(x, y')| \leqslant C\left(\dfrac{|y - y'|}{|x - y|}\right)^{\epsilon}|x - y|^{-n}$ for $x, y, y' \in \mathbb{R}^n$ with $|y - y'| < \dfrac{1}{2}|x - y|$.

Kernels with the above properties are called Calderón-Zygmund kernels. See [DJS2] for details and examples.

A complex-valued bounded function is said to be para-accretive if ([DJS2])

(1.3) There is $c$ positive such that for every cube $Q$ in $\mathbb{R}^n$, there is a subcube $I$ with

$$\left|\frac{1}{|Q|}\int_I b(x)\,dx\right| \geqslant c.$$

Note that the cube $I$ in (1.3) satisfies

$$|I| \geqslant \frac{c}{\|b\|_{L^\infty}}|Q|.$$

Finally, a linear operator $T$ from $C_0^\eta(\mathbb{R}^n)$ to $C_0^\eta(\mathbb{R}^n)'$, $0 < \eta < 1$, is said to satisfy the weak boundedness property if

$$(1.4) \qquad |(T\varphi)(\psi)| \leqslant C|Q|^{1+2\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{\mathrm{Lip}\,\eta}$$

for all cubes $Q$ and $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ with support in $Q$. In [DJS2], this definition is shown to be independent of $\eta$. We can now state the $Tb$ Theorem of G. David, J.-L. Journé and S. Semmes (see A. McIntosh and Y. Meyer [MM] for the first version of the $Tb$ Theorem).

**The $Tb$ Theorem.** ([DJS1], [DJS2]). *Suppose $b_1$ and $b_2$ are para-accretive functions on $\mathbb{R}^n$ and that $T$ is a linear operator such that $M_{b_2}TM_{b_1}$ is continuous from $C_0^\eta(\mathbb{R}^n)$ to $C_0^\eta(\mathbb{R}^n)'$ for some $0<\eta<1$, with a Calderón-Zygmund kernel $K(x,y)$, i.e.,* (1.1) *and* (1.2) (i), (ii), (iii) *hold. Then $T$ is bounded on $L^2$ if and only if*

(1.5) (i) $Tb_1 \in \mathrm{BMO}$.
(ii) $T^*b_2 \in \mathrm{BMO}$ (*where $T^*$ denotes the transpose of $T$*).
(iii) *$M_{b_2}TM_{b_1}$ satisfies the weak boundedness property.*

The reader is referred to Section 1 of [DJS2] for the definition of $Tb_1$ and $T^*b_2$- we only point out here that (1.1) and (1.2)(ii) are needed to define $Tb_1$ while (1.1) and (1.2)(iii) are needed for $T^*b_2$. We also mention in passing that the hypothesis (1.2)(i) on the size of the kernel $K(x,y)$ is not needed in the $Tb$ Theorem since it is already implied by the other hypotheses. See the end of Section 3.

Conversely, it was shown ([DJS2; Proposition 1 in Section 9]) in the case $b_1=b_2=b$ is bounded, that if every linear operator $T$ satifying (1.1), (1.2) and (1.5) is bounded on $L^2$, then $b$ is para-accretive. The main result of this paper is that this converse result holds in general-namely, the para-accretivity of both $b_1$ and $b_2$ is necessary if the $Tb$ Theorem is to hold. Two complex-valued bounded functions $b_1$ and $b_2$ are said to be jointly para-accretive if there is $c>0$ such that for every cube $Q$ in $\mathbb{R}^n$, there is a subcube $I$ with

$$\frac{1}{|Q|}\max\left\{\left|\int_I b_1(x)\,dx\right|,\left|\int_I b_2(x)\,dx\right|\right\}\geqslant c.$$

**Theorem 1.** *Suppose $b_1$ and $b_2$ are complex-valued bounded functions. If $b_2$ is not para-accretive, then there exists a linear operator $T$ with kernel $K$ satisfying* (1.1), (1.2) (i), (ii), (iii) (*with* $\epsilon=1$) *and such that*

(1.6) (i) $Tb_1 \in L^\infty$,
(ii) $T^*b_2 \in L^\infty$,
(iii) *$M_{b_2}TM_{b_1}$ has the weak boundedness property,*
(iv) *$TM_{b_1}$ fails to have the weak boundedness property if $b_1$ and $b_2$ are jointly para-accretive, while $T$ fails to have the weak boundedness property if $b_1$ and $b_2$ are not jointly para-accretive.*

Note that by (1.6)(iv), the operator $T$ in Theorem 1 is not bounded on $L^2$ and thus the para-accretivity of $b_2$ is necessary for the $Tb$ Theorem to hold. By duality, the para-accretivity of $b_1$ is also necessary.

A fairly straightforward consequence of Theorem 1 and a lemma of Y. Meyer ([M1]; Lemme 2) is the following characterization of para-accretivity in terms of the weak boundedness property. We thank Rodolfo Torres for discussions leading to this result. Let $\mathcal{C}$ denote the set of linear operators $T$ with kernel $K(x,y)$ satisfying (1.1) (with $b_1 = b_2 = 1$) and (1.2)(i) and (ii) —but not necessarily (1.2)(iii)— and $T1 = 0$.

**Theorem 2.** *A complex-valued bounded function $b$ is para-accretive if and only if for every $T$ in $\mathcal{C}$, $T$ has the weak boundedness property* (1.4) *whenever $M_b T$ does.*

*Remark.* Theorem 2 remains true if $\mathcal{C}$ is replaced by the larger class $\mathcal{C}'$ of operators $T$ with kernel satisfying (1.1) (with $b_1 = b_2 = 1$) and (1.2)(i) and (ii) and $T1 \in \mathrm{BMO}$. See Section 3.

Note by contrast, that Lemme 2 of [M1] shows that for any bounded function $b$, $M_b T$ has the weak boundedness property whenever $T$ in $\mathcal{C}$ does. We now recall a result of P. G. Lemarié [L].

**The $T1$ Theorem for Besov Spaces.** ([L]). *Suppose $T$ in $\mathcal{C}$ satisfies the weak boundedness property* (1.4). *Then $T$ is bounded on the homogeneous Besov space $\dot{B}_p^{\alpha,q}$ for $1 \leqslant p, q \leqslant \infty$ and $0 < \alpha < \epsilon$, where $\epsilon$ is the order of smoothness of $K$ in the first variable in* (1.2)(ii).

As indicated in Section 14 of [DJS2], Lemarié's Theorem yields a $Tb$ Theorem for Besov spaces —If $T$ satisfies (1.1), (1.2)(i) and (ii) and $Tb_1 = 0$, and if $M_{b_2} T M_{b_1}$ has the weak boundedness property where $b_2$ is para-accretive, then $T M_{b_1}$ is bounded on $\dot{B}_p^{\alpha,q}$ for $1 \leqslant p, q \leqslant \infty$ and $0 < \alpha < \epsilon$. Note that exactly half of the asymmetric hypotheses in the $Tb$ Theorem (with BMO replaced by 0) are needed here. The other half imply by duality that $M_{b_2} T$ is bounded on $\dot{B}_p^{\alpha,q}$ for $1 \leqslant p, q \leqslant \infty$ and $-\epsilon < \alpha < 0$. See Section 14 of [DJS2] where these results are interpolated to yield another proof of the $Tb$ Theorem for $L^2$.

In order to reduce this $Tb$ Theorem to the $T1$ Theorem of Lemarié, simply observe that $T M_{b_1}$ is in $\mathcal{C}$ and satisfies the weak boundedness property by the «only if» half of Theorem 2. The «if» half of Theorem 2 shows that the para-accretivity of $b_2$ cannot be removed.

The above considerations also apply to the homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ once we have shown that the conclusion of Lemarié's Theorem applies to $\dot{F}_p^{\alpha,q}$ in place of $\dot{B}_p^{\alpha,q}$ for $1 < p, q < \infty$ and $0 < \alpha < \epsilon$. The following result has been independently obtained by B. Jawerth, M. Taibleson and G. Weiss ([HJTW]).

