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Vector-Valued Multipliers on Stratified Groups

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Dedicated to the memory of J. L. Rubio de Francia

Introduction

Let $\mathcal{L}$ be a left invariant sublaplacian on a stratified Lie group G and denote by $\{E(\lambda): \lambda \geqslant 0\}$ its spectral resolution. The «multiplier operator» $m(\mathcal{L})$ can be defined for any Borel measurable function m on $[0, +\infty)$ by the spectral theorem according to the prescription

$$m(\mathcal{L}) = \int_0^\infty m(\lambda)\, dE(\lambda).$$

More generally, for every $t > 0$ consider the operator $m(t\mathcal{L})$ defined by the formula

$$m(t\mathcal{L}) = \int_0^\infty m(t\lambda)\, dE(\lambda)$$

and the maximal operator $m_*(\mathcal{L})$ associated to the family of operators $\{m(t\mathcal{L})\}_{t>0}$ defined by

$$m_*(\mathcal{L})f(x) = \sup_{t>0} |m(t\mathcal{L})f(x)|, \tag{0.1}$$

for all f in $\mathcal{S}$ and $x \in G$.

A natural problem is to find conditions on the function m which ensure the boundedness of the multiplier operator $m(\mathcal{L})$ or the maximal operator $m_*(\mathcal{L})$ on various spaces of distributions on G. When G is the Heisenberg group a criterion for the boundedness of the multiplier operator on the spaces L^p, $1 < p < \infty$, was given by De Michele and Mauceri [8] and Mauceri [15]. Hulanicki and Stein proved a Marcinkiewicz-type multiplier theorem for Hardy spaces on any stratified group G ([11], p. 208). Their result was improved by De Michele and Mauceri [9], who proved that if the function m satisfies a fractional order smoothness condition of order $s > Q(1/p - 1/2) + 1$, where Q is the homogeneous dimension of the group, then the multiplier operator is bounded on the Hardy space H^p, $0 < p \leqslant 1$. Later Christ ([5]) proved that smoothness of order $s > Q/2$ yields boundedness on L^p for $1 < p < \infty$. In [15] Mauceri gave also a general condition for the boundedness of the maximal operator $m_*(\mathcal{L})$ on L^p, $1 < p < \infty$, and from H^1 to L^1. The condition involves the behaviour at infinity of the Mellin transform of the function m. He also gave an application to the almost everywhere convergence of the Riesz means for the eigenfunction expansions of the sublaplacian.

In the first part of this paper we sharpen the multiplier theorem given in [9] proving that if the function m satisfies a smoothness condition of order $s > Q(1/p - 1/2)$, $0 < p \leqslant 1$, then the multiplier operator $m(\mathcal{L})$ is bounded on H^p. Christ's result follows from this by duality and interpolation. As an application of this result we study the boundedness of the strongly singular multipliers $\psi(\mathcal{L})\mathcal{L}^{-\beta}\exp(i\mathcal{L}^{\alpha})$, $\alpha > 0$, $\mathrm{Re}\,(\beta) \geqslant 0$, where ψ is a smooth «cutoff» function which vanishes in a neighborhood of the origin and is identically 1 in a neighborhood of infinity. These multipliers were investigated in the context of $\mathbb{R}^n$ by Miyachi ([17]).

In the second part of the paper we extend the multiplier theorem to a vector-valued setting. Namely we consider a Banach space $\mathfrak{X}$ of functions on $\mathbb{R}_+$ endowed with a dilation invariant norm $\|\cdot\|_{\mathfrak{X}}$. We shall wiew the family of operators $\{m(t\mathcal{L})\}_{t>0}$ as a «vector-valued multiplier» mapping a scalar valued function f to the $\mathfrak{X}$-valued function $m(\cdot\mathcal{L})f$. Our aim is to give conditions on m which imply that for every test function f on G the function $(t, x) \mapsto m(t\mathcal{L})f(x)$ satifies an a priori estimate

$$\left(\int_G \|m(\cdot\mathcal{L})f(x)\|_{\mathfrak{X}}^p\,dx\right)^{1/p} \leqslant C\|f\|_{H^p} \tag{0.2}$$

for some p in $(0, 1]$ or a corresponding estimate with $\|f\|_{H^p}$ replaced by $\|f\|_p$ if $1 < p < \infty$. Estimate (0.2) implies that the operator $f \mapsto m(\cdot\mathcal{L})f$ extends to a bounded linear operator from H^p to the Bochner-Lebesgue space $L^p(\mathfrak{X})$ of all $\mathfrak{X}$-valued p-integrable functions on G. We study this problem when $\mathfrak{X}$ is either a Besov space defined in terms of the multiplicative structure of $\mathbb{R}_+$ or the space $C_0(\mathbb{R}_+)$ of all continuous functions vanishing at infinity on $\mathbb{R}_+$. In

the first case estimate (0.2) can be viewed as a regularity result for the «means» $m(t\mathcal{L})f$, $t > 0$, of the function f. Mixed norm estimates analogous to (0.2) were studied in connection with the problem of regularity of spherical means in [4], [20], [23], [19], [18], in the Euclidean case, and in [6], [16] in the context of compact Lie groups. If $\mathfrak{X} = C_0(\mathbb{R}_+)$ estimate (0.2) implies the $H^p - L^p$ boundedness of the maximal operator $m_*(\mathcal{L})$ investigated in [15]. Our result (Theorem 2.6 below) improves the results given there. Maximal multiplier theorems in the context of $\mathbb{R}^n$ where given by J. L. Rubio de Francia in [21]. Our Corollary 2.7 is a version for stratified groups of a result of his.

