Revista Matemática Iberoamericana
Vol. 6, n.os 1 y 2, 1990

Global Existence of Solutions for the Nonlinear Boltzmann Equation of Semiconductor Physics

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Abstract

In this paper we give a proof of the existence and uniqueness of smooth solutions for the nonlinear semiconductor Boltzmann equation. The method used allows to obtain global existence in time and uniqueness for dimensions 1 and 2. For dimension 3 we only can assure local existence in time and uniqueness. First, we define a sequence of solutions for a linearized equation and then, we prove the strong convergence of the sequence in a suitable space. The method relies in the use of interpolation estimates in order to control the decay of the solution when the wave vector goes to infinity.

1. Introduction

This paper is devoted to the Boltzmann equation of semiconductors, which is the basic equation of the kinetic model of semiconductors. In this model, each type of carriers is described by a distribution function $f(x, k, t)$, where $x \in \mathbb{R}^d$ is the spatial position, $k \in \mathbb{R}^d$ the wave vector of the carriers and $t \geqslant 0$, the time. We will denote by d the space dimension, which will be equal to 1, 2 or 3.

If all the physical constants are taken equal to unity, the Boltzmann equation, which gives the evolution of the distribution function, is written

$$\begin{cases} \dfrac{\partial f}{\partial t} + v(k)\nabla_x f - E(x,t)\nabla_k f = Q(f) \\ f(x,k,0) = f_0(x,k) \end{cases} \tag{1.1}$$

where the electric field $E(x, t)$ is coupled to the distribution function by the Poisson equation

$$E(x,t) = C(d)\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\rho(y,t)\,dy \tag{1.2}$$

being

$$\rho(x,t) = \int_{\mathbb{R}^d} f(x,k,t)\,dk \tag{1.3}$$

the electric charge.

The velocity $v(k)$ is a known function deduced from the band diagram and the collision term $Q(f)$ will be given here according to

$$Q(f)(x,k,t) = \int_{\mathbb{R}^d} [S(x,k',k)f(x,k',t)(1-f(x,k,t)) - S(x,k,k')f(x,k,t)(1-f(x,k',t))]\,dk' \tag{1.4}$$

where $S(x, k, k')\,dk'$ is the transition rate for an electron at the position x to be scattered from a state k, into a state belonging to a small volume dk' around the state k'.

We refer the reader to [1, 2, 3] for the physical background of (1.1)-(1.4). We have neglected, in our model, the electron-electron scattering as well as the generation-recombination processes.

In this paper, we show that the method used by P. Degond [5], to prove the existence of solutions for the Vlasov-Fokker-Planck equation, is also applicable to the Boltzmann equation of semiconductors, where nonlinear integral operators are included. So, we give a proof of the existence and uniqueness of global in time smooth solutions of (1.1)-(1.4) when the space dimension is 1 or 2. For dimension 3 the method used only allows to prove the existence and uniqueness of local in time smooth solutions. The global in time existence for dimension 3 is still an open problem, like usually occurs in Vlasov-Poisson type equations. The outline of the proof is essentially the same as in [5]. We define a sequence of solutions (f^n, E^n) for a suitable linearized equation and then we prove the strong convergence of this sequence.

We have used the linearization proposed by F. Poupaud in [4]. In this last reference, a proof of the existence and uniqueness of the semiconductor Boltzmann equation is given, but when all the integrals on the wave vector space are taken over a bounded domain. So, the control of the electric charge is not there a problem. However, in order to control the electric charge, when the integral which defines it is taken over $\mathbb{R}^d$, we need to estimate the decay of the solution when the wave vector goes to infinity. This will be obtained following the idea of [5] and using some estimates on the collision integral.

The outline of the paper is the following: in Section 2, we state the existence and uniqueness theorem. Section 3 is devoted to the definition of the iterative sequence on which the proof is based. We also give some a priori estimates. In Section 4, we obtain the basic estimates which allow to prove the convergence of the procedure in Section 5.

2. Existence and Uniqueness Theorem

We define the functional space

$$\chi = \{\varphi(x,y,z): \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}: \varphi(x,y,z) \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_y, L^1(\mathbb{R}^d_z)) \text{ and } (1+|z|^2)^{\gamma/2}\varphi(x,y,z) \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_z, L^1(\mathbb{R}^d_y))\}$$

with $\gamma > d$.

We assume the following considerations

$$S(x,k,k') \geqslant 0; \qquad S(x,k,k') \in \chi \tag{2.1}$$

$$|\nabla_{x,k,k'} S| \in \chi \tag{2.2}$$

$$\nabla_k v(k) \in L^\infty(\mathbb{R}^d). \tag{2.3}$$

Now, we can state our main result.

Theorem 2.1. *We suppose that the initial data $f_0(x,k)$ satisfies*

(2.4) $0 \leqslant f_0 \leqslant 1$; $f_0 \in W^{1,1}(\mathbb{R}^{2d})$; $(1+|k|^2)^{\gamma/2}(|f_0| + |Df_0|) \in L^\infty(\mathbb{R}^{2d})$; $\gamma > d$

Then the semiconductor Boltzmann equation:

$$\frac{\partial f}{\partial t} + v(k)\nabla_x f - E(x,t)\nabla_k f = Q(f), \qquad f(x,k,0) = f_0(x,k) \tag{2.5}$$

$$Q(f)(x,k,t) = \int_{\mathbb{R}^d} [S(x,k',k) f(x,k',t)(1 - f(x,k,t)) - S(x,k,k') f(x,k,t)(1 - f(x,k',t))]\, dk' \tag{2.6}$$

$$E(x,t) = C(d) \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \rho(y,t)\, dy \tag{2.7}$$

$$\rho(x,t) = \int_{\mathbb{R}^d} f(x,k,t)\, dk \tag{2.8}$$

with the assumptions (2.1), (2.2) *and* (2.3), *admits a unique classical solution, in a time interval* $[0, T[$, *where*

$$T = \infty \quad \text{if} \quad d = 1 \quad \text{or} \quad 2,$$

and T is finite and depends on f_0, *S and* γ *if* $d = 3$.

