

Pointwise and Spectral Control of Plate Vibrations

Alain Haraux and Stéphane Jaffard

Résumé

On considère le problème du contrôle ponctuel (c'est-à-dire au moyen d'une masse de Dirac située en un point fixé) des vibrations d'une plaque Ω . Sous des conditions aux limites générales, incluant les plaques posées ou encastées, mais excluant (et pour cause) le cas où existent des vibrations propres multiples, nous montrons la contrôlabilité des combinaisons linéaires finies des fonctions propres en tout point de Ω qui n'est zéro d'aucune fonction propre et en tout temps strictement supérieur à la moitié de la surface de la plaque. Ce résultat est optimal car aucune combinaison linéaire finie non nulle de fonctions propres n'est ponctuellement contrôlable en un temps strictement inférieur à la moitié de la surface de la plaque. Sous la même condition sur le temps, mais pour un domaine Ω quelconque de \mathbb{R}^2 , on résout le problème du contrôle spectral interne, c'est-à-dire que pour tout disque ouvert $\omega \subset \Omega$, une combinaison linéaire finie quelconque des fonctions propres peut être ramenée à l'équilibre au moyen d'un contrôle $h \in \mathcal{D}((0, T) \times \Omega)$ tel que $\text{supp}(h) \subset (0, T) \times \omega$.

Abstract

We consider the problem of controlling pointwise (by means of a time dependent Dirac measure supported by a given point) the motion of a vibrating plate Ω . Under general boundary conditions, including the special cases of simply

supported or clamped plates, but of course excluding the cases where some multiple eigenvalues exist for the biharmonic operator, we show the controllability of finite linear combinations of the eigenfunctions at any point of Ω *where no eigenfunction vanishes* at any time greater than half of the plate's area. This result is optimal since *no* finite linear combination of the eigenfunctions other than 0 is pointwise controllable at a time smaller than half of the plate's area. Under the same condition on the time, but for an *arbitrary* domain Ω in \mathbb{R}^2 , we solve the problem of *internal* spectral control, which means that for any open disk $\omega \subset \Omega$, any finite linear combination of the eigenfunctions can be set to equilibrium by means of a control function $h \in \mathcal{D}((0, T) \times \Omega)$ supported in $(0, T) \times \omega$.

1. Introduction and Functional Setting

In order to make the theory more transparent, we shall consider the general case of a second order conservative evolution equation and apply only at the end our abstract results to the specific case of a 2-dimensional vibrating plate. Let Ω be a bounded domain of \mathbb{R}^N (or a compact N -dimensional manifold without boundary) and A a positive self-adjoint operator in $H = L^2(\Omega)$. We assume that A satisfies the following properties

- (1.1) A is *coercive* on H .
(1.2) $D(A^{1/2}) \subset C(\bar{\Omega})$ with continuous imbedding.

Given $T > 0$, $\xi \in \Omega$ and $[y^0, y^1] \in D(A^{1/2}) \times L^2(\Omega)$, we are interested in the existence of a *control function* $h \in L^2(]0, T])$ such that $\text{supp}(h) \subset]0, T]$ and for which the unique generalized solution y of

$$(1.3) \quad \begin{cases} y'' + Ay = h(t)\delta(x - \xi) & \text{in }]0, T[, \\ y(0, x) = y^0(x), \\ y'(0, x) = y^1(x) & \text{in } \Omega, \end{cases}$$

satisfies $y(T, x) = y'(T, x) = 0$ in Ω . If such a control h exists, we shall say that the state $[y^0, y^1]$ is «pointwise exactly L^2 -controllable in ξ at time T ».

The possibility of solving this «pointwise exact controllability problem» is related to the amount of information revealed by the restriction to $]0, T]$ of $t \mapsto \phi(t, \xi)$ where ϕ is an arbitrary solution of the homogeneous equation

$$(1.4) \quad \phi'' + A\phi = 0 \quad \text{in }]0, T[, \quad \phi \in C(0, T; V) \cap C^1(0, T; H)$$

with $V = D(A^{1/2})$, $H = L^2(\Omega)$. (Note that as a consequence of (1.2) we have $\phi(t, \xi) \in C(]0, T])$ for any such solution ϕ). In fact if any $[y^0, y^1]$ from a *dense* subset of $V \times H$ is exactly L^2 -controllable in ξ at time T , then any solution

ϕ of the homogeneous equation (1.4) such that $\phi(t, \xi)$ vanishes identically on $]0, T[$ is the trivial solution $\phi \equiv 0$. *Conversely*, if any solution ϕ of the homogeneous equation (1.4) such that $\phi(t, \xi)$ vanishes identically on $]0, T[$ is the trivial solution $\phi \equiv 0$, then for each $(\phi^0, \phi^1) \in V \times H$, we consider the (clearly well-defined) *norm*

$$(1.5) \quad p(\phi^0, \phi^1) = \left\{ \int_0^T \phi^2(t, \xi) dt \right\}^{1/2},$$

where ϕ is the solution of equation (1.4) with initial data (ϕ^0, ϕ^1) . The following result then follows from the general HUM method of J. L. Lions ([17, 18, 19]).

Proposition 1.1. *A given state $[y^0, y^1] \in V \times H$ is exactly L^2 -controllable at ξ in time T if and only if there exists a constant $C \geq 0$ such that for every $(\phi^0, \phi^1) \in V \times H$,*

$$(1.6) \quad \left| \int_{\Omega} (\phi^0 y^1 - \phi^1 y^0) dx \right| \leq Cp(\phi^0, \phi^1).$$

As was clearly established in [6], the set of pointwise exactly L^2 -controllable states (always a *dense* subset of $V \times H$ when p is a norm) is usually complicated and more precisely depends on the observation point ξ in a very complicated and unstable way, even in the simplest case of the standard vibrating string with fixed end! The only reasonable thing to be expected in general is that (1.6) might hold true when both y^0 and y^1 are finite linear combinations of the eigenfunctions of A , assuming that no eigenfunction vanishes at ξ . This implies in particular that all eigenvalues of A are simple, a condition that we shall assume in most of this text (Sections 2, 3 and 5). The controllability of all states for which y^0 and y^1 are finite linear combinations of the eigenfunctions of A is what we shall call «pointwise spectral controllability». Taking account of the form of the general solutions to the homogeneous equation, it is natural to apply the methods of harmonic analysis to solve this problem. Indeed, any solution of (1.4) can be written as a series

$$\phi(t, x) = \sum \{ \phi_n(x) \cos \sqrt{\lambda_n} t + \psi_n(x) \sin \sqrt{\lambda_n} t \}$$

where the functions ϕ_n, ψ_n are eigenfunctions of A associated to the eigenvalues λ_n , or in complex form

