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Unique Continuation for $|\Delta u| \leqslant V|\nabla u|$ and Related Problems

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Introduction

Much of this paper will be concerned with the proof of the following

Theorem 1. *Suppose $d \geqslant 3$, $r = \max\{d, (3d-4)/2\}$. If $V \in L^r_{\mathrm{loc}}(\mathbb{R}^d)$, then the differential inequality $|\Delta u| \leqslant V|\nabla u|$ has the strong unique continuation property in the following sense: If u belongs to the Sobolev space $W^{2,p}_{\mathrm{loc}}$ and if $|\Delta u| \leqslant V|\nabla u|$ and*

$$\lim_{R \to 0} R^{-N} \int_{|x|<R} |\nabla u|^{p'} = 0$$

for all N then u is constant.

Here we are using the notational convention

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{p} - \frac{1}{p'} = \frac{1}{r}.$$

In one sense Theorem 1 is just an ϵ improvement on a result of Y. M. Kim [10] stating the same with $r = (3d-2)/2$ ($r = d$ would be best possible; the result in [10] is itself a refinement of previous work [7], [8], [2]). However, we think it is of interest from the technical point of view. This is because of the counterexamples of Jerison and others [7], [8], [2] which show that no improvement on Kim's result can be obtained by the «Carleman method» as it is usually applied-*i.e.* as a direct consequence of a «Carleman» (weighted Sobolev) inequality.

The main point of this paper is in Section 4 where we give a variant of the Carleman method which in some circumstances lets one use the information in the Carleman inequalities more efficiently. We believe our method can be developed further, but seemingly difficult questions in real analysis come up and so far we have not been able to deal with them. See the remarks and conjecture at the end of Section 4.

In addition to this we give a partly new approach to proving the Carleman inequalities. Roughly speaking, the idea is that regardless of what weight one wants to use they just reflect properties of the Taylor remainder of the fundamental solution for Δ. So we make the main estimates with the weight $|x|^{-n}$ (the natural weight in the context of the Taylor remainder) and pass from these estimates to better estimates with respect to other weights using osculation by functions of the form $c|x-b|^{-n}$. See formula (3.1) and the proof of Proposition 3.2, and the proof of Lemma 5.1.

Here is an outline of the paper: In Section 1 we do asymptotics for the remainder term in the Taylor expansion of the fundamental solution and in Section 2 we apply this asymptotics to prove certain $L^p \to L^q$ estimates. Much of what we do in these two sections in probably not new —the methods certainly are not, although we could not find references for the actual results. In Section 3 we pass to the Carleman inequalities that we need for Theorem 1. In Section 4 we give a real variable lemma we need for our modified Carleman method, and then prove Theorem 1. The approach to Carleman inequalities in Sections 1-3 leads naturally to refinements of various known results. In Section 5 we make some observations of this type. In particular, we show how to lessen the differentiability assumptions in a result of Sogge [15] on unique continuation for variable coefficient operators.

We assume $d \geqslant 3$ throughout the paper. Theorem 1 (with $r = 2$) is known when $d = 2$. It may be derived by reading between the lines in [3] and is also a special case of [10]. We will use the notation $x \lesssim y$ to mean «x is less than or equal to a constant times y» and $x \approx y$ for «$x \lesssim y$ and $y \lesssim x$».

1. Taylor Expansion of the Fundamental Solution

Notation. $\Gamma_y(x) = \Gamma(x, y) = c_d|x - y|^{2-d}$: the solution vanishing at infinity of the equation $\Delta\Gamma_y = \delta_y$.

p_n^y: the degree $n - 1$ Taylor polynomial at the origin of the function Γ_y.

$$I_n(x,y) = |y|^{d-2}\left(\frac{|y|}{|x|}\right)^n (\Gamma(x,y) - p_n^y(x)).$$

For $x, y \in \mathbb{R}^d$ we denote

$$r = r_{xy} = \frac{|x|}{|y|} \quad \text{and} \quad \theta = \theta_{xy} = \angle_{xOy} \in [0, \pi]$$

the unoriented angle subtended by x and y at the origin.

If $n \in \mathbb{Z}^+$ it is easily seen (*cf.* [13]) that

$$|x|^{-n} f(x) = \int I_n(x,y)|y|^{-(n+d-2)} \Delta f(y)\, dy \tag{1.1}$$

for all $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. We claim the following estimates on I_n.

Proposition 1.1.

(i) $\left|\nabla_x^j\left(I_n - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^n \Gamma\right)\right| \lesssim n^{d-2+j}|y|^{-j}$ *when* $|x - y| < \frac{|y|}{2n}\cdot$

(ii) $|\nabla_x^j I_n| \lesssim n^{d-3+j} \min\{n, |1-r|^{-1}\}|x|^{-j}$ *when* $|x - y| \geqslant \frac{1}{2n}|y|$.

(iii) *We can write* $I_n(x,y) = \mathrm{Re}\,(a(r,\theta)e^{in\theta})$ *for a suitable complex valued function a satisfying*

$$\left|\frac{d^i}{dr^i}\frac{d^j}{d\theta^j}a(r,\theta)\right| \lesssim n^{d/2-2}|\sin\theta|^{1-d/2-j}(|\sin\theta| + |1-r|)^{-1-i}$$

when $|\sin\theta| \geqslant \frac{1}{2n}\cdot$

Here i and j run through $\mathbb{Z}^+ \cup \{0\}$, ∇^j means j^{th} gradient and the constants depend on d, i, j. As discussed in the introduction, it seems unlikely that Proposition 1.1 is new. We also want to note that C. Sogge's approach to unique continuation problems (*e.g.* [15], [16]) is based on related if less explicit asymptotics, and that E. Sawyer [13] had earlier used essentially (ii) of Proposition 1.1 to study unique continuation in $\mathbb{R}^3$ where the more delicate estimate (iii) is not needed.

PROOF OF (i) AND (ii). Homogeneity considerations reduce to the case where y is (say) the first standard basis element e. The Taylor expansion of Γ_e is $\sum Z_k$ where Z_k, a suitable normalization of the k^{th} zonal harmonic, satisfies $|Z_k(x)| \lesssim k^{d-3}|x|^k$ (*cf.* [21]) and therefore also $|\nabla^j Z_k| \lesssim k^{d-3+j}|x|^{k-j}$ (use that f harmonic implies $|\nabla f(x)| \lesssim r^{-1}\max\{|f(y)|: |y-a| \leqslant r\}$ with $r = |x|/k$).Thus

$$|\nabla_x^j(|x|^n(I_n - |x|^{-n}\Gamma))| = |\nabla_x^j p_n^y|$$
$$\lesssim \sum_{k=0}^{n-1} k^{d-3+j}|x|^{k-j}$$
$$\leqslant n^j|x|^{-j}\sum_{k=0}^{n-1} k^{d-3}|x|^k.$$

A calculation with the product rule gives

$$|\nabla_x^j(I_n - |x|^{-n}\Gamma)| \lesssim n^j|x|^{-j-n}\sum_{k=0}^{n-1} k^{d-3}|x|^k. \tag{1.2}$$

Statement (i) follows since $|x|^{k-j-n} \lesssim 1$ when $|x-e| \leqslant 1/2n$. We also obtain $|\nabla_x^j(I_n - |x|^{-n}\Gamma)| \lesssim n^{d-2+j}$ when $|x-e| > 1/2n$, $1 - 1/n < |x| < 1 + 1/n$. Since $|\nabla_x^j(|x|^{-n}\Gamma_e)| \lesssim n^{d-2+j}|x|^{-j}$ when $|x-e| > 1/2n$ and $|x| > 1 - 1/n$ we obtain (ii) in the region $1 - 1/n < |x| < 1 + 1/n$. When $|x| > 1 + 1/n$, (1.2) implies (ii) as follows: bound the right side of (1.2) by estimates $k^{d-3} \leqslant n^{d-3}$ and then summing a geometric series, use the triangle inequality and the bound $|\nabla_x^j(|x|^{-n}\Gamma_e)| \lesssim n^{d-2+j}|x|^{-j}$. When $|x| < 1 - 1/n$ we use instead $|x|^n I_n = \sum_{k \geq n} Z_k$ and obtain

$$|\nabla_x^j(|x|^n I_n)| \lesssim \sum_{k \geq n} k^{d-3+j}|x|^{k-j}$$
$$\lesssim n^{d-3+j}|x|^{n-j}(1-|x|)^{-1}.$$

By the product rule $|\nabla_x^j I_n| \lesssim n^{d-3+j}|x|^{-j}(1-|x|)^{-1}$, and (ii) is proved.

PROOF OF (iii). There are various methods for doing such asymptotics and we use a method based on contour integration. Similar arguments may be found in [19], p. 158, [17] and in numerous places in [20]. There are two cases depending on whether d is even or odd.

PROOF OF (iii) WHEN d IS EVEN. We write

$$c_d|y|^{d-2}|x-y|^{2-d} = c_d\left|\frac{x}{|y|} - \frac{y}{|y|}\right|^{2-d}$$
$$= c_d|(e^{-i\theta} - r)(e^{i\theta} - r)|^{1-d/2}$$
$$= f(r)$$

where $f(z) = c_d[(e^{-i\theta} - z)(e^{i\theta} - z)]^{1-d/2}$ is analytic except for poles at $e^{\pm i\theta}$. It follows that

$$I_n(x,y) = r^{-n}R_n(r)$$

where $R_n = f - p_n$ and p_n is the degree $n-1$ Taylor polynomial for f at zero. If $|z|$ is small then by elementary complex variables

$$\begin{aligned} z^{-n}R_n(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1/2} \zeta^{-n}f(\zeta)(\zeta - z)^{-1}\,d\zeta \\ &= -\Big(\operatorname*{Res}_{e^{i\theta}} + \operatorname*{Res}_{e^{-i\theta}}\Big)(\zeta^{-n}f(\zeta)(\zeta - z)^{-1}) \\ &\quad + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{|\zeta|=R} \zeta^{-n}f(\zeta)(\zeta - z)^{-1}\,d\zeta \end{aligned}$$

$$z^{-n}R_n(z) = -\Big(\operatorname*{Res}_{\zeta=e^{i\theta}} + \operatorname*{Res}_{\zeta=e^{-i\theta}}\Big)(\zeta^{-n}f(\zeta)(\zeta - z)^{-1}) \tag{1.3}$$

since the limit term is zero. By analytic continuation (1.3) is valid for all z. The residues may be evaluated by taking $d/2 - 2$ derivatives. Carrying this out with $z = r$,

$$I_n(x, y) = \operatorname{Re}(e^{in\theta}a(r, \theta))$$

where $a(r, \theta)$ is a finite sum of terms of the form

$$a_{klm}(\sin\theta)^{1-d/2-k}(e^{-i\theta} - r)^{-1-l}e^{\operatorname{Im}\theta},$$

a_{klm} being constants with $|a_{klm}| \approx n^m$, and $k + 1 + m = d/2 - 2$. If (as we are assuming) $n|\sin\theta|$ is bounded away from zero, the main term is the term with $m = d/2 - 2$ and we obtain

$$\begin{aligned} |a(r,\theta)| &\lesssim n^{d/2-2}|\sin\theta|^{1-d/2}|e^{-i\theta} - r|^{-1} \\ &\approx n^{d/2-2}|\sin\theta|^{1-d/2}(|\sin\theta| + |1-r|)^{-1}. \end{aligned}$$

Similarly we can estimate derivatives $\frac{d^i}{dr^i}\frac{d^j}{d\theta^j}a(r,\theta)$. Each r-derivative produces a factor of $(e^{-i\theta} - r)^{-1}$ while each θ derivative produces at worst a factor of $|\sin\theta|^{-1}$. Estimate (iii) follows.

PROOF OF (iii) WHEN d IS ODD. The reason things are a bit more complicated here is of course that $[(z - e^{i\theta})(z - e^{-i\theta})]^{1-d/2}$ is multivalued. We fix $\theta \in (0, \pi)$ and let θ_1 and θ_2 be sufficiently close to $-\theta$ and θ respectively. Let $\gamma = \{e^{it}: \theta_1 \leqslant t \leqslant \theta_2\}$ and let $f_{\theta_1\theta_2}$ be the branch of $[(z - e^{i\theta_1})(z - e^{i\theta_2})]^{-1/2}$ defined on $\mathbb{C}\backslash\gamma$ and with $f_{\theta_1\theta_2}(0)$ close to 1. We will write f instead of $f_{\theta_1\theta_2}$ when no confusion will result. With $q = (d-3)/2$, and C_1 a suitable constant,

$$C_1\frac{d^q}{d\theta_1^q}\frac{d^q f}{d\theta_2^q}\bigg|_{\theta_1=-\theta,\,\theta_2=\theta}$$

gives a branch of the function $[(z - e^{i\theta})(z - e^{-i\theta})]^{1-d/2}$. Any such branch changes sign across γ and therefore agrees with $|e^{i\theta} - r|^{2-d}$ for $r < 1$ and with $-|e^{i\theta} - r|^{2-d}$ for $r > 1$, or viceversa. That means $|y|^{d-2}(\Gamma_y(x) - p_n^y(x))$ has the form

$$C_2 \frac{d^q}{d\theta_1^q} \frac{d^q}{d\theta_2^q} R_n(r)$$

where C_2 is a suitable constant, $f = f_{\theta_1 \theta_2}$ is as above, p_n is the degree $n - 1$ Taylor polynomial of f at $z = 0$, and

$$R_n(r) = \begin{cases} f(r) - p_n(r) & \text{when } \quad 0 < r < 1 \\ -f(r) - p_n(r) & \text{when } \quad r > 1. \end{cases}$$

Therefore

$$I_n(x,y) = C_2 \frac{d^q}{d\theta_1^q} \frac{d^q}{d\theta_2^q} (r^{-n} R_n(r)) \Big|_{\theta_1 = -\theta, \theta_2 = \theta}.$$

We now rewrite $r^{-n}R_n(r)$ using contour integration.
Denote

$$\int_{\gamma-} f\,dz = \lim_{r \uparrow 1} \int_\gamma f\,dz$$

$$\int_{\gamma+} f\,dz = \lim_{r \downarrow 1} \int_\gamma f\,dz$$

Let

$$T_1 = \{re^{i\theta_1}: 1 < r < \infty\}$$

oriented with r decreasing and

$$T_2 = \{re^{i\theta_2}: 1 < r < \infty\}$$

oriented with r increasing. With γ oriented counterclockwise, and $|z|$ small,

$$z^{-n}(f(z) - p_n(z)) = \frac{1}{2\pi i} \int_{|\zeta| = 1/2} \zeta^{-n}(\zeta - z)^{-1} f(\zeta)\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{\gamma - \gamma +} \zeta^{-n}(\zeta - z)^{-1} f(\zeta)\, d\zeta$$

$$z^{-n}(f(z) - p_n(z)) = -\frac{1}{\pi i} \int_{\gamma+} \zeta^{-n}(\zeta - r)^{-1} f(\zeta)\, d\zeta \tag{1.4}$$

since f changes sign across γ. By analytic continuation (1.4) is valid for $z \in \mathbb{C}\backslash\gamma$. Now integration over the countour $T_1\gamma_+ T_2$ implies

$$r^{-n}(f(r) - p_n(r)) = \begin{cases} \dfrac{1}{\pi i}\displaystyle\int_{T_1+T_2} \zeta^{-n}(\zeta - r)^{-1}f(\zeta)\,d\zeta & (0 < r < 1) \\ 2r^{-n}f(r) + \dfrac{1}{\pi i}\displaystyle\int_{T_1+T_2} \zeta^{-n}(\zeta - r)^{-1}f(\zeta)\,d\zeta & (r > 1) \end{cases}$$

where the $2r^{-n}f(r)$ term comes from the residue at $\zeta = r$. In either case,

$$r^{-n}R_n(r) = \frac{1}{\pi i}\int_{T_1+T_2} \zeta^{-n}(\zeta - r)^{-1}f(\zeta)\,d\zeta$$

i.e.

$$r^{-n}R_n(r) = \frac{1}{\pi i}\left\{e^{-i(n-1)\theta_2}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1}f_{\theta_1\theta_2}(e^{i\theta_2}t)\,dt - e^{-i(n-1)\theta_1}\int_1^\infty t^{-n}(e^{i\theta_1}t - r)^{-1}f_{\theta_1\theta_2}(e^{i\theta_1}t)\,dt\right\}. \tag{1.5}$$

Now define the quantities

$$I_{klm} = \frac{d^k}{dr^k}\frac{d^l}{d\theta_1^l}\frac{d^m}{d\theta_2^m}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1}f(e^{i\theta_2}t)\,dt$$

It is clear that I_{kl0} has the following form:

$$I_{kl0} = C_{kl}e^{il\theta_1}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1-k}(e^{i\theta_2}t - e^{i\theta_1})^{-l}f(e^{i\theta_2}t)\,dt.$$

An induction on m shows that I_{klm} has the following form:

$$I_{klm} = \sum_{i+j\le m}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1-k-i}(e^{i\theta_2}t - e^{i\theta_1})^{-l-j}f(e^{i\theta_2}t)p_{ijklm}(t, \theta_1, \theta_2)\,dt,$$

p_{ijklm} being polynomials in t of degree $\leqslant i+j$ with coefficients which are smooth functions of θ_1, θ_2 and independent of n. We claim that

$$|I_{klm}(r, -\theta, \theta)| \lesssim n^{-1/2}|\sin\theta|^{-1/2-l-m}(|\sin\theta| + |1 - r|)^{-1-k}. \tag{1.6}$$

PROOF OF (1.6). Clearly

$$|I_{klm}(r, -\theta, \theta)| \lesssim \sum_{i+j\le m}\int_1^\infty t^{-n}|e^{i\theta}t - r|^{-1-k-i}(|t - 1| + |\sin\theta|)^{-1/2-l-j}(t - 1)^{-1/2}t^{i+j}\,dt.$$

