

Hermite Expansions on \mathbb{R}^{2n} for Radial Functions

S. Thangavelu

1. Introduction

Let $\Phi_\nu(x)$ stand for the normalized Hermite functions on \mathbb{R}^n which are eigen functions of the Hermite operator

$$H_n = \left(-\Delta + \frac{1}{4} |x|^2 \right).$$

Let $P_N f$ denote the projection of $L^2(\mathbb{R}^n)$ onto the space spanned by $\{\Phi_\nu: |\nu| = N\}$. Then the Riesz means of order $\delta > 0$ are defined by

$$S_L^\delta f = \sum \left(1 - \frac{2N+n}{2L} \right)_+^\delta P_N f.$$

In [9] we proved the uniform estimates

$$(1.1) \quad \|S_L^\delta f\|_p \leq c \|f\|_p, \quad 1 \leq p \leq \infty \quad (1.1)$$

whenever $\delta > (n-1)/2$. For a fixed p one is interested in finding the smallest value of δ so that the uniform estimates (1.1) will hold for that fixed p . If we define the critical index $\delta(p)$ by

$$\delta(p) = \max \left(n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right)$$

then $\delta > \delta(p)$ is known to be a necessary condition for the validity of (1.1).

This is the consequence of a transplation theorem (see [4]). As proved in [9] for $p = 1$ the condition $\delta > \delta(1)$ is also sufficient to imply the uniform estimates. But for other values of p it is not known whether $\delta > \delta(p)$ is sufficient or not.

In [1] Chris Sogge studied the Riesz means of eigen function expansions associated to elliptic differential operators on a compact manifold. Adapting an argument of Fefferman-Stein [2] he showed that for

$$(1.2) \quad \left| \frac{1}{p} - \frac{1}{2} \right| \geq \frac{1}{n+1}$$

$$\|S_L^\delta f\|_p \leq C \|f\|_p, \quad \delta > \delta(p)$$

provided that the operator is of order two. The main idea of the proof is that $L^p - L^2$ estimates of the projection operators associated with the expansions are sufficient to prove summability results if used together with the kernel estimates for large values of δ .

For the Hermite projection operators it is possible to prove the estimates

$$(1.3) \quad \|P_N f\|_2 \leq CN^{\delta(p)} \|f\|_p, \quad \left| \frac{1}{p} - \frac{1}{2} \right| \geq \frac{1}{n+1}.$$

As it was proved in [9] we also have the pointwise estimate for the kernel of S_L^δ .

$$|S_L^\delta(x, y)| \leq C \{L^{n/2}(1 + L^{1/2}|x - y|)^{-\delta-1} + L^{n/2}(1 + L^{1/2}|x + y|)^{-\delta-1}\}.$$

So, one is tempted to use the same arguments as in [1] to study the L^p mapping properties of S_L^δ . Unfortunately the arguments break down owing to the fact that the eigenvalues of H_n are not squares of integers.

But things are not so bad, if we consider on \mathbb{R}^{2n} only the radial functions then the above estimates for P_N can be improved. Indeed we will show that

$$(1.4) \quad \|P_N f\|_2 \leq CN^{\delta(p)/2 - 1/4} \|f\|_p, \quad \left| \frac{1}{p} - \frac{1}{2} \right| \geq \frac{1}{4n}.$$

Also for radial functions $S_L^\delta f$ is given by a kernel which satisfies the estimate

$$(1.5) \quad |S_L^\delta(x, y)| \leq CL^n(1 + L^{1/2}|x - y|)^{-\delta-1}.$$

By using the same arguments as in [1] we will prove that the estimates (1.4) and (1.5) imply the following result.