This solution is such that

$$0 \leqslant f \leqslant 1 \tag{2.9}$$

$$f \in L^{\infty}_{\text{loc}}([0, T[, W^{1,1}(\mathbb{R}^{2d})) \tag{2.10}$$

$$(1 + |k|^2)^{\gamma/2}(|f| + |Df|) \in L^{\infty}_{\text{loc}}([0, T[, L^{\infty}(\mathbb{R}^{2d})) \tag{2.11}$$

$$E \in L^{\infty}_{\text{loc}}([0, T[, W^{1,\infty}(\mathbb{R}^d_x)). \tag{2.12}$$

3. The Iterative Sequence

As in [4], we define the following operators

$$A(f)(x,k,t) = \int S(x,k',k) f(x,k',t)\, dk' \tag{3.1}$$

$$B(f)(x,k,t) = \int S(x,k,k')(1 - f(x,k',t))\, dk'. \tag{3.2}$$

Then, the collision term $Q(f)$ can be written

$$Q(f)(x,k,t) = A(f)(x,k,t)(1 - f(x,k,t)) - B(f)(x,k,t) f(x,k,t)$$

and the Boltzmann equation (2.5) is given by

$$\begin{cases} \dfrac{\partial f}{\partial t} + v(k)\nabla_x f - E(x,t)\nabla_k f + \lambda(f) f = A(f) \\ \lambda(f) = A(f) + B(f) \end{cases} \tag{3.3}$$

The proof of the Theorem 2.1 will be based on the following iterative sequence (f^n, E^n):

We begin with

$$f^0(x,k,t) = f_0(x,k). \tag{3.4}$$

Then, if we consider that f^n is known, we can compute the charge ρ^n and the electric field E^n according to

$$(3.5)\qquad \begin{cases} \rho^n(x,t) = \displaystyle\int f^n(x,k,t)\,dk \\ E^n(x,t) = C(d)\displaystyle\int \frac{x-y}{|x-y|^d}\rho^n(y,t)\,dy \end{cases}$$

Finally f^{n+1} is defined as the solution of the following equation:

$$(3.6)\qquad \begin{cases} \dfrac{\partial f^{n+1}}{\partial t} + v(k)\nabla_x f^{n+1} - E^n(x,t)\nabla_k f^{n+1} + \lambda(f^n)f^{n+1} = A(f^n) \\ f^{n+1}(x,k,0) = f_0(x,k) \end{cases}$$

Now we can state

Proposition 3.1. *The functions f^n of the iterative sequence defined by* (3.4), (3.5) *and* (3.6) *satisfy*:

(i) $0 \leqslant f^n \leqslant 1$.
(ii) f^n *are uniformly bounded in* $L^\infty([0,T^*];\,L^1(\mathbb{R}^{2d}))$ *for any time* T^*.

PROOF. We assume that (i) holds for f^n. In view of (2.1), (3.1) and (3.2) we have:

$$\lambda(f^n) \geqslant 0; \qquad A(f^n) \geqslant 0.$$

So since the source and the initial data for the equation (3.6) are non-negative we get

$$f^{n+1} \geqslant 0.$$

In the other hand we can also write

$$\begin{cases} \left[\dfrac{\partial}{\partial t} + v(k)\nabla_x - E^n(x,t)\nabla_k + \lambda(f^n)\right](1-f^{n+1}) = B(f^n), \\ (1-f^{n+1})(x,k,0) = 1 - f_0(x,k) \geqslant 0. \end{cases}$$

The same argument leads to

$$1 - f^{n+1} \geqslant 0, \quad \textit{i.e.} \quad f^{n+1} \leqslant 1.$$

In order to prove (ii) we integrate (3.6) and we use the non negativity of $\lambda(f^n)$ to obtain

$$\|f^{n+1}(t)\|_1 \leqslant \|f_0\|_1 + \int_0^t \|A(f^n)(s)\|_1\,ds.$$

Now, thanks to (2.1) and (2.4) we get

$$\|A(f^n)(s)\|_1 = \iint f^n(x,k',s)\left(\int S(x,k',k)\,dk\right)dx\,dk'$$
$$\leqslant C_1(S)\|f^n(s)\|_1$$
$$\|f^{n+1}(t)\|_1 \leqslant \|f_0\|_1 + C_1\int_0^t \|f^n(s)\|_1\,ds.$$

Now if we note by $\delta(t)$ the solution of the linear equation

$$\dot{\delta}(t) = C_1\delta(t), \qquad \delta(0) = \|f_0\|_1$$

it is a simple matter to check that

$$\|f^n(t)\|_1 \leqslant \delta(t) \quad \text{for every} \quad t \geqslant 0. \tag{3.7}$$

So, if we define $C = \underset{[0,T^*]}{\operatorname{Max}}\,\delta(t)$ we have

$$\|f^n(t)\|_1 \leqslant C(f_0, S, T^*) \quad \text{for every} \quad t \in [0,T^*], \qquad T^* < \infty$$

and (ii) is proved.