**Theorem 3.** *Suppose $T$ in $\mathcal{C}$ satisfies the weak boundedness property. Then $T$ is bounded on $\dot{F}_p^{\alpha,q}$ for $1 < p, q < \infty$ and $0 < \alpha < \epsilon$, where $\epsilon$ is as in* (1.2)(ii).

Theorem 3 is easily obtained by adapting the proof of the $T1$ Theorem for $L^2$ outlined in Section 2 of [DJS2] and we will sketch the relevant details in Section 4 below. We remark that M. Frazier, Y.-S. Han, B. Jawerth and G. Weiss have shown ([FHJW]) that for $T$ in $\mathcal{C}$ satisfying the weak boundedness property and the additional smoothness (1.2)(iii), $T$ maps $\dot{F}_p^{\alpha,q}$-atoms to $\dot{F}_p^{\alpha,q}$-molecules (and so is bounded on $\dot{F}_p^{\alpha,q}$) for $1 < p, q < \infty$, $0 < \alpha < \epsilon$. Theorem $m$ is proved in Section $(m + 1)$, $m = 1, 2, 3$.

## 2. Proof of Theorem 1

The proof of Theorem 1 splits into two cases.

*Case* 1. $b_1$ and $b_2$ are jointly para-accretive.

We modify the construction in Proposition 1 of Section 9 of [DJS2] (of an operator for which the *Tb* Theorem fails for a non-para-accretive function $b = b_1 = b_2$) in the spirit of a para-product. The basic idea evolves from the observation that if a Calderón-Zygmund kernel $K(x, y)$ equals $(1 + |x - y|)^{-n}$ for $|x - y| \leqslant N$, $-|x - y|^{-n}$ for $2N \leqslant |x - y| \leqslant N^2$ and zero for $|x - y| > 2N^2$ then $\| T1 \|_{L^\infty} \leqslant C$ and the weak boundedness constant $C$ in (1.4) is at least $c \log N$. Suppose there is $c > 0$ such that for every cube $Q$ in $\mathbb{R}^n$, there is a sub-cube $I$ with

$$\frac{1}{|Q|} \max \left\{ \left| \int_I b_1(x)\, dx \right|, \left| \int_I b_2(x)\, dx \right| \right\} \geqslant c. \tag{2.1}$$

If $b_1$, $b_2$ are bounded in absolute value by $M$, then (2.1) forces

$$|I| \geqslant \frac{c}{M} |Q|$$

and so the ratio of the side lengths of $I$ and $Q$ is bounded below by

$$\delta = 1 \Big/ \left[ \left( \frac{M}{c} \right)^{1/n} \right]$$

where $[x]$ denotes the greatest integer part of $x$. Since $b_2$ is not para-accretive, we can find a cube $Q_k$, for each $k > 0$, with the property

$$\sup_{\text{cubes} J \subset 3Q_k} \left| \frac{1}{|Q_k|} \int_J b_2(x)\, dx \right| \leqslant \frac{\delta^{kn}}{k}.$$

Thus

$$(2.2) \quad \left| \frac{1}{|J|} \int_J b_2(x)\, dx \right| \leqslant \frac{1}{k} \quad \text{for all cubes } J \subset 3Q_k \text{ with } |J|^{1/n} \geqslant \delta^k |Q_k|^{1/n}.$$

Momentarily fix $k$ with $\delta^{kn}/k < c$. Then (2.1) and (2.2) imply that for every cube $J \subset 3Q_k$ with side length at least $\delta^k$ times that of $Q_k$, there is a cube $I \subset J$ of side length at least $\delta$ times that of $J$ such that

$$\left| \frac{1}{|I|} \int_I b_1(x)\, dx \right| \geqslant c.$$

Let $s_k$ denote the side length of $Q_k$. For $j = 0, 1, 2, \ldots, k-1$, let $\{J_i^j\}_{i=1}^{3^n \delta^{-jn}}$ denote the «dyadic» decomposition of $3Q_k$ into $3^n\, \delta^{-jn}$ congruent subcubes of side length $\delta^j s_k$ with pairwise disjoint interiors. For each cube $J_i^j$ whose triple is contained in $3Q_k$, let $(J_i^j)'$ denote the translate of $J_i^j$ by $\delta^j s_k(1, 1, \ldots, 1)$ and then set

$$J_i^{j*} = \frac{1}{3}(J_i^j)'.$$

By (2.1), there is a subcube $I_i^j$ of $J_i^{j*}$ with side length at least $\delta^{j+1}s_k/3$ and satisfying

$$\left| \frac{1}{|I_i^j|} \int_{I_i^j} b_1(x)\, dx \right| \geqslant c$$

(we may suppose $\delta \leqslant 1/3$).

We must now smooth out these averages. We claim that there are Lipschitz functions $\varphi_i^j$ satisfying

(2.3) (i) $\operatorname{supp} \varphi_i^j \subset I_i^j$,

(ii) $|\varphi_i^j(y)| \leqslant |I_i^j|^{-1}$, for $y \in \mathbb{R}^n$,

(iii) $|\varphi_i^j(y) - \varphi_i^j(y')| \leqslant C \dfrac{|y - y'|}{|I_i^j|^{1/n}} |I_i^j|^{-1}$, for $y, y' \in \mathbb{R}^n$,

(iv) $\left| \int \varphi_i^j(y) b_1(y)\, dy \right| \geqslant c/2$,

where the constant $C$ in (2.3)(iii) depends on $M$ and $c$ in (2.1). To construct the $\varphi_i^j$, simply choose $\varphi_i^j$ to be supported in $I_i^j$ with values between 0 and

$|I_i^j|^{-1}$, and to take the value $|I_i^j|^{-1}$ on $\gamma I_i^j$ where $\gamma < 1$ is so close to 1 that

$$\left|\int \varphi_i^j(y) b_1(y)\,dy\right| = \left|\frac{1}{|I_i^j|}\int_{I_i^j} b_1(y)\,dy + \int\left(\varphi_i^j - \frac{1}{|I_i^j|}\chi_{I_i^j}\right)(y) b_1(y)\,dy\right|$$
$$\geqslant c - M|I_i^j \backslash \gamma I_i^j|/|I_i^j| > c/2$$

Property (2.3)(iii) follows if the $\varphi_i^j$ are taken to be translates and dilates of a fixed smooth $\varphi$.

Now we claim there exist Lipschitz functions $\psi_i^j$ satisfying

(2.4) (i) $\operatorname{supp}\psi_i^j \subset \frac{3}{2} J_i^j$.

(ii) $0 \leqslant \psi_i^j \leqslant 1$.

(iii) $\sum_{i=1}^{3^n\delta^{-jn}} \psi_i^j(x) = 1, \qquad x \in 3Q_k, \quad 0 \leqslant j \leqslant k-1,$

(iv) $|\psi_i^j(x) - \psi_i^j(x')| \leqslant C\dfrac{|x-x'|}{|J_i^j|^{1/n}}$.

(v) $\left|\dfrac{1}{|J_i^j|}\displaystyle\int \psi_i^j(x) b_2(x)\,dx\right| \leqslant \dfrac{C}{k}$.

Define $\beta(x)$ on $\mathbb{R}$ to equal 1 for $|x| \leqslant 1/2$, 0 for $|x| \geqslant 3/2$ and to be linear on each of the intervals $[-3/2, -1/2]$ and $[1/2, 3/2]$. If the $\psi_i^j$ are taken to be appropriate dilates and translates of

$$\psi(x) = \prod_{l=1}^{n} \beta(x_l),$$

then (2.4)(i)-(iv) hold immediately. Since $\psi$ is a positive integral of characteristic functions of parallelepipeds whose sidelengths lie between 1 and 3, property (2.4)(v) would follow from (2.2) if only the cubes $J$ in (2.2) were permitted to be parallelepipeds contained in $3Q_k$ with sidelengths at least $\delta^k|Q_k|^{1/n}$. However, it is an easy exercise to verify that one may replace the subcubes $I$ in the definition of para-accretive in (1.3) by parallelepipeds. Indeed, if

$$\left|\frac{1}{|Q|}\int_I b(x)\,dx\right| \geqslant c$$

for a parallelepiped $I$ contained in $Q$, then there is $N$ large, depending only on $\|b\|_\infty$ and $c$, such that if $\{J_i\}_{i=1}^{N^n}$ is the «dyadic» decomposition of $Q$ into congruent subcubes of sidelength $|Q|^{1/n}/N$ and

$$I^* = \cup\{J_i : J_i \cap I \neq \varnothing\},$$

then

$$\left|\frac{1}{|Q|}\int_{I^*} b(x)\,dx\right| \geqslant \left|\frac{1}{|Q|}\int_I b(x)\,dx\right| - \frac{|I^*\backslash I|}{|Q|}\,\|b\|_\infty$$
$$\geqslant \frac{c}{2}.$$

It follows that

$$\left|\frac{1}{|Q|}\int_{J_i} b(x)\,dx\right| \geqslant \frac{c}{2N^n}$$

for at least one of the cubes $J_i$. This completes the proof of (2.4).