Theorem. *Let $f \in L^p(\mathbb{R}^{2n})$ be radial, $1 < p < 4n/(2n + 1)$ and let $\delta > \delta(p)$. Then there is a C independent of L such that $\|S_L^\delta f\| \leq C \|f\|_p$ holds.*

Corollary. For f radial and $\delta > 0$ we have

$$\|S_L^\delta f\|_p \leq C \|f\|_p, \quad \frac{4n}{2n+1+\delta} < p < \frac{4n}{2n-1-2\delta}.$$

In the next section we obtain $L^p - L^2$ bounds for the projection operators and in the third section an estimate of the kernel is proved. Theorem and its corollary will be proved in the final section. The author wishes to thank Chris Sogge for clarifying certain points in the proof of the above theorem.

2. Bounds for the Projection Operators

Recall that the n -dimensional Hermite functions $\Phi_\nu(x)$ are defined by

$$\Phi_\nu(x) = (2^{|\nu|} \nu! \sqrt{2\pi})^{-n/2} e^{-|x|^2/2} \prod_{j=1}^n h_{\alpha_j}(x_j/\sqrt{2})$$

where

$$h_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2}).$$

Then $\{\Phi_\nu(x)\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$. On $L^2(\mathbb{R}^{2n})$ we have another orthonormal basis given by the special Hermite functions $\Phi_{\nu\mu}$. These functions are defined by

$$\Phi_{\nu\mu}(z) = \pi^{-n/2} \int_{\mathbb{R}^n} e^{i(xu)/\sqrt{2}} \Phi_\nu\left(u - \frac{y}{\sqrt{2}}\right) \Phi_\mu\left(u + \frac{y}{\sqrt{2}}\right) du,$$

where $z = x + iy \in \mathbb{C}^n$. If L is the operator $L = H_{2n} - iN$ where

$$N = \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right)$$

then it is not difficult to check that $\Phi_{\nu\mu}(z)$ are eigen functions of L :

$$(2.1) \quad L\Phi_{\nu\mu}(z) = (2|\nu| + n)\Phi_{\nu\mu}(z).$$

For these facts about the special Hermite functions we refer to the paper of Strichartz [7]. For the operator H_{2n} the eigen functions are $\Phi_\nu(z) = \Phi_\nu(x, y)$ and one has

$$(2.2) \quad H_{2n}\Phi_\nu(z) = (|\nu| + n)\Phi_\nu(z).$$

Let Q_N and P_N stand for the projections of $L^2(\mathbb{R}^{2n})$ onto the space spanned by $\{\Phi_{\nu\mu} : |\nu| = N\}$ and $\{\Phi_\nu : |\nu| = N\}$. We claim the following is true.

Lemma 2.1. *If f is a radial function on \mathbb{R}^{2n} then one has $P_{2N}f = Q_Nf$ and $P_{2N+1}f = 0$ for all N .*

PROOF. For radial functions f it is easily seen that $Nf = 0$ and hence $Lf = H_{2n}f$. Since $H_{2n}f$ is again radial we have $L^k f = H_{2n}^k f$ for all $k = 1, 2, \dots$. Therefore, for $t > 0$, $e^{-tL}f = e^{-tH_{2n}}f$ which translates into

$$\sum_{N=0}^{\infty} \omega^{2N} Q_N f = \sum_{N=0}^{\infty} \omega^N P_N f$$

which is true even if ω is complex and $|\omega| < 1$. The last equality immediately implies that $P_{2N}f = Q_Nf$ and $P_{2N+1}f = 0$.

The result of this Lemma is the key one to get improved estimates for the projection operators. Before proceeding further we need to recall several facts about the Weyl transform W . (A good reference for these is the paper of Mauceri [6].)

The Weyl transform W takes functions on \mathbb{C}^n into bounded operators on $L^2(\mathbb{R}^n)$. It is defined by the equation

$$W(f) = \int_{\mathbb{C}^n} f(z) W(z) dz d\bar{z}$$

where $W(z)$ is the operator valued function

$$W(z)\varphi(\xi) = e^{ix(y/2 + \xi)}\varphi(\xi + y)$$

where $z = x + iy$ and $\xi \in \mathbb{R}^n$. When f is radial the Weyl transform reduce to the Laguerre transform

$$W(f) = \sum_{N=0}^{\infty} R_N(f) \tilde{P}_N$$

where \tilde{P}_N is the projection of $L^2(\mathbb{R}^n)$ onto the N -th eigenspace of the operator $-\Delta + |x|^2$ and $R_N(f)$ are defined by

$$R_N(f) = \frac{N!}{(N+n-1)!} \int_{\mathbb{C}^n} f(z) L_N^{n-1} \left(\frac{1}{2} |z|^2 \right) e^{-|z|^2/4} dz d\bar{z}.$$