4. The Basic Estimates

In order to obtain the strong convergence of f^n we need to control the gradients of the functions. Furthermore, the use of interpolation inequalities (A.3) and (A.4) of Appendix requieres L^∞ estimates on ρ^n and $\nabla_x\rho^n$. There will be essentially obtained, as in [5], using some estimates on the decay at infinity, in the wave vector space, of f^n.

Lemma 4.1. *We assume that f_0 satisfies the hypotheses* (2.4). *Then we have for every n*

(i) *ρ^n is uniformly bounded in $L^\infty([0,T^*], L^\infty(\mathbb{R}^d))$,*
(ii) *E^n is uniformly bounded in $L^\infty([0,T^*], L^\infty(\mathbb{R}^d))$,*

where

$$T^* < \infty \quad \textit{if} \quad d = 1 \quad \textit{or} \quad 2,$$
$$T^* < T \quad \textit{finite if} \quad d = 3.$$

PROOF. Multiplying equation (3.6) by $(1 + |k|^2)^{\gamma/2}$ and defining

$$Y^n(x, k, t) = (1 + |k|^2)^{\gamma/2} f^n(x, k, t)$$

we get

$$\frac{\partial Y^{n+1}}{\partial t} + v(k)\,\nabla_x Y^{n+1} - E^n(x, t)\,\nabla_k Y^{n+1} + \lambda(f^n) Y^{n+1} = R_1^{n+1} + R_2^n \tag{4.1}$$

$$R_1^{n+1} = -\gamma(E^n, k)(1 + |k|^2)^{(\gamma-2)/2} f^{n+1}$$
$$R_2^n = A(f^n)(1 + |k|^2)^{\gamma/2}$$

From (4.1) we can obtain

$$\| Y^{n+1}(t)\|_\infty \leqslant \| Y_0\|_\infty + \int_0^t (\|R_1^{n+1}(s)\|_\infty + \|R_2^n(s)\|_\infty)\, ds. \tag{4.2}$$

But,

$$R_2^n(s) = (1 + |k|^2)^{\gamma/2} \int S(x, k', k) f^n(x, k', s)\, dk'$$
$$\leqslant \| Y^n(s)\|_\infty (1 + |k|^2)^{\gamma/2} \int S(x, k', k)\, dk'$$

and thanks to (2.1)

$$\|R_2^n(s)\|_\infty \leqslant C(S, \gamma)\| Y^n(s)\|_\infty. \tag{4.3}$$

On the other hand

$$\|R_1^{n+1}(s)\|_\infty \leqslant \gamma \|E^n(s)\|_\infty \|(1 + |k|^2)^{(\gamma-1)/2} f^{n+1}(s)\|_\infty. \tag{4.4}$$

Then, using the interpolation inequality (A.1) of Appendix and Proposition 3.1, we have

$$\begin{aligned} \|(1 + |k|^2)^{(\gamma-1)/2} f^{n+1}(s)\|_\infty &\leqslant C(\gamma)\| f^{n+1}(s)\|_\infty^{1/\gamma} \| Y^{n+1}(s)\|_\infty^{1-1/\gamma} \\ &\leqslant C(\gamma)\| Y^{n+1}(s)\|_\infty^{1-1/\gamma}. \end{aligned} \tag{4.5}$$

Now, using (A.2), (A.3), (3.7) and again Proposition 3.1

$$\|E^n(s)\|_\infty \leqslant C(d)\|\rho^n(s)\|_1^{1/d} \|\rho^n(s)\|_\infty^{(d-1)/d}$$
$$\|\rho^n(s)\|_1^{1/d} = \| f^n(s)\|_1^{1/d} \leqslant \delta(s)^{1/d} = \varphi(s)$$

$$\begin{aligned} \|\rho^n(s)\|_\infty &\leqslant C(\gamma, d)\| f^n(s)\|_\infty^{(\gamma-d)/\gamma} \| Y^n(s)\|_\infty^{d/\gamma} \\ &\leqslant C(\gamma, d)\| Y^n(s)\|_\infty^{d/\gamma}. \end{aligned} \tag{4.6}$$

Thus,

$$\|E^n(s)\|_\infty \leqslant C(\gamma, d)\varphi(s)\| Y^n(s)\|_\infty^{(d-1)/\gamma}. \tag{4.7}$$

So, (4.7) becomes

$$\|Y^{n+1}(t)\|_\infty \leqslant C_1 + C_2 \int_0^t \|Y^n(s)\|_\infty \, ds + C_3 \int_0^t \varphi(s) \|Y^{n+1}(s)\|_\infty^{1-1/\gamma} \|Y^n\|_\infty^{(d-1)/\gamma} \, ds \tag{4.8}$$

where, from now on, C_i will denote constants depending on f_0, S, γ and d.