1. The Multiplier Theorem

Some notation about stratified groups and Hardy spaces would be in order. However, since the notation used in the literature is quite standard, to avoid wasting of space, we refer the reader to the monograph by Folland and Stein [11] for all unexplained terminology and notation. For a more concise exposition, the reader may also consult Section 1 of [9]. Let G be a stratified group. Denote by $\{\delta_t : t > 0\}$ a family of dilations of the Lie algebra g of G; following a common abuse of notation we shall also denote by $\{\delta_t : t > 0\}$ the induced family of dilations of G. Let $\{E(\lambda): \lambda \geqslant 0\}$ be the spectral resolution of a left invariant sublaplacian $\mathcal{L}$ on G. Since the spectral measure of $\{0\}$ is zero we shall regard our spectral multipliers as functions defined on $\mathbb{R}_+$ rather than on $[0, +\infty)$. If m is a bounded Borel function on $\mathbb{R}_+$ the operator $m(\mathcal{L})$ is bounded on L^2, the space of square integrable functions with respect to Haar measure, and commutes with left translations. Thus, by the Schwartz kernel theorem, there exists a tempered distribution k on G such that $m(\mathcal{L})f = f * k$ for all functions in the Schwartz space $\mathcal{S}$. Moreover, for every $t > 0$ the distribution kernel of the operator $m(t\mathcal{L})$ is $k_{\sqrt{t}}$, where $k_{\sqrt{t}}$ is the distribution obtained by «dilating and normalising by $\sqrt{t}$ the distribution k», *i.e.*

$$\langle k_{\sqrt{t}}, f \rangle = \langle k, f \circ \delta_{\sqrt{t}} \rangle$$

for all f in $\mathcal{S}$ ([11], Lemma 6.29). For $s \geqslant 0$ and $1 \leqslant p, q \leqslant \infty$ let $\Lambda^s_{p,q}(\mathbb{R}_+)$ be the space of all functions m on $\mathbb{R}_+$, whose pull-back $m \circ \exp$ via the exponential map is in the Besov space $\Lambda^s_{p,q}(\mathbb{R})$. The norm of a function m in $\Lambda^s_{p,q}(\mathbb{R}_+)$ is the norm of $m \circ \exp$ in $\Lambda^s_{p,q}(\mathbb{R})$. In particular we shall denote by $H^s_2(\mathbb{R}_+)$ the Sobolev space $\Lambda^s_{2,2}(\mathbb{R}_+)$. Throughout this paper we shall denote by ϕ a function in $C^\infty_c(\mathbb{R}_+)$ supported in $(1/2, 2)$, such that $\sum_{j\in\mathbb{Z}} \phi(2^j\lambda) = 1$ for every $\lambda > 0$. The main result of this section is

Theorem 1.1. *Suppose that m is a function on $\mathbb{R}_+$ which satisfies*

$$\sup_{k\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H_2^s} < \infty \tag{1.1}$$

for some $s > Q(1/p - 1/2)$ and $0 < p \leqslant 1$. Then the multiplier operator $m(\mathcal{L})$ extends to a bounded operator on H^q, if $p \leqslant q \leqslant 1$, on L^q if $1 < q < \infty$, and on BMO.

The proof of Theorem 1.1 is merely a refinement of the arguments of [9]. We shall begin by establishing some weighted norm inequalities for the distribution kernel of the operator $m(\mathcal{L})$, when m is a function with compact support in $\mathbb{R}_+$. The key step is the following lemma, which is an improved version of Lemma 3.1 in [9].

Lemma 1.2. *Suppose that $\alpha \geqslant 0$, $1 \leqslant p \leqslant 2$. Let m be a function in $H_2^s(\mathbb{R}_+)$ supported in $(1/2, 2)$, where $s > \alpha/p + Q(1/p - 1/2)$. Let k be the distribution kernel of $m(\mathcal{L})$. Then*

$$\int_G |x|^\alpha |Y^I k(x)|^p \, dx \leqslant C \|m\|_{H_2^s} \tag{1.2}$$

for every multiindex I.

PROOF. We first prove (1.2) in the case $p = 2$. A slight refinement of the argument of Lemma 3.1 in [9] shows that (1.2) holds for $s > (\alpha + 1)/2$ (we only need to recall that the Fourier coefficients of a function F in $H_2^s(\mathbb{T})$ satisfy the estimate $\sum_n (1 + |n|)^{s-\epsilon-1/2} |\hat{F}(n)| \leqslant C_\epsilon \|F\|_{H_2^s}$ for every $\epsilon > 0$). Let ψ be a function in $C_c^\infty(\mathbb{R}_+)$ supported in $(1/2, 2)$ such that $\psi(\lambda) = 1$ for every λ in the support of m. Given a bounded measurable function f on $\mathbb{R}_+$, denote by k_f the distribution kernel of the operator $\psi(\mathcal{L})f(\mathcal{L})$. The previous argument shows that $f \mapsto k_f$ is a bounded linear map from $H_2^s(\mathbb{R}_+)$ into $L^2(G, |x|^\alpha \, dx)$ for every s, α such that $\alpha \geqslant 0$ and $s > (\alpha + 1)/2$. On the other hand, by the Plancherel formula ([9], formula (3.7)) the map $f \mapsto k_f$, $f \in L^\infty \cap L^2(\mathbb{R}_+)$, extends to a bounded linear map from $L^2(\mathbb{R}_+)$ into L^2. Now fix β, $r > 0$ such that $r > \beta/2$. By interpolating between the $H_2^s(\mathbb{R}_+) - L^2(G, |x|^\alpha \, dx)$ estimate, which holds for $s > (\alpha + 1)/2$, and the $L^2(\mathbb{R}_+) - L^2$ estimate, and letting α tend to infinity, we get that $f \mapsto k_f$ maps $H_2^s(\mathbb{R}_+)$ into $L^2(G, |x|^\beta \, dx)$ continuously. This proves estimate (1.2) for $p = 2$ and $I = 0$. The estimate for the other values of p can be obtained by Hölder's inequality. The estimate for $I \neq 0$ follows from that for $I = 0$ as in the proof of Lemma 3.1 in [9]. This completes the proof.