If we let

$$\varphi_N(z) = (2\pi)^{-n} L_N^{n-1} \left(\frac{1}{2} |z|^2 \right) e^{-|z|^2/4},$$

then it is clear that $W(\varphi_N) = \tilde{P}_N$ due to the orthogonality properties of the Laguerre functions.

Another important fact about the Weyl transform which we need is its action on Lf . If we define the operators Z_j and \bar{Z}_j by

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4} \bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} z_j$$

then a simple calculation shows that

$$-\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) = \frac{1}{4} L.$$

Since

$$W\left(-\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) f\right) = W(f)H,$$

where $H = -\Delta + |x|^2$ we have the equation

$$W(Lf) = 4W(f)H.$$

From this we conclude that $W(Q_N f) = W(f)\tilde{P}_N$ and when f is radial the above formula becomes $W(Q_N f) = R_N(f)\tilde{P}_N$. Since $W(\varphi_N) = \tilde{P}_N$ we have the result $Q_N f = R_N(f)\varphi_N$ for radial functions. Thus for radial functions $P_{2N} f = R_N(f)\varphi_N$ and we are ready to prove the following proposition.

Proposition 2.1. *When f is radial and $1 \leq p < 4n/(2n+1)$ we have*

$$\|P_N f\|_2 \leq CN^{n(1/p-1/2)-1/2} \|f\|_p.$$

PROOF. Since $\|\varphi_N\|_2 = CN^{(n-1)/2}$ we have

$$\begin{aligned} \|P_{2N} f\|_2 &= CN^{-(n-1)/2} \left| \int_{\mathbb{C}^n} f(z) \varphi_N(z) dz d\bar{z} \right| \\ &\leq CN^{-(n-1)/2} \|\varphi_N\|_q \|f\|_p \end{aligned}$$

where $1/p + 1/q = 1$. So we need to estimate the L^q norm of φ_N . A simple calculation shows that

$$(2.3) \quad \|\varphi_N\|_q = CN^{(n-1)/2} \|\mathcal{L}_N^{\alpha+\beta}(r)r^{-\beta/2}\|_q$$

where

$$\alpha + \beta = n - 1, \quad \beta = 2(n-1)\left(\frac{1}{2} - \frac{1}{q}\right)$$

and \mathcal{L}_N^δ are the normalized Laguerre functions of type δ .

Now we will make use of the following estimates proved in [5]. Assume that $\alpha + \beta > -1$ and $\alpha > -2/q$. Then

$$(2.4) \quad \|\mathcal{L}_N^{\alpha+\beta}(r)r^{-\beta/2}\|_q \leq CN^{\beta/2-1/q}$$

if $1 \leq q \leq 4$ and $\beta > 2/q - 1/2$ or $q > 4$ and $\beta > 4/3q - 1/3$.

When $1 \leq p < 4/3$, $q > 4$ and $4/3q - 1/3 < 0$ so that $\beta > 4/3q - 1/3$.

When $4/3 \leq p < 4n/(2n+1)$, $4n/(2n-1) < q \leq 4$ and it is easily checked that $\beta > 2/q - 1/2$. Hence in view of the estimate (2.4) we immediately get

$$\|P_{2N}f\|_2 \leq CN^{(n-1)(1/2-1/q)-1/q} \|f\|_p.$$

This proves the proposition.