In order to estimate $\|Y^n(t)\|_\infty$ we define

$$y_n(t) = \operatorname{Max}\{1, \|Y^n(t)\|_\infty\}.$$

If $d < 2$, (4.8) simplifies into

$$y_{n+1}(t) \leqslant C_1 + C_2 \int_0^t y_n(s)\, ds + C_3 \int_0^t \varphi(s) y_{n+1}^{1-1/\gamma}(s) y_n^{1/\gamma}(s)\, ds.$$

Now if we consider the solution $\alpha(t)$ of the linear equation

$$\dot\alpha(t) = [C_2 + C_3\varphi(t)]\alpha(t); \qquad \alpha(0) = C_1$$

it is easy to prove [5] by induction on n that

$$y_n(t) \leqslant \alpha(t) \quad \text{for every} \quad t \geqslant 0 \quad \text{and} \quad n \in \mathbb{N}$$

and thus

$$\|Y^n(t)\|_\infty \leqslant C(f_0, S, T^*, d, \gamma), \qquad t \in [0, T^*], \quad T^* < \infty; \quad n \in \mathbb{N}. \tag{4.9}$$

If $d = 3$, (4.8) leads to

$$y_{n+1}(t) \leqslant C_1 + C_2 \int_0^t y_n^{1+1/\gamma}(s)\, ds + C_3 \int_0^t \varphi(s) y_{n+1}^{1-1/\gamma}(s) y_n^{2/\gamma}(s)\, ds.$$

Now, considering the solution $\alpha(t)$ of the equation

$$\dot\alpha(t) = [C_2 + C_3\varphi(t)]\alpha(t)^{1+1/\gamma}; \qquad \alpha(0) = C_1$$

which exists in a time interval $[0, T[$, where T depends on f_0, S and γ (through C_1, C_2 and C_3), the same reasoning can be applied to obtain

$$\|Y^n(t)\|_\infty \leqslant C(f_0, S, T^*, \gamma) \qquad t \in [0, T^*], \quad T^* < T; \qquad n \in \mathbb{N}. \tag{4.10}$$

Now, the propositions (i) and (ii) are obvious from (4.6) and (4.7).

Lemma 4.2. *We assume that f_0 satisfies the hypotheses* (2.4). *Then, considering Df as a vector ${}^t(\nabla_x f, \nabla_k f)$, we have*:

(i) *Df^n is uniformly bounded in $L^\infty([0, T^*], L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}))$,*
(ii) *$\nabla_x \rho^n$ is uniformly bounded in $L^\infty([0, T^*], L^\infty(\mathbb{R}^d))$,*
(iii) *$\nabla_x E^n$ is uniformly bounded in $L^\infty([0, T^*], L^\infty(\mathbb{R}^d))$,*

where

$$T^* < \infty \quad \text{if} \quad d = 1 \quad \text{or} \quad 2,$$
$$T^* < T \quad \text{finite if} \quad d = 3.$$

PROOF. If we differentiate equation (3.6) with respect to (x, k), we obtain

$$(4.11) \quad \frac{\partial}{\partial t}(Df^{n+1}) + v(k)\nabla_x(Df^{n+1}) - E^n(x,t)\nabla_k(Df^{n+1}) + \lambda(f^n)Df^{n+1}$$
$$= DA(f^n) - f^{n+1}D\lambda^n + \mathfrak{M}^n Df^{n+1}$$
$$= DA(f^n)(1 - f^{n+1}) - f^{n+1}DB(f^n) + \mathfrak{M}^n Df^{n+1}$$

where $\mathfrak{M}^n$ is the following matrix decomposed in 3×3 blocks

$$\mathfrak{M}^n = \begin{bmatrix} 0 & \nabla_x E^n \\ -\nabla_k v & 0 \end{bmatrix}.$$

Now, multiplying equation (4.11) by $(1 + |k|^2)^{\gamma/2}$ and defining

$$Z^n(x,k,t) = (1 + |k|^2)^{\gamma/2} Df^n(x,k,t)$$

we get

$$(4.12) \quad \frac{\partial Z^{n+1}}{\partial t} + v(k)\nabla_x Z^{n+1} - E^n(x,t)\nabla_k Z^{n+1} + \lambda(f^n)Z^{n+1}$$
$$= P_1^n + P_2^n + P_3^{n+1} + P_4^{n+1}$$

$$P_1^n = DA(f^n)(1 - f^{n+1})(1 + |k|^2)^{\gamma/2}$$
$$P_2^n = -f^{n+1}DB(f^n)(1 + |k|^2)^{\gamma/2}$$
$$P_3^{n+1} = -\gamma(E^n, k)(1 + |k|^2)^{(\gamma-2)/2} Df^{n+1}$$
$$P_4^{n+1} = \mathfrak{M}^n Z^{n+1}.$$

From (4.12) we obtain

$$(4.13) \quad \|Z^{n+1}(t)\|_\infty$$
$$\leqslant \|Z_0\|_\infty + \int_0^t [\|P_1^n(s)\|_\infty + \|P_2^n(s)\|_\infty + \|P_3^{n+1}(s)\|_\infty + \|P_4^{n+1}(s)\|_\infty]\, ds.$$

From now on, we will denote by $\psi_i(t)$ some known functions depending on f_0, S, γ and d; and obtained from the functions $\alpha(t)$ defined in Lemma 4.1. So the functions $\psi_i(t)$ will satisfy

$$\psi_i(t) \in L^\infty([0, T^*])$$

where $T^* < \infty$ if $d = 1$ or 2, $T^* < T$ finite if $d = 3$, and T depends on f_0, S and γ.