$$\phi(t, x) = \sum \varphi_j(x) e^{i\mu_j t}$$

where the μ_j stand for the (positive or negative) square roots of the eigenvalues λ_n . Thus for fixed x , it is a linear combination of some complex exponentials, the properties of which will be the key point of this work. Our main result

will, maybe surprisingly, turn out to be a consequence of one among the deepest classical results on harmonic analysis from the «sixties», namely the *Beurling Malliavin criterion* for computing the *completeness radius of a family of complex exponentials*. The application of this powerful machinery to our problem is the object of Sections 2 and 3 of this paper. The case of a vibrating plate with constant Lamé coefficients is a special case of our abstract result obtained for $N = 2$ and $A = \Delta^2$ with relevant boundary conditions. In Section 4, we shall combine the result of Section 2 with, essentially, a biorthogonality technique in the spirit of [2, 16, 22] to solve the easier problem of «internal spectral controllability» under slightly relaxed conditions on the domain. However we feel that much more should be done in this last direction, as already strongly suggested by the special cases considered in [6, 11, 15]. Finally in Section 5, we consider some additional examples and we discuss the relationship between spectral controllability and some uniqueness questions.

2. Some Properties of the Completeness Radius of a Family of Complex Exponentials

The main tool from harmonic analysis that we shall use in this paper is the notion of completeness radius and its characterization by some estimates.

2.1. Definition and some properties of the Completeness Radius

Definition 2.1.1. *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of distinct real numbers. Consider all the functions of the form*

$$(2.1) \quad f(t) = \sum_{k \in J} f_k e^{i\lambda_k t},$$

J being any finite subset of \mathbb{Z} . The completeness radius of Λ is defined as

$$R(\Lambda) = \sup \{A > 0: \text{the functions } f \text{ of the form (2.1) are dense in } C([-A, A])\}.$$

In particular, if the functions f of the form (2.1) are dense in $C([-A, A])$ for all $A > 0$, we set $R(\Lambda) = \infty$. On the other hand, if the density fails for all $A > 0$, we set $R(\Lambda) = 0$.

Remark 2.1.2. A classical result from the theory of nonharmonic Fourier series (*cf. e.g.* [24, Theorem 8 p. 129]) asserts that either the functions of the form (2.1) are dense in $C([a, b])$, or no complex exponential of the form $e^{i\nu t}$ with ν different from all λ_n can be obtained as a limit of functions of the form (2.1) in $C([a, b])$. This interesting alternative is the main idea for the proof of Proposition 2.2.1 below.

Remark 2.1.3. For all $p \in [1, +\infty)$ we also have

$$R(\Lambda) = \sup \{A > 0: \text{the functions } f \text{ of the form (2.1)} \\ \text{are dense in } L^p([-A, A])\},$$

with the same conventions in the limiting cases $R(\Lambda) = 0, \infty$ (see [24]). In the sequel we shall be especially concerned by the case $p = 2$.

2.2. A «Density-Controllability» Alternative

The main result of this section is the following

Proposition 2.2.1. *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Then we have the following properties*

(1) *For each $T > 2R(\Lambda)$ and for each $n \in \mathbb{Z}$ there exists a constant C_n such that*

$$(2.2) \quad |f_n| \leq C_n \left\{ \int_0^T |f(t)|^2 dt \right\}^{1/2},$$

for each function f of the form (2.1) with $n \in J$.

(2) *On the other hand for each $T < 2R(\Lambda)$ and for each finite sequence $\{\alpha_n\}_{n \in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that*

$$(2.2)' \quad \left| \sum_{n \in F} \alpha_n f_n \right| \leq C \left\{ \int_0^T |f(t)|^2 dt \right\}^{1/2}$$

for each f of the form (2.1) with $F \subset J$.

As a first step we will establish the following lemma.

Lemma 2.2.2. *Let $I = (0, T)$ and let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that the set of functions of the form (2.1) is not dense in $L^2(I)$. Then, for each $n \in \mathbb{Z}$ there exist a constant C_n such that (2.2) holds for each function f of the form (2.1) with $n \in J$.*

PROOF. If (2.2) is not satisfied for some n , we can find a sequence of functions $\{f^p\}$ of the form (2.1) such that $|f_n^p| = 1$ and $\int_I |f^p(x)|^2 dx \rightarrow 0$ as $p \rightarrow +\infty$. It follows that the constant 1 is the limit in $L^2(I)$ of some functions g of the form

$$(2.3) \quad g(t) = \sum_{k \in J} g_k e^{i\mu_k t},$$

where J is a finite subset of $\mathbb{Z} - \{n\}$, and: $\mu_k = \lambda_k - \lambda_n$. By repeated integration in t , we deduce that all polynomials of t with complex coefficients are also limits in $L^2(I)$ of some functions g of the form (2.3). Indeed, let $p \in \mathbb{N}$, $\epsilon > 0$ and g of the form (2.3) be such that

$$\left\| t^p - \sum_{k \in J} g_k e^{i\mu_k t} \right\|_2 \leq \epsilon$$

where $\| \cdot \|_2$ stands for the norm in $L^2(I)$. By integrating in t , we deduce easily the estimate

$$\left\| t^{p+1} - (p+1) \sum_{k \in J} g_k \frac{e^{i\mu_k t}}{\mu_k} + (p+1) \sum_{k \in J} \frac{g_k}{\mu_k} \right\|_\infty \leq (p+1)\epsilon T^{1/2}$$

where $\| \cdot \|_\infty$ stands for the norm in $L^\infty(I)$. Hence, in particular

$$\left\| t^{p+1} - (p+1) \sum_{k \in J} g_k \frac{e^{i\mu_k t}}{\mu_k} + (p+1) \sum_{k \in J} \frac{g_k}{\mu_k} \right\|_2 \leq (p+1)\epsilon T.$$

Then by approximating the constant $(p+1) \sum_{k \in J} (g_k/\mu_k)$ in $L^2(I)$ by functions of the form (2.3), we find a sequence of coefficients $\{g'_k\}_{k \in J'}$ for which

$$\left\| t^{p+1} - \sum_{k \in J'} g'_k e^{i\mu_k t} \right\|_2 \leq 2(p+1)\epsilon T.$$

This proves the claim by induction on p since it has been proved already for $p = 0$. Finally by the Stone-Weierstrass density theorem, the functions g of the form (2.3) are dense in $L^2(I)$, and the same property follows at once for functions f of the form (2.1).