We wish to define an operator $T_k$ with kernel of the form

$$K_k(x,y) = \sum_{j,i} \beta_i^j \psi_i^j(x)\varphi_i^j(y) \tag{2.5}$$

where the $\beta_i^j$ are bounded constants so chosen that $\|T_k b_1\|_{L^\infty} \leqslant C$ and the weak boundedness constant for $T_k M_{b_1}$ (the best $C$ in (1.4) with $T = T_k M_{b_1}$) is of the order of $k$. We will see that the size and smoothness estimates (1.2) for $K_k$, the boundedness of $|T_k^* b_2|$ by $C$ and the weak boundedness of $M_{b_2} T_k M_{b_1}$ with constant of the order of 1, all follow independently of the particular choice of bounded $\beta_i^j$'s.

In order to define the constants $\beta_i^j$, let

$$\Omega_0 = Q_k,\quad \Omega_1 = 2Q_k,\quad \Omega_2 = \frac{5}{2}Q_k, \ldots,\quad \Omega_j = (3 - 2^{1-j})Q_k$$

for $1 \leqslant j \leqslant \left[\frac{k-1}{2}\right]$. In the case $0 \leqslant j \leqslant \left[\frac{k-1}{2}\right]$, we define

$$\beta_i^j = \begin{cases} 1\Big/\int \varphi_i^j(y) b_1(y)\,dy & \text{if } J_i^j \subset \Omega_j \\ 0 & \text{otherwise} \end{cases}$$

In the case $\left[\frac{k-1}{2}\right] + 1 \leqslant j \leqslant k-1$, we define

$$\beta_i^j = \begin{cases} -1\Big/\int \varphi_i^j(y) b_1(y)\,dy & \text{if } J_i^j \subset \Omega_{k-1-j} \\ 0 & \text{otherwise} \end{cases}$$

With this choice of $\beta_i^j$ we calim the following properties:

(2.6) (i) $\| T_k b_1 \|_{L^\infty} \leqslant C$.

(ii) $\| T_k^* b_2 \|_{L^\infty} \leqslant C$.

(iii) $|K_k(x, y)| \leqslant C|x - y|^{-n}$.

(iv) $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leqslant C \dfrac{|x - x'|}{|x - y|^{n+1}}$ whenever $|x - x'| < \dfrac{1}{2}|x - y|$,

(v) $M_{b_2} T_k M_{b_1}$ satisfies the weak boundedness property (1.4) with constant $C$ independent of $k$,

(vi) $T_k M_{b_1}$ satisfies the weak boundedness property (1.4) only with constant $C \geqslant c'k$.

We begin by proving the key properties (i) and (vi) that rely on our particular choice of $\beta_i^j$. To see (i), fix $x$ in $\Omega_l \backslash \Omega_{l-1}$ for some $0 \leqslant l \leqslant [(k-1)/2]$ (where $\Omega_{-1} = \varnothing$). We have

$$\begin{aligned} T_k b_1(x) &= \sum_{j=0}^{k-1} \sum_i \beta_i^j \psi_i^j(x) \int \varphi_i^j(y) b_1(y)\, dy \\ &= \sum_{j=0}^{k-1} A_j(x). \end{aligned}$$

For $0 \leqslant j \leqslant l - 2$ and $k - l + 1 \leqslant j \leqslant k - 1$, $\beta_i^j \psi_i^j(x) = 0$ for all $i$ since supp $\psi_i^j \cap \Omega_{l-2} = \varnothing$ if $\psi_i^j(x) \neq 0$ and so then $\beta_i^j = 0$ by definition. Thus $A_j(x) = 0$ for these ranges of $j$. For

$$l + 1 \leqslant j \leqslant \left[ \frac{k-1}{2} \right], \qquad \beta_i^j = 1 \Big/ \int \varphi_i^j(y) b_1(y)\, dy$$

whenever $\psi_i^j(x) \neq 0$ and so $A_j(x) = \sum_i \psi_i^j(x) = 1$ by (2.4)(iii). Similarly,

$$A_j(x) = -1 \quad \text{for} \quad \left[ \frac{k-1}{2} \right] + 1 \leqslant j \leqslant k - l - 2.$$

Finally, if $j$ is one of the four remaining cases, $j = l - 1, l, k - l - 1$ or $k - l$, we simply use the crude estimate

$$|A_j(x)| \leqslant \sum_i \psi_i^j(x) = 1$$

which follows from

$$\left| \beta_i^j \int \varphi_i^j(y) b_1(y)\, dy \right| \leqslant 1.$$

Altogether we obtain

$$(2.7)\qquad |T_k b_1(x)| = \left|\sum_{j=0}^{k-1} A_j(x)\right|$$
$$\leqslant 4 + \left|\sum_{j=l+1}^{[(k-1)/2]} A_j(x) + \sum_{j=[(k-1)/2]+1}^{k-l-2} A_j(x)\right|$$
$$= 4 + \left|\left[\frac{k-1}{2}\right] - l - \left[k-l-2-\left[\frac{k-1}{2}\right]\right]\right|$$
$$= \begin{cases} 4 & \text{if } k \text{ even} \\ 5 & \text{if } k \text{ odd} \end{cases}$$

Since, by the same argument, $|T_k b_1(x)| \leqslant 2$ for $x$ outside $\Omega_{[(k-1)/2]}$, we have proved (2.6)(i).

To see (vi), let $J$ denote one of the cubes $J_i^j$ with $j = \left[\frac{k-1}{2}\right] + 1$ and such that the triple of $J_i^j$ lies in $Q_k$. For any bounded functions $\varphi$ and $\psi$ we have

$$(2.8)\qquad \langle T_k \varphi, \psi \rangle = \sum_{j,i} \beta_i^j \iint \psi(x)\psi_i^j(x)\varphi_i^j(x)\varphi(y)\,dx\,dy.$$

If $\varphi$ and $\psi$ are both supported in $5J$, then all the integrals in the sum on the right side of (2.8) vanish for $0 \leqslant j \leqslant \left[\frac{k-1}{2}\right]$ since the cube $5J$ cannot simultaneously intersect the supports of $\psi_i^j$ and $\varphi_i^j$ if $\delta$ is small enough (*e.g.* $\delta < 1/60$) by (2.3)(i), (2.4)(i), the definition of $J_i^{j*}$ and some elementary geometry. In particular, if $\psi = \chi_J$ and $\varphi = \chi_{5J} b_1$, then in addition, $\operatorname{supp} \varphi_i^j \subset 5J$ whenever $\operatorname{supp} \psi_i^j \cap J \neq \varnothing$ and so

$$(2.9)\qquad \langle T_k \chi_{5J} b_1, \chi_J \rangle = \sum_{j=[(k-1)/2]+1}^{k-1} \sum_i \int_J \psi_i^j(x)\beta_i^j \int_{5J} \varphi_i^j(y) b_1(y)\,dy\,dx$$
$$= -\sum_{j=[(k-1)/2]+1}^{k-1} \sum_i \int_J \psi_i^j(x)\,dx$$
$$= -\left(k-1-\left[\frac{k-1}{2}\right]\right)|J|,$$

by (2.4)(iii). We note in passing that (2.9) shows that the norm of $T_k$ as an operator on $L^2$ is of the order at least $k$. Now choose $\psi$ Lipschitz with support in $\gamma J$, taking the value 1 on $J$ and choose $\varphi$ Lipschitz with support in $5J$, taking the value 1 on $(5/\gamma)J$. Take $\gamma$ so close to 1 that for $0 < \eta < 1$,

$$(2.9)' \qquad |\langle T_k M_{b_1}\varphi, \psi\rangle| \geqslant C_\eta \left[\frac{k-1}{4}\right] |J|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta}\,.$$

where $C_\eta$ is independent of $k$. This proves (2.6)(vi).