3. Estimating the Riesz Kernel

To get a good estimate for the kernel of the Riesz means we need to recall the definition of twisted convolution. The twisted convolution of two functions f and g both defined on \mathbb{C}^n is defined by

$$(3.1) \quad f \times g(z) = \int_{\mathbb{C}^n} f(z-v)g(v)\bar{\omega}(z,v) dv d\bar{v}$$

where

$$\omega(z,v) = \exp\left(-\frac{i}{2} \operatorname{Im} z\bar{v}\right).$$

In terms of the real variables

$$(3.2) \quad f \times g(x) = \int_{\mathbb{R}^{2n}} f(x-y)g(y)e^{iP(x,y)} dy$$

where $P(x,y)$ is a real polynomial. An important result we need is the fact

$$(3.3) \quad W(f \times g) = W(f)W(g).$$

In the previous section we have observed that

$$W(Q_N f) = W(f)\tilde{P}_N = W(f)W(\varphi_N)$$

and hence in view of (3.3) we have $Q_N f = f \times \varphi_N$. Now for a radial function f

$$\begin{aligned} S_L^\delta f &= \sum \left(1 - \frac{k+n}{L}\right)_+^\delta P_k f \\ &= \sum \left(1 - \frac{2k+n}{L}\right)_+^\delta Q_k f = f \times s_L^\delta \end{aligned}$$

where the kernel s_L^δ is defined by

$$(3.4) \quad s_L^\delta(z) = \sum \left(1 - \frac{2k+n}{L}\right)_+^\delta \varphi_k(z)$$

Thus we have

$$S_L^\delta f(x) = \int S_L^\delta(x, y) f(y) dy$$

where the kernel is given by

$$(3.5) \quad S_L^\delta(x, y) = s_L^\delta(x - y) e^{iP(x, y)}.$$

We can now prove the following estimate.

Proposition 3.1.

$$(3.6) \quad |S_L^\delta(x, y)| \leq CL^n (1 + L^{1/2}|x - y|)^{-\delta - n - 1/3}$$

PROOF. From (3.5) what we need is to prove

$$|s_L^\delta(z)| \leq CL^n (1 + L^{1/2}|z|)^{-\delta - n - 1/3}.$$

Consider the Cesaro means σ_L^δ defined by

$$(3.7) \quad \sigma_L^\delta f(z) = \frac{1}{A_L^\delta} \sum_{k=0}^L A_{L-k}^\delta Q_k f(z)$$

which is given by twisted convolution with

$$(3.8) \quad \sigma_L^\delta(z) = \frac{1}{A_L^\delta} \sum_{k=0}^L A_{L-k}^\delta \varphi_k(z).$$

There is a formula (see Gergen [3]) connecting $s_L^\delta(z)$ and $\sigma_L^\delta(z)$ viz.,

$$s_L^\delta(z) = \frac{1}{L^\delta} \sum_{k=0}^L V(L - k) A_k^\delta \sigma_k^\delta(z)$$

where the function V satisfies the estimate $|V(t)| \leq C(1 + t^2)^{-1}$. In view of this formula it is enough to prove

$$|\sigma_k^\delta(z)| \leq Ck^n (1 + k^{1/2}|z|)^{-\delta - n - 1/3}.$$

The following formula is true for Laguerre polynomials (see [5]):

$$\sum_{k=0}^N A_{N-k}^\alpha L_k^\beta(r) = L_N^{\alpha+\beta+1}(r).$$

Since

$$\sigma_k^\delta(z) = \frac{1}{A_k^\delta} \sum_{j=0}^k A_{k-j}^\delta L_j^{n-1} \left(\frac{1}{2} |z|^2 \right) e^{-|z|^2/4}$$

we immediately get the formula

$$(3.9) \quad \sigma_k^\delta(z) = \frac{1}{A_k^\delta} L_k^{\delta+n} \left(\frac{1}{2} |z|^2 \right) e^{-|z|^2/4}.$$

Now we can make use of the asymptotic estimates for the Laguerre Polynomials $L_k^{\delta+n} \left(\frac{1}{2} |z|^2 \right)$ (see for example [5]). Indeed, if we let

$$\mathfrak{L}_N^\alpha(r) = N^{-\alpha/2} L_N^\alpha(r) e^{-r/2} r^{\alpha/2}$$

then the following estimates are valid.