Let k be a function in $C^\infty(G)$. For every positive integer N, denote by $P_x^{(N)}(k; \cdot)$ the right Taylor polynomial of k at x of homogeneous degree N ([11], p. 26). Set

$$\Delta^{(N)}k(x, y) = k(x, y) - P_x^{(N)}(k; y).$$

For r, $R > 0$ define

$$\omega_{N,r}(k; R) = R^{-Q-2r} \int_{|y|<R} dy \int_{|x|\geq 2R} dx\, |\Delta^{(N)}k(x, y)|^2 |x|^{Q+2r}.$$

By using Lemma 1.2 instead of Lemma 3.1 in [9], we obtain the following refinement of Lemma 3.2 in [9].

Lemma 1.3. *Let m be a function in $H_2^s(\mathbb{R}_+)$ supported in $(1/2, 2)$. Denote by k the distribution kernel of $m(\mathfrak{L})$. Let r be a positive number, N a nonnegative integer such that $N \leqslant r < N + 1$. If $s > Q/2 + r$, then there exist positive constants δ, η and C such that*

$$\omega_{N,r}(k; R) \leqslant C\|m\|_{H_2^s}^2 \min\{R^{\delta}, R^{-\eta}\}. \tag{1.3}$$

PROOF OF THEOREM 1.1. Argue as in the proof of Theorem 1.1 in [9], using lemmata 1.2 and 1.3 instead of lemmata 3.1 and 3.2 therein to get the result for $0 < q \leqslant 1$. The case $q > 1$ follows by interpolation between the H^1 and the BMO estimates.

As an application of Theorem 1.1 we discuss the boundedness of the two-parameter family of operators $m_{\alpha,\beta}(\mathfrak{L}) = \psi(\mathfrak{L})\mathfrak{L}^{-\beta}\exp(i\mathfrak{L}^{\alpha})$, $\alpha > 0$, $\mathrm{Re}(\beta) \geqslant 0$, where ψ is a smooth «cutoff» function which vanishes in a neighborhood of the origin and is identically 1 in a neighborhood of infinity. In the Euclidean context sharp H^p boundedness results have been given by Miyachi ([17]) in the case $\alpha \neq 1$ and by Peral ([20]) in the case $\alpha = 1$. See also [10].

Corollary 1.4. *Suppose that $\alpha > 0$, $\mathrm{Re}(\beta) \geqslant 0$, and $\mathrm{Re}(\beta)/\alpha > Q|1/p - 1/2|$. Then the operator $m_{\alpha,\beta}(\mathfrak{L})$ is bounded on H^p if $0 < p \leqslant 1$, on L^p if $1 < p < \infty$ and on* BMO *if $p = \infty$.*

PROOF. A direct calculation shows that for every nonnegative integer n and large R

$$\|\phi(\cdot)m_{\alpha,\beta}(R\cdot)\|_{H_2^n} \leqslant C_{\alpha,\beta}R^{-\mathrm{Re}(\beta)+n\alpha}$$

(notice that the left hand side vanishes if R is small). By interpolation, a similar estimate holds with a nonnegative s in place of n; hence the result for $0 < p \leqslant 1$ and BMO follows directly from Theorem 1.1. The result for $1 < p < \infty$ follows by applying Stein's complex interpolation theorem to the analytic family of operators $\{m_{\alpha,\beta}: 0 \leqslant \mathrm{Re}(\beta) < Q/2 + \epsilon\}$, $\epsilon > 0$.

2. Vector-Valued Multipliers

If A is a Banach space we denote by $L^p(A)$, $0 < p \leqslant \infty$, the Bochner-Lebesgue space of all strongly measurable A-valued functions F on G, for which

$$\|F\|_{L^p(A)} = \left(\int_G \|F(x)\|_A^p \, dx \right)^{1/p}$$

(with the usual modification when $p = \infty$). If $0 < p \leqslant 1$ we denote by $H^p(A)$ the atomic Hardy space defined in terms of A-valued atoms (A-valued atoms are defined as in the scalar case except that absolute values are to be replaced by the norm in A). For a locally integrable A-valued function F we define the maximal function

$$F^{\#}(x) = \sup_{x \in B} \frac{1}{|B|} \int_B \|F(y) - F_B\|_A \, dy,$$

where B stands for an arbitrary ball in G and F_B is the average of F over B. Then BMO(A) is the space $\{F \in L^1_{\mathrm{loc}} : \|F\|_{\mathrm{BMO}(A)} = \|F^{\#}\|_\infty < \infty\}$. We recall that $H^1(B)^*$ imbeds isometrically into BMO(B^*) for every Banach space B. The imbedding is surjective if B is reflexive (more generally if B^* has the Radon-Nikodym property ([2], [3]).

Let $\mathcal{H}$ be a separable Hilbert space. By a result of Marcinkiewicz and Zygmund ([13]) if T is a bounded linear operator on L^2 the operator $T \otimes Id$ on $L^2 \otimes \mathcal{H}$ has a bounded extension $T_{\mathcal{H}}$ to $L^2(\mathcal{H})$. In particular we shall denote by $\{dE_{\mathcal{H}}(\lambda)\}$ the $L^2(\mathcal{H})$ projection-valued measure on $\mathbb{R}_+$ associated to the resolution of the identity $\{dE(\lambda)\}$ of the sublaplacian $\mathcal{L}$. If $\mathcal{K}$ is another Hilbert space we shall denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, endowed with the operator norm $\|\cdot\|_{\mathcal{H},\mathcal{K}}$. Notice that for every bounded continuous function M on $\mathbb{R}_+$, with values in $\mathcal{B}(\mathcal{H}, \mathcal{K})$, $ME_{\mathcal{H}}(\Omega) = E_{\mathcal{K}}(\Omega)M$ for each Borel subset Ω of $\mathbb{R}_+$. Thus for every function F in $L^2(\mathcal{H})$ the improper Riemann integral

$$M(\mathcal{L})F = \int_0^\infty M(\lambda) \, dE_{\mathcal{H}}(\lambda) F$$

converges in $L^2(\mathcal{K})$ and defines a bounded linear operator $M(\mathcal{L})$ from $L^2(\mathcal{H})$ into $L^2(\mathcal{K})$. Moreover $\|M(\mathcal{L})\|_{L^2(\mathcal{H}),L^2(\mathcal{K})} = \sup_{\lambda>0} \|M(\lambda)\|_{\mathcal{H},\mathcal{K}}$.