We can estimate $t \lesssim |\sin\theta|^{-1}|e^{i\theta}t - r|$ and $t \lesssim |\sin\theta|^{-1}((t-1) + |\sin\theta|)$, and therefore

$$|I_{klm}(r, -\theta, \theta)| \lesssim$$

$$\sum_{i+j\leq m} |\sin\theta|^{-i-j} \int_1^\infty t^{-n}|e^{i\theta}t - r|^{-1-k}(t - 1 + |\sin\theta|)^{-1/2-l}(t-1)^{-1/2}\,dt$$

$$\lesssim |\sin\theta|^{-1/2-l-m} \int_1^\infty t^{-n}(t-1)^{-1/2}|e^{i\theta}t - r|^{-1-k}\,dt. \tag{1.7}$$

If $r \leqslant 1 + |\sin\theta|$ we can estimate $|e^{i\theta}t - r| \gtrsim |\sin\theta| + |1 - r|$, and (1.6) follows since

$$\int_1^\infty (t-1)^{-1/2}t^{-n}\,dt \lesssim n^{-1/2}.$$

If $r \geqslant 1 + |\sin\theta|$ we split the integral in (1.7) into $\int_1^{(1+r)/2}$ and $\int_{(1+r)/2}^\infty$. When $t < (1+r)/2$ we have $|e^{i\theta}t - r| \gtrsim |\sin\theta| + |1 - r|$ and can argue as before. On the other hand

$$\int_{(1+r)/2}^\infty t^{-n}|e^{i\theta}t - r|^{-1-k}(t-1)^{-1/2}\,dt$$

$$\lesssim (r|\sin\theta|)^{-1-k} \int_{(1+r)/2}^\infty t^{-n}(t-1)^{-1/2}\,dt$$

$$\lesssim (r|\sin\theta|)^{-1-k} n^{-1/2}\left[\frac{1+r}{2}\right]^{1-n}$$

$$= (r-1)^{-1-k}\left[\frac{r-1}{r|\sin\theta|}\right]^{1+k} n^{-1/2}\left[\frac{1+r}{2}\right]^{1-n}$$

$$\lesssim n^{-1/2}(r-1)^{-1-k}$$

$$\approx n^{-1/2}(|1-r| + |\sin\theta|)^{-1-k}$$

and (1.6) follows; we used $r - 1 \geqslant |\sin\theta| \geqslant 1/2n$ and to derive the next to last line, the fact that $(tx)^\alpha(1+x)^{-t}$ is uniformly bounded over $t \geqslant 0$ and $0 \leqslant x \leqslant 1$ for any fixed $\alpha > 0$.

A term

$$e^{i(n-1)\theta} \frac{d^q}{d\theta_1^q}\frac{d^q}{d\theta_2^q}\bigg|_{\substack{\theta_1 = -\theta \\ \theta_2 = \theta}} e^{-i(n-1)\theta_2} \int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1} f(e^{i\theta_2}t)\,dt \tag{1.8}$$

is a sum at terms $(-i(n-1))^p I_{0qm}(r, -\theta, \theta)$ with $m + p = q$. The result of taking $ir-$ and $j\theta-$ derivatives of a term (1.8) is a sum of bounded constants times terms $(-in)^p I_{i,q+t,m+s}(-\theta,\theta)$ with $s + t = j$ and is therefore

$$\lesssim \sum_{m+p=q} \sum_{t+s=j} n^{p-1/2} |\sin\theta|^{-(m+q+t+s+1/2)} |e^{i\theta} - r|^{-1-i}$$

For $|\sin\theta| \gtrsim 1/n$ the worst terms here are the terms with $p = q$ and we conclude that $d^{i+j}/dr^i\, d\theta^j$ of a term (1.8) is

$$\lesssim n^{d/2-2} |\sin\theta|^{-(q+j+1/2)} |e^{i\theta} - r|^{-1-i}.$$

This and a similar estimate for the contribution from the second term in (1.5) prove (iii), so we are done with Proposition 1.1.

Now we let α be a multiindex. We derive an expression like (1.1) for $|x|^{-n} D^\alpha f(x)$. If α and β are multiindices then $\beta \leqslant \alpha$ means that $\beta_j \leqslant \alpha_j$ for each $j \in \{1, \ldots, d\}$.

Lemma 1.1. *If $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ then*

(1.9) $|x|^{-n} D^\alpha f(x)$

$$= \sum_{0 \le \beta \le \alpha} n^{|\beta|} u_{\alpha\beta n}(x) |x|^{|\alpha-\beta|} \left[\int D_x^{\alpha-\beta} \left(I_{n+|\alpha|}(x,y) - |y|^{d-2} \left(\frac{|y|}{|x|} \right)^{n+|\alpha|} \Gamma(x,y) \right) \right.$$

$$\left. \cdot |y|^{-(n+d-2+|\alpha|)} \Delta f(y)\, dy + D^{\alpha-\beta} \int \Gamma(x,y) |x|^{-(n+|\alpha|)} \Delta f(y)\, dy \right]$$

where $u_{\alpha\beta n}$ are fixed (i.e. independent of f) functions homogeneous of degree zero and smooth on the unit sphere with bounds independent of n, and $u_{\alpha 0 n} = 1$.

Remarks. (1) The convergence of the first set of integrals follows from Proposition 1.1(i).

(2) We are mainly interested in the case $|\alpha| \leqslant 2$, and in this case (1.9) may be written in simplified form. If $|\alpha - \beta| = 1$ one can differentiate under the integral sign in the second integral. If $|\alpha - \beta| = 2$ and $D^{\alpha-\beta}$ is a mixed partial one can still do this provided one interprets the resulting integral as a principal value. If $D^{\alpha-\beta} = \dfrac{d^2}{dx_j^2}$ then one obtains an extra δ-function term. See *e.g.* [5] p. 99. We obtain the following:

Corollary 1.1. *If $|\alpha| \leqslant 1$, or if $|\alpha| = 2$ and $D^\alpha = \dfrac{d^2}{dx_i\, dx_j}$ with $i \neq j$, then*

(1.10) $|x|^{-n}D^\alpha f(x)$

$$= \sum_{0\le\beta\le\alpha} n^\beta u_{\alpha\beta n}(x)|x|^{|\alpha-\beta|}\int D_x^{\alpha-\beta}I_{n+|\alpha|}(x,y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)\,dy$$

where we interpret the integral as a principal value if $|\alpha-\beta| = 2$. *The formula remains valid when*

$$D^\alpha = \frac{d^2}{dx_j^2},$$

provided the left hand side is replaced by $|x|^{-n}\left(D^\alpha f(x) - \frac{1}{d}\Delta f(x)\right)$.

PROOF OF LEMMA 1.1. We use induction on $|\alpha|$, and when $\alpha = 0$ it reduces to (1.1). Suppose it is proved for multiindices of length less than $|\alpha|$. Write down (1.1) with $n+|\alpha|$ instead of n and take the D^α derivative using the product rule:

$$|x|^{-(n+|\alpha|)}f(x) = \int I_{n+|\alpha|}(x,y)|y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy,$$

$$\sum_{0\le\beta\le\alpha} n^{|\beta|}v_{\alpha\beta n}(x)|x|^{-(n+|\alpha|+|\beta|)}D^{\alpha-\beta}f(x)$$

$$= \int D_x^\alpha\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right)|y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy$$

$$+ D^\alpha\int \Gamma(x,y)|x|^{-(n+|\alpha|)}\Delta f(y)\,dy$$

where the $v_{\alpha\beta n}$ satisfy the same conditions as the $u_{\alpha\beta n}$ and $v_{\alpha 0n} = 1$. As mentioned above, the differentiation under the integral sign is justified because of Proposition 1.1(i). Isolate the $\beta = 0$ term on the left hand side and multiply by $|x|^{|\alpha|}$:

$$(1.11)\quad \frac{D^\alpha f(x)}{|x|^n} = -\sum_{0<\beta\le\alpha} n^{|\beta|}v_{\alpha\beta n}(x)|x|^{-(n+|\beta|)}D^{\alpha-\beta}f(x)$$

$$+ |x|^{|\alpha|}\left[\int D_x^\alpha\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right)\right.$$

$$\cdot |y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy$$

$$\left.+ D^\alpha\int \Gamma(x,y)|x|^{-(n+|\alpha|)}\Delta f(y)\,dy\right].$$

The $|x|^{|\alpha|}[\ ]$ term has the form of the $\beta = 0$ term in (1.9). The quantity $|x|^{-(n+|\beta|)} D^{\alpha-\beta} f(x)$ may be evaluated using the inductive hypothesis with α replaced by $\alpha - \beta$ and n by $n + |\beta|$. The sum over β becomes

$$\sum_{0<\beta\leq\alpha} \sum_{0\leq\gamma\leq\alpha-\beta} n^{|\beta|} v_{\alpha\beta n}(x) n^{|\gamma|} u_{\alpha-\beta,\gamma,n+|\beta|}(x) |x|^{|\alpha-\beta-\gamma|}$$

$$\left[\int D_x^{\alpha-\beta-\gamma}\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|} \Gamma(x,y)\right)|y|^{-(n+|\alpha|+d-2)} \Delta f(y)\, dy\right.$$

$$\left. + D^{\alpha-\beta-\gamma} \int \Gamma(x,y)|x|^{-(n+|\alpha|)} \Delta f(y)\, dy\right].$$

Let

$$\sigma = \beta + \gamma, \qquad u_{\alpha\sigma n} = -\sum_{\beta+\gamma=\sigma} v_{\alpha\beta n} u_{\alpha-\beta,\gamma,n+|\beta|},$$

then the above expression is

$$\sum_{0<\sigma\leq\alpha} n^{|\sigma|} u_{\alpha\sigma n}(x)|x|^{|\alpha-\sigma|}\left[\int D_x^{\alpha-\sigma}\left(I_{n+|\alpha|} - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|} \Gamma(x,y)\right)\right.$$

$$\cdot |y|^{-(n+|\alpha|+d-2)} \Delta f(y)\, dy$$

$$\left. + D^{\alpha-\sigma} \int \Gamma(x,y)|x|^{-(n+|\alpha|)} \Delta f(y)\, dy\right]$$

and we obtain (1.9).

Lemma 1.2. *Let* $A(x,y)$ *be any term* $n^{|\beta|} u_{\alpha\beta n}(x)|x|^{|\alpha-\beta|} D_x^{\alpha-\beta} I_{n+|\alpha|}(x,y)$, $0 \leqslant \beta \leqslant \alpha$, $u_{\alpha\beta n}$ *as above.*

(i) $\left|A(x,y) - n^{|\beta|} u_{\alpha\beta n}(x)|x|^{|\alpha-\beta|} D_x^{\alpha-\beta} |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|} \Gamma(x,y)\right| \lesssim n^{d-2+|\alpha|}$

when $|x-y| < \dfrac{1}{2n}|y|$.

(ii) $|A(x,y)| \lesssim n^{d-3+|\alpha|} \min\{n, |1-r|^{-1}\}$ *when* $|x-y| > \dfrac{1}{2n}|y|$.

(iii) $A(x,y) = \operatorname{Re}(e^{in\theta} q(x,y))$ *where the amplitude* q *satisfies*

$$|D_y^\tau D_x^\sigma q| \lesssim n^{d/2-2+|\alpha|}|y|^{-|\tau|}|x|^{-|\sigma|}|\sin\theta|^{1-d/2-|\sigma|-|\tau|}(|\sin\theta| + |1-r|)^{-1}$$

when $|\sin\theta| > \dfrac{1}{2n}$.

PROOF. Parts (i) and (ii) follow immediately from the corresponding parts of Proposition 1.1. Part (iii) also follows this way but there is some calculation involved, which we sketch for the reader's convenience. First consider any function $b(r,\theta)$. Denote by δ^k any derivative of the form $\left(r\frac{d}{dr}\right)^i\left(\frac{d}{d\theta}\right)^{k-i}$. Then for any σ and τ, $D_y^\tau D_x^\sigma b(r,\theta)$ is a sum of terms of the form

$$|x|^{-|\sigma|}|y|^{-|\tau|}u(x,y)\delta^k b$$

where $k \leqslant |\sigma| + |\tau|$ and u denotes any fixed (*i.e.* independent of b) function homogeneous of degree zero in each variable ($u(\lambda x, \mu y) = u(x,y)$) and smooth on $S^{d-1} \times S^{d-1}$. This may be proved easily by induction on $|\sigma| + |\tau|$. Consider now the expression for $I_{n+|\alpha|}$ in (iii) of Proposition 1.1. Using the product rule to calculate $\delta^k(e^{i(n+|\alpha|)\theta}a)$, we may write $D_x^{\alpha-\beta}I_{n+|\alpha|}$ as a sum of terms

$$\mathrm{Re}\left(\frac{u(x,y)}{|x|^{|\alpha-\beta|}}e^{i(n+|\alpha|)\theta}n^l\delta^k a\right)$$

with $k+1 \leqslant |\alpha-\beta|$ and u as above. Absorbing $e^{i|\alpha|\theta}$ as well as relevant factors $u_{\alpha\beta n}$ into $u(x,y)$, we may therefore write $A(x,y)$ as a sum of terms

$$\mathrm{Re}\,(u(x,y)e^{in\theta}n^l\delta^k a)$$

where now $k + l \leqslant |\alpha|$.

I.e. $q(x,y)$ is a sum of terms $n^l u(x,y)\delta^k a$. If we take $D_y^\tau D_x^\sigma$ of such a term we obtain terms of the form

$$n^l|y|^{-|\tau|}|x|^{-|\sigma|}u(x,y)\delta^{k+m}a$$

where $k + l \leqslant |\alpha|$, $m \leqslant |\sigma| + |\tau|$. Proposition 1.1 implies (after a calculation) that such a term is

$$\lesssim n^{l+d/2-2}|\sin\theta|^{1-d/2-(k+m)}(|\sin\theta| + |1-r|)^{-1}|y|^{-|\tau|}|x|^{-|\sigma|}.$$

The worst terms here are the terms with $k = 0$, $l = |\alpha|$, $m = |\sigma| + |\tau|$, and we obtain (iii).

Proposition 1.2. *When* $|\alpha| \leqslant 1$, *or* $|\alpha| = 2$ *and* $D^\alpha \neq \frac{d^2}{dx_j^2}$ *we have*

$$|x|^{-n}D^\alpha f(x) = \int I_n^{(\alpha)}(x,y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)\,dy$$

where $I_n^{(\alpha)}$ *satisfies* (i)-(iii) *below. When* $D^\alpha = \frac{d^2}{dx_j^2}$ *we have*

$$|x|^{-n}\left(D^\alpha f(x) - \frac{1}{d}\Delta f(x)\right) = \int I_n^{(\alpha)}(x,y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)\,dy$$

where $I_n^{(\alpha)}$ *satisfies* (i)-(iii).

(i) *If* $|x-y| < \dfrac{|y|}{2n}$ *then*

$$|I_n^{(\alpha)}(x,y)| \lesssim |y|^{d-2+|\alpha|}|x-y|^{-(d-2+|\alpha|)}$$

when $|\alpha| \leqslant 1$, *and*

$$|I_n^{(\alpha)}(x,y) - |y|^d D_x^\alpha(\Gamma(x,y))| \lesssim n|y|^{d-1}|x-y|^{-(d-1)}$$

when $|\alpha| = 2$.

(ii) *If* $|x-y| \geqslant \dfrac{|y|}{2n}$ *then*

$$|I_n^{(\alpha)}(x,y)| \lesssim n^{d-3+|\alpha|}\min\left\{n, \left|1-\frac{|x|}{|y|}\right|^{-1}\right\}.$$

(iii) *Choose a coordinate system on the unit sphere* S^{d-1}, *let* e *and* f *be variables on* S^{d-1}, D_e^σ, D_f^τ *denote differentiation in the given coordinate system. Then for* $|\sin\theta| > \dfrac{1}{2n}$,

$$I_n^{(\alpha)}(x,y) = \mathrm{Re}\,(e^{in\theta}q(x,y))$$

where

$$|D_e^\sigma D_f^\tau I_n^{(\alpha)}(se,tf)| \lesssim n^{|\alpha|+d/2-2}|\sin\theta|^{1-d/2-|\sigma|-|\tau|}\left(|\sin\theta| + \left|1-\frac{s}{t}\right|\right)^{-1}.$$

Remarks. (1) The form of the estimates in (iii) (the fact that the right hand side increases with $|\sigma|$ and $|\tau|$) shows they are independent of the coordinate system.

(2) A similar expression of the form

$$|x|^{-n}(D^\alpha f(x) - p_\alpha(D)\Delta f(x)) = \int I_n^{(\alpha)}(x,y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)$$

could be given for higher derivatives.

(3) The kernel $I_n^{(\alpha)}$ is of course obtained from Corollary 1.1. Thus it is a sum of terms of the type in Lemma 1.2. Part (ii) of Proposition 1.2 then follows immediately from (ii) of Lemma 1.2. Parts (i) and (iii) follow the same way after some manipulations with the product rule and (for (iii)) to compare derivatives in $\mathbb{R}^d$ with derivatives on the sphere. We leave them to the reader.

2. Estimates from Spheres to Spheres

Notation. If K is a function on $\mathbb{R}^d \times \mathbb{R}^d$, $s, t > 0$ then K^{st} is the function on $S^{d-1} \times S^{d-1}$ defined by $K^{st}(e, f) = K(se, tf)$.