$$|\mathfrak{L}_N^\alpha(r)| \leq C \begin{cases} (rv)^{\alpha/2} & \text{if } 0 \leq r < \frac{1}{v} \\ (rv)^{-1/4} & \text{if } \frac{1}{v} \leq r < \frac{v}{2} \\ [\nu(v^{1/3} + |\nu - r|)]^{-1/4} & \text{if } \frac{v}{2} \leq r < \frac{3v}{2} \\ e^{-r\nu} & \text{if } r \geq \frac{3v}{2} \end{cases}$$

where $\nu = 4N + 2\alpha + 2$. In view of these estimates it is an easy matter to show that

$$|\sigma_k^\delta(z)| \leq Ck^n (1 + k^{1/2}|z|)^{-\delta - n - 1/3}.$$

This completes the proof of the proposition

Remark. Incidentally, using the above estimate one can prove a pointwise convergence result for radial functions. When f is radial

$$S_L^\delta f(x) = \int S_L^\delta(x-y) e^{iP(x,y)} f(y) dy$$

shows that when $\delta > n - \frac{1}{3}$ we have

$$\text{Sup } |S_L^\delta f(x)| \leq C \Lambda f(x)$$

where Λ is the Hardy-Littlewood maximal function. So when f is in $L^p(\mathbb{R}^{2n})$ and radial, $S_L^\delta f(x) \rightarrow f(x)$ a.e. as L tends to infinity.

We further remark that in [9] we have proved the pointwise convergence for any L^p function when $\delta > n - 1/3$. That required a considerable amount of work because there estimating the Riesz kernel was much difficult.

4. L^p Bounds for Riesz Means

In this section we will prove that when

$$1 < p < \frac{4n}{2n+1} \quad \text{and} \quad \delta > \delta(p)$$

the following uniform estimates are valid

$$\|S_L^\delta f\|_p \leq C \|f\|_p, \quad f \text{ is radial.}$$

Following [1] we take a partition of unity

$$\sum_{-\infty}^{\infty} \varphi(2^\nu t) = 1$$

for $t > 0$ where φ is a C_0^∞ function supported in $(1/2, 2)$. For each ν set

$$(4.1) \quad \varphi_{L,\nu}^\delta(t) = \varphi\left(2^\nu\left(1 - \frac{t}{L}\right)\right)\left(1 - \frac{t}{L}\right)^\delta.$$

Furthermore, for $\nu = 1, 2, \dots$ define

$$(4.2) \quad S_{L,\nu}^\delta f = \sum \varphi_{L,\nu}^\delta(\lambda_k) P_k f$$

$$(4.3) \quad S_{L,0}^\delta f = \sum \varphi_0\left(1 - \frac{\lambda_k}{L}\right)\left(1 - \frac{\lambda_k}{L}\right)^\delta P_k f$$

where $\lambda_k = k + n$ and

$$\varphi_0(t) = 1 - \sum_{\nu=1}^{\infty} \varphi(2^\nu t).$$

Then we have

$$(4.4) \quad S_L^\delta f = S_{L,0}^\delta f + \sum_{\nu=1}^{[\log \sqrt{L}]} S_{L,\nu}^\delta f + R_L^\delta f.$$

It is easily seen that $S_{L,0}^\delta$ satisfies the conditions of the Marcinkiewicz

multiplier theorem (see [8]) and hence

$$(4.5) \quad \|S_{L,0}^\delta f\|_p \leq C \|f\|_p, \quad 1 < p < \infty.$$

So what we need to prove is the existence of an $\epsilon > 0$ such that

$$(4.6) \quad \|S_{L,\nu}^\delta f\|_p \leq C 2^{-\nu\epsilon} \|f\|_p,$$

$$(4.7) \quad \|R_{L,\nu}^\delta f\|_p \leq C 2^{-\nu\epsilon} \|f\|_p.$$

As remarked in [1] it is enough to prove (4.6).