PROOF OF PROPOSITION 2.2.1. It follows clearly from the definition of $R(\Lambda)$ that for each $T > 2R(\Lambda)$ the functions f of the form (2.1) are not dense in $L^2(0, T)$, and therefore assertion 1) is an immediate consequence of Lemma 2.2.2. On the other hand for each *finite set* $F \subset \mathbb{Z}$, if we denote by Λ' the set $\{\lambda_n\}_{n \in \mathbb{Z} - F}$, then classically $R(\Lambda') = R(\Lambda)$. As a consequence for each $T < 2R(\Lambda)$, the set of functions f of the form (2.1) with $J \cap F = \emptyset$ is dense in $L^2(0, T)$, and therefore for each non trivial sequence $\{\alpha_n\}_{n \in F}$ of complex numbers, the function

$$a(t) = \sum_{k \in F} \alpha_k e^{i\lambda_k t}$$

can be approached in $L^2(0, T)$ by functions f of the form (2.1) with $J \cap F = \emptyset$. By taking the difference we find a sequence of functions of the form (2.1) tending to 0 in $L^2(0, T)$ and for which the left-hand side in (2.2)' is *constant and positive*. This clearly establishes assertion 2).

2.3. Computation of a Beurling-Malliavin Density

The main result of this section is the following

Theorem 2.3.1. *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d^+, d^- \geq 0$ and $0 \leq \alpha < 1$*

$$(2.4) \quad \#\{\lambda \in \Lambda: 0 \leq \lambda \leq t\} = d^+ t + O(t^\alpha)$$

and

$$\#\{\lambda \in \Lambda: -t \leq \lambda \leq 0\} = d^- t + O(t^\alpha).$$

Then we have

$$(2.5) \quad R(\Lambda) = \pi d, \quad d = \max\{d^+, d^-\}.$$

Theorem 2.3.1. will be a consequence of the famous Beurling-Malliavin Theorem. In the important special case where $d^+ = d^-$, it will be sufficient to verify the following lemma.

Lemma 2.3.2. *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d \geq 0$ and $0 \leq \alpha < 1$*

$$(2.6) \quad \#\{\lambda \in \Lambda: 0 \leq \lambda \leq t\} = dt + O(t^\alpha) \text{ and } \#\{\lambda \in \Lambda: -t \leq \lambda \leq 0\} = dt + O(t^\alpha).$$

Let us represent the generic compact interval of \mathbb{R} by $\omega = [\omega_1, \omega_2]$ and define for each $\epsilon > 0$ the set

$$(2.7) \quad \Omega_\epsilon = \{\omega: |\omega|^{-1} \#(\Lambda \cap \omega) - d \geq \epsilon\}.$$

Then if we represent each interval ω by a point in the upper half-plane through the formulas

$$(2.8) \quad T(\omega) = (x, y) \quad \text{with} \quad x = (\omega_1 + \omega_2)/2 \quad \text{and} \quad y = |\omega| = \omega_2 - \omega_1,$$

we have

$$(2.9) \quad \forall \epsilon > 0, \quad \iint_{T(\Omega_\epsilon)} \frac{dx dy}{1 + x^2 + y^2} < \infty,$$

for every $\epsilon > 0$.

PROOF. As a consequence of hypothesis (2.6) we have immediately

$$\#(\Lambda \cap \omega) - d|\omega| = O\{|\omega_1|^\alpha + |\omega_2|^\alpha\}.$$

therefore we only need to check that for each $K \geq 0$, the set

$$A(K) = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq K(|x + y|^\alpha + |x - y|^\alpha)\}$$

satisfies

$$\iint_{A(K)} \frac{dx dy}{1 + x^2 + y^2} < \infty.$$

But obviously

$$A(K) \subset B = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq M(1 + x^2)^{\alpha/2}\}$$

for some constant M related to K . Finally we have

$$\begin{aligned} \int_B \frac{dx dy}{1 + x^2 + y^2} &\leq \int_{-\infty}^{+\infty} dx \int_0^{M(1+x^2)^{\alpha/2}} \frac{dy}{1 + x^2 + y^2} \\ &\leq M \int_{-\infty}^{+\infty} (1 + x^2)^{\alpha/2 - 1} dx < \infty. \end{aligned}$$

The result follows immediately.

In order to complete the proof of Theorem 2.3.1, it will be useful to recall the main concepts required to formulate the general Beurling-Malliavin Theorem.

Definition 2.3.3. *A sequence of distinct real numbers $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is said to be regular with Beurling-Malliavin density equal to $d \geq 0$ if for each $\epsilon > 0$, the set Ω_ϵ given by (2.7) satisfies (2.9) with T given by (2.8).*

We now recall the main result of [1].

Theorem 2.3.4. (Beurling-Malliavin.) *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of distinct real numbers. Then*

(a) *If Λ is regular with Beurling-Malliavin density equal to $d \geq 0$, we have*

$$R(\Lambda) = \pi d.$$

(b) *If Λ is not regular, then*

$$R(\Lambda) = \pi d,$$

where d is the infimum of all Beurling-Malliavin densities of regular sequences of distinct real numbers containing Λ .

PROOF OF THEOREM 2.3.1. (a) If $d^+ = d^-$, the result of Lemma 2.3.2. precisely means that Λ is regular with Beurling-Malliavin density equal to d , and (a) from the statement of Theorem 2.3.4. gives exactly (2.5).

(b) Otherwise, one easily finds that any regular sequence of distinct real numbers containing Λ has a density at least equal to d . On the other hand by «completing» Λ it is rather straightforward to build a sequence of distinct real numbers containing Λ and satisfying (2.6). As a consequence of Lemma 2.3.2., such a sequence must be regular with Beurling-Malliavin density equal to d . Then (2.5) follows at once from (b) in the statement of Theorem 2.3.4.

By combining the results of Proposition 2.2.1 and Theorem 2.3.1, we obtain

Corollary 2.3.5. *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d^+ \geq 0$, $d^- \geq 0$ and $0 \leq \alpha < 1$*

$$(2.4) \quad \begin{aligned} \#\{\lambda \in \Lambda: 0 \leq \lambda \leq t\} &= d^+ t + O(t^\alpha) \quad \text{and} \\ \#\{\lambda \in \Lambda: -t \leq \lambda \leq 0\} &= d^- t + O(t^\alpha). \end{aligned}$$

Then letting $d = \max\{d^+, d^-\}$, we have the following properties

(1) For each $T > 2\pi d$ and for each $n \in \mathbb{Z}$ there exists a constant C_n such that

$$|f_n| \leq C_n \left\{ \int_0^T |f(t)|^2 dt \right\}^{1/2},$$

for each function f of the form (2.1) with $n \in J$.

(2) On the other hand for each $T < 2\pi d$ and for each finite sequence $\{\alpha_n\}_{n \in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that

$$\left| \sum_{n \in F} \alpha_n f_n \right| \leq C \left\{ \int_0^T |f(t)|^2 dt \right\}^{1/2}$$

for each f of the form (2.1) with $F \subset J$.