We fix a coordinate system S^{d-1} and sufficiently large constants $\{C_j\}$ and, for $\lambda \in (0, 1)$, denote by χ_λ any function from $S^{d-1} \times S^{d-1}$ to $[0, 1]$ satisfing the following: $\chi_\lambda(x, y) = 0$ if $|\sin \theta(x, y)| < \lambda/100$ or $|\sin \theta(x, y)| > 100\lambda$, and $|D^\alpha \chi_\lambda| \leqslant C_{|\alpha|} \lambda^{-|\alpha|}$. We denote by χ^λ any function from $S^{d-1} \times S^{d-1}$ to $[0, 1]$ satisfying: $\chi^\lambda(x, y) = 0$ if $|\sin \theta(x, y)| > 100\lambda$, and $|D^\alpha \chi^\lambda| \leqslant C_{|\alpha|} \lambda^{-|\alpha|}$. That is, χ_λ is smooth cutoff to $|\sin \theta| \approx \lambda$ and χ^λ is a smooth cutoff to $|\sin \theta| \lesssim \lambda$.

We will identify a kernel with the operator it induces, and denote the norm of an operator acting from $L^p(X, \mu)$ to $L^p(Y, \nu)$ by $\| T \|_{L^p(X, \mu) \to L^q(Y, \mu)}$, or $\| T \|_{p \to q}$ if there is no confusion.

Proposition 2.1. *Let ψ be a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$ with $\psi(x, y) = 1$ when $|x - y| > \frac{1}{n}|y|$, $\psi(x, y) = 0$ when $|x - y| < \frac{1}{2n}|y|$, and $|\nabla^k \psi| \lesssim \left(\frac{|y|}{n}\right)^{-k}$ Let $I_n^{(\alpha)}$ be as in Proposition 1.2 and $K_n^{(\alpha)} = \psi I_n^{(\alpha)}$. Then for $1 \leqslant p \leqslant 2$, $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$, $\lambda > \frac{1}{200n}$, $\chi_\lambda K_n^{(\alpha)s,t}$ maps $L^p(S^{d-1})$ to $L^{p'}(S^{d-1})$ with norm $\lesssim n^{d/2r - 1/r - 1 + |\alpha|} \lambda^{-d/2r + 1} \left(\left| 1 - \frac{s}{t} \right| + \lambda \right)^{-1}$. If in addition $\lambda \leqslant \frac{200}{n}$ then χ_λ then χ_λ may be replaced by χ^λ here.*

Remarks. (1) This will be proved by applying oscillating integral lemmas to the asymptotics in Proposition 1.2 (iii). This type of argument is very standard by now and has been used in closely related contexts by C. Sogge [15, 16]. We want to point out that only the most simpleminded mapping properties of oscillating integrals are used in our version, namely the «variable coefficient Plancherel» of Hormander [6] (or see [18], p. 347).

(2) In proving Theorem 1, we use only the case $|\alpha| = 1$. The other cases are used in Section 5.

Lemma 2.1. *Suppose $1 < p, q < \infty$, (X, μ), (Y, ν), (Z, σ), (W, τ) are measure spaces, $\{T_{wy}\}_{y \in Y, w \in W}$ is a measurable family of operators from $L^p(X, \mu)$ to $L^q(Z, \nu)$ and the kernel*

$$n(w, y) = \| T_{wy} \|_{L^p(X, \mu) \to L^q(Z, \sigma)}$$

defines a bounded operator $f \to \int n(w, y) f(y)\, d\nu(y)$ from $L^p(Y, \nu)$ to $L^q(W, \tau)$ with norm N. For $f\colon X \times Y \to \mathbb{C}$ define $f_y(x) = f(x, y)$ etc. Then T defined by

$$(Tf)_w = \int T_{wy}(f_y)\,d\nu(y)$$

is a bounded operator from $L^p(X \times Y, \mu \times \nu) \to L^q(Z \times W, \sigma \times \tau)$ with norm $\leqslant N$.

PROOF.
$$\begin{aligned}\|Tf\|_q &= \left(\int \left\| \int T_{wy} f_y \,d\nu(y)\right\|^q_{L^q(Z)} d\tau(w)\right)^{1/q}\\ &\leqslant \left(\int \left(\int \|T_{wy} f_y\|_{L^q(Z)}\,d\nu(y)\right)^q d\tau(w)\right)^{1/q}\\ &\leqslant \left(\int \left(\int n(w,y)\|f_y\|_{L^p(X)}\,d\nu(y)\right)^q d\tau(w)\right)^{1/q}\\ &\leqslant N\left(\int \|f_y\|^p_{L^p(X)}\,d\nu(y)\right)^{1/p}\\ &= N\|f\|_p.\end{aligned}$$

Lemma 2.2. *Suppose $1 \leqslant p \leqslant 2$, $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$, (X, μ) and (Y, ν) are measure spaces, $K: X \times Y \to \mathbb{C}$ and $u: X \to \mathbb{R}^+$, $v: Y \to \mathbb{R}^+$. Define*

$$A = \sup_x \|(u(x)v(y))^{-1/p'}K(x,y)\|_{L^{r'}(Y,\, v(y)\nu)}$$

$$B = \sup_y \|(u(x)v(y))^{-1/p'}K(x,y)\|_{L^{r'}(X,\, u(x)\mu)}$$

Then the norm of $f \to \int K(x,y) f(y)\,d\nu(y)$ as an operator from $L^p(Y,\nu)$ to $L^{p'}(X,\mu)$ is $\leqslant (AB)^{1/2}$.

PROOF. If $u = v = 1$ this follows by interpolation: the norm of K from L^1 to $L^{r'}$ is $\leqslant B$ and the norm from L^r to L^∞ is $\leqslant A$. The general case may be reduced to the case $u = v = 1$ by observing that the norm of the operator in question is the same as the norm from $L^p(Y, v\nu)$ to $L^{p'}(X, u\mu)$ of the operator

$$f \to \int u(x)^{-1/p'} K(x,y) v(y)^{1/p} f(y)\,d\nu(y)$$

$$\left(= f \to \int u(x)^{-1/p'} K(x,y) v(y)^{-1/p'} f(y)\,d(v\nu)(y)\right).$$

Lemma 2.3. *Suppose A, B are dics in $\mathbb{R}^{d-1}$ with radii δ, ϵ respectively, $\delta \leqslant \epsilon$. Suppose $\theta: A \times B \to \mathbb{R}$ is C^∞, $a \in C_0^\infty(A \times B)$. Assume that on* supp a *we have $|D_x^\alpha D_y^\beta \theta| \leqslant C_{\alpha\beta}\epsilon^{1-|\beta|}\delta^{-|\alpha|}$ and at each point the matrix $\nabla_x \nabla_y \theta$ has at least m eigenvalues with magnitude $\geqslant C^{-1}\delta^{-1}$. Furthermore suppose $|D_x^\alpha D_y^\beta a| \leqslant C_{\alpha\beta}\epsilon^{-|\beta|}\delta^{-|\alpha|}$. Then the kernel $a(x,y)e^{in\theta(x,y)}$ is $L^p(dy) \to L^{p'}(dx)$ bounded with norm $\lesssim (\delta\epsilon)^{(d-1)/p'}(n\epsilon)^{-m/p'}$ for $1 \leqslant p \leqslant 2$, where the implicit constant depends on $d, C, \{C_{\alpha\beta}\}$.*

PROOF. It is enough to do the $p = 2$ case since the $p = 1$ case is easy and the rest follows by interpolation. We can assume A, B centered at zero. Consider instead the scaled kernel $\tilde{K}(x,y) = a(\delta x, \epsilon y)e^{in\theta(\delta x, \epsilon y)}$ on $D(0,1) \times D(0,1)$ and write it as $\tilde{a}(x,y)e^{i\tilde{n}\tilde{\theta}(x,y)}$ where $\tilde{n} = n\epsilon$, $\tilde{a}(x,y) = a(\delta x, \epsilon y)$ and $\tilde{\theta}(x,y) = \epsilon^{-1}\theta(\delta x, \epsilon y)$. This reduces us to the case $\delta = \epsilon = 1$. If $m = d - 1$ we would now be done by [18], p. 347. In general, we can assume by a partition of unity and linear change of coordinates that all eigenvalues of $\nabla_{\bar{x}}\nabla_{\bar{y}}\tilde{\theta}$ are $\gtrsim 1$, where we let $\bar{x}$ (respectively, $\bar{y}$) denote the first m coordinates of $x(y)$ and $\bar{\bar{x}}(\bar{\bar{y}})$ be the last $d - 1 - m$ coordinates. If we fix $\bar{\bar{x}}, \bar{\bar{y}}$ and let $K_{\bar{\bar{x}}\bar{\bar{y}}}$ be the operator $f \to \int e^{i\tilde{n}\tilde{\theta}(x,y)}\tilde{a}(x,y)f(y)\,d\bar{y}$ acting from $L^2(\mathbb{R}^m, d\bar{y})$ to $L^2(\mathbb{R}^m, d\bar{x})$ we have a norm bound $\tilde{n}^{-m/2}$ by [18], p. 347. By (for example) Lemma 2.1 the norm of $\tilde{K}$ is also $\lesssim \tilde{n}^{-m/2}$ and the lemma follows.

Lemma 2.4. *Suppose* $\theta: \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$, $a \in C_0^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$. *Assume* $|x - y| \approx \rho$ *for all* $x, y \in \operatorname{supp} a$, *and on* supp a, *we have* $|D_x^\alpha D_y^\beta \theta| \lesssim C_{\alpha\beta}\rho^{1-|\alpha|-|\beta|}$, *and* $\nabla_x\nabla_y\theta$ *has at least m eigenvalues with magnitude* $\geqslant (C\rho)^{-1}$. *Assume moreover that* $|\nabla_x^\alpha \nabla_y^\beta a| \leqslant C_{\alpha\beta}\rho^{-|\alpha|-|\beta|}$. *Then* $a(x,y)e^{in\theta(x,y)}$ *is* $L^p \to L^{p'}$ *bounded with norm* $\lesssim \rho^{2(d-1)/p'}(n\rho)^{-m/p'}$, $1 \leqslant p \leqslant 2$.

PROOF. Again need only be done when $p = 2$. Let T be the operator in question. Let $\{q_j\}$ be a partition of unity subordinate to a covering by discs of radius ρ and with each point belonging to a bounded number of them. Let T_j be the operator with kernel $q_j(x)\, a(x,y)e^{in\theta(x,y)}$. For each j there are a bounded number of k such that $T_j^*T_k$ or $T_jT_k^*$ or $T_k^*T_j$ or $T_kT_j^*$ is nonzero and it follows (*e.g.* Cotlar's lemma) that $\|T\| \lesssim \sup_j \|T_j\|$. On the other hand T_j satisfies the hypothesis of Lemma 2.3 ($\delta = \epsilon = \text{const} \cdot \rho$)-the result follows.

PROOF OF PROPOSITION 2.1. We will assume $|\alpha| = 1$ to simplify the notation. This is no loss of generality because both estimates (ii) and (iii) in Proposition 1.2 (which are the basis of the proof) depend on α through the factor $n^{|\alpha|}$ and the same is true of the estimate in Proposition 2.1. We then fix α and drop the α superscripts, *e.g.* we write K_n^{st} for $K_n^{(\alpha)st}$. We also define

$$D(n,\lambda) = n^{d/2r - 1/r}\lambda^{-d/2r+1}. \tag{2.1}$$

There are two cases $\frac{1}{200n} \leqslant \lambda \leqslant \frac{200}{n}$ and $\lambda > \frac{200}{n}$. In the first case we use Proposition 1.2(ii) to conclude that

$$\|\chi^\lambda K_n^{st}\|_\infty \leqslant n^{d-2}\min\left\{\left|1 - \frac{s}{t}\right|^{-1}, n\right\} \approx n^{d-2}\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}.$$

For fixed e, the set $\{f \in S^{d-1} : \chi^\lambda K_n(e,f) \neq 0\}$ has measure $\approx n^{-(d-1)}$, so

$$\sup_e \|\chi^\lambda K_n^{st}\|_{L^{r'}(df)} \lesssim n^{d-2-(d-1)/r'}\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}.$$

Likewise

$$\sup_f \|\chi^\lambda K_n^{st}\|_{L^{r'}(de)} \lesssim n^{d-2-(d-1)/r'}\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}$$

so we get (*e.g.* by Lemma 2.2)

$$\begin{aligned}\|\chi^\lambda K_n^{st}\|_{p\to p'} &\lesssim n^{d-2-(d-1)/r'}\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}\\ &= D\left(n, \frac{1}{n}\right)\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}\\ &\approx D(n,\lambda)\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}\end{aligned}$$

as claimed. For the case $\lambda > 200/n$, we first observe following *e.g.* C. Sogge [14] that the phase function $\theta : S^{d-1} \times S^{d-1}$ in Proposition 1.2 has the property that $\nabla_x\nabla_y\theta$ (relative to a coordinate system) has $d-2$ eigenvalues with magnitude $\approx |\sin\theta|^{-1}$. If we let a be the amplitude function in Proposition 1.2 (iii) and

$$\tilde{a}(e,f) = n^{1-d/2}\lambda^{d/2-1}\left(\lambda + \left|1 - \frac{s}{t}\right|\right)a(se, tf)$$

then we have $|D_e^\alpha D_f^\beta \tilde{a}| \lesssim \lambda^{-(|\alpha|+|\beta|)}$. It follows by the product rule that also $|D_e^\alpha D_f^\beta(\chi_\lambda \tilde{a})| \lesssim \lambda^{-(|\alpha|+|\beta|)}$. Thus in local coordinates on S^{d-1}, $\chi_\lambda \tilde{a} e^{in\theta}$ satisfies the hypothesis of Lemma 2.4 with $m = d-2$, $\rho = \text{const}\cdot\lambda$. We obtain

$$\|\chi_\lambda \tilde{a} e^{in\theta}\|_{p\to p'} \lesssim \lambda^{2(d-1)/p'}(n\lambda)^{-(d-2)/p'} \tag{2.2}$$

$$\|\chi_\lambda K_n^{st}\|_{p\to p'} \lesssim n^{d/2-1}\lambda^{-(d/2-1)}\left(\lambda + \left|1 - \frac{s}{t}\right|\right)^{-1}\lambda^{2(d-1)/p'}(n\lambda)^{(d-2)/p'}$$

which works out to

$$\|\chi_\lambda K_n^{st}\|_{p\to p'} \lesssim D(n,\lambda)\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}.$$

Proposition 2.2. *Fix α with $|\alpha| \leqslant 2$. For sufficiently large $\nu \in \mathbb{R}^+$ there is a kernel $L_\nu^{(\alpha)}$ such that (with the same modification as in Proposition 1.2 when $D^\alpha = \frac{d^2}{dx_j^2}$)*

$$|x|^{-\nu} D^\alpha f(x) = \int_{\mathbb{R}^d} L_\nu^{(\alpha)}(x, y)|y|^{-\nu} \Delta f(y)\, dy$$

and $L_\nu^{(\alpha)} = M_\nu^{(\alpha)} + N_\nu^{(\alpha)}$ where for suitable C,

(i) *if $1 < p, q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{2 - |\alpha|}{d}$ then $M_\nu^{(\alpha)}$ maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ with norm bounded independently of ν.*

(ii) *$N_\nu^{(\alpha)}(x, y) = 0$ if $|x - y| < (200\nu)^{-1}|y|$, and for any $\lambda > (200\nu)^{-1}$,*

$$\|\chi_\lambda N_\nu^{(\alpha)st}\|_{p \to p'} \lesssim (st)^{-(d-2+|\alpha|)/2} \nu^{|\alpha|-1} D(\nu, \lambda)$$

$$\cdot \begin{cases} \left(\frac{s}{t}\right)^\rho, & \text{if} \quad s < \frac{t}{2}, \\ \left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}, & \text{if} \quad \frac{t}{2} \leqslant s < 2t, \\ \left(\frac{t}{s}\right)^{1-\rho}, & \text{if} \quad s > 2t. \end{cases}$$

Here ρ is defined to be the number in $[0, 1)$ such that $\nu - \frac{d - 2 + |\alpha|}{2} + \rho$ is a integer. χ_λ may be replaced by χ^λ if $\lambda < \frac{200}{\nu}$.