The proofs of (ii) and (v), to which we now turn, are essentially the same as those given for Proposition 1 in Section 9 of [DJS2] to prove the analogous statements for their counterexample in the case $b_1 = b_2$. In fact,

$$|T_k^* b_2(y)| = \left|\sum_{j,i} \beta_i^j \varphi_i^j(y) \int \psi_i^j(x) b_2(x)\, dx\right|$$

$$\leqslant \sum_{j,i} \frac{2}{c} |I_i^j|^{-1} \chi_{I_i^j}(y) \frac{C}{k} |J_i^j|$$

by (2.3)(iv), (2.3)(i), (ii), and (2.4)(v). Since $I_i^j$ has side length at least $\frac{1}{3}\delta^{j+1}s_k$ and $J_i^j$ has side length $\delta^j s_k$, it follows that

$$|T_k^* b_2(y)| \leqslant \frac{2C}{ck} \sum_{j,i} 3\delta^{-1} \chi_{I_i^j}(y)$$

$$\leqslant \frac{6C}{c\delta},$$

which is (2.6)(ii).

To establish (v), we must show that

$$(2.10) \qquad |\langle M_{b_2} T_k M_{b_1}\varphi, \psi\rangle| \leqslant C|Q|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta}$$

for all $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ with support in the cube $Q$. We use the argument in Section 9 of [DJS2]. The point is that we only need small integrals for one of the $b_i$, in this case $b_2$. Fix a cube $Q$ of side length $s$ and Lip $\eta$ functions $\varphi, \psi$ with support in $Q$. Then

$$(2.11) \qquad \langle M_{b_2} T_k M_{b_1}\varphi, \psi\rangle = \sum_{j,i} \beta_i^j \iint \psi(x) b_2(x) \psi_i^j(x) \varphi_i^j(y) b_1(y) \varphi(y)\, dx\, dy$$

$$= \sum_j \mathrm{B}_j.$$

If $\delta^j s_k \geqslant s$, then we estimate $B_j$ directly by

$$(2.12) \qquad |B_j| \leqslant \frac{2}{c} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty} M^2 s^{2n} \left(\frac{3\delta^{-j-1}}{s_k}\right)^n$$

$$\leqslant C \frac{s^{2n}}{(\delta^j s_k)^n} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty}$$

using (2.3)(iv) and (ii). If $\delta^j s_k < s$ and $x_i^j$ denotes the centre of $J_i^j$, then

$$(2.13)\qquad B_J = \sum_i \beta_i^j \iint [\psi(x) - \psi(x_i^j)] b_2(x)\psi_i^j(x)\varphi_i^j(y) b_1(y)\varphi(y)\,dx\,dy$$
$$+ \sum_i \psi(x_i^j)\beta_i^j \iint b_2(x)\psi_i^j(x)\varphi_i^j(y) b_1(y)\varphi(y)\,dx\,dy$$
$$= C_j + D_j.$$

Now

$$(2.14)\qquad |C_j| \leqslant \frac{2}{c}(\delta^j s_k)^\eta \|\psi\|_{\mathrm{Lip}\,\eta} M^2 \|\varphi\|_{L^\infty} \sum_i \int_{3Q}\int_Q \psi_i^j(x)\varphi_i^j(y)\,dy\,dx$$
$$\leqslant Cs^n(\delta^j s_k)^\eta \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{L^\infty}$$

and

$$(2.15)\qquad |D_j| \leqslant \|\psi\|_{L^\infty} \frac{2}{c} M \|\varphi\|_{L^\infty} \sum_i \int_Q \left| \int \psi_i^j(x) b_2(x)\,dx \right| \varphi_i^j(y)\,dy$$
$$\leqslant C\frac{s^n}{k} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty}$$

using (2.3)(iv), (ii) and (2.4)(v). Altogether then, (2.11)-(2.15) yield

$$|\langle M_{b_2} T_k M_{b_1}\varphi, \psi\rangle| \leqslant \sum_{j:\,\delta^j s_k \geq s} C\frac{s^{2n}}{(\delta^j s_k)^n} \|\psi\|_{L^\infty}\|\varphi\|_{L^\infty}$$
$$+ \sum_{j:\,\delta^j s_k < s} \left[ Cs^n(\delta^j s_k)^\eta \|\psi\|_{\mathrm{Lip}\,\eta}\|\varphi\|_{L^\infty} + C\frac{s^n}{k}\|\psi\|_{L^\infty}\|\varphi\|_{L^\infty} \right]$$
$$\leqslant Cs^n \|\psi\|_{L^\infty}\|\varphi\|_{L^\infty} + Cs^{n+\eta}\|\psi\|_{\mathrm{Lip}\,\eta}\|\varphi\|_{L^\infty}$$
$$\leqslant Cs^{n+2\eta}\|\psi\|_{\mathrm{Lip}\,\eta}\|\varphi\|_{\mathrm{Lip}\,\eta}$$

and this completes the proof of (2.10) and hence that of (2.6)(v).

Finally, the kernel $K_k(x, y)$ satisfies

$$|K_k(x,y)| \leqslant \frac{2}{c} \sum_{j:\,|x-y| \leq 3\delta^j s_k} \frac{3}{(\delta^{j+1}s_k)^n}$$
$$\leqslant C|x-y|^{-n}$$

by (2.3)(i), (ii), (iv) and (2.4)(i) which proves (2.6)(iii). If $|x - x'| < \frac{1}{2}|x - x|$, then

$$|K_k(x,y) - K_k(x',y)| \leqslant \frac{2}{c} \sum_{j:\,|x-y| \leq 3\delta^j s_k} \sum_i |\psi_i^j(x) - \psi_i^j(x')|\,|\varphi_i^j(y)|$$
$$\leqslant C|x-x'| \sum_{j:\,|x-y| \leq 3\delta^j s_k} (\delta^j s_k)^{-1}(\delta^{j+1}s_k)^{-n}$$
$$\leqslant C|x-x'|\,|x-y|^{-n-1}$$

by (2.3)(i), (ii), (iv) and (2.4)(i)-(iv). If $|y-y'| < \frac{1}{2}|x-y|$, then

$$|K_k(x,y) - K_k(x,y')| \leqslant \frac{2}{c} \sum_{j:\,|x-y| \leq 3\delta^j s_k} \sum_i \psi_i^j(x)|\varphi_i^j(y) - \varphi_i^j(y')|$$
$$\leqslant C|y-y'| \sum_{j:\,|x-y| \leq 3\delta^j s_k} (\delta^{j+1}s_k)^{-n-1}$$
$$\leqslant C|y-y'|\,|x-y|^{-n-1}$$

by (2.3)(i)-(iii) and (2.4)(i)-(iii). This proves (2.6)(iv) and completes the proof of the properties (2.6).

Before assembling the operators $T_k$ to form an operator $T$ satisfying the conclusions (1.6)(i)-(iv) of Case 1, it is convenient to arrange for an additional property of the cubes $Q_k$:

(2.16) If $3Q_k \cap 3Q_l \neq \varnothing$, then either $s_k \leqslant \delta^l s_l$ or $s_l \leqslant \delta^k s_k$.