The special case where $d^+ = d^-$ is especially important for the sequel (Sections 3 and 4) and therefore we state it separately for the reader's convenience

Corollary 2.3.6. *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d \geq 0$ and $0 \leq \alpha < 1$*

$$(2.4) \quad \#\{\lambda \in \Lambda: 0 \leq \lambda \leq t\} = dt + O(t^\alpha) \quad \text{and} \quad \#\{\lambda \in \Lambda: -t \leq \lambda \leq 0\} = dt + O(t^\alpha).$$

Then we have the following properties

(1) For each $T > 2\pi d$ and for each $n \in \mathbb{Z}$ there exists a constant C_n such that

$$|f_n| \leq C_n \left\{ \int_0^T |f(t)|^2 dt \right\}^{1/2},$$

for each function f of the form (2.1) with $n \in J$.

(2) On the other hand for each $T < 2\pi d$ and for each finite sequence $\{\alpha_n\}_{n \in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that

$$\left| \sum_{n \in F} \alpha_n f_n \right| \leq C \left\{ \int_0^T |f(t)|^2 dt \right\}^{1/2}$$

for each f of the form (2.1) with $F \subset J$.

3. Application to Spectral Pointwise Control of some Plate Models

3.1. An Abstract Controllability Result

The main result of the section is the following.

Theorem 3.1.1. *Let Ω be a bounded domain of \mathbb{R}^N (or a compact N -dimensional manifold without boundary) and A a positive self-adjoint operator in $H = L^2(\Omega)$ satisfying conditions (1.1) and (1.2) with A^{-1} compact. We denote by $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ the increasing sequence of eigenvalues of $A^{1/2}$. We assume that all the eigenvalues λ_j are simple and that we have for some $d \geq 0$ and $0 \leq \alpha < 1$*

$$(3.1) \quad \#\{\lambda \in \Lambda^+ : \lambda \leq t\} = dt + O(t^\alpha).$$

Let $\xi \in \Omega$ be any point at which no eigenfunction of A vanishes, and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of A . Then

- (1) or every $T > 2\pi d$, and $(y^n, y^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\text{supp}(h) \subset [0, T]$ and such that the unique solution y of (1.3) satisfies $y(T, x) = y'(T, x) = 0$ in Ω .
- (2) This result is optimal: as soon as $T < 2\pi d$, there is no $(y^0, y^1) \in D \times D$ except the trivial state $(0, 0)$ for which such a control h exists.

PROOF. This is a straightforward consequence of Proposition 1.1 and Corollary 2.3.6 applied with $\Lambda = \Lambda^+ \cup (-\Lambda^+)$. Indeed any solution of (1.4) with

initial data in $D \times D$ has the form

$$u(t, x) = \sum_{j \in J} \{u_j \cos \lambda_j t + v_j \sin \lambda_j t\} \varphi_j(x),$$

where the functions φ_j denote an orthonormal sequence of eigenfunctions of A and the coefficients u_j and v_j and given by the formulas

$$u_j = \int_{\Omega} u(0, x) \varphi_j(x) dx, \quad v_j = \frac{1}{\lambda_j} \int_{\Omega} u'(0, x) \varphi_j(x) dx.$$

It is then clear that a direct application of Corollary 2.3.6 to the function $f(t) = u(t, \xi)$ with u as above provides exactly the result by taking into account Proposition 1.1.

3.2. Application to Simply Supported Plates

Let Ω be a bounded domain in \mathbb{R}^2 . We denote by $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ the increasing sequence of eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$: it is known that under very general assumptions on Ω , for instance if $\partial\Omega$ is smooth, the counting function $n(t) = \#\{\lambda \in \Lambda^+ : \lambda \leq t\}$ where each $\lambda \in \Lambda^+$ is repeated according to its multiplicity satisfies the so called Weyl formula:

$$(3.2) \quad n(t) = dt + O(t^{1/2}) \quad \text{with} \quad d = (1/4\pi) \text{vol}(\Omega).$$

As a special consequence of Theorem 3.1.1 we find

Theorem 3.2.1. *Let Ω be a bounded domain in \mathbb{R}^2 satisfying the «Weyl formula», assume that all eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ are simple and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta)$ in $H_0^1(\Omega)$. Let finally $\xi = (\xi_1, \xi_2) \in \Omega$ be any point at which no eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$ vanishes. Then*

For every $T > (1/2) \text{vol}(\Omega)$, and $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\text{supp}(h) \subset [0, T]$ such that the unique solution Ψ of

$$\begin{cases} \Psi_{tt} + \Delta^2 \Psi = h(t) \delta_{\xi}(x, y) & \text{in }]0, T[\times \Omega, \\ \Psi = \Delta \Psi = 0 & \text{on } [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & \text{in } \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & \text{in } \Omega, \end{cases}$$

satisfies $\Psi(T, \bullet) = \Psi_t(T, \bullet) = 0$.

This result is optimal: more precisely if $T < (1/2) \text{vol}(\Omega)$, no non-zero finite linear combination of the eigenfunctions of $(-\Delta)$ in $H_0^1(\Omega)$ is pointwise L^2 -controllable.

When Ω is a rectangle of the form $(0, \pi) \times (0, L)$, with $(L/\pi)^2 \notin \mathbb{Q}$, let D denote the vector space of finite linear combinations of the basic eigenfunctions $\sin mx \sin(n\pi y/L)$, $m \in \mathbb{N}$, $n \in \mathbb{N}$. We have the following result:

Proposition 3.2.2. *Let $\xi = (\xi_1, \xi_2) \in \Omega$ be fixed with $\xi_1/\pi \notin \mathbb{Q}$, $\xi_2/L \notin \mathbb{Q}$. For each $T > (1/2)\pi L$ and each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\text{supp}(h) \subset [0, T]$ and such that the unique solution Ψ of*

$$\begin{cases} \Psi_{tt} + \Delta^2 \Psi = h(t)\delta_\xi(x, y) & \text{in }]0, T[\times \Omega, \\ \Psi = \Delta \Psi = 0 & \text{on } [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & \text{in } \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & \text{in } \Omega, \end{cases}$$

satisfies $\Psi(T, \cdot) = \Psi_t(T, \cdot) = 0$.