PROOF. We fix α and drop the α superscripts. We also assume for notational purposes that $D^\alpha \neq \frac{d^2}{dx_j^2}$. Write $\nu = n + \frac{d - 2 + |\alpha|}{2} - \rho$, where $n \in \mathbb{Z}$, and ρ is as above. Define

$$L_\nu(x, y) = |x|^{-(d-2+|\alpha|)/2+\rho} |y|^{-(d-2+|\alpha|)/2-\rho} I_n(x, y).$$

Then

$$\begin{aligned} \int L_\nu(x, y)|y|^{-\nu} \Delta f(y)\, dy &= |x|^{-(d-2+|\alpha|)/2+\rho} \int I_n(x, y)|y|^{-(n+d-2+|\alpha|)} \Delta f(y) \\ &= |x|^{-(d-2+|\alpha|)/2+\rho} |x|^{-n} D^\alpha f(x) \\ &= |x|^{-\nu} D^\alpha f(x). \end{aligned}$$

Also define $M_\nu = (1-\psi)L_\nu$, $N_\nu = \psi L_\nu$, ψ as in Proposition 2.1. Then (i) follows from (i) of Proposition 1.2 which implies $|M_\nu| \lesssim |x-y|^{-(d-2+|\alpha|)}$ when $|\alpha| \leqslant 1$ and may be treated as a fractional integral, and that M_ν may be treated as a (truncated) singular integral when $|\alpha| = 2$. As for (ii), Proposition 2.1. implies

$$\|\chi_\lambda N_\nu^{st}\|_{p\to p'} \lesssim n^{|\alpha|-1} D(n,\lambda)\left(\frac{s}{t}\right)^{\rho}(st)^{-(d-2+|\alpha|)/2}\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}$$

and we have

$$\left(\frac{s}{t}\right)^{\rho}\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1} \lesssim \begin{cases} \left(\dfrac{s}{t}\right)^{\rho}, & \text{if} \quad s<\dfrac{t}{2}, \\ \left(\left|1-\dfrac{s}{t}\right|+\lambda\right)^{-1}, & \text{if} \quad \dfrac{t}{2}<s<2t, \\ \left(\dfrac{t}{s}\right)^{1-\rho}, & \text{if} \quad s>2t. \end{cases}$$

In Section 5 we will also want a certain variant on Proposition 2.2. Let e_0 be a point of S^{d-1} and, for given ν, let $\phi\colon S^{d-1}\to\mathbb{R}$ be such that $\phi(e)=1$ if $|e-e_0|<C\nu^{-1/2}$, $\phi(e)=0$ if $|e-e_0|>2C\nu^{-1/2}$ and $|D^\alpha\phi| \lesssim (\nu^{-1/2})^{-|\alpha|}$ (here C is a suitable constant). For $\lambda > 100C\nu^{-1/2}$ consider the kernel

$$\phi(e)\chi_\lambda(e,f)N_\nu^{(\alpha)}(se,tf)$$

where $N_\nu^{(\alpha)}$ is as in Proposition 2.2. We claim

Proposition 2.3. *With notation as in Proposition* 2.2

$$\|\phi(e)\chi_\lambda(e,f)N_\nu^{(\alpha)}(se,tf)\| \lesssim \left(\lambda\sqrt{\nu}\right)^{-(d-1)/p'}(st)^{-(d-2+|\alpha|)/2}\nu^{|\alpha|-1}D(\nu,\lambda)$$

$$\cdot\begin{cases} \left(\dfrac{s}{t}\right)^{\rho}, & \textit{if} \quad s<\dfrac{t}{2}, \\ \left(\left|1-\dfrac{s}{t}\right|+\lambda\right)^{-1}, & \textit{if} \quad \dfrac{t}{2}\leqslant s<2t, \\ \left(\dfrac{t}{s}\right)^{1-\rho}, & \textit{if} \quad s>2t. \end{cases}$$

PROOF. Exactly as for Proposition 2.2 except that we use Lemma 2.3 instead of 2.4, with $\epsilon \approx \lambda$, $\delta \approx n^{-1/2}$. The effect is that instead of (2.2) we have

$$\|\phi\chi_\lambda \tilde{a} e^{in\theta}\|_{p\to p'} \lesssim n^{-(d-1)/2p'}\lambda^{(d-1)/p'}(n\lambda)^{-(d-2)/p'}.$$

The extra factor of $(n^{1/2}\lambda)^{-(d-1)/p'}$ remains throughout the proof and we end up with Proposition 2.3.

3. Carleman Inequalities

Notation. Fix p and r with $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$ and a multiindex α with $|\alpha| = 1$, and let $L_\nu = L_\nu^{(\alpha)}$ be the kernels in Proposition 2.2.

If $x, y \in \mathbb{R}^d$ we let $s = |x|$, $t = |y|$, $\sigma = \log(1/s)$, $\tau = \log(1/t)$. If $\gamma \subset \mathbb{R}$ then $\gamma_* = \{s \in \mathbb{R}: \log(1/s) \in \gamma\}$, $A(\gamma) = \{x \in \mathbb{R}^d: |x| \in \gamma_*\}$.

The characteristic function of a set E will be denoted 1_E. We keep the notation from Section 2, *e.g.* the functions χ_λ.

The purpose of this section is to prove Carleman type inequalities needed for Theorem 1. We actually prove two inequalities-the first will be used when $d \leqslant 4$ and the second when $d \geqslant 5$.

Proposition 3.1. *Suppose $r = d$. If ν is large enough, $\nu - (d-1)/2$ is not an integer, and $\beta, \gamma \subset \mathbb{R}$ are intervals with* $\min\{|\beta|, |\gamma|\} \geqslant \nu^{-1}$ *then, with the notation* $|\gamma|' = \min\{|\gamma|, 1\}$, *we have*

$$\| 1_{A(\gamma)} L_\nu I_{A(\beta)} \|_{p \to p'} \lesssim (\nu \min\{|\gamma|', |\beta|'\})^{1/2 - 1/d}.$$

The implicit constant depends on $\operatorname{dist}\left(\nu - \frac{d-1}{2}, \mathbb{Z}\right)$.

To state the other inequality we fix $\psi: \mathbb{R}^+ \to \mathbb{R}$ increasing and convex. We have then (for $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$)

$$e^{\nu\psi(\sigma)} D^\alpha f(x) = \int P_\nu(x, y) e^{\nu\psi(\tau)} \Delta f(y)\, dy \tag{3.1}$$

where $P_\nu(x, y) = e^{-\nu(\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma))} L_{\nu\psi'(\sigma)}(x, y)$.

This is because of the following calculation:

$$\begin{aligned} e^{\nu\psi(\sigma)} D^\alpha f(x) &= e^{\nu(\psi(\sigma) - \psi'(\sigma)\sigma)} |x|^{-\nu\psi'(\sigma)} D^\alpha f(x) \\ &= \int e^{\nu(\psi(\sigma) - \psi'(\sigma)\sigma)} L_{\nu\psi'(\sigma)}(x, y) |y|^{\nu\psi'(\sigma)} \Delta f(y)\, dy \\ &= \int e^{\nu(\psi(\sigma) - \psi'(\sigma)\sigma)} L_{\nu\psi'(\sigma)}(x, y) e^{-\nu(\psi(\tau) - \tau\psi'(\sigma))} e^{\nu\psi(\tau)} \Delta f(y)\, dy \\ &= \int P_\nu(x, y) e^{\nu\psi(\tau)} \Delta f(y)\, dy. \end{aligned}$$

Proposition 3.2. *Suppose $d < r < \infty$, and ψ: $\mathbb{R}^+ \to \mathbb{R}$ is C^2 and satisfies the following conditions: there is $C > 0$ such that $C^{-1} < \psi'(\sigma) < C$ for all σ, and for any $\delta > 0$ there is $C_\delta > 0$ such that $\psi''(\sigma) \geqslant C_\delta e^{-\delta\sigma}$. Then for ν large enough and β, γ intervals with* $\min\{|\beta|, |\gamma|\} \geqslant \nu^{-1}$ *and with the notation* $|\gamma|'' = \min\{|\gamma|, \nu^{-1/2}\}$, *and* $\delta = \min\{|\beta|'', |\gamma|''\}$, $\epsilon = \max\{|\beta|'', |\gamma|''\}$, *we have*

$$\| 1_{A(\psi^{-1}\gamma)} P_\nu 1_{A(\psi^{-1}\beta)} \|_{p \to p'} \lesssim \nu^{(d-2)/2r} \delta^{1/2r'} \begin{cases} \delta^{(1-(d+1)/r)/2} & (r < d+1) \\ 1 + \log(\epsilon/\delta) & (r = d+1) \\ \epsilon^{(1-(d+1)/r)/2} & (r > d+1) \end{cases}$$

Remarks. (1) Note that our intervals $\psi^{-1}\gamma, \psi^{-1}\beta$ are contained in $\mathbb{R}^+$, *i.e.*, Proposition 3.2 is actually an estimate on functions supported in $D(0, 1)$.

(2) The assumptions on ψ are not the most general possible. For example, instead of the upper bound on ψ' we could assume that for every $\delta > 0$ there exists $C_\delta > 0$ such that $\psi' \leqslant C_\delta e^{\delta\sigma}$. What is significant (as in previous work on similar problems) is to have some kind of «strict convexity» hypothesis, *i.e.* lower bound on ψ''/ψ'. The first example in [8] or [2] shows that Proposition 3.2 fails when $\psi(\sigma) = \sigma$, $r \geqslant (3d-2)/2$, $|\gamma| = |\beta| = \infty$.

(3) A version of Proposition 3.1 could be proved also when $\nu - (d-1)/2$ is an integer, but there would be a dependence on $|\gamma|$, $|\beta|$ which blows up as $|\gamma| \to \infty$ or $|\beta| \to \infty$. This of course is the same phenomenon as appeared in [9]. Actually Proposition 3.1 and 3.2 are just convenient ways of recording the information in Section 2 and (in Section 4) we probably could have worked instead with Proposition 2.2 directly.

(4) The crucial point for us will be the dependence on $|\gamma|$ and $|\beta|$ when $|\gamma|$ and $|\beta|$ are less than the critical numbers 1 in Proposition 3.1, $\nu^{-1/2}$ in Proposition 3.2. See Corollary 3.1 below where we state what we actually use.

(5) As far as why we need both Propositions 3.1 and 3.2: 3.2 is a much stronger inequality and we need that when $d \geqslant 5$. On the other hand, nothing like 3.2 can be true when $r = d$ (as is the case when $d \leqslant 4$) because the problem is the scale invariant, and a scale invariant «strict convexity» hypothesis would have to be of the form $\psi''/\psi' \geqslant$ const., which is incompatible with ψ' being bounded. (Of course one needs ψ' bounded for the application to the SUCP.)

PROOFS. Propositions 3.1 and 3.2 both follow by integrating out Proposition 2.2 with respect to the radial variable. The following fact will be useful.

Lemma 3.1. *Suppose $\lambda > 0$, $\rho \geqslant 0$. Then for intervals $\gamma \subset \mathbb{R}$,*

$$\| e^{-\rho x^2} (|x| + \lambda)^{-1} \|_{L^{r'}(\gamma)} \lesssim \lambda^{-1} \min\{\lambda, |\gamma|, \rho^{-1/2}\}^{1/r'}$$

where the implicit constant only depends on $r' \in (1, \infty)$.

This seems to be most easily proved by splitting into six cases according to the relative sizes of λ, $|\gamma|$, and $\rho^{-1/2}$.

PROOF OF PROPOSITION 3.1. Since the right hand side of the inequality is always larger than or equal to 1 it will suffice to prove it for N_ν in place of L_ν. Choose a partition of unity on $S^{d-1} \times S^{d-1}$ consisting of functions $\{\gamma_{2^j\nu^{-1}}\}$ and $\chi^{\nu^{-1}}$ where j runs from 1 to $\log_2 \nu$. Fix j and let $\lambda = 2^j\nu^{-1}$ and consider

$$\left\| 1_{A(\gamma)}(x)\chi_\lambda\left(\frac{x}{|x|}, \frac{y}{|y|}\right)N_\nu(x,y)1_{A(\beta)}(y)\right\|_{p\to p'} \tag{3.2}$$

Also let $n(s,t)$ be the $L^p(S^{d-1}) \to L^{p'}(S^{d-1})$ norm of the kernel $(\chi_\lambda N_\nu)^{st}$. Regard $\mathbb{R}^d$ as $S^{d-1} \times \mathbb{R}^+$ with measure $d\theta \times s^{d-1}\,ds$ and apply Lemma 2.1, then Lemma 2.2 with $u(s) = s^{-d}$, $v(t) = t^{-d}$. This gives

$$\begin{aligned}(3.2) &\leqslant L^p(\beta_*, t^{d-1}\,dt) \to L^{p'}(\gamma_*, s^{d-1}\,ds) \text{ norm of } n(s,t)\\ &\leqslant (AB)^{1/2},\end{aligned}$$

where

$$A = \sup_{s\in\gamma_*} \|s^{d/p'}t^{d/p'}n(s,t)\|_{L^{d'}(\beta_*,\,dt/t)}$$

$$B = \sup_{t\in\beta_*} \|s^{d/p'}t^{d/p'}n(s,t)\|_{L^{d'}(\gamma_*,\,ds/s)}$$

Using Proposition 2.2 to estimate n we have (with $\delta = \operatorname{dist}(n - (d-1)/2, \mathbb{Z})$)

$$A \lesssim D(\nu,\lambda)\sup_s \|K(s,t)\|_{L^{d'}(\beta_*,\,dt/t)},$$

where

$$K(s,t) = \begin{cases}(|\sigma-\tau|+\lambda)^{-1} & \text{if } |\sigma-\tau|<1\\ e^{-\delta|\sigma-\tau|} & \text{if } |\sigma-\tau|>1\end{cases}$$

and $D(\nu,\lambda) = \nu^{1/2-1/d}\lambda^{1/2}$. Using Lemma 3.1 with $\rho = 0$,

$$\begin{aligned}\|(|\sigma-\tau|+\lambda)^{-1}\|_{L^{d'}(\beta_*,\,dt/t)} &= \|(|\sigma-\tau|+\lambda)^{-1}\|_{L^{d'}(\beta,\,d\tau)}\\ &\lesssim \lambda^{-1}\min\{\lambda, |\beta|\}^{1/d'}\end{aligned}$$

and of course $\|e^{-\delta|\sigma-\tau|}\|_{L^{d'}(\beta_*,\,dt/t)} \lesssim \min\{1, |\beta|\}^{1/d'}$. We conclude

$$\|K(s,t)\|_{L^{d'}(\beta_*,\,dt/t)} \lesssim \begin{cases}\lambda^{-1}\min\{\lambda,|\beta|\}^{1/d'} & \text{if } |\beta|\leqslant 1\\ \lambda^{-1/d} & \text{if } |\beta|>1\end{cases}$$

$$A \lesssim \begin{cases} \nu^{1/2-1/d}\lambda^{-1/2}\min\{\lambda, |\beta|\}^{1/d'} & \text{if } \ |\beta| \leqslant 1 \\ (\nu\lambda)^{1/2-1/d} & \text{if } \ |\beta| > 1 \end{cases}$$

so in fact

$$A \lesssim \nu^{1/2-1/d}\lambda^{-1/2}\min\{|\lambda|, |\beta|'\}^{1/d'} \tag{3.3}$$

for all β.

There is an analogous estimate for B where $|\gamma|'$ substitutes for $|\beta|'$ in (3.3), hence an estimate for (3.2). Also we have the same estimate for

$$\| 1_{A(\gamma)}\chi^{\lambda}N_\nu 1_{A(\beta)}\|_{p\to p'}$$

when $\lambda \approx \nu^{-1}$. Summing over λ we obtain

$$\begin{aligned} &\| 1_{A(\gamma)}N_\nu 1_{A(\beta)}\|_{p\to p'} \\ &\quad \leqslant \nu^{1/2-1/d}\sum_{\lambda=2^j\nu^{-1},\, 0\leq j\leq \log_2\nu}\lambda^{-1/2}\min\{\lambda, |\beta|'\}^{1/2d'}\min\{\lambda, |\gamma|'\}^{1/2d'} \\ &\quad \lesssim [\nu\min\{|\beta|', |\gamma|'\}]^{1/2-1/d} \end{aligned}$$

and Proposition 3.1 follows.

PROOF OF PROPOSITION 3.2. Convexity of ψ implies that $\psi(\tau)-\psi(\sigma)-\psi'(\sigma)(\tau-\sigma)$ is positive and in fact $\geqslant k(\sigma)\min\{1, (\tau-\sigma)^2\}$ where $k(\sigma)$ denotes any lower bound for $\psi''/2$ on the interval $(\sigma-1, \sigma+1)$. We are assuming there is such a bound of the form $C_\delta e^{-\delta\sigma}$ for any given $\delta > 0$.

Write $P_\nu = Q_\nu + R_\nu$ where Q_ν(respectively, R_ν) comes from substituting $M_\nu(N_\nu)$ for L_ν in the definition of P_ν. Then $|Q_n| \lesssim |M_\nu| \lesssim |x-y|^{-(d-1)}$ and since Q_ν vanishes for $|x-y| > \nu^{-1}$ it follows (*e.g.* from Lemma 2.2) that Q_ν is $L^p \to L^{p'}$ bounded with norm $\lesssim \nu^{d/r-1}$. So it suffices to prove Proposition 3.2 for R_ν instead of P_ν. Introduce the same partition of unity as in the proof of Proposition 3.1 and consider

$$\| 1_{A(\psi^{-1}\gamma)}\chi_\lambda R_\nu 1_{A(\psi^{-1}\beta)}\|_{p\to p'}. \tag{3.4}$$

We bound (3.4) as in the proof of Proposition 3.1, *i.e.* use Lemma 2.1, then 2.2 with $u(s) = s^{-d}$, $v(t) = t^{-d}$ to obtain (3.4) $\leqslant (AB)^{1/2}$

$$A = D(\nu, \lambda)\sup_s \| e^{-\nu(\psi(\tau)-\psi(\sigma)-\psi'(\sigma)(\tau-\sigma))}n(s,t)(st)^{d/p'}\|_{L^{r'}(\psi^{-1}\beta,\, d\tau)},$$

$$B = D(\nu, \lambda)\sup_t \| e^{-\nu(\psi(\tau)-\psi(\sigma)-\psi'(\sigma)(\tau-\sigma))}n(s,t)(st)^{d/p'}\|_{L^{r'}(\psi^{-1}\gamma,\, d\sigma)}.$$

To calculate A, fix s and consider separately the contributions to the $L^{r'}$ norm from $\psi^{-1}\beta \cap \{|\sigma-\tau| > 1\}$ and $\psi^{-1}\beta\cap\{|\sigma-\tau| < 1\}$, using the lower bounds

$\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma) \gtrsim C_\delta s^\delta \min\{1, (\sigma - \tau)^2\}$. Estimating the exponential factor by its value when $|\sigma - \tau| = 1$ and using Proposition 2.2 we see that the contribution from $\psi^{-1}\beta \cap \{|\sigma - \tau| > 1\}$ is $\lesssim D(\nu, \lambda) e^{-C_\delta \nu s^\delta} s^{(1 - d/r)/2}$ which may be made $\lesssim \nu^{-T}$ for any given T by choosing δ small. The other is

$$(3.5) \qquad \lesssim s^{(1 - d/r)/2} \left(\int_{\psi^{-1}\beta} e^{-C_\delta \nu s^\delta (\sigma - \tau)^2} (|\sigma - \tau| + \lambda)^{-r'} d\tau \right)^{1/r'}.$$

Here $|\psi^{-1}\beta| \approx |\beta|$ because of the boundedness assumptions on ψ' so by Lemma 3.1,

$$\begin{aligned}(3.5) &\lesssim s^{(1 - d/r)/2} \lambda^{-1} \min\{(\nu s^\delta)^{-1/2}, \lambda, |\beta|\}^{1/r'} \\ &\lesssim \lambda^{-1} \min\{\nu^{-1/2}, \lambda, |\beta|\}^{1/r'}\end{aligned}$$

provided $\delta \leqslant 1 - d/r$. It follows that $A \lesssim \lambda^{-1} \min\{\nu^{-1/2}, \lambda, |\beta|\}^{1/r'} D(\nu, \lambda)$. Similar (not identical!) estimates can be made for B leading to the same bound except that $|\beta|$ is replaced with $|\gamma|$. Therefore

$$\begin{aligned}&\| 1_{A(\psi^{-1}\gamma)} R_\nu 1_{A(\psi^{-1}\beta)} \|_{p \to p'} \\ &\quad \lesssim \nu^{(d-2)/2r} \sum_{\substack{\lambda = 2^j \nu^{-1} \\ 0 \le j \le \log_2 \nu}} \lambda^{-d/2r} [\min\{\nu^{-1/2}, \lambda, |\beta|\} \min\{\nu^{-1/2}, \lambda, |\gamma|\}]^{1/2r'}.\end{aligned}$$

Proposition 3.2 follows by doing the sum separately over $\lambda < \min\{|\beta|'', |\gamma|''\}$, $\min\{|\beta|'', |\gamma|''\} < \lambda < \max\{|\beta|'', |\gamma|''\}$, $\lambda > \max\{|\beta|'', |\gamma|''\}$.