Definition 2.1. *Let A and B be two Banach spaces such that $A \cap \mathcal{H}$ is dense in A. If the operator $M(\mathcal{L})$ extends to a bounded operator from the Hardy space $H^p(A)$ to $L^p(B)$ for some $p \in (0, 1]$ we say that M is a vector-valued multiplier of $H^p(A)$ into $L^p(B)$. In this definition the pair $(H^p(A), L^p(B))$ should be replaced by $(L^p(A), L^p(B))$ if $1 < p < \infty$ and by $(L^\infty(A), \mathrm{BMO}(B))$ if $p = \infty$.*

As in the scalar case, given $p \in (0, 1]$ and spaces A, B one wishes to find conditions on the operator-valued function M that guarantee that M is a multiplier of $H^p(A)$ into $L^p(B)$. In this paper we shall consider this problem in the following context. Let $\mathfrak{X}$ denote the Banach space $\Lambda^s_{2,q}(\mathbb{R}_+)$, $s > 0$, $1 \leqslant q \leqslant \infty$, of all measurable functions f on $\mathbb{R}_+$ whose pull back $f \circ \exp$ via the exponential map is in the usual Besov space $\Lambda^s_{2,q}(\mathbb{R})$ ([1]). Thus

$$\|f\|_{\mathfrak{X}} = \|f \circ \exp\|_{\Lambda^s_{2,q}(\mathbb{R})}$$

is a dilation invariant norm on $\mathfrak{X}$. If $m \in \mathfrak{X}$ and $\lambda \in \mathbb{R}_+$ we shall denote by $m(\cdot\lambda)$ the λ-dilate of the function m, namely the function $t \mapsto m(t\lambda)$. Thus the map $\lambda \mapsto m(\cdot\lambda)$ is a bounded, continuous $\mathfrak{X}$-valued function on $\mathbb{R}_+$, provided that $q < \infty$. We shall also view it as a $\mathcal{B}(\mathbb{C}, \mathfrak{X})$ and a $\mathcal{B}(\mathfrak{X}^*, \mathbb{C})$-valued function, via the natural isometric identifications of $\mathfrak{X}$ with $\mathcal{B}(\mathbb{C}, \mathfrak{X})$ and with a subspace of $\mathcal{B}(\mathfrak{X}^*, \mathbb{C})$. Thus if $\mathfrak{X}$ is the Hilbert space $H^s_2(\mathbb{R}_+) = \Lambda^s_{2,2}(\mathbb{R}_+)$ we shall denote by $m(\cdot\mathcal{L})$ both the operator of L^2 into $L^2(\mathfrak{X})$ defined by

$$m(\cdot\mathcal{L})f = \int_0^\infty m(\cdot\lambda)\, dE(\lambda) f, \qquad f \in L^2,$$

and the operator of $L^2(\mathfrak{X}^*)$ into L^2 defined by

$$\langle m(\cdot\mathcal{L}), F\rangle = \int_0^\infty \langle m(\cdot\mathcal{L}), dE_{\mathfrak{X}}(\lambda) F\rangle, \qquad F \in L^2(\mathfrak{X}^*).$$

Our multiplier theorem is then

Theorem 2.1. *Suppose that* $0 < p \leqslant \infty$, $\beta > 0$ *and* $s > Q\left|\frac{1}{p} - \frac{1}{2}\right| + \beta$. *If*

$$\sum_{k \in \mathbb{Z}} \|\phi(\cdot) m(2^k \cdot)\|_{H^s_2} < \infty$$

then $m(\cdot\mathcal{L})$ *extends to a bounded operator*

(i) *from* H^p *to* $L^p(\Lambda^\beta_{2,1}(\mathbb{R}_+))$ *if* $0 < p \leqslant 1$;

(ii) *from* L^p *to* $L^p(\Lambda^\beta_{2,1}(\mathbb{R}_+))$ *if* $1 < p < \infty$;

(iii) *from* L^∞ *to* $\mathrm{BMO}(\Lambda^\beta_{2,1}(\mathbb{R}_+))$ *if* $p = \infty$.

If $p = 2$ one can actually prove a sharper result. Indeed one has

Lemma 2.2. *If* $m \in \Lambda^s_{2,q}(\mathbb{R}_+)$, $s \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$, *then* $m(\cdot\mathcal{L})$ *extends to a bounded operator from* L^2 *to* $L^2(\Lambda^s_{2,q}(\mathbb{R}_+))$ *and from* $L^2(\Lambda^{-s}_{2,q}(\mathbb{R}_+))$ *to* L^2, *whose norm does not exceed* $C\|m\|_{\Lambda^s_{2,q}}$.