Estimate on P_1^n

$$\|P_1^n(t)\|_\infty \leqslant \|1 - f^{n+1}(t)\|_\infty \|(1 + |k|^2)^{\gamma/2} DA(f^n)(t)\|_\infty \\ \leqslant \|(1 + |k|^2)^{\gamma/2} DA(f^n)(t)\|_\infty.$$

But

$$(1 + |k|^2)^{\gamma/2} DA(f^n) = (1 + |k|^2)^{\gamma/2} \int \nabla_{x,k} S(x, k', k) f^n(x, k', t)\, dk' \\ + (1 + |k|^2)^{\gamma/2} \int S(x, k', k)\, \nabla_{x,k} f^n(x, k', t)\, dk'.$$

So, we have

$$\|P_1^n(t)\|_\infty \leqslant \|Y^n(t)\|_\infty \left\| (1 + |k|^2)^{\gamma/2} \int |\nabla_{x,k} S(x, k', k)|\, dk' \right\|_\infty + \\ + \|Z^n(t)\|_\infty \left\| (1 + |k|^2)^{\gamma/2} \int S(x, k', k)\, dk' \right\|_\infty$$

and using hypotheses (2.1) and (2.2)

$$\|P_1^n(t)\|_\infty \leqslant C(\gamma, S)(\alpha(t) + \|Z^n(t)\|_\infty) \\ \leqslant \psi_1(t)(1 + \|Z^n(t)\|_\infty) \tag{4.14}$$

Estimate on P_2^n

$$\|P_2^n(t)\|_\infty \leqslant \|Y^{n+1}(t)\|_\infty \|DB(f^n)(t)\|_\infty.$$

But

$$DB(f^n) = \int \nabla_{x,k} S(x, k, k')(1 - f^n(x, k', t))\, dk' - \int S(x, k, k') \nabla_{x,k} f^n(x, k', t)\, dk'.$$

Thus,

$$\|DB(f^n)(t)\|_\infty \leqslant \left\| \int |\nabla_{x,k} S(x, k, k')|\, dk' \right\|_\infty + \|Z^n(t)\|_\infty \left\| \int S(x, k, k')\, dk' \right\|_\infty$$

and again from (2.1) and (2.2)

$$\|DB(f^n)(t)\|_\infty \leqslant C(S)(1 + \|Z^n(t)\|_\infty). \tag{4.15}$$

So, we have

$$\|P_2^n(t)\|_\infty \leqslant C(S)\alpha(t)(1 + \|Z^n(t)\|_\infty) \\ \leqslant \psi_2(t)(1 + \|Z^n(t)\|_\infty). \tag{4.16}$$

Estimate on P_3^{n+1}

$$\|P_3^{n+1}(t)\|_\infty \leqslant \gamma\|E^n(t)\|_\infty\|(1+|k|^2)^{\gamma/2}Df^{n+1}(t)\|_\infty$$

and thanks to (4.7)

$$\|P_3^{n+1}(t)\|_\infty \leqslant C(\gamma, d)\varphi(t)\alpha(t)^{(d-1)/\gamma}\|Z^{n+1}(t)\|_\infty .$$

We recall that the function $\varphi(t) = \delta(t)^{1/d}$, where $\delta(t)$ was obtained in Proposition 3.1, belongs to $L^\infty([0, T^*])$, for any $T^* < \infty$.

So, we get

$$\|P_3^{n+1}(t)\|_\infty \leqslant \psi_3(t)\|Z^{n+1}(t)\|_\infty . \tag{4.17}$$

Estimate on P_4^{n+1}

$$\|P_4^{n+1}(t)\|_\infty \leqslant \|\mathfrak{M}^n(t)\|_\infty\|Z^{n+1}(t)\|_\infty .$$

We use (2.3) and the interpolation inequality (A.4) to obtain

$$\|\mathfrak{M}^n(t)\|_\infty \leqslant C(d)[1 + \|\rho^n(t)\|_\infty[1 + \mathrm{Log}\,(1 + \|\nabla_x\rho^n(t)\|_\infty)] + \|\rho^n(t)\|_1].$$

Now, from (3.7) and (4.6)

$$\begin{aligned}\|\mathfrak{M}^n(t)\|_\infty &\leqslant C(\gamma, d)[1 + \alpha(t)^{d/\gamma}[1 + \mathrm{Log}\,(1 + \|\nabla_x\rho^n(t)\|_\infty)] + \delta(t)] \\ &\leqslant \psi_4(t)[1 + \mathrm{Log}\,(1 + \|\nabla_x\rho^n(t)\|_\infty)].\end{aligned} \tag{4.18}$$

But we have

$$\nabla_x\rho^n(t) = \int \nabla_x f^n(x, k, t)\, dk.$$

So, applying the interpolation inequality (A.2) we get

$$\|\nabla_x\rho^n(t)\|_\infty \leqslant \left\|\int |\nabla_x f^n(x, k, t)|\, dk\right\|_\infty \leqslant C(\gamma, d)\|Z^n(t)\|_\infty \tag{4.19}$$

and thus,

$$\|P_4^{n+1}(t)\|_\infty \leqslant \psi_5(t)[1 + \mathrm{Log}\,(1 + \|Z^n(t)\|_\infty)]\|Z^{n+1}(t)\|_\infty . \tag{4.20}$$

So, from (4.14), (4.16), (4.17) and (4.20), equation (4.13) becomes

$$\begin{aligned}\|Z^{n+1}(t)\|_\infty \leqslant \|Z_0\|_\infty &+ \int_0^t \psi_6(s)(1 + \|Z^n(s)\|_\infty + \|Z^{n+1}(s)\|_\infty)\, ds \\ &+ \int_0^t \psi_5(s)\|Z^{n+1}(s)\|_\infty \mathrm{Log}\,(1 + \|Z^n(s)\|_\infty)\, ds.\end{aligned} \tag{4.21}$$