We record the following formal consequence which is what is actually used in the proof of Theorem 1.

Corollary 3.1. *Let* $r = \max\{d, (3d - 4)/2\}$. *If* $d \geqslant 5$ *then* $\| 1_{A(\psi^{-1}\gamma)} P_\nu \|_{p \to p'} \lesssim (\nu |\gamma|'')^{1/r}$. *If* $d = 3$ *or* 4 *then* $\| 1_{A(\psi^{-1}\gamma)} L_\nu \|_{p \to p'} \lesssim (\nu |\gamma|')^{(d-2)/2r}$ *provided* $\nu - (d - 1)/2$ *is kept bounded away from the integers.*

PROOF. This is just index juggling. The $d \leqslant 4$ case is the easiest. The $d \geqslant 5$ case splits into subcases $d = 5$, 6, or $\geqslant 7$ corresponding to the three alternatives in Proposition 3.2. We explain only the $d \geqslant 7$ case. We have (since here $r = (3d - 4)/2 > d + 1$, $|\beta| = \infty$, $|\beta|'' = \nu^{-1/2}$, $\delta = |\gamma|''$, $\epsilon = \nu^{-1/2}$)

$$\begin{aligned}\| 1_{A(\psi^{-1}\gamma)} P_\nu \|_{p \to p'}^r &\lesssim \nu^{(3d-2)/8} (|\gamma|'')^{(3d-6)/4} \\ &= \nu |\gamma\|'' (\nu^{1/2} |\gamma|'')^{(3d-10)/4} \\ &\leqslant \nu |\gamma|''.\end{aligned}$$

Remark. The proof of Proposition 3.1 also proves the Jerison-Kenig result: If we take $\alpha = 0$, $r = d/2$, and keep $\nu - (d - 2)/2$ bounded away from the integers, then we obtain (instead of (3.3))

$$A \lesssim \nu^{1-2/d}\lambda^{-1}\min\{\lambda, |\beta|'\}^{1-2/d}.$$

Using this estimate and continuing as before, we eventually obtain

$$\| 1_{A(\gamma)} L_\nu 1_{A(\beta)} \|_{p \to p'} \leqslant C,$$

which is Jerison-Kenig if $\gamma = \beta = \mathbb{R}$ (and gives nothing better for other $|\gamma|$, $|\beta|$, as one must expect anyway since the Jerison-Kenig bound is the same, as the bound for fractional integrals).

Conversely, it seems likely that other known proofs of the Jerison-Kenig result could be modified to give Proposition 3.1.

4. Proof of Theorem 1

The main point is the following lemma.

Lemma 4.1. *Suppose μ is a positive measure on $\mathbb{R}$ without atoms and such that*

$$\lim_{T\to\infty} \frac{1}{T} \log \mu(\{x: |x| > T\}) = -\infty.$$

Define μ_k for $k \in \mathbb{R}$ by $d\mu_k(x) = e^{kx}\, d\mu(x)$. Suppose $N \in \mathbb{R}^+$. Then there are disjoint intervals $I_j \subset \mathbb{R}$ and numbers $k_j \in [N, 2N]$ such that (with C a positive universal constant)

(i) $\mu_{k_j}(I_j) \geqslant \frac{1}{2} \|\mu_{k_j}\|$

(ii) $\sum |I_j|^{-1} \geqslant CN.$

PROOF. Suppose μ satisfies the hypotheses and fix $k \in \mathbb{R}$. Let a_k be a number such that $\mu_k((-\infty, a_k)) = \|\mu_k\|/4$, and let b_k be a number such that $\mu_k((b_k, \infty)) = \|\mu_k\|/4$. These a_k and b_k exist since μ_k is finite and continuous. Define $\gamma_k = [a_k, b_k]$.

Claim. $$|\gamma_k \cap \gamma_j| \lesssim \frac{1}{|k-j|}.$$

In the proof we assume $j > k$. We may suppose $b_k > a_j$; else there is nothing to prove. If $x > b_k$ and $y < a_j$ then $x > |\gamma_k \cap \gamma_j| + y$, so that

$$\frac{\mu_j((b_k,\infty))}{\mu_k((b_k,\infty))} = \frac{1}{\mu_k((b_k,\infty))}\int_{b_k}^{\infty} e^{(j-k)x}\,d\mu_k(x)$$

$$\geqslant e^{(j-k)|\gamma_k\cap\gamma_j|}\frac{1}{\mu_k((-\infty,a_j))}\int_{-\infty}^{a_j} e^{(j-k)y}\,d\mu_k(y)$$

$$= e^{(j-k)|\gamma_k\cap\gamma_j|}\frac{\mu_j((-\infty,a_j))}{\mu_k((-\infty,a_j))}$$

$$\|\mu_k\|\,\|\mu_j\| \geqslant \mu_j((b_k,\infty))\mu_k((-\infty,a_j))$$

$$> e^{(j-k)|\gamma_k\cap\gamma_j|}\mu_k((b_k,\infty))\mu_j((-\infty,a_j))$$

$$= \frac{1}{16}e^{(j-k)|\gamma_k\cap\gamma_j|}\|\mu_k\|\,\|\mu_j\|$$

and $|\gamma_k\cap\gamma_j| < 4\log 2/(j-k)$, proving the claim.

Now we restrict k, j, etc. to lie in $[N, 2N]$. If there is k with $|\gamma_k| \leqslant 1/N$ there is nothing to prove. Otherwise define γ_k to be minimal if $|\gamma_j\cap\gamma_k| > |\gamma_j|/2$ implies $|\gamma_j| > |\gamma_k|/2$. Consider the collection of all minimal intervals and take a subcover with the Besicovitch property (the subcover is the union of two families of pairwise disjoint intervals, and every minimal interval is contained in the union of two subcover intervals). Such a subcover exists because the faster-than-exponential decay implies an upper bound on the lengths of the γ_k. We will be done if we show the subcover intervals have property (ii).

For any γ_k we can find a chain $k = k_1, k_2, \ldots$ with $|\gamma_{k_{j+1}}| \leqslant |\gamma_{k_j}|/2$ and $|\gamma_{k_{j+1}}\cap\gamma_{k_j}| \geqslant |\gamma_{k_j}|/2$. Such a chain must terminte at a minimal interval γ_{k_m} since we are assuming a lower bound $|\gamma_k| \geqslant 1/N$. The claim shows $|k_{j+1}-k_j| \lesssim |\gamma_{k_j}\cap\gamma_{k_{j+1}}|^{-1} \leqslant 2|\gamma_{k_{j+1}}|^{-1}$. The geometric decrease of the $|\gamma_{k_j}|$ then shows $|k-k_m| \lesssim |\gamma_{k_m}|^{-1}$. There must be a subcover interval γ_j with $|\gamma_{k_m}\cap\gamma_j| \geqslant |\gamma_{k_m}|/2$, and minimality of γ_{k_m} implies $|\gamma_{k_m}| \geqslant |\gamma_j|/2$ so that $|k_m - j| \lesssim |\gamma_j|^{-1}$, $|k-j| \leqslant |k-k_m| + |k_m - j| \lesssim |\gamma_{k_m}|^{-1} + |\gamma_j|^{-1} \lesssim |\gamma_j|^{-1}$.

So we associated a subcover interval γ_j to each interval γ_k, in such a way that

$$|\{k\colon \gamma_j \text{ associated to } \gamma_k\}| \lesssim |\gamma_j|^{-1}.$$

Then

$$N = |[N,2N]| \leqslant \sum |\{k\colon \gamma_j \text{ associted to } \gamma_k\}| \lesssim \sum |\gamma_j|^{-1}.$$

In proving Theorem 1 in the $d \leqslant 4$ case we will need to restrict the possible values of k_j. Hence the following

Corollary 4.1. *Suppose $m \in \mathbb{R}$, $b \in \mathbb{R}$. Then Lemma 4.1 remains true if the following modifications are made: the numbers k_j are required to belong to the arithmetic progression $\{mn + b: n \in \mathbb{Z}\}$; and (ii) is replaced by*

(ii)′
$$\sum_j \max\{|I_j|^{-1}, m\} \gtrsim N.$$

PROOF. This is actually a corollary of the *proof* of Lemma 4.1. We carry out the same argument requiring all the values k, j, etc., to belong to the given arithmetic progression. The only change is in the last paragraph of the proof where we now have associated a subcover interval γ_j to each γ_k in such a way that

$$\operatorname{card}\{k: \gamma_j \text{ associated to } \gamma_k\} \lesssim \max\{1, (m|\gamma_j|)^{-1}\}$$

since the right-hand side is the cardinality of the set of arithmetic progression elements lying within $|\gamma_j|^{-1}$ of j.

In finishing the proof we may assume $m < N$; otherwise there is nothing to prove as we may take $\{I_j\}$ to be a singleton. When $m < N$, we have

$$\begin{aligned} N &\lesssim m \operatorname{card}\{mn + b: N \leqslant mn + b \leqslant 2N\} \\ &\lesssim m \sum_{\text{subcover intervals } \gamma_j} \operatorname{card}\{k: \gamma_k \text{ associated to } \gamma_j\} \\ &\lesssim m \sum \max\{1, (m|\gamma_j|)^{-1}\} \\ &= \sum \max\{m, |\gamma_j|^{-1}\}. \end{aligned}$$

We now finish Theorem 1.

First of all, if $\psi: (0, \infty) \to \mathbb{R}$ is increasing and convex and ψ' is bounded then the formula

$$e^{\nu\psi(\sigma)} D^\alpha f(x) = \int L_\nu(x, y) e^{\nu\psi(\tau)} \Delta f(y)\, dy \tag{4.1}$$

derived for $C_0^\infty(D(0,1)\setminus\{0\})$ functions in Section 3 extends to functions in $W^{2,p}$ with support in $D(0, 1)$ and such that $\|\nabla f\|_{L^{p'}(D(0,r))}$ and $\|\nabla f\|_{L^p(D(0,r))}$ vanish faster than any power of r as $r \to 0$. This is standard. First, if supp f does not contain the origin then it follows using a mollifier. The general case follows using a cutoff function near 0 and controlling the error terms by the infinite order vanishing (and the fact that $\psi(\tau) \leqslant C\tau$).

In particular (4.1) is valid for ϕu if u is as in Theorem 1 and $\phi \in C_0^\infty$ with $\phi = 1$ in a neighborhood of 0 since the infinite order vanishing of Δu follows from that of ∇u using Hölder's inequality.

Let $r = \max\{d, (3d - 4)/2\}$. As in, *e.g.* [9], Theorem 1 will follow if we show there is $\epsilon_0 > 0$ such that $0 \leqslant S_0 \leqslant 1$ and $\|V\|_{L^r(D(0, S_0))} < \epsilon_0$ imply u vanishes identically on $D(0, S_0)$. So let ϵ_0 be small enough and suppose $\| V \|_{L^r(D(0,S_0))} < \epsilon_0$ but ∇u does not vanish identically on $D(0, S_0)$. Let $S_1 < S_0$ be such that ∇u does not vanish identically on $D(0, S_1)$ and choose $\phi \in C_0^\infty$ with $\phi = 1$ on $D(0, S_1)$ and $\operatorname{supp} \phi \subset D(0, S_0)$. Let $f = \phi u$.

We first consider the $d \geqslant 5$ case.

Let $\psi(\sigma) = \sigma - (\sigma + 1)^{1/2}$ (any other function satisfying the hypotheses of Proposition 3.2 would do as well). Define a measure μ on $\mathbb{R}$ by

$$\mu(\gamma) = \int_{A(\psi^{-1}\gamma)} (V|\nabla f|)^p.$$

The infinite order vanishing implies μ has the faster-than-exponential decay property assumed in Lemma 4.1. With notation as in Lemma 4.1 we have

$$\mu_k(\gamma) = \int_{A(\psi^{-1}\gamma)} (e^{\nu\psi} V|\nabla f|)^p$$

where $\nu = k/p$. For N sufficiently large, we let $\{I_j\}$ be the intervals from Lemma 4.1. We may assume they have length $\geqslant 1/N$ (else drop all but one of them and expand that one to length $1/N$). Fix j and denote I_j by I, k_j by k, and $\nu = k/p$. We have

$$\begin{aligned}
\|\mu_k\| &\leqslant 2 \int_{A(\psi^{-1}I)} (e^{\nu\psi} V|\nabla f|)^p \\
&\leqslant 2\Big(\int_{A(\psi^{-1}I)} V^r\Big)^{p/r} \Big(\int_{A(\psi^{-1}I)} (e^{\nu\psi}|\nabla f|^{p'}\Big)^{p/p'} \\
&\lesssim \Big(\int_{A(\psi^{-1}I)} V^r\Big)^{p/r} (N|I|)^{p/r} \| e^{\nu\psi} \Delta f \|_p^p.
\end{aligned}$$

The last line follows from Corollary 3.1 since $|\psi^{-1}I| \approx |I|$ (as ψ' is bounded away from 0 and ∞) and $\nu \approx N$. We calculate $\Delta f = \Delta(\phi u)$ by the product rule:

$$\begin{aligned}
|\Delta f| &= |\phi \, \Delta u + 2 \nabla\phi \, \nabla u + u \, \Delta\phi| \\
&\leqslant \phi V |\nabla u| + |2 \nabla\phi \, \nabla u + u \, \Delta\phi| \\
&\leqslant V|\nabla f| + |u \nabla\phi| + |2 \nabla\phi \, \nabla u + u \, \Delta\phi| \\
&= V|\nabla f| + E,
\end{aligned}$$

where $E \in L^p$ is supported in $\{x : S_1 < |x| < S_0\}$. Thus

$$\|\mu_k\| \lesssim \Big(\int_{A(\psi^{-1}I)} V^r\Big)^{p/r} (N|I|)^{p/r} \Big(\|\mu_k\| + \int |e^{\nu\psi} E|^p \Big). \tag{4.2}$$

Now we use the usual trick: $\|\mu_k\|$ grows faster than $e^{k\psi(\sigma_1)}$ as $k \to \infty$ while the last term in (4.2) is $O(e^{k\psi(\sigma_1)})$. So for large enough N the last term may be absorbed leading to

$$\|\mu_k\| \lesssim \left(\int_{A(\psi^{-1}I)} V^r\right)^{p/r} (N|I|)^{p/r} \|\mu_k\|$$

$$\int_{A(\psi^{-1}I)} V^r \geqslant (N|I|)^{-1}.$$

We now sum over j. Using (ii) of Lemma 4.1,

$$\int_{\cup A(\psi^{-1}I_j)} V^r \gtrsim \sum_j (N|I_j|)^{-1} \gtrsim 1$$

which is a contradiction if ϵ_0 is small. This finishes the high dimensional case.

If $d = 3$ or 4 the argument is similar. Now we take $\psi(\sigma) = \sigma$. We define μ as above, but now instead of choosing the I_j by Lemma 4.1 we use Corollary 4.1. Thus there are disjoint intervals I_j such that for each j there is k_j with

$$p^{-1}k_j - \frac{d-1}{2} = \frac{1}{2} \bmod 1$$

and

$$\mu_{k_j}(I_j) \geqslant \frac{1}{2}\|\mu_{k_j}\|, \qquad \sum_j \max\{1, |I_j|^{-1}\} \geqslant N.$$

We now fix a value of j and argue as above. The only difference is that in applying Corollary 3.1 we obtain a factor $(N\min\{|I|, 1\})^{(d-2)/2r}$ instead of $(N|\mathrm{I}|)^{1/r}$. We end up with

$$\int_{A(I)} V^r \gtrsim (N\min\{1, |I|\})^{-(d-2)/2}$$

and therefore

$$\begin{aligned}\int_{\cup_j A(\psi^{-1}I_j)} V^r &\gtrsim \sum_j (N\min\{1, |I_j|\})^{-(d-2)/2} \\ &\geqslant \left\{\sum_j (N\min\{1, |I_j|\})^{-1}\right\}^{(d-2)/2} \\ &\gtrsim 1\end{aligned}$$

by Corollary 4.1. This is a contradiction as before.