Proposition 4.1.

$$(4.8) \quad \|S_{L,\nu}^\delta f\|_2 \leq C(2^{-\nu}\sqrt{L})^{1/2} 2^{-\delta\nu}(\sqrt{L})^{\delta(p)} \|f\|_p.$$

PROOF. Note that $\varphi_{L,\nu}^\delta(t) = 0$ unless t satisfies $2^{-\nu-1} \leq 1 - t/L \leq 2^{-\nu+1}$ and on the support $\varphi_{L,\nu}^\delta(t)$ has the bound $|\varphi_{L,\nu}^\delta(t)| \leq C 2^{-\nu\delta}$. Consequently, by the orthogonality of the projections

$$\|S_{L,\nu}^\delta f\|_2^2 \leq C 2^{-2\delta\nu} \sum \|P_k f\|_p^2$$

where the sum is extended over all k satisfying $2^{-\nu-1} \leq 1 - \frac{k+n}{L} \leq 2^{-\nu+1}$. Since we have

$$\|P_k f\|_2^2 \leq C k^{2n(1/p-1/2)-1} \|f\|_p^2$$

a simple calculation shows that

$$\|S_{L,\nu}^\delta f\|_2^2 \leq C 2^{-2\nu\delta} (2^{-\nu}\sqrt{L})(\sqrt{L})^{2\delta(p)} \|f\|_p^2.$$

This proves the proposition.

Once we have the estimate (4.8) and the kernel estimate (3.6) we can proceed as in [1] to prove the theorem mentioned in the introduction. For the sake of completeness and to clarify certain points we give a somewhat detailed proof of the theorem.

Proceeding as in [1] one can show that given a $\gamma > 0$, there is an $\epsilon > 0$ such that

$$(4.9) \quad \int_{|x-y| > 2^{\nu(1+\gamma)}/\sqrt{L}} |S_L^\delta(x,y)| dy \leq C 2^{-\nu\epsilon}$$

holds uniformly in x . The proof of this we will not elaborate since it is already given in full details in [1]. The proof makes use of the kernel estimate (3.6).

Let B be a ball of radius $2^{\nu(1+\gamma)}/\sqrt{L}$, then

$$\|S_{L,\nu}^\delta f\|_{L^p(B)} \leq |B|^{1/p-1/2} \|S_{L,\nu}^\delta f\|_{L^2(B)}.$$

If we use the bounds (4.8) we will get

$$\|S_{L,\nu}^\delta f\|_{L^p(B)} \leq C 2^{-\nu(\delta+1/2)} 2^{\nu(1+\gamma)(\delta(p)+1/2)} \|f\|_p.$$

Since $\delta > \delta(p)$ we can choose γ so that $\delta + 1/2 > (1 + \gamma)(\delta(p) + 1/2)$ and with that choice of γ it is clear that there is an $\epsilon > 0$ such that the following is true

$$(4.10) \quad \|S_{L,\nu}^\delta f\|_{L^p(B)} \leq C 2^{-\nu\epsilon} \|f\|_p.$$

Split the kernel $S_{L,\nu}^\delta(x, y)$ into two parts by setting

$$\begin{aligned} K_1(x, y) &= S_{L,\nu}^\delta(x, y), \quad \text{if } |x - y| \leq 2^{\nu(1+\gamma)}/\sqrt{L} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$K_2(x, y) = S_{L,\nu}^\delta(x, y) - K_1(x, y).$$

Here γ is the positive number already chosen. The estimate (4.9) immediately proves that

$$\|K_2 f\|_p \leq C 2^{-\nu\epsilon} \|f\|_p.$$

To prove a similar estimate for K_1 we proceed as follows.