PROOF. Assume first that $q = 2$. Since $\lambda \to m(\cdot\lambda)$ is a bounded, continuous $\Lambda^s_{2,2}(\mathbb{R}_+)$-valued function, $m(\cdot\mathcal{L})$ is a bounded operator from L^2 to $L^2(\Lambda^s_{2,2}(\mathbb{R}_+))$ and from $L^2(\Lambda^{-s}_{2,2}(\mathbb{R}_+))$ to L^2, whose norm is $\sup_\lambda \|m(\cdot\lambda)\|_{\Lambda^s_{2,2}} = \|m\|_{\Lambda^s_{2,2}}$. The result for $q \neq 2$ follows by applying the $[\ ,\ ]_{\theta,q}$ interpolation method to the bilinear maps $(m,f) \mapsto m(\cdot\mathcal{L})f$ from $\Lambda^s_{2,2} \times L^2$ into $L^2(\Lambda^s_{2,2}(\mathbb{R}_+))$ and $(m,F) \mapsto \langle m(\cdot\mathcal{L}), F\rangle$ from $\Lambda^s_{2,2}(\mathbb{R}_+) \times L^2(\Lambda^{-s}_{2,2}(\mathbb{R}_+))$ into L^2 for different values of s ([1]).

We turn now to a description of the $H^p - L^p$ results. We begin by stating a result for vector-valued singular integrals. Let A and B be two Banach spaces. We consider kernels K which are strongly measurable functions defined on G and with values in the space $\mathcal{B}(A, B)$ of all bounded linear operators from A to B. We suppose that $\|K\|_{A,B}$ is locally integrable away from the origin. On $G \times G$ we shall consider the measure $d\mu_r(x,y) = |x|^{Q+2r}\,dx\,dy$, $r > 0$, and the sets $S_R = \{(x,y)\colon |x| \geqslant 2R,\ |y| < R\}$, $R > 0$.

Definition 2.2. *If N is a nonnegative integer and $r > 0$ we say that K is a kernel of type $\mathfrak{N}_{N,r}(A,B)$ if there exists a polynomial $P_x(y) = \sum_{|I| \leq N} a_I(x)y^I$, $x, y \in G$, of homogeneous degree N, whose coefficients a_I are strongly measurable $\mathcal{B}(A,B)$-valued functions on G such that*

$$\mathfrak{N}_{N,r}(K) = \sup_{R>0}\left(R^{-Q-2r}\iint_{S_R}\|K(xy) - P_x(y)\|^2_{A,B}\,d\mu_r(x,y)\right)^{1/2} < \infty.$$

Definition 2.3. *A linear operator T mapping A-valued functions into B-valued functions is called a singular integral operator of type $\mathfrak{N}_{N,r}(A,B)$ if the following two conditions are satisfied*:

(i) *T is a bounded operator from $L^2(A)$ to $L^2(B)$;*
(ii) *there exists a kernel K of type $\mathfrak{N}_{N,r}(A,B)$ such that*

$$TF(x) = \int_G K(y^{-1})F(xy)\,dy$$

for every F in $L^2(A)$ with compact support and for almost every x in the complement of the support of F.

Theorem 2.3. *Suppose that $0 < p \leqslant 1$. If T is a singular integral operator of type $\mathfrak{N}_{N,r}(A,B)$ for some noninteger $r > Q(1/p - 1)$ and for $N = [r]$ then T can be extended to a bounded operator from $H^p(A)$ to $L^p(B)$. Moreover*

$$\|T\|_{H^p(A),L^p(B)} \leqslant C(\|T\|_{L^2(A),L^2(B)} + \mathfrak{N}_{N,r}(K)).$$

Remarks. The proof of Theorem 2.3 is a simple adaptation to the vector-valued case of proof of Theorem 2.1 in [9]. Related results on operator-valued singular integrals can be found in [22].

We shall apply Theorem 2.3 in the following context. Let $\mathfrak{X}$ denote the Sobolev space $H_2^s(\mathbb{R}_+)$, $s \geqslant 0$. If m is a function in $\mathfrak{X}$ by Lemma 2.2 the operator $m(\cdot\mathfrak{L})$ is bounded from L^2 to $L^2(\mathfrak{X})$ and from $L^2(\mathfrak{X}^*)$ to L^2. Moreover it commutes with left translations. Thus, by the Schwartz kernel theorem, there exists a $\mathfrak{X}$-valued distribution K on G (*i.e.* a bounded linear map from $\mathcal{S}$ into $\mathfrak{X}$) such that $m(\cdot\mathfrak{L})f = f * K$ for all f in $\mathcal{S}$. We shall show, essentially, that K is a locally integrable function away from the origin and satisfies conditions $\mathfrak{N}_{N,r}(\mathbb{C}, \mathfrak{Y})$ and $\mathfrak{N}_{N,r}(\mathfrak{Y}^*, \mathbb{C})$ for suitable N and r and for certain Besov spaces $\mathfrak{Y}$ of functions on $\mathbb{R}_+$.

Remarks. If the function M is bounded on $\mathbb{R}_+$ and k is the distribution kernel of the operator $m(\mathfrak{L})$ then the $\mathfrak{X}$-valued distribution kernel K of the operator $m(\cdot\mathfrak{L})$ is the continuous linear map $f \mapsto \langle k_{\sqrt{t}}, f\rangle$ from $\mathcal{S}$ into $\mathfrak{X}$, because $k_{\sqrt{t}}$ is the kernel of the operator $m(t\mathfrak{L})$ for every $t > 0$.