In order to estimate $\|Z^n(t)\|_\infty$ we define the function

$$z_n(t) = \text{Max}\,\{e, \|Z^n(t)\|_\infty\}$$

and we obtain from (4.21)

$$z_{n+1}(t) \leqslant C_4 + \int_0^t \psi_7(s)(z_{n+1}(s) + z_n(s))\,\text{Log}\,z_n(s)\,ds.$$

Now, we consider the differential equation

$$\dot\beta(t) = 2\psi_7(t)\beta(t)\,\text{Log}\,\beta(t); \qquad \beta(0) = C_4$$

whose solution

$$\beta(t) = \exp\left[(\text{Log}\,C_4)\exp\int_0^t 2\psi_7(s)\,ds\right]$$

exists in a time interval $[0, T[$ with $T = \infty$ if $d = 1$ or 2, T finite and depending on f_0, S and γ if $d = 3$.

So, the same argument as for Lemma 4.1 proves that

$$\|Z^n(t)\|_\infty \leqslant \beta(t) \leqslant C(f_0, S, T^*, \gamma, d); \tag{4.22}$$

$$\text{for every} \quad t \in [0, T^*] \quad \begin{cases} T^* < \infty & \text{if} \quad d = 1 \quad \text{or} \quad 2 \\ T^* < T & \text{if} \quad d = 3 \end{cases}, \quad \text{and} \quad n \in \mathbb{N}.$$

Now, propositions (ii) and (iii) are obvious from (4.18), (4.19) and (4.22).

In order to finish the proof of lemma, we have to estimate $\|Df^n(t)\|_1$.

From (4.11) we can write

$$\begin{aligned} (4.23)\quad & \|Df^{n+1}(t)\|_1 \\ & \leqslant \|Df_0\|_1 + \int_0^t [\|DA(f^n)(s)\|_1 + \|f^{n+1}D\lambda(f^n)(s)\|_1 + \|\mathcal{M}^n Df^{n+1}(s)\|_1]\,ds. \end{aligned}$$

By considering the estimates (4.22) obtained above, we have

$$\|f^{n+1}D\lambda(f^n)(t)\|_1 \leqslant [\|DA(f^n)(t)\|_\infty + \|DB(f^n)(t)\|_\infty]\,\|f^{n+1}(t)\|_1$$

and thanks to (4.14), (4.15) and (3.7)

$$\|f^{n+1}D\lambda(f^n)(t)\|_1 \leqslant \psi_8(t). \tag{4.24}$$

Using (4.18) and (4.19),

$$\begin{aligned} \|\mathcal{M}^n Df^{n+1}(t)\|_1 &\leqslant \|\mathcal{M}^n(t)\|_\infty \|Df^{n+1}(t)\|_1 \\ &\leqslant \psi_2(t)\|Df^{n+1}(t)\|_1. \end{aligned} \tag{4.25}$$

$$\|DA(f^n)(t)\|_1 \leqslant \iint f^n(x,k',t)\,dx\,dk' \int |\nabla_{x,k}S(x,k',k)|\,dk$$
$$+ \iint |\nabla_{x,k}f^n(x,k',t)\,dx\,dk'| \int S(x,k',k)\,dk.$$

Applying hypotheses (2.1) and (2.2), and thanks to (3.7) we have

$$\begin{aligned}\|DA(f^n)(t)\|_1 &\leqslant C(S)[\|f^n(t)\|_1 + \|Df^n(t)\|_1]\\ &\leqslant \psi_{10}(t)[1 + \|Df^n(t)\|_1].\end{aligned} \tag{4.26}$$

So, from (4.23), (4.24), (4.25) and (4.26) we obtain the following Gronwall inequality

$$\|Df^{n+1}(t)\|_1 \leqslant \|Df_0\|_1 + \int_0^t \psi_{11}(s)(1 + \|Df^{n+1}(s)\|_1 + \|Df^n(s)\|_1)\,ds.$$

Now if we use the same reasoning as for estimates on $Y^n(t)$ and $Z^n(t)$ we can write that

$$\|Df^{n+1}(t)\|_1 \leqslant C(f_0, S, T^*, \gamma, d); \tag{4.27}$$

$$\text{for every} \quad t \in [0, T^*] \quad \begin{cases} T^* < \infty & \text{if} \quad d = 1 \quad \text{or} \quad 2 \\ T^* < T & \text{if} \quad d = 3 \end{cases}; \quad \text{and} \quad n \in \mathbb{N}.$$

Thus, proposition (i) is proved from (4.22) and (4.27), and the proof of Lemma 4.2 is finished.