3.3. Application to the Case of Clamped Plates and Other Boundary Conditions

Let Ω be a rectangle or a bounded domain in \mathbb{R}^2 with a smooth boundary. We denote by $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ the increasing sequence of the square roots of the eigenvalues of Δ^2 with relevant homogeneous boundary conditions: under very general assumptions on these boundary conditions, the counting function

$$n'(t) = \#\{\lambda \in \Lambda^+ : \lambda \leq t\}$$

where each $\lambda \in \Lambda^+$ is repeated according to its multiplicity still satisfies the Weyl formula (3.1) with the same value of d . As a consequence of Theorem 3.1.1 we find for instance

Theorem 3.3.1. *Let Ω be a bounded smooth domain or a rectangle in \mathbb{R}^2 for which all the eigenvalues of Δ^2 in $H_0^2(\Omega)$ are simple, and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of Δ^2 in $H_0^2(\Omega)$. Let finally $\xi = (\xi_1, \xi_2) \in \Omega$ be any point at which no eigenfunction of Δ^2 in $H_0^2(\Omega)$ vanishes. Then*

For every $T > (1/2)\text{vol}(\Omega)$, and $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\text{supp}(h) \subset [0, T]$ such that the unique solution Ψ of

$$\begin{cases} \Psi_{tt} + \Delta^2 \Psi = h(t)\delta_\xi(x, y) & \text{in }]0, T[\times \Omega, \\ \Psi = |\nabla \Psi| = 0 & \text{on } [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & \text{in } \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & \text{in } \Omega, \end{cases}$$

satisfies $\Psi(T, \cdot) = \Psi_t(T, \cdot) = 0$.

This result is optimal: more precisely if $T < (1/2) \text{vol}(\Omega)$, no non zero finite linear combination of the eigenfunctions of Δ^2 in $H_0^2(\Omega)$ is pointwise L^2 -controllable.

PROOF. Let Ω be a bounded smooth domain or a rectangle in \mathbb{R}^2 for which all the eigenvalues of Δ^2 in $H_0^2(\Omega)$ are simple. Then Ivrii [10] asserts that under general positivity conditions, the fact that Δ^2 with the given boundary conditions is elliptic in the sense of Shapiro-Lopatinskii implies that the counting function $n'(t)$ satisfies (3.1). It is rather easy to check (cf. e.g. Wloka [23]) that the operator Δ^2 in $H_0^2(\Omega)$ satisfies the Shapiro-Lopatinskii condition, therefore Theorem 3.1.1. is applicable. (For a related weaker property cf. also Plejel [21].)

Remark 3.3.2. Of course the difficulty in general will be to determine the «strategic points» $\xi = (\xi_1, \xi_2)$ at which no eigenfunction of Δ^2 in $H_0^2(\Omega)$ vanishes. Even when Ω is a rectangle of the form $(0, \pi) \times (0, L)$, the eigenfunctions of Δ^2 in $H_0^2(\Omega)$ become more complicated than in the case of simply supported plates, and it is probably not so easy to find the strategic points. We know, however, that in the absence of multiple eigenvalues, almost every point is strategic.

Remark 3.3.3. In Section 5, the case of variable Lamé coefficients will be treated.

4. Some Applications to Spectral Internal Control

Let Ω be a bounded domain of \mathbb{R}^N (or a compact N -dimensional manifold without boundary) and A a positive self-adjoint operator with compact resolvent in $H = L^2(\Omega)$. Let $\{\lambda_j\}_{1 \leq j \leq +\infty}$ be the increasing (without taking care of multiplicity) sequence of eigenvalues of A and for each j , let

$$F_j = \{u \in L^2(\Omega): Au = \lambda_j u\}.$$

Then we have the following

Theorem 4.1. *Assume that A has the following properties*

- (1) *For every j , the conditions $u \in F_j$ and $u \equiv 0$ on some non-empty open set imply $u \equiv 0$.*
- (2) *There is a finite $T_0 > 0$ for which the functions*

$$\sum_{j \in J} \{u_j e^{i\sqrt{\lambda_j}t} + v_j e^{-i\sqrt{\lambda_j}t}\}, \quad J \text{ finite subset of } \mathbb{N} - \{0\}$$

where the u_j and v_j are complex coefficients that are not dense in $L^2(0, T_0; \mathbb{C})$.

Then for each $T > T_0$, there are functions $\{f_j\}_{1 \leq j \leq +\infty}$ of class C^∞ with compact support in $(0, T)$ such that we have the following properties

(a) For every $\Psi \in L^2(\Omega)$, the unique solution u of

$$(4.1) \quad u'' + Au = f_j(t)\Psi(x) \text{ in } \mathbb{R} \times \Omega, \quad u(0, x) = u'(0, x) = 0 \text{ in } \Omega$$

fulfills $u(T, \cdot) \in F_j$ and $u'(T, \cdot) = 0$.

(b) For every ω non-empty open subset of Ω , and every $\varphi \in F_j$, there exists $\Psi \rightarrow D(\Omega)$ with compact support in ω such that the unique solution u of (3.1) satisfies $u(T) = \varphi$ and $u'(T) = 0$. Moreover the solution v of

$$(4.2) \quad v'' + Av = f'_j(t)\Psi(x) \text{ in } \mathbb{R} \times \Omega, \quad u(0, x) = u'(0, x) = 0 \text{ in } \Omega$$

fulfills $v(T) = 0$ and $v'(T) = -\lambda_j \varphi$.

PROOF. Let $\mu_j = \lambda_j^{1/2}$ for all j . We shall prove the result in five steps.

(1) First of all it follows from (2) that for each j fixed, the function $\sin(\mu_j t)$ is not a limit in $L^2(0, T_0; \mathbb{C})$ of finite linear combinations of the functions $\exp(\pm i\mu_k t)$ for $k \neq j$. Indeed in such a case, the function $\exp(i\mu_j t)$ would be a limit in $L^2(0, T_0; \mathbb{C})$ of finite linear combinations of the functions $\exp(\pm i\mu_k t)$ for $k \neq j$ and of the function $\exp(-i\mu_j t)$, which by an argument similar to the proof of Lemma 2.2.2 would contradict property (2).

(2) In particular, there exists $h_j \in L^2(0, T_0; \mathbb{C})$ for which

$$\int_0^{T_0} h_j(t) e^{\pm i\mu_k t} dt = 0 \quad \text{for } k \neq j; \quad \int_0^{T_0} h_j(t) \sin(\mu_j t) dt \neq 0.$$

Replacing h_j by either its real or its imaginary part, we may assume $h_j \in L^2(0, T_0; \mathbb{R})$.