If $m \in C_c^\infty(\mathbb{R}_+)$ the kernel k is in $\mathcal{S}$ ([15], Proposition 2.7). Thus $K = k_{\sqrt{\cdot}}$ is a smooth $\mathfrak{X}$-valued function away from the origin. As in Section 1 we denote by $P_x^{(N)}$ the right Taylor polynomial of K at x of homogeneous degree N. Notice that the coefficients of $P_x^{(N)}$ are smooth $\mathfrak{X}$-valued functions of x on $G\setminus\{0\}$. We also denote by $\Delta^{(N)}K(x, y)$ the difference $K(xy) - P_x^{(N)}(y)$, for x, y in G, $x \neq 0$. For every $R > 0$ and x in $G\setminus\{0\}$ let $K_R(x)$ denote the R-dilate of $K(x)$ (as a distribution on $\mathbb{R}_+$). Then

$$\Delta^{(N)}K_R(x, y) = R^{-Q}(\Delta^{(N)}K)(R^{-1}x, R^{-1}y)$$

and, by using the invariance of the norm in $\mathfrak{X}$, it is an easy matter to show that

$$R^{-Q-2r}\iint_{S_R} \|\Delta^{(N)}K(x, y)\|_{\mathfrak{X}}^2 \, d\mu_R(x, y)$$

is independent of R. Thus

$$\mathfrak{N}_{N,r}(K)^2 = \iint_{S_1} \|\Delta^{(N)}K(x, y)\|_{\mathfrak{X}}^2 \, d\mu_r(x, y). \tag{2.2}$$

Lemma 2.4. *Suppose that* $0 < p \leqslant 1$. *Let* $\mathfrak{Y}$ *denote the Besov space* $\Lambda_{2,q}^\beta(\mathbb{R}_+)$, $\beta \geqslant 0$, $1 \leqslant q \leqslant \infty$. *If* $m \in H_2^s(\mathbb{R}_+)$ *is supported in* $(1/2, 2)$ *and* $s > Q(1/p - 1/2) + \beta$ *then* $m(\cdot\mathfrak{L})$ *is a bounded operator from* H^p *to* $L^p(\mathfrak{Y})$ *and from* $H^p(\mathfrak{Y}^*)$ *to* L^p, *whose norm does not exceed* $C\,\|m\|_{H_2^s}$.

PROOF. Assume first than $m \in C_c^\infty((1/2, 2))$ and $\mathfrak{Y} = \Lambda_{2,2}^n(\mathbb{R}_+)$, where n is a nonnegative integer. We shall prove that $m(\cdot\mathfrak{L})$ is a singular integral operator of types $\mathfrak{N}_{N,r}(\mathbb{C}, \mathfrak{Y})$ and $\mathfrak{N}_{N,r}(\mathfrak{Y}^*, \mathbb{C})$ for some $r > Q(1/p - 1)$ and for $N = [r]$.

Let k and K be the kernels of $m(\mathfrak{L})$ and $m(\cdot\mathfrak{L})$, respectively. By the previous remarks $k \in \mathcal{S}$, $K \in C^\infty(G\setminus\{0\})$ and, for every $x \neq 0$, $K(x)$ is the function $t \to k_{\sqrt{t}}(x)$. Therefore $K \in L^1_{\mathrm{loc}}(\mathcal{Y})$ away from the origin. Since by Lemma 2.2 $m(\cdot\mathfrak{L})$ is a bounded operator from L^2 to $L^2(\mathcal{Y})$ and from $L^2(\mathcal{Y}^*)$ to L^2 whose norm is $\|m\|_\infty \leqslant C\|m\|_{H^s_2}$, by Theorem 2.3 and (2.2) we only need to show that

$$\mathfrak{N}_{N,r}(K)^2 = \iint_{S_1} \|\Delta^{(N)}K(x,y)\|^2_{\mathcal{Y}}\, d\mu_r(x,y) \leqslant C\|m\|^2_{H^s_2}.$$

Let ρ denote the dilation invariant differential operator $t(d/dt)$ on $\mathbb{R}_+$. Then

$$\|\Delta^{(N)}K(x,y)\|^2_{\mathcal{Y}} = \sum_{j=0}^{n} \int_0^\infty |\rho^j \Delta^{(N)} k_{\sqrt{t}}(x,y)|^2 \frac{dt}{t}.$$

By the spectral theorem

$$\rho^j f * k_{\sqrt{t}} = \int_0^\infty \rho^j m(t\lambda)\, dE(\lambda) f.$$

Let $k^{(j)}$ be the kernel of the operator $\rho^j m(\mathfrak{L})$. Then, by a straightforward, albeit tedious, computation

$$\rho^j \Delta^{(N)} k_{\sqrt{t}}(x,y) = \Delta^{(N)} k^{(j)}_{\sqrt{t}}(x,y) = t^{-Q/2} \Delta^{(N)} k^{(j)}(t^{-1/2}x, t^{-1/2}y).$$

Therefore, interchanging the order of integration, performing the change of variables $(t^{-1}x, t^{-1}y) = (\xi, \eta)$ and applying Lemma 1.3, we get that

$$\begin{aligned}
\iint_{S_1} \|\Delta^{(N)}K(x,y)\|^2_{\mathcal{Y}}\, d\mu_r(x,y) &= \sum_{j=0}^{n} \iint_{S_1} \int_0^\infty |\rho^j \Delta^{(N)} k_{\sqrt{t}}(x,y)|^2\, d\mu_r(x,y) \frac{dt}{t} \\
&\leqslant C \sum_{j=0}^{n} \int_0^\infty \omega_{N,r}(k^{(j)}, t^{-1}) \frac{dt}{t} \\
&\leqslant C\|m\|^2_{\Lambda^\sigma_{2,2}}
\end{aligned}$$