To achieve this by induction, suppose $Q_1, \ldots, Q_k$ are dyadic cubes satisfying (2.2) and (2.16). Let $\mathcal{S}_k$ consist of the (finitely many) dyadic cubes of side length at least $\delta^j s_j$ and at most $\delta^{-k-1}s_j$ whose triples intersect $3Q_j$, $j = 1, 2, \ldots, k$. Then choose $Q_{k+1}$ to be a dyadic cube not in $\mathcal{S}_k$, and satisfying (2.2). Now define, as in [DJS2],

$$T = \sum_{k=1}^{\infty} \frac{1}{k^2} T_{k^3}. \tag{2.17}$$

The estimates (1.2)(i)-(iii) and (1.6)(i)-(iii) all follow easily from (2.6)(i)-(v) and (2.17) and it remains only to check that $TM_{b_1}$ fails to have the weak boundedness property. For this, fix $l^3$ and $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ supported in $5J$ (associated to $Q_{l^3}$ as above) so that (2.9)′ holds with $l^3$ in place of $k$. Then

$$\langle TM_{b_1}\varphi, \psi\rangle = l^{-2}\langle T_{l^3}M_{b_1}\varphi, \psi\rangle + \sum_{k \neq l} k^{-2}\langle T_{k^3}M_{b_1}\varphi, \psi\rangle. \tag{2.18}$$

If $3Q_{k^3}$ intersects $3Q_{l^3}$ and $s_{l^3} \leqslant \delta^{k^3} s_{k^3}$, then the separation of the supports of

$\psi_i^j$ and $\varphi_i^j$ associated to $Q_{k^3}$ (see (2.3)(i) and (2.4)(i)) shows that

$$\langle T_{k^3} M_{b_1} \varphi, \psi \rangle = 0.$$

Of course $\langle T_{k^3} M_{b_1} \varphi, \psi \rangle$ also vanishes if $3Q_{k^3} \cap 3Q_{l^3} = \varnothing$. Thus if

$$\langle T_{k^3} M_{b_1} \varphi, \psi \rangle \neq 0,$$

then by (2.16), $s_{k^3} \leqslant \delta^{l^3} s_{l^3}$. Suppose that $3Q_{k^3}$ intersects $5J$. From $s_{k^3} \leqslant \delta^{l^3} s_{l^3}$ and (2.6)(i) we have

$$\| T_{k^3} M_{b_1}(\chi_{6J}) \|_{L^\infty} = \| T_{k^3} b_1 \|_{L^\infty} \leqslant C.$$

Thus

$$\begin{aligned} \langle T_{k^3} M_{b_1} \varphi, \psi \rangle &= \iint K_{k^3}(x, y) \psi(x) b_1(y) \varphi(y)\, dx\, dy \\ &= \iint K_{k^3}(x, y) \psi(x) b_1(y) [\varphi(y) - \chi_{6J}(y) \varphi(x)]\, dx\, dy \\ &\quad + \iint K_{k^3}(x, y) \psi(x) b_1(y) \chi_{6J}(y) \varphi(x)\, dx\, dy \\ &= A + B. \end{aligned}$$

Now

$$\begin{aligned} |A| &\leqslant C \int_{6J} \int_{5J} |x - y|^{-n} \| \psi \|_{L^\infty} M |x - y|^{\eta} \| \varphi \|_{\mathrm{Lip}\,\eta}\, dx\, dy \\ &\leqslant C |J|^{1 + \eta/n} \| \psi \|_{L^\infty} \| \varphi \|_{\mathrm{Lip}\,\eta} \\ &\leqslant C |J|^{1 + \eta/n} \| \psi \|_{\mathrm{Lip}\,\eta} \| \varphi \|_{\mathrm{Lip}\,\eta}, \end{aligned}$$

and

$$\begin{aligned} |B| &= \left| \int T_{k^3} M_{b_1}(\chi_{6J})(x) \psi(x) \varphi(x)\, dx \right| \\ &\leqslant C |J|\ \| \psi \|_{L^\infty} \| \varphi \|_{L^\infty} \\ &\leqslant C |J|^{1 + 2\eta/n} \| \psi \|_{\mathrm{Lip}\,\eta} \| \varphi \|_{\mathrm{Lip}\,\eta}. \end{aligned}$$

Summing in $k$ yields

$$\left| \sum_{k \neq l} k^{-2} \langle T_{k^3} M_{b_1} \varphi, \psi \rangle \right| \leqslant C_\eta |J|^{1 + 2\eta/n} \| \varphi \|_{\mathrm{Lip}\,\eta} \| \psi \|_{\mathrm{Lip}\,\eta},$$

$0 < \eta < 1$, and since (1.9)′ holds for $l^3$, *i.e.*

$$| l^{-2} \langle T_{l^3} M_{b_1} \varphi, \psi \rangle | \geqslant C l |J|^{1 + 2\eta/n} \| \varphi \|_{\mathrm{Lip}\,\eta} \| \psi \|_{\mathrm{Lip}\,\eta},$$

(2.18) shows that

$$| \langle T M_{b_1} \varphi, \psi \rangle | \geqslant C l |J|^{1 + 2\eta/n} \| \varphi \|_{\mathrm{Lip}\,\eta} \| \psi \|_{\mathrm{Lip}\,\eta}.$$

Letting $l$ tend to infinity shows that $TM_{b_1}$ fails to have the weak boundedness property and this completes the proof of Theorem 1 in the case $b_1$ and $b_2$ are jointly para-accretive.

*Case* 2. $b_1$ and $b_2$ are not jointly para-accretive.

In this case, the proof is simply a discrete version of the proof of Proposition 1 in Section 9 of [DJS2]. If Case 1 fails, then we can find a cube $Q_k$, for each $k > 0$, with the property

$$\sup_{\text{cubes } J \subset 3Q_k} \frac{1}{|Q_k|} \max\left\{\left|\int_J b_1\right|, \left|\int_J b_2\right|\right\} \leqslant \frac{(1/2)^{kn}}{k}.$$

With $k$ momentarily fixed, and $J_i^j$ and $\psi_i^j$ as in Case 1 (but with $\delta = 1/2$), define the kernel of $T_k$ by

$$K_k(x, y) = \sum_{j, i: J_i^j \subset Q_k} (\delta^j s_k)^{-n} \psi_i^j(x) \psi_i^j(y).$$

Properties (2.6)(ii)-(v) hold for $T_k$ just as in Case 1. Property (2.6)(i) now has the same proof as (2.6)(ii) and choosing nonnegative $\psi, \varphi \in C^\infty(\mathbb{R}^n)$ to be 1 on $Q_k$ with support in $2Q_k$, we have

$$\langle T_k \varphi, \psi \rangle \geqslant \int_{Q_k} \int_{Q_k} K_k(x, y)\, dx\, dy \geqslant Ck|Q_k|,$$

*i.e.* $T_k$ satisfies the weak boundedness property (1.4) only with a constant $C \geqslant c'k$. With

$$T = \sum_{k=1}^{\infty} \frac{1}{k^2} T_{k^3},$$

(1.6)(i)-(iii) hold and $T$ fails to have the weak boundedness property. This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

The «only if» half of Theorem 2 is a simple consequence of Theorem 1. If $b$ is not para-accretive, then Theorem 1, with $b_1 = 1$ and $b_2 = b$ produces a linear operator $T$ with kernel $K$ satisfying (1.2)(i), (ii) and (iii) such that $T1 \in L^\infty$, and $M_b T$, but not $T$, satisfies the weak boundedness property. This operator $T$ satisfies the requirements for membership in $\mathcal{C}$ except for $T1 = 0$. This however can be remedied by considering instead $\tilde{T} = T - \Pi_{T1}$ where $\Pi_\beta$ denotes the para-product operator

$$\Pi_\beta(f)(x) = \int_0^\infty \psi_t * \{(\psi_t * \beta)(\varphi_t * f)\}(x) \frac{dt}{t}, \tag{3.1}$$

where $\psi, \varphi \in \mathfrak{D}$ with

$$\int \varphi = 1, \qquad \int \psi = 0, \qquad \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{for} \quad \xi \neq 0,$$

and

$$\eta_t(x) = t^{-n}\eta\left(\frac{x}{t}\right).$$

Since, for $\beta \in \mathrm{BMO}$, the kernel of $\Pi_\beta$ satisfies (1.2)(i), (ii), (iii) and $\Pi_\beta(1) = \beta$ and $\Pi_\beta$ is bounded on $L^2$(*cf.* [CM]), it follows that $\tilde{T} \in \mathcal{C}$ and $M_b\tilde{T}$, but not $\tilde{T}$, has the weak boundedness property. This proves the «only if» half of Theorem 2.