We now have a question: does Lemma 4.1 extend to $\mathbb{R}^d$ in the following form? Suppose μ is a measure on $\mathbb{R}^d$ with faster —than— exponential decay (say, absolutely continuous) and define $d\mu_k(x) = e^{kx}\,d\mu(x)$. Then there should be rectangles $\{R_j\}$ such that

(i) the R_j are disjoint,
(ii) for each j there is $k \in [-N, N] \times \cdots \times [-N, N]$ such that $\mu_k(R_j) \gtrsim \|\mu_k\|/2$ for all $T > 0$,
(iii) $\sum |R_j|^{-1} \geqslant C^{-1}N^d$ where C only depends on d.

Remarks. (1) We believe that an affirmative answer should lead to a proof of the WUCP for $|\Delta u| \leqslant V|\nabla u|$ with $V \in L^d$, although we do not have a reduction of one problem to the other.

(2) Natural examples are a Gaussian and surface measure on the unit sphere. For the Gaussian, the R_j are cubes with equal side length and in the surface measure case they are the covering of the sphere by rectangles with dimensions $N^{-1} \times (N^{-1/2} \times \cdots \times N^{-1/2})$ familiar in connection with Stein's restriction problem and so forth. In both examples, the order N^d is attained. The second example shows that the R_j cannot in general be taken with sides parallel to the axes.

(3) Weaker disjointness conditions than (i) would also be of interest, *e.g.*

(i)′ $$\|\Sigma\chi_{R_j}\|_p^p \approx \|\Sigma\chi_{R_j}\| \quad \text{for all} \quad p < \infty,$$

or

(i)″ the number of R_j containing any given point is bounded by a power of $\log N$.

We can answer the question affirmatively with (i)″ replacing (i) if $d = 2$. Of course, the answer is also affirmative when $d = 1$ (Lemma 4.1). The two dimensional result and partial results in higher dimensions will appear in a subsequent paper. The author can now prove the WUCP for $|\Delta u| \leqslant A|u| + B|\nabla u|$, $A \in L^{\alpha/2}$, $B \in L^r$, $r > d$, along the lines described above.

5. Further Results

The effect of the «osculation by $|x|^{-n}$» argument in the proof of (3.1) was that by using weights with a sufficient amount of «convexity» one could localize in the radial variable to intervals of (logarithmic) length $n^{-1/2}$. One can ask whether it is possible to localize also in the other variables (*i.e.*, to discs of radius $n^{-1/2}$) by the same kind of argument. It turns out that this is possible in some cases and we will present consequences for the weak unique continuation problem. Our main goal is the following refinement of a result of C. Sogge [15].

Theorem 5.1. *Suppose*

$$L = \sum a_{ij} \frac{d^2}{dx_i dx_j}$$

is an elliptic operator with $C^{1+\eta}$ coefficients ($0 < \eta \leqslant 1 - 2/d$), and $V \in L^r$ where $r = (d-2)/2\eta$. Suppose $u \in W^{2,p}$ satisfies $|Lu| \leqslant V|u| + C|\nabla u|$ and vanishes on an open set. Then $u = 0$.

Remarks. (1) This result with $\eta = 1 - 2/d$ implies Sogge's (he proved the same assuming L has C^∞ coefficients). As far as the minimal regularity is concerned, nothing better than Lip 1 is possible even if $r = \infty$, as was shown by Plis [11]. Lip 1 is known to be sufficient when $r = \infty$ through work of Aronszajn, Cordes and Hormander in the 1950's. The argument below can be adapted to give this (only a linear change of variables is used in place of Lemma 5.6) but so can many other arguments. The new point is that there are L^r results with less than C^∞ coefficients. It appears likely that the optimal result for the WUCP will be that it holds provided L has Lip 1 coefficients and $V \in L^{d/2}_{\text{loc}}$. Sogge pointed out that if the main conjecture on $L^p \to L^p$ behavior of oscillating integrals could be established (*i.e.*, an affirmative answer given to the first question at the end of [6] when $r = q$) then the argument below could be used to prove the WUCP with Lip 1 coefficients and $V \in L^r_{\text{loc}}$ for any given $r > d/2$. This is because the $L^p \to L^p$ conjecture would improve the estimate in (iii) of Lemma 5.2 below. The general conjecture in [6] has been disproved by Bairgain.

(2) The argument will be based on freezing of coefficients —this is made possible by the localization effect in Lemma 5.1 below. Sogge used ψDO calculus which is naturally less efficient if one cares about the minimal regularity of the coefficients. On the other hand it must be pointed out that Sogge also treated the SUCP. One would expect this to be possible by our method also, but not without significant changes.

(3) The same kind of localization effect may be used to refine some results of Chanillo-Sawyer [4]. We discuss this at the end of the section.

We let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$ be the upper half space.

Lemma 5.1. *Suppose $d/2 \leqslant r \leqslant \infty$. Let ν be sufficiently large. For $a \in \mathbb{R}^d_+$ denote $D_a = D(a, \nu^{-1/2} a_d)$ and $\Gamma_a(x) = 1 + (|x-a|^2/4a_d x_d)$. Then for $f \in C_0^\infty(\mathbb{R}^d_+)$ we have*

(i) $\|x_d^{-\nu} f\|_{L^{p'}(D)} \lesssim \|\Gamma_a^{-\nu} x_d^{2-d/r-\nu} \Delta f\|_{L^p}$

(ii) $\|x_d^{-\nu} \nabla f\|_{L^p(D)} \lesssim \nu^{(d-2)/4r - 1/2} \|\Gamma_a^{-\nu} x_d^{1-\nu} \Delta f\|_{L^p}$

(iii) $\|x_d^{-\nu} H_f\|_{L^p(D)} \lesssim \nu^{(d-2)/4r + 1/2} \|\Gamma_a^{-\nu} x_d^{-\nu} \Delta f\|_{L^p}$.

We recall that $1/p - 1/p' = 1/r$. Also, we remark that the point of the functions $\Gamma_a^{-\nu}$ is that $\Gamma_a^{-\nu} \approx 1$ on D_a (since $|x-a|^2/4a_d x_d \lesssim \nu^{-1}$) and $\Gamma_a^{-\nu}$ dies off very fast outside D_a so that *e.g.* Lemma 5.5 below is valid. H_f is of course the Hessian matrix.

Lemma 5.1 will be a consequence of the following estimate with respect to the «natural» weights.

Lemma 5.2. *Suppose $d/2 \leqslant r \leqslant \infty$. Fix $e \in \mathbb{R}^n$ and denote $D = D(e, \nu^{-1/2}|e|)$, $\Omega = \{x \in \mathbb{R}^d: |x| > |e|/10,\ x \cdot e > 0\}$. Then for $f \in C_0^\infty(\Omega)$,*

(i) $\| |x|^{-\nu} f \|_{L^{p'}(D)} \lesssim \| |x|^{2-d/r-\nu} \Delta f \|_p$

(ii) $\| |x|^{-\nu} \nabla f \|_{L^p(D)} \lesssim \nu^{(d-2)/4r-1/2} \| |x|^{1-\nu} \Delta f \|_p$

(iii) $\| |x|^{-\nu} H_f \|_{L^p(D)} \lesssim \nu^{(d-2)/4r+1/2} \| |x|^{-\nu} \Delta f \|_p$.

PROOF OF LEMMA 5.1. There are three identical arguments for (i), (ii), (iii). We will do (iii).

Define $a^* = (a_1, \dots, a_{d-1}, -a_d)$. Apply (iii) of Lemma 5.2 taking the origin to be at a^*, and with 2ν instead of ν. Observe that the assumption that $f \in C_0^\infty(\Omega)$ is in fact satisfied. We get

$$\| |x - a^*|^{-2\nu} H_f \|_{L^p(D)} \lesssim \nu^{(d-2)/2r+1/2} \| |x - a^*|^{-2\nu} \Delta f \|_p.$$

Now $|x - a^*|^{-2} = (4a_d)^{-1} x_d^{-1} \Gamma_a(x)^{-1}$. Accordingly

$$(4a_d)^{-\nu} \| \Gamma_a^{-\nu} x_d^{-\nu} H_f \|_{L^p(D)} \lesssim \nu^{(d-2)/2r+1/2} (4a_d)^{-\nu} \| \Gamma_a^{-\nu} x_d^{-\nu} \Delta f \|_p$$

and now we are done, since $\Gamma_a^{-\nu} \approx 1$ on D.

PROOF OF LEMMA 5.2. The reader will observe that part (i) can be derived from [9]. All three parts follow readily from our Propositions 2.2 and 2.3 by the same kind of arguments as we used in Section 3, only somewhat easier. Accordingly we will omit the details of the calculations.

The estimates in Lemma 5.2 scale correctly, so we may assume e is a unit vector. Recall the kernels L^α, M^α, N^α from Section 2. For any fixed m, the kernel $|x|^m L_{t+m}^\alpha |y|^{-m}$ maps $|y|^{-t} \Delta f$ to $|x|^{-t} D^\alpha f$ (or to $|x|^{-t}(D^\alpha f - (1/d)\Delta f)$ if $D^\alpha = d^2/dx_j^2$). So to prove Lemma 5.2 it will suffice to show that for some fixed m and sufficiently large ν we have bounds

$$\begin{aligned} &\| |x|^{m+d/r-2} L_\nu^\alpha |y|^{-m} \|_{L^p(\Omega) \to L^{p'}(D)} \leqslant 1 && (\alpha = 0) \\ (5.1) \qquad &\| |x|^{m-1} L_\nu^\alpha |y|^{-m} \|_{L^p(\Omega) \to L^p(D)} \lesssim \nu^{(d-2)/4r-1/2} && (|\alpha| = 1) \\ &\| |x|^m L_\nu^\alpha |y|^{-m} \|_{L^p(\Omega) \to L^p(D)} \lesssim \nu^{(d-2)/4r+1/2} && (|\alpha| = 2) \end{aligned}$$

In fact it will suffice to prove the bounds (5.1) with the powers of x dropped from the left hand side (they are ≈ 1 on D) and with L_ν^α replaced by N_ν^α (the corresponding bounds for M_ν^α are easy estimates on fractional integrals when $|\alpha| < 2$ or singular integrals when $|\alpha| = 2$). Thus we will prove that

$$
\begin{aligned}
&\|N_\nu^\alpha |y|^{-m}\|_{L^p(\Omega)\to L^{p'}(D)} \lesssim 1 && (\alpha = 0)\\
(5.2)\qquad &\|N_\nu^\alpha |y|^{-m}\|_{L^p(\Omega)\to L^p(D)} \lesssim \nu^{(d-2)/4r-1/2} && (|\alpha| = 1)\\
&\|N_\nu^\alpha |y|^{-m}\|_{L^p(\Omega)\to L^p(D)} \lesssim \nu^{(d-2)/4r+1/2} && (|\alpha| = 2)
\end{aligned}
$$

provided the fixed positive number m is sufficiently large.

Let ψ be a smooth function which is 1 on the double of D, 0 outside the triple and has the natural bounds. Fix α and write

$$
\begin{aligned}
S(x,y) &= N_\nu^\alpha(x,y)\psi(x)\\
T(x,y) &= N_\nu^\alpha(x,y)(1-\psi(x))
\end{aligned}
$$

Consider first S. Let $\psi_\lambda(x,y)$ denote smooth cutoffs to $|x-y| \approx \lambda$, such that

$$1 = \sum_{\substack{\lambda = 2^j\nu^{-1}\\ 1\le 2^j \le \nu^{1/2}}} \psi_\lambda$$

on supp S. Then with the χ_λ, K^{st} notation used in Sections 2 and 3, and with C a suitable constant, $(\psi_\lambda S)^{st}$ will be

$$
\begin{aligned}
&0, && \text{if} \quad |s-t| > C\lambda,\\
&\chi_\lambda N_\nu^{\alpha st}, && \text{if} \quad |s-t| > C\lambda,\\
&\chi^{\nu^{-1}} N_\nu^{\alpha st} + \sum_{\substack{\mu = 2^j\nu^{-1}\\ 1\le 2^j\le \nu\lambda}} \chi_\mu S^{st}, && \text{if} \quad C^{-1}\lambda < |s-t| < C\lambda.
\end{aligned}
$$

Using Proposition 2.2, and summing a geometric series in the case $C^{-1}\lambda < |s-t| < C\lambda$, we obtain

$$\|(\psi_\lambda S)^{st}\|_{p\to p'} \lesssim \nu^{|\alpha|-1} D(\nu,\lambda)\left(\left|1-\frac{s}{t}\right| + \lambda\right)^{-1},$$

with $D(\nu,\lambda)$ as in (2.1). Lemmas 2.1 and 2.2 may thus be applied (take $u = v = 1$ in Lemma 2.2) to give (use Lemma 3.1 with $\rho = 0$, $|\gamma| \approx \lambda$ to calculate the relevant $L^{r'}$ norms)

$$\text{(5.2a)}\qquad \|\psi_\lambda S\|_{p\to p'} \lesssim \nu^{|\alpha|-1} D(\nu,\lambda)\lambda^{-1/r}$$

and therefore also

$$\text{(5.3)}\qquad \|\psi_\lambda S|y|^{-m}\|_{p\to p'} \lesssim \nu^{|\alpha|-1} D(\nu,\lambda)\lambda^{-1/r}$$

since $|y|^{-m} = 1$ on supp S.

Now we need the following observation.

Lemma 5.3. *Suppose* $S: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is such that* $S(x,y) = 0$ *for* $|x-y| > \epsilon$. *Then* $(1 \leqslant p \leqslant 2, 1/r = 1/p - 1/p')$

$$\|S\|_{p\to p} \lesssim \epsilon^{d/r}\|S\|_{p\to p'}.$$

To prove this (easier than finding a reference!) let $\{B_j\}$ be a covering of $\mathbb{R}^d$ by balls with radius ϵ and finite overlap (*i.e.*, no point belongs to more than C of them where C is a fixed constant). Let $\tilde{B}_j$ be the double of B_j. The $\{\tilde{B}_j\}$ still have finite overlap, and $S(x,y) = 0$ if $x \in B_j$, $y \notin \tilde{B}_j$. So

$$\begin{aligned}\|Sf\|_p^p &\leqslant \sum_j \int_{B_j} |Sf|^p \\ &\leqslant \sum_j \int_{B_j} |S(\chi_{\tilde{B}_j} f)|^p \\ &\leqslant \epsilon^{dp/r} \sum_j \|S(\chi_{\tilde{B}_j} f)\|_{p'}^p \quad \text{(Hölder)} \\ &\leqslant \epsilon^{dp/r} \sum_j \|\chi_{\tilde{B}_j} f\|_p^p \\ &\approx \epsilon^{dp/r}\|f\|_p^p.\end{aligned}$$

Applying this in our situation when $|\alpha| = 1$ or 2 we obtain from (5.3)

$$\begin{aligned}\alpha = 0: &\quad \|\psi_\lambda S|y|^{-m}\|_{p\to p'} \lesssim \nu^{-1}D(\nu,\lambda)\lambda^{-1/r}\ , \\ \alpha = 1: &\quad \|\psi_\lambda S|y|^{-m}\|_{p\to p} \lesssim D(\nu,\lambda)\lambda^{-1/r}\lambda^{d/r}\ , \\ \alpha = 2: &\quad \|\psi_\lambda S|y|^{-m}\|_{p\to p} \lesssim \nu D(\nu,\lambda)\lambda^{-1/r}\lambda^{d/r}.\end{aligned}$$

Now we sum over $\lambda = 2^j\nu^{-1}$, $1 \leqslant 2^j \leqslant \nu^{1/2}$. For $\alpha = 1$ or 2 we have a convergent geometric series with the main term being the $\lambda = \nu^{-1/2}$ term, and we obtain

$$\|S|y|^{-m}\|_{p\to p} \lesssim \nu^{(d-2)/4r - 1/2 + |\alpha| - 1}.$$

For $\alpha = 0$ there are several cases according to the relative sizes of r and $(d+2)/2$, but we always get $\|S|y|^{-m}\|_{p\to p'} \leqslant 1$ (in fact, it is clear that only the case $r = d/2$ has to be considered).