(3) Let now $T > T_0$, $0 < \eta \leq (T - T_0)/2$ and

$$\begin{aligned} h_j(t, \eta) &= 0 && \text{on } (-\infty, \eta), \\ h_j(t, \eta) &= h_j(t - \eta) && \text{on } (\eta, T_0 + \eta), \\ h_j(t, \eta) &= 0 && \text{on } (T_0 + \eta, +\infty). \end{aligned}$$

For every $k \neq j$, we clearly have

$$\int_0^T h_j(t, \eta) e^{\pm i\mu_k t} dt = 0.$$

On the other hand, if η is small enough we have

$$\int_0^T h_j(t, \eta) \sin(\mu_j t) dt \neq 0.$$

For such a fixed η , let $h_j(t, \eta) = h_j(t)$. Define also $\rho_\epsilon \in D(0, \epsilon)$ with $\rho_\epsilon \geq 0$ and $\int \rho_\epsilon(x) dx = 1$, and let us introduce $h_j * \rho_\epsilon = h_{j, \epsilon}$. Then for each $\epsilon \in (0, \eta)$ we have $h_{j, \epsilon} \in D(0, T)$. In addition,

$$\int_0^T h_{j, \epsilon}(t) e^{\pm i\mu_k t} dt = 0 \quad \text{for } k \neq j,$$

$$\lim_{\epsilon \rightarrow 0} \int_0^T h_{j, \epsilon}(t) \sin(\lambda_j t) dt = \int_0^T h_j(t) \sin(\lambda_j t) dt \neq 0.$$

By selecting $\epsilon > 0$ small enough and replacing $h_{j, \epsilon}$ by some proportional function, we obtain $g_j \in D(0, T)$ such that

$$(4.3) \quad \int_0^T g_j(t) e^{\pm i\mu_k t} dt = 0 \quad \text{for } k \neq j \quad \text{and} \quad \int_0^T g_j(t) \sin(\mu_j t) dt = 1.$$

We can, in fact, also assume

$$(4.4) \quad \int_0^T g_j(t) \cos \mu_j t dt = 0.$$

As a matter of fact, if

$$\int_0^T g_j(t) \cos \mu_j t dt = I \neq 0,$$

let $g_j^*(t) = g_j(t + \alpha) + c g_j(t)$, $\alpha \neq 0$ being taken small enough. Then

$$\begin{aligned} \int_0^T g_j^*(t) \cos \mu_j t dt &= cI + \int_0^T g_j(t) \cos \mu_j(t - \alpha) dt \\ &= (c + \cos(\alpha\mu_j))I + \sin(\alpha\mu_j), \end{aligned}$$

vanishes for

$$c = -\cos(\alpha\mu_j) - \sin(\alpha\mu_j)/I.$$

Taking c as above we have

$$\begin{aligned} \int_0^T g_j^*(t) \sin \mu_j t dt &= c + \int_0^T g_j(t) \sin \mu_j(t - \alpha) dt \\ &= c + \cos(\alpha\mu_j) - I \sin(\alpha\mu_j) \\ &= -(I + 1/I) \sin(\alpha\mu_j) \neq 0 \end{aligned}$$

for α small.

We can then replace g_j by λg_j^* with $\lambda \neq 0$ properly chosen.

(4) Let $f_j(t) = g_j(T - t)$ for $t \in [0, T]$. The solution of (4.1) is given by

$$u(t, x) = \int_0^t f_j(t - s) \left\{ \sum_m \sin(\mu_m s) \Psi_m(x) \right\} ds$$

with

$$\Psi_m = \text{Proj}_{F_m}(\Psi) = P_m \Psi, \quad \text{for } m \in \{1, 2, \dots\}.$$

In particular we have

$$\begin{aligned} u(T, x) &= \int_0^T g_j(s) \left\{ \sum_m \sin(\mu_m s) \Psi_m(x) \right\} ds \\ &= \sum_m \left\{ \int_0^T g_j(s) \sin(\mu_m s) \right\} \Psi_m(x) \\ &= \Psi_j(x). \end{aligned}$$

On the other hand for t close to T we have

$$u'(t, x) = \int_0^t f'_j(t - s) \left\{ \sum_m \sin(\mu_m s) \Psi_m(x) \right\} ds.$$

Therefore

$$\begin{aligned} u'(T, x) &= \int_0^T f'_j(T - s) \left\{ \sum_m \sin(\mu_m s) \Psi_m(x) \right\} ds \\ &= \sum_m \left\{ \int_0^T g'_j(s) \sin(\mu_m s) ds \right\} \Psi_m(x) = 0, \end{aligned}$$

since integration by parts gives

$$\int_0^T g'_j(s) \sin(\mu_m s) ds = -\lambda_m \int_0^T g_j(s) \cos(\mu_m s) ds = 0.$$

for every $m \in \mathbb{N} - \{0\}$. This establishes (a) with $u(T, \bullet) = \Psi_j$. Moreover we notice that $u' = v$ is the solution of (4.2) with initial data $(0, 0)$ and satisfies $v(T, \bullet) = 0$; $v'(T, \bullet) = u''(T, \bullet) = -Au(T, \bullet) = -\lambda_j \Psi_j$.

(5) To establish (b), we now use hypothesis (1). Indeed, for a fixed integer j , we consider an orthonormal basis $\{\varphi_1, \dots, \varphi_r\}$ of F_j . To finish the proof we just need to show that we can find $\Psi \in \mathcal{D}(\Omega)$ with support in ω such that $P_j \Psi = \varphi_1$ (say). In the opposite case, the linear form defined by

$$\Psi \in \mathcal{D}(\omega) \mapsto \int_{\omega} \Psi(x) \varphi_1(x) dx$$

would vanish on the intersection of the kernels of the linear forms defined by

$$\Psi \in \mathfrak{D}(\omega) \mapsto \int_{\omega} \Psi(x) \varphi_k(x) dx, \quad k \geq 2.$$

By a standard result of linear algebra we would deduce the existence of real coefficients $\{\alpha_k\}_{k \geq 2}$ for which

$$\int_{\omega} \Psi(x) \varphi_1(x) dx = \sum_{k \geq 2} \alpha_k \int_{\omega} \Psi(x) \varphi_k(x) dx,$$

for all $\Psi \in \mathfrak{D}(\omega)$. This immediately implies that for every $x \in \omega$,

$$\varphi_1(x) = \sum_{k \geq 2} \alpha_k \varphi_k(x).$$

This is in contradiction with hypothesis (1) and the linear independence of $\{\varphi_k\}_{k \geq 1}$.

Let us now denote by D the vector space of all (finite) linear combinations of the eigenfunctions of A and assume that all hypotheses of Theorem 4.1 are satisfied. Then by an immediate calculation we obtain the following result.

Corollary 4.2. *For each $(y^0, y^1) \in D \times D$, there exists $h \in \mathfrak{D}((0, T) \times \Omega)$ with $\text{supp}(h) \subset (0, T) \times \omega$ and such that the unique solution y of*

$$(4.5) \quad \begin{cases} y'' + Ay = h(t, x) & \text{in } (0, T) \times \Omega, \\ y(0, x) = y^0(x) & \text{in } \Omega, \\ y'(0, x) = y^1(x) & \text{in } \Omega \end{cases}$$

satisfies $y(T, \bullet) = y'(T, \bullet) = 0$.