We now prove the «if» half of Theorem 2 for the larger class of operators $\mathcal{C}'$. Suppose $b$ is para-accretive, $T \in \mathcal{C}'$, so that $T1 \in \mathrm{BMO}$, and $M_bT$ has the weak boundedness property. We follow the idea of the proof of Meyer's Lemme 2 in [M1]. Fix for the moment a cube $Q$ with center $x_0$ and $\theta \in \mathfrak{D}$ with $\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n : |x_i| \leqslant 4, 1 \leqslant i \leqslant n\}$ and $\theta = 1$ on $\{x \in \mathbb{R}^n : |x_i| \leqslant 2, 1 \leqslant i \leqslant n\}$. Let

$$\chi_0(x) = \theta\left(\frac{x - x_0}{\frac{1}{2}|Q|^{1/n}}\right)$$

and $\chi_1 = 1 - \chi_0$. Then $\varphi = \varphi\chi_0$ for all $\varphi$ in $C_0^\eta(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset Q$ and so we have

$$\begin{aligned} M_bT\varphi(x) &= b(x)\int K(x,y)[\varphi(y) - \varphi(x)]\chi_0(y)\,dy \\ &\quad + \varphi(x)b(x)\int K(x,y)\chi_0(y)\,dy \\ &= p(x) + q(x) \end{aligned} \tag{3.2}$$

where the equalities hold in the distribution sense.

Using the weak boundedness property of $M_bT$ and the size condition

$$|b(x)K(x,y)| \leqslant C|x-y|^{-n},$$

the proof of Lemme 3 in [M1] shows that

$$p(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} b(x)K(x,y)[\varphi(y) - \varphi(x)]\chi_0(y)\,dy$$

is actually a bounded function with

$$(3.3)\qquad |p(x)| \leqslant \overline{\lim_{\epsilon\to 0}} \int_{|x-y|>\epsilon} |b(x)|\,|K(x,y)|\,|\varphi(y)-\varphi(x)|\,|\chi_0(y)|\,dy$$

$$\leqslant C\|b\|_{L^\infty}\|\chi_0\|_{L^\infty}\int_{4Q}|x-y|^{\eta-n}\|\varphi\|_{\mathrm{Lip}\,\eta}\,dy$$

$$\leqslant C\|b\|_{L^\infty}|Q|^{\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}.$$

To estimate $q(x)$, let $\tilde{q}(x)$ denote the restriction of the distribution

$$T\chi_0(x) = \int K(x,y)\chi_0(y)\,dy$$

to the open cube

$$U = \frac{3}{2}Q.$$

In analogy with the argument on the bottom of page 246 of [M1], let $a(x)$ be a smooth $H^1$-atom with support in $U$. Then

$$\left|\int \tilde{q}(x)a(x)\,dx\right| = \left|\int T1(x)a(x)\,dx - \iint [K(x,y)-K(x_0,y)]a(x)\chi_1(y)\,dx\,dy\right|$$

since $T1 = T\chi_0 + T\chi_1$ and $\int a = 0$, and so

$$(3.4)\qquad \left|\int \tilde{q}(x)a(x)\,dx\right| \leqslant \|T1\|_{\mathrm{BMO}}\|a\|_{H^1} + C\|a\|_{L^1} \leqslant C\|a\|_{H^1},$$

by (1.2)(ii).

Inequality (3.4) shows that $\tilde{q}\in \mathrm{BMO}\,(U)$. We will now use the para-accretivity of $b$ to estimate the average

$$w_Q = \frac{1}{|Q|}\int_Q \tilde{q}(x)\,dx.$$

For this we need

**Lemma 3.5.** *Suppose $b$ is para-accretive and $Q$ is a cube in $\mathbb{R}^n$. Then there is $\rho\in\mathfrak{D}$ with*

$$\operatorname{supp}\rho \subset Q,\quad \|\rho\|_{\mathrm{Lip}\,\eta} \leqslant C_1|Q|^{-1-\eta/n} \quad \textit{and} \quad \left|\int_Q b(x)\rho(x)\,dx\right| \geqslant C_2 > 0,$$

*where $C_1$ and $C_2$ are constants independent of $Q$.*

Assuming the lemma, we have for $\rho$ as above,

$$\begin{aligned} C_2|w_Q| &\leqslant |\langle bw_Q, \rho\rangle| \\ &= |\langle M_b T\chi_0, \rho\rangle + \langle b(w_Q - \tilde{q}), \rho\rangle| \\ &\leqslant C|Q|^{1+2\eta/n}\|\chi_0\|_{\mathrm{Lip}\,\eta}\|\rho\|_{\mathrm{Lip}\,\eta} + C\|b\|_{L^\infty}\|\tilde{q}\|_{\mathrm{BMO}} \end{aligned}$$

since $M_b T$ satisfies (1.4) and $\|\rho\|_{L^\infty} \leqslant |Q|^{-1}$ with $\operatorname{supp}\rho \subset Q$, and so

$$C_2|w_Q| \leqslant C|Q|^{1+2\eta/n}|Q|^{-\eta/n}|Q|^{-1-\eta/n} + C \leqslant C.$$

Thus $|w_Q| \leqslant C$ where $C$ is independent of $Q$ and if $\psi \in C_0^\eta(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset Q$, then

$$\begin{aligned} |\langle q, M_{b^{-1}}\psi\rangle| &= |\langle \tilde{q}, \varphi\psi\rangle| \\ &\leqslant |\langle w_Q, \varphi\psi\rangle| + |\langle \tilde{q} - w_Q, \varphi\psi\rangle| \\ &\leqslant C|Q|\,\|\varphi\|_{L^\infty}\|\psi\|_{L^\infty}, \end{aligned}$$

since $\tilde{q} \in \mathrm{BMO}\,(U)$, and

$$|\langle p, M_{b^{-1}}\psi\rangle| \leqslant C\|b\|_{L^\infty}\|b^{-1}\|_{L^\infty}|Q|^{1+\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{L^\infty}$$

by (3.3). Using these inequalities and (3.2) we obtain

$$\begin{aligned} |\langle T\varphi, \psi\rangle| &= |\langle p, M_{b^{-1}}\psi\rangle + \langle q, M_{b^{-1}}\psi\rangle| \\ &\leqslant C\|b\|_{L^\infty}\|b^{-1}\|_{L^\infty}|Q|^{1+2\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{\mathrm{Lip}\,\eta} \end{aligned}$$

which is (1.4) since $b^{-1}$ is bounded if $b$ is para-accretive.

It remains to prove Lemma 3.5. Since $b$ is para-accretive, there is a cube

$$I \subset \frac{1}{2}Q$$

such that

$$\left|\int_I b(x)\,dx\right| \geqslant \delta|Q|$$

where $\delta > 0$ depends only on $b$. Fix $\theta \in \mathfrak{D}$ with

$$\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n : |x_i| \leqslant 1 + \epsilon, 1 \leqslant i \leqslant n\}, \qquad 0 \leqslant \theta \leqslant 1$$

and

$$\theta(x) = 1 \quad \text{if} \quad |x_i| \leqslant 1, \qquad 1 \leqslant i \leqslant n.$$

Let

$$\rho(x) = |Q|^{-1}\theta\left(\frac{x - x_I}{\frac{1}{2}|I|^{1/n}}\right)$$

where $x_I$ is the centre of $I$ and $\epsilon > 0$ is sufficiently small that $\operatorname{supp}\rho \subset 2I \subset Q$, and

$$\begin{aligned}\left|\int \rho(x)b(x)\,dx\right| &= \left|\int_I \rho(x)b(x)\,dx + \int_{(1+\epsilon)I\setminus I}\rho(x)b(x)\,dx\right| \\ &\geqslant \delta - \|b\|_{L^\infty}\frac{|(1+\epsilon)I\setminus I|}{|Q|} > \frac{\delta}{2} = C_2.\end{aligned}$$

For such $\rho$ we have

$$\|\rho\|_{\mathrm{Lip}_\eta} \leqslant C|Q|^{-1}|I|^{-\eta/n}\|\theta\|_{\mathrm{Lip}_\eta} \leqslant C_1|Q|^{-1-\eta/n}$$

since

$$\delta|Q| \leqslant \left|\int_I b(x)\,dx\right| \leqslant \|b\|_{L^\infty}|I|$$

and this completes the proof of Lemma 3.5 and so also Theorem 2.