Let $B(h)$ be the ball

$$|x - h| \leq \frac{1}{4} \frac{2^{\nu(1+\gamma)}}{\sqrt{L}},$$

and let $B^*(h)$ be the ball

$$|x - h| \leq \frac{3}{4} \frac{2^{\nu(1+\gamma)}}{\sqrt{L}},$$

and $B^{**}(h)$ the ball

$$|x - h| \leq \frac{5}{4} \frac{2^{\nu(1+\gamma)}}{\sqrt{L}}.$$

We split the function f into three parts viz

$$f_1 = f\chi_{B^*}, \quad f_2 = f\chi_{B^{**} \setminus B^*}, \quad f_3 = f - f_1 - f_2.$$

Since the kernel K_1 is supported in the region $|x - y| \leq \frac{2^{\nu(1+\gamma)}}{\sqrt{L}}$ we have

$$K_1 f = K_1 f_1 + K_1 f_2.$$

We will prove that

$$\int_{B(h)} |K_1 f(x)|^p dx \leq C 2^{-\nu \epsilon p} \int_{B^{**}(h)} |f(y)|^p dy$$

Integration with respect to h will then prove the bound

$$\|K_1 f\|_p \leq C 2^{-\nu \epsilon} \|f\|_p.$$

When $x \in B(h)$ and y is in the support of f_1 we have $|x - y| \leq 2^{\nu(1+\gamma)}/\sqrt{L}$ so that $K_1 f_1 = S_{L,\nu}^\delta f_1$. In view of (4.10) we get

$$\int_{B(h)} |K_1 f_1(x)|^p dx \leq C 2^{-\nu \epsilon} \int_{B^{**}(h)} |f(y)|^p dy.$$

And when $x \in B(h)$ and y is in the support of f_2 we have

$$|x - y| \geq \frac{\frac{1}{2} 2^{\nu(1+\gamma)}}{\sqrt{L}},$$

and hence in view of (4.9) we get

$$\int_{B(h)} |K_1 f_2(x)|^p dx \leq C 2^{-\nu \epsilon} \int_{B^{**}(h)} |f(y)|^p dy.$$

This completes the proof of the theorem.

We conclude this section with a proof of the corollary. Observe that when $p = \frac{4n}{2n+1}$, $\delta(p) = 0$. Given $\delta > 0$, a simple calculation shows that $\delta > \delta(p_0)$ for any

$$p_0 > \frac{4n}{2n+1+2\delta}$$

and hence

$$\|S_L^\delta f\|_{p_0} \leq C \|f\|_{p_0}.$$

By self adjointness and duality we also have

$$\|S_L^\delta f\|_{p'_0} \leq C \|f\|_{p'_0}.$$

An interpolation proves that $\|S_L^\delta f\|_p \leq C \|f\|_p$ for $p_0 < p < p'_0$. Hence the corollary.

Note added in the proof. Recently the author has proved the main theorem of this paper on \mathbb{R}^{2n+1} also.

References

- [1] Sogge C. D. On the convergence of Riesz means on compact manifolds. *Annals of Math.* **126**(1987), 439-447.
- [2] Fefferman, C. A note on spherical summation multipliers, *Israel J. Math.* **15**(1973), 44-52.
- [3] Gergen, J. J. Summability of double Fourier series, *Duke Math. J.* **3**(1937), 133-148.
- [4] Kenig, C. E., Stanton, R. and Tomas, P. Divergence of eigen function expansions, *J. Funct. Anal.* **46**(1982), 28-44.
- [5] Markett, C. Mean Cesaro summability of Laguerre expansions and norm estimates with shifted parameters, *Analysis mathematica* **8**(1982), 19-37.
- [6] Mauceri, G. The Weyl transform and bounded operators on $L^p(\mathbb{R}^n)$, *J. Funct. Anal.* **39**(1980), 408-429.
- [7] Strichartz, R. S. Harmonic analysis as spectral theory of Laplacians, *J. Funct. Anal.* **87**(1989), 51-148.
- [8] Thangavelu, S. Multipliers for Hermite expansions, *Rev. Mat. Iberoamericana*, **3**(1987), 1-24.
- [9] Thangavelu, S. Summability of Hermite expansions II, *Trans. A.M.S.* **314**(1989), 143-170.

Recibido: 28 de marzo de 1989.

S. Thangavelu
T.I.F.R. Centre
I.I.Sc Campus
Bangalore-560 012
INDIA