Now we consider $T|y|^{-m}$. Choose a smooth cutoff $J(x,y) = J(\theta_{xy})$ with $J = 1$ when θ_{xy} is less than $C^{-1}\nu^{-1/2}$ (for suitable large C) and $J = 0$ when $\theta_{xy} > 2C^{-1}\nu^{-1/2}$. Write $T = JT + (1-J)T$. $(JT)^{st}$ will vanish if $|s-t|$ is small compared with $\nu^{-1/2}$, and when $|s-t| \gtrsim \nu^{-1/2}$ we can write

$$(JT)^{st} = \chi^{\nu^{-1}} N_\nu^{\alpha st} + \sum_{\substack{\lambda = 2^j\nu^{-1} \\ 1 \leq 2^j \leq \nu^{1/2}}} \chi_\lambda N_\nu^{\alpha st}.$$

We may apply Proposition 2.2 to the summands and then sum a geometric series obtaining (remember we can assume $|s - t| \gtrsim \nu^{-1/2}$)

$$\|(JT)^{st}\|_{p\to p'} \lesssim t^{-(d-2+|\alpha|)/2}\nu^{|\alpha|-1}D(\nu, \nu^{-1/2})\left(\left|1 - \frac{s}{t}\right| + \nu^{-1/2}\right)^{-1}.$$

Likewise $((1 - J)T)^{st}$ may be written as

$$\sum_{\substack{\lambda = 2^j\nu^{-1/2}\\ 1 \le 2^j \le \nu^{1/2}}} \chi_\lambda N_\nu^{\alpha st}.$$

We then apply Proposition 2.3 (which was included for this purpose) to estimate the action of $\chi_\lambda N_\nu^{\alpha st}$ from L^p to $L^{p'}(D_s)$, where $D_s = \{e \in S^{d-1}: se \in D\}$, and sum a geometric series to obtain

$$\|((1 - J)T)^{st}\|_{p\to p'} \lesssim t^{-(d-2+|\alpha|)/2}\nu^{|\alpha|-1}D(\nu, \nu^{-1/2})\left(\left|1 - \frac{s}{t}\right| + \nu^{-1/2}\right)^{-1}$$

and therefore the same estimate for T^{st}. Thus

$$\|(T|y|^{-m})^{st}\|_{p\to p'} \lesssim t^{-(d-2+|\alpha|)/2-m}\nu^{|\alpha|-1}\nu^{(3d-4)/4r-1/2}\left(\left|1 - \frac{s}{t}\right| + \nu^{-1/2}\right)^{-1}.$$

Also $(T|y|^{-m})^{st} = 0$ if $|s - 1|$ is large compared with $\nu^{-1/2}$ or if $|t| < 1/10$. Denoting (for suitable C)

$$Q(s, t) = \begin{cases} t^{-(d-2+|\alpha|)/2-m}\left(\left|1 - \frac{s}{t}\right| + \nu^{-1/2}\right)^{-1} & \left(\text{if } |t| > \frac{1}{10},\ |s - 1| < C\nu^{-1/2}\right)\\ 0 & (\text{otherwise}),\end{cases}$$

we have

$$\|Q(s, t)\|_{L^r(s^{d-1}ds)} \lesssim \nu^{1/2r} \quad \text{for any } t,$$
$$\|Q(s, t)\|_{L^r(t^{d-1}dt)} \lesssim \nu^{1/2r} \quad \text{for any } s,$$

provided m has been chosen large enough that there is no trouble at infinity. Lemmas 2.1, 2.2 therefore imply

$$\|T|y|^{-m}\|_{L^p(\Omega)\to L^{p'}(D)} \lesssim \nu^{|\alpha|-1}\nu^{(3d-2)/4r-1/2} \tag{5.4}$$

and therefore also (by Hölder's inequality)

$$\|T|y|^{-m}\|_{L^p(\Omega)\to L^p(D)} \lesssim \nu^{|\alpha|-1}\nu^{(d-2)/4r-1/2}.$$

Lemma 5.2 follows.

In Lemma 5.2 we needed the assumption $r \geqslant d/2$ because otherwise the «fractional integration» part M_ν^α was unbounded if $\alpha = 0$. On the other hand the estimates in the above argument work perfectly well if $r < d/2$. We record what we proved in case $\alpha = 0$ for use in connection with the Chanillo-Sawyer results at the end of the section.

Lemma 5.2. *Let m be a sufficiently large fixed positive number. If e is a unit vector,* $D = D(e, \nu^{-1/2})$, $\Omega = \{x : x \cdot e > 0, |x| > 1/10\}$, ψ *and* ψ_λ *are as above,* $S(x,y) = \psi(x,y)N_\nu^0(x,y)$, $T(x,y) = (1-\psi(x,y))N_\nu^0(x,y)$, *then for* $f \in L^p(\Omega)$, $1 \leqslant 2^j \leqslant \nu^{1/2}$

(i) $\|(\psi_\lambda S)f\|_{p'} \lesssim \nu^{-1}D(\nu,\lambda)\lambda^{-1/r}\|f\|_p$,

(ii) $\|Tf\|_{p'} \lesssim \nu^{(3d-2)/4r-3/2}\||y|^m f\|_p$.

PROOF. (i) Follows directly from (5.3) and (ii) from (5.4).

To prove Theorem 5.1 we need a version of Lemma 5.1 with the Laplacian replaced by a variable coefficient operator. This will work out because the disc D is small enough to permit a good approximation of the variable coefficient operator by constant coefficient one. Since the coefficients are $C^{1+\eta}$ and the linear term can be eliminated by changing coordinates (geodesic normal coordinates or something similar) we expect an approximation on D to within $O(\nu^{-1/2(1+\eta)})$, *i.e.*, the following lemma.

Lemma 5.4. *There is constant* $\beta > 0$ *so that if L is as in Theorem* 1, *B is a suitable constant depending on ellipticity bounds for L in* $\|x\| < 1$, *then for any* $\tau > 0$, *if* $a \in \mathbb{R}^d$ *is such that* $\|a\| < 1$ *and* a_d *is sufficiently small (depending on bounds for ellipticity and* $C^{1+\eta}$ *bounds for coefficients when* $|x| < 1$), *and if* ν *is sufficiently large and if we set* $\bar{D}_a = D(a, a_d/B\sqrt{\nu})$ *then*

(i) $\|x_d^{-\nu}H_f\|_{L^p(\bar{D}_a)} \lesssim \nu^{(d-2)/4r+1/2}(\|\Gamma_a^{-\beta\nu}x_d^{-\nu}Lf\|_p + E_a)$

(ii) $\|x_d^{-\nu}\nabla f\|_{L^p(\bar{D}_a)} \lesssim \nu^{(d-2)/4r-1/2}(\|\Gamma_a^{-\beta\nu}x_d^{-\nu}Lf\|_p + E_a)$

(iii) $\|x_d^{-\nu}f\|_{L^{p'}(\bar{D}_a)} \lesssim \|\Gamma_a^{-\beta\nu}x_d^{-\nu}Lf\|_p + E_a$

where

$$E_a = (\nu^{-1/2}\tau)^{1+\eta}\|x_d^{-\nu}\Gamma_a^{-\beta\nu}H_f\|_p + \|x_d^{-\nu}\Gamma_a^{-\beta\nu}\nabla f\|_p.$$

The proof will be by a suitable change of variables and approximation by the Laplace operator as described above. Before getting into this, let us use Lemma 5.4 to prove Theorem 5.1. We need one other (elementary) lemma.

Lemma 5.5. *Let $\{a^j\}$ be a collection of points such that the discs*

$$\bar{D}_j = D\left(a^j, \frac{a_d^j}{B\sqrt{\nu}}\right)$$

cover $\mathbb{R}_+^d$ and have finite overlap. Then for any $\alpha > 0$, $\sum_j \Gamma_{a^j}(x)^{-\alpha\nu} \leqslant C_\alpha$ independently of $x \in \mathbb{R}_+^d$ and ν, provided ν is sufficiently large.

PROOF. We first observe that if $|a - b| < a_d/\sqrt{\nu}$ then $\Gamma_a(x)^{-\nu} \leqslant C\Gamma_b(x)^{-\alpha_0\nu}$ for suitable constants C and α_0. This may be seen as follows: If $|x - a| < 2a_d/\sqrt{\nu}$ then both sides are ≈ 1. If

$$|x - a| > \frac{2a_d}{\sqrt{\nu}} \quad \text{then} \quad \frac{|x-a|^2}{a_d x_d} \approx \frac{|x-b|^2}{b_d x_d}$$

so for suitable $\alpha_0 < 1$,

$$\begin{aligned} 1 + \frac{|x-a|^2}{4a_d x_d} &\geqslant 1 + \alpha_0 \frac{|x-b|^2}{4b_d x_d} \\ &\geqslant \left(1 + \frac{|x-b|^2}{4b_d x_d}\right)^{\alpha_0}, \end{aligned}$$

and

$$\left(1 + \frac{|x-a|^2}{4a_d x_d}\right)^{-\nu} \leqslant \left(1 + \frac{|x-b|^2}{4b_d x_d}\right)^{-\alpha_0\nu}.$$

It follows that the discrete sum in Lemma 5.5 may be replaced by an integral *i.e.*

$$\sum_j \Gamma_{a^j}(x)^{-\alpha\nu} \lesssim \int_{\mathbb{R}_+^d} \Gamma_a(x)^{-\alpha_0\alpha\nu} \frac{da}{(a_d/\sqrt{\nu})^d}.$$

The integral is scale invariant so we may assume $x_d = 1$. Then (let $\beta = \alpha_0\alpha$)

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}} \left(1 + \frac{|x-a|^2}{4a_d x_d}\right)^{-\beta\nu} da_1 \cdots da_{d-1} \\ &\qquad = \left[\frac{4a_d}{(1+a_d)^2}\right]^{\beta\nu} \int_{\mathbb{R}^{d-1}} \left(1 + \left(\frac{t}{1+a_d}\right)^2\right)^{-\beta\nu} dt_1 \cdots dt_{d-1} \\ &\qquad \approx \left(\frac{4a_d}{(1+a_d)^2}\right)^{\beta\nu} \left(\frac{1+a_d}{\sqrt{\nu}}\right)^{d-1} \end{aligned}$$

$$\int_{\mathbb{R}^d_+} \Gamma_a^{-\beta\nu} \frac{da}{(a_d/\sqrt{\nu})^d} \approx \nu^{1/2} \int_0^\infty \left(\frac{4a_d}{(1+a_d)^2}\right)^{\beta\nu} (1+a_d)^{d-1} \frac{da_d}{a_d^d}$$
$$\lesssim \nu^{1/2} \int_0^\infty \left(\frac{4a_d}{(1+a_d)^2}\right)^{\beta\nu - d + 1} \frac{da_d}{a_d}$$
$$\lesssim 1$$

for large ν, since the integrand dies rapidly when $|a_d - 1| > \nu^{-1/2}$.

PROOF OF THEOREM 5.1. By the usual Carleman argument it will suffice to prove that there is $\rho > 0$ depending only on ellipticity and $C^{1+\eta}$ bounds for L inside the unit disc, such that

$$(5.5) \qquad \|x_d^{-\nu} f\|_{p'} + \nu^\epsilon \|x_d^{-\nu} \nabla f\|_p \lesssim \|x_d^{-\nu} Lf\|_p$$

for all $f \in C_0^\infty(D(0,\rho) \cap \mathbb{R}^d_+)$, where $\epsilon = 1/2 - (d-2)/4r > 0$. We remark that in addition to (5.5) we will also prove

$$(5.5)' \qquad \|x_d^{-\nu} H_f\|_p \lessdot \nu^{1-\epsilon} \|x_d^{-\nu} Lf\|_p$$

for $f \in C_0^\infty(D(0,\rho) \cap \mathbb{R}^d_+)$, with ϵ as above. To choose ρ, we first choose a small enough τ and then choose ρ according to the «sufficiently small» in Lemma 5.4.

To prove (5.5), (5.5)′, cover $D(0,\rho)$ with discs $\bar{D}_j = (a^j, a_d^j/B\sqrt{\nu})$ with finite overlap, write down the conclusion of Lemma 5.4(i) for each a^j, raise to the power p, and sum over j, obtaining

$$\sum_j \int_{\bar{D}_j} |x_d^{-\nu} H_f|^p \lesssim \nu^{p(d-2)/4r + p/2} \left(\sum_j \int |x_d^{-\nu} Lf|^p \Gamma_{aj}^{-\beta\nu p} + \sum_j E_{aj}^p \right).$$

Do the same for the conclusions of Lemma 5.4(ii) and (iii) after multiplying through by ν and $\nu^{(d-2)/4r + 1/2}$ respectively. Writing out E_{aj} in longhand, we get

$$\sum_j \int_{\bar{D}_j} |x_d^{-\nu} H_f|^p + \nu^p \sum_j \int_{\bar{D}_j} |x_d^{-\nu} \nabla f|^p + \sum_j \nu^{p(d-2)/4r + p/2} \left(\int_{\bar{D}_j} |x_d^{-\nu} f|^{p'} \right)^{p/p'}$$
$$\lesssim \nu^{p(d-2)/4r + p/2} \sum_j \left\{ \int |x_d^{-\nu} Lf|^p \Gamma_{aj}^{-\beta\nu p} + (\nu^{-1/2} t)^{p(1+\eta)} \int |x_d^{-\nu} H_f|^p \Gamma_{aj}^{-\beta\nu p} \right.$$
$$\left. + \int |x_d^{-\nu} \nabla f|^p \Gamma^{-\beta\nu p} \right\}.$$

The $\bar{D}_j$ cover the support of f so the first sum on the left hand side dominates $\|x_d^{-\nu} H_f\|_p^p$. Likewise the second sum dominates $\nu^p \|x_d^{-\nu} \nabla f\|_p^p$, and (using that $\ell^1 \subset \ell^{p/p'}$) the third sum dominates $\nu^{p(d-2)/4r + p/2} \|x_d^{-\nu} f\|_{p'}^p$. On the right side we use Lemma 5.5 to bound $\sum \Gamma_{aj}^{-\beta\nu p}$ by a constant, and we also use $a_d \leqslant \rho$. The resulting inequality is

$$\|x_d^{-\nu} H_f\|_p^p + \nu^p \|x_d^{-\nu} \nabla f\|_p^p + \nu^{p(d-2)/4r+p/2} \|x_d^{-\nu} f\|_{p'}^p$$
$$\lesssim \nu^{p(d-2)/4r+p/2} (\|x_d^{-\nu} Lf\|_p^p + (\nu^{-1/2}\tau)^{(1+\nu)p} \|x_d^{-\nu} H_f\|_p^p + \|x_d^{-\nu} \nabla f\|_p^p).$$

Our assumption on r says that $\nu^{p(d-2)/4r+p/2} \nu^{-(1+\nu)p/2} = 1$. So for small τ we can bootstrap the second term on the right side. We can also bootstrap the third term obtaining

$$\|x_d^{-\nu} H_f\|_p^p + \nu^p \|x_d^{-\nu} \nabla f\|_p^p + \nu^{p(d-2)/4r+p/2} \|x_d^{-\nu} f\|_{p'}^p$$
$$\lesssim \nu^{p(d-2)/4r+p/2} \|x_d^{-\nu} Lf\|_p^p$$

which is equivalent to (5.5), (5.5)′.

It remains to prove Lemma 5.4. The following lemma provides the necessary change of coordinates.

Lemma 5.6. *Suppose L is as in Theorem* 5.1. *Then for $p \in \mathbb{R}^d$ and $\epsilon > 0$ small enough there is a C^2 (actually C^∞) diffeomorphism $T: \mathbb{R}^d \to \mathbb{R}^d$ with*

(i) *$T(p) = p$, and there is a linear map $S: \mathbb{R}^d \to \mathbb{R}^d$ with $(Sx)_d = x_d$ and such that $Tx = p + S(x - p)$ when $|x - p| > \epsilon$.*
(ii) $|DT(x) - S| \lesssim |x - p|$, $\quad |D^2T(x)| \lesssim 1$, $\quad |DT^{-1}(x) - S^{-1}| \lesssim |x - p|$, $|D^2T^{-1}(x)| \lesssim 1$.
(iii) *$L(f \circ T) = (\Delta f) \circ T + O(|x - p|^{1+\eta}|H_f| \circ T) + O(|\nabla f| \circ T)$ when $|x - p| < \epsilon/2$.*

All bounds depend only on ellipticity and $C^{1+\eta}$ bounds for L near p (and not on ϵ).

PROOF. Let us first do the case where $a_{ij}(p) = \delta_{ij}$ (Kronecker delta). We would like to start by choosing geodesic normal coordinates. We do not actually have geodesic normal coordinates unless a_{ij} is C^2, but we can use a second order approximation to them. We give this construction for the reader's convenience:

By assumption we have $a_{ij}(x) = \delta_{ij} = \ell_{ij}(x - p) + O(|x - p|^{1+\eta})$ where ℓ_{ij} are linear functions with $\ell_{ij} = \ell_{ji}$. We claim there are second order homogeneous polynomials Q_i with

$$\ell_{ij} + \frac{dQ_i}{dx_j} + \frac{dQ_j}{dx_i} = 0.$$

To see this let $a_{ijk} = \dfrac{1}{2}\left(\dfrac{d\ell_{jk}}{dx_i} - \dfrac{d\ell_{ik}}{dx_j} - \dfrac{d\ell_{ij}}{dx_k}\right)$. Then $a_{ijk} = a_{ikj}$ and

$$a_{ijk} + a_{jik} + \frac{d\ell_{ij}}{dx_k} = 0.$$

It follows that there are homogeneous quadratics Q_i with $\frac{d^2Q_i}{dx_j dx_k} = a_{ijk}$ and that they satisfy

$$\frac{d^2Q_i}{dx_j dx_k} + \frac{d^2Q_j}{dx_i dx_k} + \frac{d\ell_{ij}}{dx_k} = 0$$

for all i, j, k and $\frac{d^2Q_j}{dx_j} + \frac{d^2Q_j}{dx_i} + \ell_{ij} = 0$.

Now define $T(x) = x + \phi(x)Q(x-p)$ where $Q = (Q_1 \cdots Q_d)$ and $\phi(x) = 1$ when $|x-p| < \epsilon/2$, $\phi(x) = 0$ when $|x-p| > \epsilon$ and $|D^\alpha \phi| \lesssim \epsilon^{-|\alpha|}$.