We close this section by proving the remark made in the introduction that the size condition (1.2)(i) on the kernel of $T$ is not needed in the *Tb* Theorem for $L^2$. Suppose (1.1) and (1.2)(ii), (iii) hold and that $M_{b_2}TM_{b_1}$ satisfies the weak boundedness property with $b_1$ and $b_2$ para-accretive. Fix $x$ and $y$ in $\mathbb{R}^n$ and let $s = |x - y| > 0$. By Lemma 3.5, there are $\rho_1$ and $\rho_2$ in $\mathfrak{D}$ with

$$\operatorname{supp}\rho_1 \subset B\left(y, \frac{s}{3}\right), \qquad \operatorname{supp}\rho_2 \subset B\left(x, \frac{s}{3}\right), \quad \|\rho_i\|_{\mathrm{Lip}_\eta} \leqslant Cs^{-n-\eta}$$

and

$$\left|\int b_i(u)\rho_i(u)\,du\right| \geqslant c > 0 \quad \text{for} \quad i = 1, 2.$$

Thus

$$\begin{aligned}(3.6)\qquad c^2|K(x,y)| &\leqslant \left|\iint \rho_2(u)b_2(u)K(x,y)b_1(v)\rho_1(v)\,du\,dv\right| \\ &\leqslant \left|\iint \rho_2(u)b_2(u)[K(x,y) - K(u,v)]b_1(v)\rho_1(v)\,du\,dv\right| \\ &\quad + \left|\iint \rho_2(u)b_2(u)K(u,v)b_1(v)\rho_1(v)\,du\,dv\right| \\ &= A + B.\end{aligned}$$

The smoothness conditions (1.2)(ii), (iii) yield

$$\begin{aligned}|K(x,y)-K(u,v)| &\leqslant |K(x,y)-K(u,y)| + |K(u,y)-K(u,v)|\\ &\leqslant C|x-y|^{-n}\end{aligned}$$

for $u \in \operatorname{supp} \rho_2$ and $v \in \operatorname{supp} \rho_1$, and it follows that

$$\begin{aligned}A &\leqslant C|x-y|^{-n}\|b_1\|_{L^\infty}\|b_2\|_{L^\infty}\|\rho_1\|_{L^1}\|\rho_2\|_{L^1}\\ &\leqslant C|x-y|^{-n}.\end{aligned}$$

Since (1.1) holds and $M_{b_2}TM_{b_1}$ has the weak boundedness property,

$$\begin{aligned}B &= |\langle M_{b_2}TM_{b_1}\rho_1, \rho_2\rangle|\\ &\leqslant Cs^{n+2\eta/n}\|\rho_1\|_{\mathrm{Lip}\,\eta}\|\rho_2\|_{\mathrm{Lip}\,\eta}\\ &\leqslant Cs^{-n}\\ &= C|x-y|^{-n}.\end{aligned}$$

Combining the estimates for $A$ and $B$ with (3.6) yields (1.2)(ii) as required.

The above argument can easily be modified to show the same conclusion if one of the smoothness conditions (1.2)(ii), (iii) is replaced by a Hörmander condition. If both (1.2)(ii) and (iii) are replaced by Hörmander conditions, then the conclusion is that the integral size estimates

$$\int_{r<|x-v|<2r} |K(x,v)|\,dv \leqslant C$$

and

$$\int_{r<|u-y|<2r} |K(u,y)|\,du \leqslant C$$

hold for all $x$ and $y$.

## 4. Proof of Theorem 3

As mentioned in the introduction, Theorem 3 is easily proved by adapting the proof of the $T1$ Theorem for $L^2$ that is outlined in Section 2 of [DJS2]-the main tool being the Calderón reproducing formula. The following sketch will highlight the main differences.

Choose $\phi \in C^\infty(\mathbb{R}^n)$ with support in the unit ball and mean value zero so that the identity operator is given by the Calderón reproducing formula

$$I = \int_0^\infty \phi_s \phi_s \frac{ds}{s} \tag{4.1}$$

where

$$\phi_s(x) = s^{-n}\phi(s^{-1}x)$$

and in the context of an operator, the symbol $\phi_s$ means convolution with $\phi_s(x)$. Formally (4.1) is

$$1 = \int_0^\infty |\hat{\phi}(s\xi)|^2 \frac{ds}{s}$$

and it is an easy matter to find $\phi$ with this property. For $f$ and $g$ test functions we have

$$\begin{aligned} \langle Tf, g\rangle &= \langle ITIf,\ g\rangle \\ &= \int_0^\infty \int_0^\infty \langle \phi_s\phi_s T\phi_t\phi_t f, g\rangle \frac{ds}{s}\frac{dt}{t} \\ &= \int_0^\infty \int_0^\infty \langle [\phi_s T\phi_t]\phi_t f, \phi_s g\rangle \frac{ds}{s}\frac{dt}{t} \end{aligned} \tag{4.2}$$

and thus we need to estimate the kernel of the operator $\phi_s T\phi_t$, which we denote by $\phi_s T\phi_t(x,y)$. We have

**Lemma 4.3.** *Suppose $T$ has kernel $K(x,y)$ satisfying* (1.2)(i) *and* (ii) (*but not necessarily smoothness in the second variable,* (1.2)(iii)) *and that* $T1 = 0$ *and $T$ has the weak boundedness property* (1.4). *Then*

$$|\phi_s T\phi_t(x,y)| \leqslant C\left[1 + \left|\log\frac{s}{t}\right|\right]\left[\left(\frac{s}{t}\right)^\epsilon \wedge 1\right]\frac{(s\vee t)^\epsilon}{[(s\wedge t) + |x-y|]^{n+\epsilon}}$$

*where the symbols* $\wedge$ *and* $\vee$ *mean minimum and maximum respectively.*

Assuming the lemma for the moment, we have

$$|(\phi_s T\phi_t)(\phi_t f)(x)| \leqslant C\omega\left(\frac{s}{t}\right)M(\phi_t f)(x)$$

where

$$\omega(r) = (1 + |\log r|)(r^\epsilon \wedge 1)$$

and $M$ denotes the Hardy-Littlewood maximal operator. Setting

$$\theta(r) = r^{-\alpha}\omega(r),$$

we obtain

$$(4.4)\quad |\langle Tf, g\rangle| \leqslant \int_0^\infty\int_0^\infty \langle M(t^{-\alpha}\phi_t f), s^\alpha|\phi_s g|\rangle \theta\left(\frac{s}{t}\right)\frac{ds}{s}\frac{dt}{t}$$

$$\leqslant \left\langle \left\{\int_0^\infty\int_0^\infty M(t^{-\alpha}\phi_t f)(x)^q \theta\left(\frac{s}{t}\right)\frac{ds}{s}\frac{dt}{t}\right\}^{1/q}, \left\{\int_0^\infty\int_0^\infty |s^\alpha\phi_s g(x)|^{q'}\theta\left(\frac{s}{t}\right)\frac{dt}{t}\frac{ds}{s}\right\}^{1/q'}\right\rangle$$

$$\leqslant \left\langle \left\{\int_0^\infty M(t^{-\alpha}\phi_t f)(x)^q\frac{dt}{t}\right\}^{1/q}, \left\{\int_0^\infty |s^\alpha\phi_s g(x)|^{q'}\frac{ds}{s}\right\}^{1/q'}\right\rangle$$

since $\displaystyle\int_0^\infty \theta\left(\frac{s}{t}\right)\frac{ds}{s} + \int_0^\infty \theta\left(\frac{s}{t}\right)\frac{dt}{t} < \infty$ for $0 < \alpha < \epsilon$,

$$\leqslant \left\|\left(\int_0^\infty [M(t^{-\alpha}\phi_t f)]^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p}\left\|\left(\int_0^\infty |s^\alpha\phi_s g|^{q'}\frac{ds}{s}\right)^{1/q'}\right\|_{L^{p'}}$$

$$\leqslant \left\|\left(\int_0^\infty |t^{-\alpha}\phi_t f|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p}\left\|\left(\int_0^\infty |s^\alpha\phi_s g|^{q'}\frac{ds}{s}\right)^{1/q'}\right\|_{L^{p'}}$$

by the Fefferman-Stein vector-valued inequality for $1 < p, q < \infty$ ([FS]),

$$= C\|f\|_{\dot{F}_p^{\alpha,q}}\|g\|_{\dot{F}_{p'}^{-\alpha,q'}}.$$

Since $(\dot{F}_p^{\alpha,q})' = \dot{F}_{p'}^{-\alpha,q'}$, it follows from (4.4) that $T$ is bounded on $\dot{F}_p^{\alpha,q}$ for $1 < p, q < \infty$ and $0 < \alpha < \epsilon$.