provided that $\sigma \geqslant Q/2 + r + n$. Choose $r > Q(1/p - 1)$ and σ such that $s > \sigma > Q/2 + r + n$. Since $H^s_2(\mathbb{R}_+)$ imbeds continuously in $\Lambda^\sigma_{2,2}(\mathbb{R}_+)$ we have proved that $\mathfrak{N}_{N,r}(K) \leqslant C\|m\|_{H^s_2}$. Thus, by Theorem 2.3, the norm of $m(\cdot\mathfrak{L})$, qua operator from H^p to $L^p(\Lambda^n_{2,2}(\mathbb{R}_+))$ and from $H^p(\Lambda^{-n}_{2,2}(\mathbb{R}_+))$ to L^p, is bounded by $C\|m\|_{H^s_2}$ provided that $s > Q(1/p - 1/2) + n$. By interpolation, via the $[\ ,\ ]_{\theta,q}$, $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, method, we obtain that if $\sigma > Q(1/p - 1/2) + \beta$ and $m \in \Lambda^\sigma_{2,q}(\mathbb{R}_+)$ then the norm of $m(\cdot\mathfrak{L})$, qua operator from H^p to $L^p(\Lambda^\beta_{2,q}(\mathbb{R}_+))$ and from $H^p(\Lambda^{-\beta}_{2,q}(\mathbb{R}_+))$ to L^p, is bounded by $C\|m\|_{\Lambda^\sigma_{2,q}}$. Since $H^s_2(\mathbb{R}_+)$ imbeds continuously in $\Lambda^\sigma_{2,q}(\mathbb{R}_+)$ for $s > \sigma$ the lemma is proved for m in $C^\infty_c((1/2,2))$. In general, if m is a function in $H^s_2(\mathbb{R}_+)$ supported in $(1/2,2)$, we pick a sequence $\{m_n\}$ of functions in $C^\infty_c((1/2,2))$ which converges to m in $H^s_2(\mathbb{R}_+)$. Then $m_n(\cdot\mathfrak{L})$ converges to $m(\cdot\mathfrak{L})$ in the strong operator topology,

qua operator from L^2 to $L^2(\mathcal{Y})$ and from $L^2(\mathcal{Y}^*)$ to L^2. Since $\{m_n(\bullet\mathcal{L})\}$ is also a Cauchy sequence in $\mathcal{B}(H^p, L^p(\mathcal{Y}))$ and in $\mathcal{B}(H^p(\mathcal{Y}^*), L^p)$, the lemma is proved.

The following corollary extends the result of Lemma 2.4 to the range $0 < p \leqslant \infty$.

Corollary 2.5. *Let $\mathcal{Y}$ denote the Besov space $\Lambda^\beta_{2,q}(\mathbb{R}_+)$, $\beta \geqslant 0$, $1 \leqslant q \leqslant \infty$. If m is a function in $H^s_2(\mathbb{R}_+)$ supported in $(1/2, 2)$ and $s > Q|1/p - 1/2| + \beta$, then the operator $m(\bullet\mathcal{L})$ is bounded from L^p to $L^p(\mathcal{Y})$, if $1 < p < \infty$, and from L^∞ to BMO$(\mathcal{Y})$, if $p = \infty$.*

PROOF. Assume first that $1 < q < \infty$, so that $\mathcal{Y}$ is reflexive. If $s > Q/2 + \beta$ the operator $m(\bullet\mathcal{L})$ is bounded from $H^1(\mathcal{Y}^*)$ to L^1 by Lemma 2.4. Thus its transpose is a bounded operator from L^∞ to BMO $(\mathcal{Y})$ with the same norm. Since the heat kernel ω satisfies $\omega(x) = \omega(x^{-1})$ for every x in G, the same property holds for the kernel of the operator $m(\bullet\mathcal{L})$. Thus the transpose of $m(\bullet\mathcal{L})$ is still $m(\bullet\mathcal{L})$. The result for $1 < p < \infty$ *follows by interpolation between the* $H^1 - L^1$ and the $L^2 - L^2$ estimates (Lemma 2.2) if $p < 2$ and the $L^2 - L^2$ and the $L^\infty -$ BMO estimates if $p > 2$. To remove the restriction $q > 1$ choose s_0, $s_1 > Q|1/p - 1/2| + \beta$ and $\theta \in (0, 1)$ such that $s = (1 - \theta)s_0 + \theta s_1$ and apply the interpolation functor $[\ ,\ ]_{\theta,1}$ to the bilinear map $(m, f) \to m(\bullet\mathcal{L})f$ from $H^{s_i}_2(\mathbb{R}_+) \times L^p$ to $L^p(\Lambda^\beta_{2,\infty}(\mathbb{R}_+))$, $i = 0, 1$. Since $\Lambda^\beta_{2,1}(\mathbb{R}_+)$ imbeds continuously into $\Lambda^\beta_{2,\infty}(\mathbb{R}_+)$, the result holds for $q = \infty$.

PROOF OF THEOREM 2.1. We prove (i). The proofs of (ii) and (iii) are similar. Let m be a function on $\mathbb{R}_+$ satisfying (2.1) and ϕ be as in Section 1. Set $m_j(\lambda) = \phi(2^{-j}\lambda)m(\lambda)$ and $\mu_j(\lambda) = m_j(2^j\lambda)$. Notice that $\|m_j\|_{H^s_2} = \|\mu_j\|_{H^s_2}$ and that each μ_j is supported in $(1/2, 2)$. Hence the norm of $\mu_j(\bullet\mathcal{L})$ qua operator from H^p to $L^p(\Lambda^\beta_{2,1}(\mathbb{R}_+))$, $0 < p \leqslant 1$, is dominated by $C\|\mu_j\|_{H^s_2}$ (C independent of j), by Corollary 2.5. Also, the dilation invariance of the $\Lambda^\beta_{2,1}(\mathbb{R}_+)$ norm implies that $m_j(\bullet\mathcal{L})$ has the same norm as $\mu_j(\bullet\mathcal{L})$. Since m decomposes into the sum $\sum_{j\in\mathbb{Z}} m_j$, the norm of $m(\bullet\mathcal{L})$ can be estimated by $\sum_{j\in\mathbb{Z}} \|m_j\|_{H^s_2}$, which is convergent by (2.1). The proof of (i) is complete.

We now discuss some applications of the above results to maximal operators. Notice that the boundedness of the maximal operator $m_*(\mathcal{L})$ (see (0.1) for the definition) from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 \leqslant p \leqslant \infty$ is equivalent to the boundedness of the vector-valued multiplier $m(\bullet\mathcal{L})$ from H^p to $L^p(l^\infty(\mathbb{R}_+))$ if $0 < p \leqslant 1$ and from L^p to $L^p(l^\infty(\mathbb{R}_+))$ if $1 < p \leqslant \infty$. Our main result concerning maximal operators is the following.