Then $T(x) = x$ if $x = p$ or $|x-p| > \epsilon$, and a short calculation proves $|D^2T| \lesssim 1$, $|DT(x) - I| \lesssim |x-p|$. This implies if ϵ is small that T is a diffeomorphism of $\mathbb{R}^d$ and T^{-1} satisfies the same bounds. Thus it remains to prove (iii). By choice of Q we have

$$\sum_{i,j} (\delta_{ij} + \ell_{ij}(x-p))\left(\delta_{im} + \frac{dQ_m}{dx_i}(x-p)\right)\left(\delta_{jn} + \frac{dQ_n}{dx_j}(x-p)\right)$$
$$= \delta_{mn} + O(|x-p|^2).$$

On the other hand, if we denote $y = Tx$ then with the summation convention

$$L(f \circ T)(x) = a_{ij}\frac{dy_m}{dx_i}\frac{dy_n}{dx_j}\frac{d^2f}{dy_m dy_n} + a_{ij}\frac{d^2y_k}{dx_i dx_j}\frac{df}{dy_k}. \tag{5.6}$$

Up to terms of order $|x-p|^{1+\eta}$ we have $a_{ij} = \delta_{ij} + \ell_{ij}(x-p)$ and

$$\frac{dy_m}{dx_i} = \delta_{im} + \frac{dQ_m}{dx_i}(x-p)$$

when $|x-p| < \epsilon$. It follows that the first term on the right in (5.6) is $O(|x-p|^{1+\eta}|H_f \circ T|)$. The second term is clearly $O(|\nabla f| \circ T)$ so we are done with the case where $a_{ij}(p) = \delta_{ij}$. We can reduce the general case to this case by a preliminary affine change of variables: explictly, let $A = (a_{ij}(p))$ and choose S so that $S^*S = A^{-1}$ and $(Sx)_d = x_d$ (this is possible for any positive symmetric A^{-1}). The preceding argument then applies to the operator $L(f \circ S) \circ S^{-1}$ in place of L giving a change of variable T_0, and we let $T = S \circ T_0$.

Now, in the situation of Lemma 5.4, we can apply Lemma 5.6 with $p = a$ and $\epsilon = a_d/2$ (provided $|a| < 1$ and a_d is small enough). We obtain a change of variables T_a. In order to proceed we now need some further elementary properties of our weights especially as to how they behave under T_a.

Lemma 5.7.

(i) $\Gamma_a^{-\nu} \circ T_a \lesssim \Gamma_a^{-\gamma\nu}$ *for a suitable fixed constant* γ.
(ii) *For any* $\epsilon > 0$ *we will have* $x_d^{-\nu} \circ T_a \lesssim x_d^{-\nu}\Gamma_a^{\epsilon\nu}$ *provided* a_d *is sufficiently small.*
Moreover (i) *and* (ii) *also hold for* T_a^{-1}.
(iii) *For any given* k, $|x - a|^k\Gamma_a^{-\nu} \lesssim (\nu^{-1/2}a_d)^k\Gamma_a^{-\nu/2}$ *provided* ν *is large enough.*

PROOF. Part (iii) follows from the rapid decay of Γ_a outside D_a. To prove (i) and (ii) we consider the regions $|x - a| < a_d/2$ and $|x - a| > a_d/2$ separately. We abbreviate T_a by T.

In the first region, we have $(Tx)_d = x_d + O(|x - a|^2)$ by (ii) of Lemma 5.6 (and the fact that $(Sx) = x_d$) and therefore

$$(Tx)_d \geqslant x_d\left(1 - C\frac{|x - a|^2}{4x_d}\right).$$

If ϵ is given then by making a_d small we make this larger than or equal to $x_d(1 - C\epsilon\,|x - a|^2/4a_dx_d) \geqslant x_d \cdot \Gamma_a(x)^{-C'\epsilon}$ which gives (ii). As for (i), we have Lipschitz bounds on T and therefore $\Gamma_a(Tx) \geqslant (1 + C\,|x - a|^2/4a_dx_d)$ for suitable C hence $\Gamma_a(Tx) \geqslant \Gamma_a(x)^{C'}$. In the region $|x - a| > a_d/2$ (ii) is a tautology and (i) again follows from the Lipschitz property of T. Since all of this followed from (i) and (ii) of Lemma 5.6 it also holds for T^{-1}.

PROOF OF LEMMA 5.4. We will prove (i) and then indicate the modifications necessary to get (ii) and (iii). Where B was unspecified, we now specify it so be an upper bound for the Lipschitz norms of the T_a and T_a^{-1}. σ below is another fixed constant. $D = D(a, a_d/\sqrt{\nu})$ and $\bar{D} = D(a, a_d/B\sqrt{\nu})$.

We justify the following string of inequalities below.

$$\begin{aligned}
\|x_d^{-\nu}H_f\|_{L^p(\bar{D})} &\lesssim \|(x_d^{-\nu}H_f) \circ T^{-1}\|_{L^p(D)} \\
&\lesssim \|(T^{-1}x)_d^{-\nu}H_{f\circ T^{-1}}\|_{L^p(D)} + \|(T^{-1}x)_d^{-\nu}\nabla(f \circ T^{-1})\|_{L^p(D)} \\
&\lesssim \|x_d^{-\nu}H_{f\circ T^{-1}}\|_{L^p(D)} + \|x_d^{-\nu}\nabla(f \circ T^{-1})\|_{L^p(D)} \\
&\lesssim \nu^{(d-2)/4r+1/2}\|\Delta(f \circ T^{-1})x_d^{-\nu}\Gamma_s^{-\nu}\|_p \qquad (5.7)\\
&\lesssim \nu^{(d-2)/4r+1/2}(\|(Lf) \circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \qquad (5.8)\\
&\qquad + \|\,|x - a|^{1+\eta}H_f \circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \\
&\qquad + \|(\nabla f) \circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \\
&\qquad + \|x_d^{-\nu}\Gamma_a^{-\nu}H_f \circ T^{-1}\|_{L^p(|x-a|>(a_d/4B^2))})
\end{aligned}$$

$$\begin{aligned}\|x_d^{-\nu}H_f\|_{L^p(\bar D)} &\lesssim \nu^{(d-2)/4r+1/2}(\|(Lf)\circ T^{-1}x_d^{-\nu}\Gamma_a^{\nu}\|_p \\ &\qquad + (\nu^{-1/2}a_d)^{1+\eta}\|H_f\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu/2}\|_p \\ &\qquad + \|(\nabla f)\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \\ &\qquad + \nu^{-100}\|x_d^{-\nu}\Gamma_\alpha^{-\nu/2}H_f\circ T^{-1}\|_p) \\ &\lesssim \nu^{(d-2)/4r+1/2}(\|x_d^{-\nu}\Gamma_a^{-\sigma\nu}Lf\|_p \\ &\qquad + [(\nu^{-1/2}a_d)^{1+\eta}+\nu^{-100}]\|x_d^{-\nu}\Gamma_a^{-\sigma\nu}H_f\|_p \\ &\qquad + \|x_d^{-\nu}\Gamma_a^{-\sigma\nu}\nabla f\|_p).\end{aligned}$$

Justification. First inequality: T is bilipshitz and $T^{-1}D\subseteq\bar D$. Second inequality: $|H_f\circ T^{-1}|\lesssim|H_{f\circ T^{-1}}|+|\nabla(f\circ T^{-1})|$ if T has bounded first and second derivatives. Third inequality: Lemma 5.7 (ii) and the fact that $\Gamma_a^{-\nu}\approx 1$ on D. Forth inequality: Lemma 5.1. Fifth inequality (5.7) $\lesssim$ (5.8): split $\mathbb{R}_+^d$ in the two regions $|x-a|<a_d/4B$ and $|x-a|>a_d/4B$. The last two terms in (5.8) bound the L^p norm in (5.7) over the region $|x-a|>a_d/4B$ by the same calculation that was used to justify the second inequality. When $|x-a|<a_d/4B$ we know that $|T_ax-a|<a_d/4$, so we can write down (iii) of Lemma 5.6 for the function $f\circ T^{-1}$ and then compose with T^{-1} obtaining

$$(Lf)\circ T^{-1}=\Delta(f\circ T^{-1})+O(|T^{-1}x-a|^{1+\eta}|H_{f\circ T^{-1}}|)+O(|\nabla(f\circ T^{-1})|).$$

Also $|T^{-1}x-a|$ is comparable with $|x-a|$ and $|H_{f\circ T^{-1}}|$ may be replaced by $|H_f\circ T^{-1}|+|(\nabla f)\circ T^{-1}|$ and $|\nabla(f\circ T^{-1})|$ by $|\nabla f\|\circ T^{-1}$ as in the second inequality. This shows (5.7) $\leqslant$ (5.8). To pass from (5.8) to the next line we estimate the second term in the parenthesis using Lemma 5.7 (iii) and the last term using that $\Gamma_a^{-\nu/2}$ is $\lesssim\nu^{-k}$ for any fixed k when $|x-a|>a_d/4B$. Finally the last inequality follows by reversing the change of variables and using Lemma 5.7(i) and (ii). One can take $\sigma=\gamma/5$ with γ as in Lemma 5.7(i). Lemma 5.4 follows since $(\nu^{-1/2}a_d)^{1+\eta}+\nu^{-100}$ may be made less than $(\tau\nu^{-1/2})^{1+\eta}$ for any given τ by making ν large and a_d small.

To obtain (ii) of Lemma 5.4, we start with $\|x_d^{-\nu}\nabla f\|_{L^p(\bar D)}$, make the change of variables T^{-1} and then use (ii) of Lemma 5.1 to obtain an expression like (5.7) but with $\nu^{(d-2)/4r-1/2}$ in place of $\nu^{(d-2)/4r+1/2}$. To obtain (iii) of Lemma 5.4 we start with $\|x_d^{-\nu}f\|_{L^{p'}(\bar D)}$ and proceed the same.

Now we prove a refinement of a result of Chanillo-Sawyer [4]. Recall (see [4] and references there) that the «Fefferman-Phong class» F_r is defined by the condition $V\in F_r$ if and only if $\|V\|_{L^r(B)}\leqslant C_V t^{d/r-2}$ for all t and all balls B of radius t. The F_r norm is the smallest possible C_V.

Theorem 5.2. *Suppose $d\geqslant 4$, $r>d/2-1$ and V is a function with sufficiently small F_r norm. Then the inequality $|\Delta u|\leqslant V|u|$ has the WUCP in the sense that if $u\in W^{2,p}_{\mathrm{loc}}$, $|\Delta u|\leqslant V|u|$, u vanishes on an open set then $u=0$.*

Remark. Chanillo-Sawyer required $r > d/2 - 1/2$ but they also treated the SUCP. An identical result to Theorem 5.2 has been proved independently by Ruiz-Vega [12] by a different argument. It should be pointed out that their version was circulated several months before ours. We include the result here only because it follows very easily from what we have been doing.

PROOF. By an argument in [4], it suffices to prove the following:

Lemma 5.8. *If $V \in F_r$ then we have $\|x_d^{-\nu} V u\|_2 \lesssim \|x_d^{-\nu} V^{-1} \Delta u\|_2$ for all $u \in C_0^\infty(\mathbb{R}_+^d)$ and sufficiently large ν.*

Covering $\mathbb{R}_+^d$ by a family of discs $D(a^j, \nu^{-1/2} a_d^j)$ and using Lemma 5.5 as in the proof of Theorem 5.1, we see it will suffice to prove

Lemma 5.8′. *If $V \in F_r$ then $\|x_d^{-\nu} V u\|_{L^2(D_a)} \lesssim \|\Gamma_a^{-\nu/2} x_d^{-\nu} V^{-1} \Delta u\|_{L^2}$ uniformly in $a \in \mathbb{R}_+^d$ and ν sufficiently large, and $u \in C_0^\infty(\mathbb{R}_+^d)$.*

This is a variant on Lemma 5.1 and again we reduce to a lemma in terms of the weights $|x|^{-\nu}$.

Lemma 5.8″. *If k is a large enough fixed positive number then, letting e be a unit vector, ν large enough, and defining*

$$D = D(e, \nu^{-1/2}), \qquad \Omega = \{x: |x| > 1/10, x \cdot e > 0\}$$

we have

$$\| |x|^{-\nu} V u \|_{L^2(D)} \lesssim \|(1 + \nu^{1/2}|x - e|)^k |x|^{-\nu} V^{-1} \Delta u\|_{L^2(\Omega)}$$

for all $u \in C_0^\infty(\Omega)$.

PROOF OF LEMMA 5.8′. (assuming 5.8″). We may take a to be such that $a_d = 1/2$.

Using Lemma 5.8″ and the argument in the proof of Lemma 5.1, we obtain

$$\|x_d^{-\nu} V u\|_{L^2(D_a)} \lesssim \|\Gamma_s^{-\nu}(1 + \nu^{1/2}|x - e|)^k x_d^{-\nu} V^{-1} \Delta u\|_{L^2}.$$

However, $(1 + \nu^{1/2}|x - e|)^k \Gamma_a^{-\nu} \lesssim \Gamma_a^{-\nu/2}$ by part (iii) of Lemma 5.6.

PROOF OF LEMMA 5.8″. It will suffice to show

$$\|V^{1/2} L_\nu V^{1/2} f\|_{L^2(D)} \lesssim \|(1 + \nu^{1/2}|x - e|)^k f\|_{L^2}. \tag{5.9}$$

If $f \in L^2(\Omega)$, where $L_\nu = L_\nu^0$. It is known (see references in [4]) that this kind of inequality is true if L_ν is replaced by $|x - y|^{2-d}$ (and therefore also if L_ν is replaced by M_ν) provided $V \in F_r$ for some $r > 1$. So it will suffice to prove (5.9) with L_ν replaced by N_ν. We write $N_\nu = S + T$ as in the proof of Lemma 5.2 and will prove (5.9) with L_ν replaced by N_ν. We write $N_\nu = S + T$ as in the proof of Lemma 5.2 and will prove (5.9) separately for S and for T. To deal with S, write $S = \sum_{\substack{\nu^{-1} \leq \lambda \leq \nu^{-1/2} \\ \lambda = 2^j \nu}} \psi_\lambda S$ as in the proof of Lemma 5.2. To estimate

$$\| V(x)^{1/2} \psi_\lambda(x, y) S(x, y) V(y)^{1/2} \|_{L^2(\Omega) \to L^2(D)}. \tag{5.10}$$

It will suffice (by a partition of unity and Cotlar's Lemma as in the proof of Lemma 2.4) to estimate the action on fuctions supported in a ball of radius λ. If B is such a ball, f vanishes off B and $\tilde{B}$ is a suitable fixed multiple of B then, using Hölder's inequality and Lemma 5.2′,

$$\begin{aligned} \| V^{1/2}(\psi_\lambda S) V^{1/2} f \|_2 &\leqslant \| V \|_{L^r(\tilde{B})}^{1/2} \| (\psi_\lambda S) V^{1/2} f \|_{p'} \\ &\lesssim \| V \|_{L^r(\tilde{B})}^{1/2} \| V^{1/2} f \|_{p'} \nu^{-1} D(\nu, \lambda) \lambda^{-1/r} \\ &\lesssim \| V \|_{L^r(\tilde{B})} \| f \|_2 \nu^{-1} D(\nu, \lambda) \lambda^{-1/r}. \end{aligned}$$

Now use the definition of F_r to estimate $\| V \|_{L^r(\tilde{B})} \lesssim \lambda^{d/r - 2}$. Substituting this in we obtain an estimate (5.10) $\lesssim (\nu\lambda)^{(d-2)/2r - 1}$. By our assumption on r the power of $\nu\lambda$ here is negative and we may sum over λ to obtain

$$\| V^{1/2} S V^{1/2} \|_{L^2(\Omega) \to L^2(D)} \lesssim 1.$$

Now we have to consider T. Write

$$T(x, y) = \sum_{j=0}^{\infty} T(x, y) \phi_j(y) = \sum_{j=0}^{\infty} T_j(x, y),$$

where

$$\phi_j = \begin{cases} 1, & \text{if } \ 2^j \nu^{-1/2} < |y - e| < 2^{j+1} \nu^{-1/2}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider a given T_j. We know from Lemma 5.2′ that T is bounded from $L^p(\Omega, |y|^m\, dy)$ to $L^{p'}(D, dx)$ with norm $\lesssim \nu^{(3d-2)/4r - 3/2}$. Therefore T_j is bounded from $L^p(\Omega, |y|^m\, dy)$ to $L^{p'}(D, dx)$ with norm $\lesssim \nu^{(3d-2)/4r - 3/2}$. Therefore T_j is bounded from $L^p(\Omega, dy)$ to $L^{p'}(D, dx)$ with norm

$$\lesssim (1 + 2^j \nu^{-1/2})^m \nu^{(3d-2)/4r - 3/2}.$$

Using Hölder's inequality as in the estimation for S, $V^{1/2} T V^{1/2}$ is bounded from $L^2(\Omega, dy)$ to $L^2(D, dx)$ with norm

$$
\begin{aligned}
&\lesssim (1 + 2^j \nu^{-1/2})^m \nu^{(3d-2)/4r-3/2} \|V\|^{1/2}_{L^r(D(e,2^{j+1}\nu^{-1/2}))} \|V\|^{1/2}_{L^r(D(e,\nu^{-1/2}))} \\
&\lesssim (1 + 2^j \nu^{-1/2})^m \nu^{(3d-2)/4r-3/2} (2^j \nu^{-1/2})^{d/2r-1} \nu^{-(d/2r-1)/2} \\
&= 2^{j(d/2r-1)} (1 + 2^j \nu^{-1/2})^m \nu^{(d-2)/4r-1/2} \\
&\leqslant 2^{j(d/2r-1)} (1 + 2^j \nu^{-1/2})^m
\end{aligned}
$$

by choice of r. The last expression may be bounded by $2^{-j}(1 + \nu^{1/2}|y - e|)^k$ for appropriate k if y is such that $T_j(x, y)$ is nonzero for some x, and therefore we obtain

$$\|V^{1/2} T_j V^{1/2} f\|_2 \lesssim 2^{-j} \|(1 + \nu^{1/2}|y - e|)^k f\|_2.$$

We may now sum over j to obtain

$$\|V^{1/2} T V^{1/2} f\|_2 \lesssim \|(1 + \nu^{1/2}|y - e|)^k f\|_2$$

and we are done.

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