In particular, in the case of vibrating plates we obtain

Corollary 4.3. *Let Ω be a bounded smooth domain or a rectangle in \mathbb{R}^2 and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta)$ in $H_0^1(\Omega)$. Then for any $T > (1/2) \text{vol}(\Omega)$ and each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h \in \mathfrak{D}((0, T) \times \Omega)$ with $\text{supp}(h) \subset (0, T) \times \omega$ and such that the unique solution Ψ of*

$$(4.6) \quad \begin{cases} \Psi_{tt} + \Delta^2 \Psi = h(t, x, y) & \text{in } (0, T) \times \Omega, \\ \Psi = \Delta \Psi = 0 & \text{on } [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & \text{in } \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & \text{in } \Omega, \end{cases}$$

satisfies $\Psi(T, \bullet) = \Psi_t(T, \bullet) = 0$. In addition the same result is valid for the equation

$$(4.6)' \quad \begin{cases} \Psi_{tt} + \Delta^2 \Psi = h(t, x, y) & \text{in } (0, T) \times \Omega, \\ \Psi = |\nabla \Psi| = 0 & \text{on } [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & \text{in } \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x, y) & \text{in } \Omega. \end{cases}$$

PROOF. Property (1) is clearly satisfied in both cases. It is therefore sufficient to check (2) for all $T_0 > (1/2) \text{vol}(\Omega)$. In the case of (4.6), when all the eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ are simple, this follows at once from the Weyl formula and Corollary 2.3.6. In fact, by considering for instance some artificial additional frequencies or by using a generalization of the Beurling-Malliavin theory for exponential-polynomial series, it is possible to extend Corollary 2.3.6 in the more general situation of a counting function allowing arbitrary finite repetitions of the frequencies. Then the Weyl formula implies (2) without requiring the eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ to be simple. The rest is clear. The same proof works for (4.6)'.

Remark 4.4. In the case of (4.6) in a rectangle, by using some results of J. P. Kahane [13], S. Jaffard [11] established the (in a sense stronger) result of exact internal controllability of any state with a finite energy, for any $T > 0$. However this result does not seem to imply immediately the existence of a C^∞ control for states $(\Psi^0, \Psi^1) \in D \times D$. The result of [11] has been recently generalized in arbitrarily many dimensions (for a product of intervals) by V. Komornik [15]. On the other hand, no such internal controllability result seems to be known for the clamped plate equation.

This theory is also applicable to cases where the open set Ω is replaced by a compact manifold without boundary. We obtain for instance the following result, valid in any dimension $N \geq 1$.

Corollary 4.5. *Let Σ be the unit sphere of \mathbb{R}^N , and let us denote by $(-\Delta_\Sigma)$ the Laplace-Beltrami operator on Σ and by D the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta_\Sigma)$. For all $T > 0$, and all ω non-empty open subset of Σ , and for each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h \in \mathcal{D}((0, T) \times \Sigma)$ with $\text{supp}(h) \subset (0, T) \times \omega$ and such that the unique solution Ψ of*

$$(4.7) \quad \begin{cases} \Psi_{tt} + \Delta_{\Sigma^2} \Psi = h(t, \sigma) & \text{in } (0, T) \times \Sigma, \\ \Psi(0; \sigma) = \Psi^0(\sigma) & \text{on } \Sigma \\ \Psi_t(0; \sigma) = \Psi^1(\sigma) & \text{on } \Sigma \end{cases}$$

satisfies $\Psi(T, \bullet) = \Psi_t(T, \bullet) = 0$.

PROOF. Property (1) is clearly satisfied. Property (2) is a rather immediate consequence of the fact that the inverses of the positive eigenvalues of $(-\Delta_{\Sigma})$ are summable. (Cf. e.g. [24], Theorem 15 p. 139.)

Remark 4.6. The control functions constructed here are of a special type. Their construction ultimately relies on the existence of a sequence of functions «bior-orthogonal» to some complex exponentials, a technique already widely used (cf. e.g., H. O. Fattorini [2], J. Lagnese [16], D. L. Russel [22]) in control theory.

Remark 4.7. It seems rather reasonable to conjecture that the internal spectral controllability for the above plate equation is valid for general 2-dimensional domains and for every $T > 0$. The study of this conjecture and some related problems will be the object of further research.

5. Possible Extensions and Additional Remarks

5.1. One-Dimensional Vibrating Systems

We consider first the equation of vibrating strings

$$\begin{cases} u_{tt} - (a(x)u_x)_x = 0 & \text{on } \mathbb{R} \times (0, L) \\ u(t, 0) = u(t, L) = 0 & \text{on } \mathbb{R}, \end{cases}$$

where a is smooth and bounded from below by a positive constant. Let A be the (strongly elliptic) unbounded operator on $L^2(0, L)$ defined by

$$D(A) = H^2 \cap H_0^1(0, L); \quad Av = -(a(x)v_x)_x \quad \text{for } v \in D(A).$$

The solutions are of the form

$$u(t, x) = \sum_{n \in \mathbb{Z} - \{0\}} u_n e^{i\lambda_n t} w_n(x)$$

where $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is given by $\Lambda^+ \cup (-\Lambda^+)$ and $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ is the increasing sequence of eigenvalues of $A^{1/2}$ in $H_0^1(0, L)$. Here Weyl's formula implies (cf. e.g. Hörmander [8], p. 273)

$$n(t) = ct + O(t^{1/2})$$

for some $c > 0$. Hence, by Theorem 3.1.1 we obtain that, appart from the nodal points (zeroes of the eigenfunctions of A) pointwise spectral controllability holds true for all times $T > 2\pi c$. This result extends to pointwise control some previous result of J. Lagnese [16] concerning internal exact controllability of strings.

Similarly we can consider vibrating beams given by

$$u_{tt} + (a(x)u_{xx})_{xx} = 0 \quad \text{on } \mathbb{R} \times (0, L)$$

with either of the following boundary conditions

$$u(t, 0) = u(t, L) = u_x(t, 0) = u_x(t, L) = 0 \quad \text{on } \mathbb{R},$$

or

$$u(t, 0) = u(t, L) = u_{xx}(t, 0) = u_{xx}(t, L) = 0 \quad \text{on } \mathbb{R}.$$

Here of course $a(x)$ is assumed smooth and bounded from below by some positive constant. Here Theorem 3.1.1 provides pointwise spectral controllability for all times $T > 0$ since the completeness radius of the corresponding complex exponentials is obviously 0. Actually in such cases the result can also be deduced by means of a variant of Ingham's Lemma (*cf.* [9.5]).