Theorem 2.6. *Let m be a function on $\mathbb{R}_+$ such that*

$$\sum_{k\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H^s_2} = D < \infty. \tag{2.2}$$

Then

(i) *if $s > Q(1/p - 1/2) + 1/2$, $m(\cdot\mathcal{L})$ extends to a bounded operator from H^p to $L^p(C_0(\mathbb{R}_+))$, if $0 < p \leqslant 1$, and from L^p to $L^p(C_0(\mathbb{R}_+))$, if $1 < p \leqslant 2$, with norm not exceeding CD;*

(ii) *if $2 \leqslant p \leqslant \infty$ and $s > (Q-1)(1/2 - 1/p) + 1/2$, $m(\cdot\mathcal{L})$ extends to a bounded operator from L^p to $L^p(l^\infty(\mathbb{R}_+))$, with norm not exceeding CD.*

PROOF. Since $\Lambda^\beta_{2,1}(\mathbb{R}_+)$ imbeds continuously into $C_0(\mathbb{R}_+)$ by Bernstein's Theorem ([12], Theorem 1), (i) is an immediate consequence of Theorem 2.1.

To prove (ii), assume first that $s > Q/2$. Then the distribution kernel of the operator $m(\mathcal{L})$ is in L^1, by Lemma 1.2. Therefore the associated maximal operator $m_*(\mathcal{L})$ is bounded on L^∞, *i.e.* $m(\cdot\mathcal{L})$ is bounded from L^∞ to $L^\infty(l^\infty(\mathbb{R}_+))$. Assume now that $s = 1/2$. By Lemma 2.2 $m(\cdot\mathcal{L})$ extends to a bounded operator from L^2 to $L^2(\Lambda^{1/2}_{2,1}(\mathbb{R}_+))$, hence from L^2 to $L^2(C_0(\mathbb{R}_+))$. An easy interpolation argument concludes the proof.

Remark. In the Euclidean setting Dappa and Trebels ([7]) proved that the maximal operator $m_*(\mathcal{L})$ is bounded on L^p, $1 < p \leqslant 2$, and of weak type $1-1$, provided that

$$\sup_{j\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H^s_2} < \infty$$

for some $s > (Q+1)/2$.

We now turn to some applications of Theorem 2.1 and Theorem 2.6.

Corollary 2.7. *Suppose that n is an integer larger than $Q/2$ and m is a function in $C^{(n)}(\mathbb{R}_+)$ which vanishes if $\lambda \leqslant 1$ and satisfies the estimate*

$$|m^{(j)}(\lambda)| \leqslant C\lambda^{-a}. \qquad j = 0, 1, \ldots, n$$

for some $a > 1/2$. Then the maximal operator $m_*(\mathcal{L})$ extends to a bounded operator from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 < p \leqslant \infty$ provided that

$$\frac{1}{r_a} = \frac{Q-2a}{2(Q-1)} < \frac{1}{p} < \frac{Q+2a-1}{2a} = \frac{1}{q_a}$$

(it must be understood that $r_a = \infty$ if $a \geqslant Q/2$).

PROOF. We retain the notation used in the proof of Theorem 2.1. An easy computation shows that

$$\|\mu_j\|_{H_2^0} \leqslant C2^{-ja} \quad \text{and} \quad \|\mu_j\|_{H_2^n} \leqslant C2^{j(n-a)}.$$

The desired result is an immediate consequence of an interpolation argument, Theorem 2.1 and Theorem 2.6.

Remark. Related results have been obtained in the Euclidean setting by J. L. Rubio de Francia ([21], Theorem B).

We now discuss the boundedness of the maximal operator associated to the Riesz means of $\mathfrak{L}$. We improve some results obtained by the first author in [15].

For every $z \in \mathbb{C}$, with $\mathrm{Re}\,(z) > 0$, set

$$m_z(\lambda) = (1-\lambda)_+^z, \qquad \lambda > 0.$$

Corollary 2.8. *Let m_z be as above. Then*

(i) *if* $\mathrm{Re}\,(z) > Q(1/p - 1/2)$, $(m_z)_*(\mathfrak{L})$ *extends to a bounded operator from* H^p *to* L^p *if* $0 < p \leqslant 1$ *and on* L^p *if* $1 < p \leqslant 2$;

(ii) *if* $\mathrm{Re}\,(z) > (Q-1)(1/2 - 1/p)$ *and* $2 \leqslant p \leqslant \infty$, $(m_z)_*(\mathfrak{L})$ *extends to a bounded operator on* L^p.

PROOF. Let ψ a smooth «cutoff» fuction on $\mathbb{R}_+$ which equals one if $\lambda < 1/2$ and vanishes if $\lambda > 3/4$. Write $m_z = m_z^1 + m_z^2$ where $m_z^1 = \psi m_z$. Since m_z^1 is the restriction to $\mathbb{R}_+$ of a function in the Schwartz class of $\mathbb{R}$, the distribution kernel of the operator $m_z^1(\mathfrak{L})$ is in the Schwartz class on the group G ([15], Proposition 2.7); hence the associated maximal operator $(m_z^1)_*(\mathfrak{L})$ is bounded from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 < p \leqslant \infty$.

Notice that m_z^2 is a compactly supported function in $\Lambda_{2,\infty}^{\mathrm{Re}\,(z)+1/2}(\mathbb{R}_+)$ (hence in $H_2^{\mathrm{Re}\,(z)+1/2+\epsilon}(\mathbb{R}_+)$ for every $\epsilon > 0$). The desired result follows at once from Theorem 2.6.

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