Returning now to the lemma, we prove the estimate (4.3) in the crucial case where $s < t$ and $|x-y| \leqslant Ct$. The three remaining cases: $s < t$ and $|x-y| > Ct$, $t < s$ and $|x-y| \leqslant Cs$, $t < s$ and $|x-y| > Cs$, are similar but easier. Let $\eta_0 \in C^\infty(\mathbb{R}^n)$ be 1 on the unit ball and 0 outside its double. Set $\eta_1 = 1 - \eta_0$. Then following the proof of Lemma 7 in Section 6 of [DJS2], we have

$$(4.5)\qquad \phi_s T\phi_t(x,y) = \iint \phi_s(x-u)K(u,v)\phi_t(v-y)\,du\,dv$$

$$= \iint \phi_s(x-u)K(u,v)[\phi_t(v-y) - \phi_t(x-y)]\,du\,dv,$$

since $T1 = 0$, and so

$$\phi_s T\phi_t(x,y) = \iint \phi_s(x-u)K(u,v)[\phi_t(v-y) - \phi_t(x-y)]\eta_0\left(\frac{v-x}{s}\right)du\,dv$$

$$+ \iint \phi_s(x-u)[K(u,v) - K(x,v)][\phi_t(v-y) - \phi_t(x-y)]\eta_1\left(\frac{v-x}{s}\right)du\,dv = A + B$$

since $1 = \eta_0 + \eta_1$ and $\phi_s 1 = 0$. Now with $\psi(u) = \phi_s(x-u)$ and

$$\varphi(v) = [\phi_t(v-y) - \phi_t(x-y)]\eta_0\left[\frac{v-x}{s}\right],$$

$$|A| = |\langle T\varphi, \psi\rangle| \leqslant Cs^{n+2\eta}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{\mathrm{Lip}\,\eta},$$

by the weak boundedness property (1.4),

$$\leqslant Cs^{n+2\eta}\left\{\left(\frac{s}{t}\right)t^{-n}s^{-\eta}\right\}\{s^{-n}s^{-\eta}\}$$

$$\leqslant C\left(\frac{s}{t}\right)t^{-n}$$

which is dominated by the right side of (4.3) when $s < t$, $|x-y| \leqslant Ct$ and $0 < \epsilon < 1$. Using the smoothness of $K(x,y)$ in $x$, (1.2)(ii), together with

$$\|\phi_t(v-y) - \phi_t(x-y)| \leqslant C\left(\frac{|v-x|}{t+|v-x|}\right)^{\epsilon} t^{-n},$$

$$|B| \leqslant C\iint_{|v-x|\geq s}|\phi_s(x-u)|\left(\frac{s}{|v-x|}\right)^{\epsilon}|v-x|^{-n}\left(\frac{|v-x|}{t+|v-x|}\right)^{\epsilon}t^{-n}\,du\,dv$$

$$\leqslant Cs^{\epsilon}t^{-n}\int_{|v-x|\geq t}|v-x|^{-n-\epsilon}\,dv + C\left(\frac{s}{t}\right)^{\epsilon}t^{-n}\int_{t\geq|v-x|\geq s}|v-x|^{-n}\,dv$$

$$\leqslant C\left(1+\log\frac{t}{s}\right)\left(\frac{s}{t}\right)^{\epsilon}t^{-n}$$

which is again dominated by the right side of (4.3) when $s < t$ and $|x-y| \leqslant Ct$. This proves (4.3) for this case and completes our sketch of the proof of Theorem 3.

*Remark.* Theorem 3 remains true in the case $p = q = 2$ if the condition $T1 = 0$ is relaxed to the condition that $|s^{-\alpha}\phi_s(T1)(x)|^2\frac{dx\,ds}{s}$ is an $\alpha$-Carleson measure for $L^2$ (*i.e.* (4.7) below holds). More precisely, if $T1 \neq 0$, then we must add to (4.5) the correction term

$$\chi_{\{s\leq Ct\}}\iint\phi_s(x-u)K(u,v)\phi_t(x-y)\,du\,dv = \phi_s(T1)(x)\phi_t(x-y)\chi_{\{s\leq Ct\}}$$

and estimate in (4.2) the new term

$$(4.6)\quad \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\{s \le Ct\}} \phi_s(T1)(x)\phi_t(x-y)\phi_t f(y)\phi_s g(x)\,dx\,dy\,\frac{ds}{s}\,\frac{dt}{t}$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \phi_s(T1)(x)\left\{\int_{C^{-1}s}^\infty \int_{\mathbb{R}^N} \phi_t(x-y)\phi_t f(y)\,dy\,\frac{dt}{t}\right\}\phi_s g(x)\,dx\,\frac{ds}{s}$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \phi_s(T1)(x) P_s f(x)\phi_s g(x)\,dx\,\frac{ds}{s}$$

where

$$P_s = \int_{C^{-1}s}^\infty \phi_t \phi_t \frac{dt}{t}$$

satisfies $|P_s(x)| \leqslant Cs^{-n}$ and, if $\phi * \phi(x) = \theta(|x|)$ is radial, then

$$P_s(x) = \int_{C^{-1}s}^\infty t^{-n}\theta\left(\frac{|x|}{t}\right)\frac{dt}{t} = |x|^{-n}\int_0^{Cs^{-1}|x|} \theta(r) r^{n-1}\,dr = 0$$

for $|x| > 2C^{-1}s$, since $\phi * \phi$ is supported in double the unit ball and has mean value zero.

The integral in (4.6) is at most

$$\left(\iint_{\mathbb{R}^{n+1}_+} |P_s f(x)|^2 |s^{-\alpha}\phi_s(T1)(x)|^2\,dx\,\frac{ds}{s}\right)^{1/2}\left(\iint_{\mathbb{R}^{n+1}_+} |s^\alpha \phi_s g(x)|^2\,dx\,\frac{ds}{s}\right)^{1/2}$$

and since the second factor is $\|g\|_{\dot F_2^{-\alpha,2}}$, duality shows that $T$ will be bounded on $\dot F_2^{\alpha,2}$ provided (with $f = I_\alpha h$)

$$(4.7)\quad \iint_{\mathbb{R}^{n+1}_+} |P_s I_\alpha h(x)|^2 |s^{-\alpha}\phi_s(T1)(x)|^2\,dx\,\frac{ds}{s} \leqslant C\int_{\mathbb{R}^n} |h(x)|^2\,dx$$

for all $h$ in $L^2(\mathbb{R}^n)$. Characterizations of (4.7) can be found in [KS] and [NRS].

Finally, since the integral in (4.6) is

$$\int_{\mathbb{R}^n}\int_0^\infty \phi_s\{[\phi_s(T1)]P_s f\}(x)\,\frac{ds}{s}\,g(x)\,dx = \int_{\mathbb{R}^n} \Pi_{T1} f(x) g(x)\,dx$$

by (3.1), it follows that $T$ is bounded on $\dot F_2^{\alpha,2}$ if and only if $\Pi_{T1}$ is bounded on $\dot F_2^{\alpha,2}$. (If $T$ is bounded on $\dot F_2^{\alpha,2}$, then $T$ has the weak boundedness property and so $T1 \in \dot B_\infty^{0,\infty}$ by [M2]. It then follows that $\Pi_{T1}$ satisfies the standard estimates (1.2) (see [CM]) and the weak boundedness property is easily checked to hold. Thus $T - \Pi_{T1}$ is bounded on $\dot F_2^{\alpha,2}$ by Theorem 3.) While (4.7) is sufficient for this, we do not know if it is necessary.

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