5. Convergence of the Iterative Sequence

We consider an arbitrary finite time T^* ($T^* < T(f_0, S, \gamma)$ if $d = 3$). Thanks to Proposition 3.1 and Lemmas 4.1 and 4.2, we have got the following convergences of subsequences

$$f^n \rightharpoonup f \quad \text{in} \quad L^\infty([0,T^*], L^\infty(\mathbb{R}^{2d})) \quad \text{weak-*} \tag{5.1}$$
$$(1+|k|^2)^{\gamma/2} f^n \rightharpoonup (1+|k|^2)^{\gamma/2} f \quad \text{in} \quad L^\infty([0,T^*], L^\infty(\mathbb{R}^{2d})) \quad \text{weak-*} \tag{5.2}$$
$$(1+|k|^2)^{\gamma/2} Df^n \rightharpoonup (1+|k|^2)^{\gamma/2} Df \text{ in } L^\infty([0,T^*], L^\infty(\mathbb{R}^{2d})) \quad \text{weak-*} \tag{5.3}$$
$$\rho^n \rightharpoonup \rho \quad \text{in} \quad L^\infty([0,T^*], W^{1,\infty}(\mathbb{R}^d)) \quad \text{weak-*} \tag{5.4}$$
$$E^n \rightharpoonup E \quad \text{in} \quad L^\infty([0,T^*], W^{1,\infty}(\mathbb{R}^d)) \quad \text{weak-*}. \tag{5.5}$$

To pass to the limit in the non-linear terms of equation (2.5) we need a strong convergence. So, we state

Lemma 5.1. *The functions* (f^n, E^n) *of the iterative sequence defined by* (3.4), (3.5) *and* (3.6) *satisfy*:

$$f^n \to f \quad in \quad L^\infty([0, T^*], L^1(\mathbb{R}^{2d})) \quad strong \tag{5.6}$$

$$E^n \to E \quad in \quad L^\infty([0, T^*], L^\infty(\mathbb{R}^d)) \quad strong \tag{5.7}$$

PROOF. The function $f^{n+1} - f^n$ satisfies the following equation

$$\begin{aligned} \frac{\partial}{\partial t}(f^{n+1} - f^n) + v(k)\nabla_x(f^{n+1} - f^n) - E^n(x, t)\nabla_k(f^{n+1} - f^n) \\ + \lambda(f^n)(f^{n+1} - f^n) = \\ A(f^n) - A(f^{n-1}) + (E^n(x, t) - E^{n-1}(x, t))\nabla_k f^n + (\lambda(f^{n-1}) - \lambda(f^n))f^n. \end{aligned} \tag{5.8}$$

Now, integrating (5.8) we can obtain

$$\|(f^{n+1} - f^n)(t)\|_1 \leqslant \int_0^t [\|Q_1^n(s)\|_1 + \|Q_2^n(s)\|_1 + \|Q_3^n(s)\|_1]\, ds \tag{5.9}$$

$$\begin{aligned} Q_1^n &= (E^n - E^{n-1})\nabla_k f^n \\ Q_2^n &= A(f^n) - A(f^{n-1}) \\ Q_3^n &= (\lambda(f^{n-1}) - \lambda(f^n))f^n \end{aligned}$$

Estimate on Q_1^n

We can state

$$\|Q_1^n(s)\|_1 \leqslant C_5 \|(f^n - f^{n-1})(s)\|_1. \tag{5.10}$$

The proof of (5.10) can be found in [5] and is omitted here. The idea of the proof will be used in the proof of uniqueness.

Estimate on Q_2^n

Using (2.1) we get

$$\begin{aligned} \|Q_2^n(s)\|_1 &\leqslant \iint |f^n - f^{n-1}|(x, k', s)\, dx\, dk' \int S(x, k', k)\, dk \\ &\leqslant C(S)\|(f^n - f^{n-1})(s)\|_1. \end{aligned} \tag{5.11}$$

Estimate on Q_3^n

$$\|(\lambda(f^{n-1}) - \lambda(f^n))f^n(s)\|_1 \leqslant \|A(f^{n-1}) - A(f^n)\|_1 + \|(B(f^{n-1}) - B(f^n))(s)\|_1.$$

But

$$\|(B(f^{n-1}) - B(f^n))(s)\|_1 \leqslant \iint |f^n - f^{n-1}|(x, k', s)\, dx\, dk' \int S(x, k, k')\, dk.$$

So, applying (2.1) and (5.11) we have

$$\|Q_3^n(s)\|_1 \leqslant C(S)\|(f^n - f^{n-1})(s)\|_1. \tag{5.12}$$

Now, from (5.9), (5.10), (5.11) and (5.12) we have

$$\|(f^{n+1} - f^n)(t)\|_1 \leqslant C_6 \int_0^t \|(f^n - f^{n-1})(s)\|_1 \, ds$$

$$\leqslant \frac{C_6^n t^n}{n!} \operatorname*{Max}_{t \in [0, T^*]} \|(f^1 - f^0)(t)\|_1$$

which proves (5.6).

Now, it is clear that

$$\rho^n \to \rho \quad \text{in} \quad L^\infty([0, T^*], L^1(\mathbb{R}^d)) \quad \text{strong} \tag{5.13}$$

and using the interpolation inequality (A.3) and (5.4)

$$\|(E^n - E)(t)\|_\infty \leqslant C(d)\|(\rho^n - \rho)(t)\|_\infty^{(d-1)/d}\|(\rho^n - \rho)(t)\|_1^{1/d}$$
$$\leqslant C_7\|(\rho^n - \rho)(t)\|_1^{1/d}$$

which proves (5.7) and that the electric field E found in (5.5) satisfies (2.7).

From the convergences (5.1)-(5.7), it is easy and classical to prove that the function f found in (5.1) satisfies equation (2.5) for almost every time, and that, since T^* is arbitrary, Propositions 2.10, 2.11 and 2.12 are satisfied. Proposition 2.9 is obvious from Proposition 3.1(i). In order to finish the proof of Theorem 2.1 it remains to prove the uniqueness.