5.2. The Case of Plates with Nonconstant Lamé Coefficients

We deal with a similar case as in Section 3.3 except that the bilaplacian is now replaced by

$$A = \sum_{i,j=1,2} \partial_{i,j} m_{i,j} \quad \text{with} \quad m_{i,j} = \frac{4\mu}{3} \left(\partial_{i,j} + \frac{\lambda}{\lambda + 2\mu} \delta_{i,j} \Delta \right).$$

The functions λ and μ are the Lamé coefficients which we suppose to be non-constant in the plate, but C^∞ . In order to obtain a «Weyl's formula» for this operator, we have to check that the assumptions given in [10] are fulfilled. The operator is symmetric since

$$\langle Au, v \rangle = \sum_{i,j} \int \frac{4\mu}{3} \partial_{i,j} u \overline{\partial_{i,j} v} + \frac{4\lambda\mu}{3(\lambda + 2\mu)} \int \Delta u \overline{\Delta v}.$$

The principal symbol of A is $(4\mu/3)(1 + \lambda/(\lambda + 2\mu))|\xi|^4$ which is positive definite. Thus we only have to check the Shapiro-Lopatinskii condition on the boundary. It is a condition on the principal part of the operator which must hold at each point x_0 of the boundary. Here, the principal part of A is a biaplacian multiplied by the smooth function $a = (4\mu/3)(1 + \lambda/(\lambda + 2\mu))$. Thus, up to the multiplicative factor $a(x_0)$, the condition to check is exactly the same as if we had $A = \Delta^2$ with the corresponding boundary conditions, and the conclusion will be the same as for the bilaplacian; namely, for simply supported or clamped plates, formula (3.1) will hold with $d = (1/4\pi) \text{vol}(\Omega)$, and thus, also the conclusion of the analog of Theorem 3.2.1.

5.3 Uniqueness and the Schrödinger Equation

In Section 3, we have given precisely the minimal time for pointwise spectral controllability. It is clear that a time T of pointwise spectral controllability is also a uniqueness time in the sense that the trace of a solution at the observation point on $(0, T)$ determines the solution. In the case of second order problems (1.3)-(1.4), it is conjectured (*cf.* Kahane [14]) that the minimal uniqueness time is equal to the minimal time for pointwise spectral controllability.

Now let Ω be a bounded domain of \mathbb{R}^2 and A a positive self-adjoint operator in $H = L^2(\Omega)$. We assume that A satisfies the properties (1.1) and (1.2). Given $T > 0$, $\xi \in \Omega$ and y^0 a finite linear combination of the eigenfunctions of A , we are interested in the existence of a control function $h \in L^2(]0, T[)$ such that $\text{supp}(h) \subset]0, T[$ and for which the unique generalized solution y of the Schrödinger type equation

$$y - iAy = h(t)\delta(x - \xi) \quad \text{in }]0, T[, \quad y(0, x) = y^0(x) \quad \text{in } \Omega,$$

satisfies

$$y(T, x) = y'(T, x) = 0 \quad \text{in } \Omega.$$

The solutions of the homogeneous equation $\phi' - iA\phi = 0$ are here given by

$$\phi(t, x) = \sum_{n \geq 1} \phi_n e^{i\lambda_n t} w_n(x),$$

where the numbers λ_n are the eigenvalues of A and the functions w_n are the associated eigenfunctions. For a given $\xi = (\xi_1, \xi_2) \in \Omega$, let

$$f(t) = \phi(t, \xi) = \sum_{n \geq 1} \alpha_n e^{i\lambda_n t},$$

Assume that the Weyl formula (3.1) holds for A with $d > 0$: then by Corollary 2.3.5 the minimal time for spectral controllability is easily seen to be also positive (more precisely equal to $2\pi d$).

On the other hand, let us show that *any positive time* T is in fact a «uniqueness time». From the Weyl formula (3.1), the properties of the initial data and the standard estimates on $\|w_n\|_\infty$ we deduce that the coefficients α_n have at most polynomial growth. Suppose that f vanishes identically on $[0, T]$ and let φ be a C^∞ nonnegative function supported inside $[-T/2, 0]$ with integral 1. Then the convolution product $\varphi * f$ vanishes identically on $[0, T/2]$ and we have

$$(\varphi * f)(t) = \sum_{n \geq 1} \alpha_n \hat{\varphi}(\lambda_n) e^{i\lambda_n t} = \sum_{n \geq 1} c_n e^{i\lambda_n t},$$

where the sequence (c_n) is quickly decreasing, hence $\varphi * f$ is C^∞ . Let

$$\psi(t) = (\varphi * f)(t)\omega(t)$$

where $\omega(t)$ is in the Schwartz class with a compactly supported Fourier transform. Then ψ is in the Schwartz class with

$$\widehat{\psi}(\xi) = [(\varphi * f) * \widehat{\omega}](\xi) = \left[\sum_{n \geq 1} c_n \delta_{\lambda_n} \right] * \widehat{\omega}(\xi).$$

Since the numbers λ_n are all positive, and ω has a compactly supported Fourier transform, the Fourier transform of ψ vanishes on $(-\infty, \beta]$ for some β . Hence $g(t) = e^{i\beta t}\psi(t)$ is a C^∞ function which vanishes identically on $[0, T/2]$ and whose Fourier transform is supported by $[0, +\infty)$. A well known theorem of Helson and Szegö (cf. [7]) asserts that if $g \in L^2(\mathbb{R})$ has its Fourier transform supported in $[0, +\infty)$, then either $g \equiv 0$ or $\text{Log}|g(t)|/(1+t^2) \in L^1(\mathbb{R})$: in particular if g vanishes on an interval we must conclude that $g \equiv 0$. In our case we conclude that $\psi \equiv 0$ for any choice of functions φ, ω as above. It then follows immediately that $f \equiv 0$, hence uniqueness is established for any $T > 0$.

As a conclusion, in the case of the above Schrödinger equation, for all times between 0 and the minimal spectral controllability time $T_0 = 2\pi d$, there exists a dense family of pointwise controllable states, but none of them is a finite linear combination of the eigenfunctions of A . It would be, of course, of interest to decide what happens for our plate models for small positive times, and in particular to settle Kahane's conjecture.

5.4. The Plate Equation in Higher Dimensions

In dimensions higher than or equal to 3, the calculations are very similar to those in dimension 2 and we shall not repeat them. The results, on the other hand, are quite different: for instance, for a 3-dimensional «plate» the Weyl formula now gives

$$N(\lambda) = c\lambda^{3/2} + O(\lambda),$$

hence the Beurling-Malliavin density of the λ_n is infinity and for no finite $T > 0$ we have pointwise spectral controllability. The uniqueness problem is also open in this case. However, if we consider the associated Schrödinger equation, then the eigenvalues are positive and the proof of Section 5.2 is still applicable. Thus here, any positive time is a uniqueness time, while there is no finite time for pointwise spectral controllability!

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Alain Haraux
Analyse Numérique,
T. 55-65, 5ème étage,
Université Pierre et Marie Curie,
4, Place Jussieu,
75252 Paris Cedex 05,
FRANCE

Stéphane Jaffard
C.E.R.M.A.
(Laboratoire de modélisation
mathématique) E.N.P.C.,
La Courtine,
93167 Noisy-le-Grand,
FRANCE