Let (f_1, E_1) and (f_2, E_2) be two solutions of (2.4)-(2.8) satisfying (2.9)-(2.12). The function $f_1 - f_2$ satisfies the following equation

$$\frac{\partial}{\partial t}(f_1 - f_2) + v(k)\nabla_x(f_1 - f_2) - E_1\nabla_k(f_1 - f_2) + \lambda(f_1)(f_1 - f_2)$$
$$= A(f_1) - A(f_2) + (E_1 - E_2)\nabla_k f_2 + (\lambda(f_2) - \lambda(f_1))f_2 \tag{5.14}$$

and integrating (5.14) we get

$$\|(f_1 - f_2)(t)\|_1 \leqslant \int_0^t [\|(A(f_1) - A(f_2))(s)\|_1$$
$$+ \|((E_1 - E_2)\nabla_k f_2)(s)\|_1 + \|(\lambda(f_2) - \lambda(f_1))f_2(s)\|_1] \, ds \tag{5.15}$$

But, as in the proof of Lemma 5.1

$$\|(A(f_1) - A(f_2))(s)\|_1 + \|(\lambda(f_1) - \lambda(f_2))f_2(s)\|_1 \leqslant C_8\|(f_1 - f_2)(s)\|_1, \tag{5.16}$$

and

$$\|((E_1 - E_2)\nabla_k f_2)(s)\|_1 \leqslant \iint |\nabla_k f_2(x,k,s)|\,|(E_1 - E_2)(x,s)|\,dx\,dk$$

$$\leqslant \int |\rho_1(y,s) - \rho_2(y,s)|\,dy \iint \frac{1}{|x-y|^{d-1}} |\nabla_k f_2(x,k,s)|\,dx\,dk.$$

Now, using (A.3), (A.2),

$$\iint \frac{1}{|x-y|^{d-1}} |\nabla_k f_2(x,k,s)|\,dx\,dk$$

$$\leqslant \left(\iint |\nabla_k f_2(x,k,s)|\,dx\,dk\right)^{1/d} \left(\left\| \int |\nabla_k f_2(\cdot,k,s)|\,dk \right\|_\infty\right)^{(d-1)/d}$$

$$\leqslant C(\gamma,d)\|Df_2(s)\|_1^{1/d} \|(1+|k|^2)^{\gamma/2} Df_2(s)\|_\infty^{(d-1)/d}$$

and thanks to (2.10) and (2.11)

$$\|((E_1 - E_2)\nabla_k f_2)(s)\|_1 \leqslant C_9 \int |\rho_1(y,s) - \rho_2(y,s)|\,dy \leqslant C_9 \|(f_1 - f_2)(s)\|_1. \tag{5.17}$$

So, from (5.15), (5.16) and (5.17) we get

$$\|(f_1 - f_2)(t)\|_1 \leqslant C_{10} \int_0^t \|(f_1 - f_2)(s)\|_1\,ds$$

which proves that

$$\|(f_1 - f_2)(t)\|_1 = 0 \quad \text{for every } t \text{ and so } \quad f_1 = f_2$$

and the proof of Theorem 2.1 is finished.

APPENDIX. Interpolation Inequalities

Lemma A.1. *For a function* $f(k)\colon \mathbb{R}^d \to \mathbb{R}$, *we have*

$$\|(1+|k|^2)^{(\gamma-1)/2} f\|_\infty \leqslant C(\gamma)\|f\|_\infty^{1/\gamma} \|(1+|k|^2)^{\gamma/2} f\|_\infty^{1-1/\gamma} \tag{A.1}$$

$$\int |f(k)|\,dk \leqslant C(\gamma,d)\|f\|_\infty^{(\gamma-d)/\gamma} \|(1+|k|^2)^{\gamma/2} f\|_\infty^{d/\gamma} \quad \textit{for} \quad \gamma > d. \tag{A.2}$$

PROOF. See [5].

Lemma A.2. *Let $\rho(x)$ be a function which belongs to $L^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ and let $E(x)$ be such that*

$$E(x) = \int \frac{x-y}{|x-y|^d} \rho(y)\, dy.$$

Then we have

$$\|E\|_\infty \leqslant C(d) \|\rho\|_1^{1/d} \|\rho\|_\infty^{(d-1)/d} \tag{A.3}$$

$$\|\nabla_x E\|_\infty \leqslant C(d)[1 + \|\rho\|_\infty [1 + \operatorname{Log}(1 + \|\nabla_x \rho\|_\infty)] + \|\rho\|_1]. \tag{A.4}$$

PROOF. See [5, 6, 7].

Acknowledgements. I would like express my gratitude to P. Degond for many helpful discussions and remarks.

References

[1] Reggiani, L. *Hot-Electron Transport in Semiconductors.* Springer-Verlag, Berlin, 1985.

[2] Rode, D. L. Low field electron transport. *Semiconductor and Semimetals*, **10**(1975).

[3] Conwell, E. M. High field transport in semiconductors, *Solid State Physics*, **9**(1967).

[4] Poupaud, F. On a system of non linear Boltzmann equations of Semiconductor Physics, Internal report, **175**(1988), CMAP, École Polytechnique, France.

[5] Degond, P. Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimensions, *Ann. Scient. École Norm. Sup.* **19**(1986), 519-542.

[6] Bardos, C. and Degond, P. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data, *Ann. Ins. H.-Poincaré. Analyse non linéaire*, **2**(1985), 101-118.

[7] Ukai, S. and Okabe, T. On the classical solution in large in time of the two-dimensional Vlasov equation, *Osaka J. Math.* **15**(1978), 245-261.

Recibido: 1 de febrero de 1990.

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