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On Pseudospheres

John L. Lewis and Andrew Vogel
Dedicated to the memory of Allen Shields

1. Introduction

Denote points in Euclidean space, $\mathbb{R}^n$, by $x = (x_1, \ldots, x_n)$ and let $\bar{E}$, ∂E, denote the closure and boundary of $E \subset \mathbb{R}^n$, respectively. Put $B(x, r) = \{y: |y - x| < r\}$ when $r > 0$. Define k dimensional Hausdorff measure, $1 \leqslant k \leqslant n$, in $\mathbb{R}^n$ as follows: for fixed $\delta > 0$ and $E \subset \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subset \bigcup B(x_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \ldots$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \sum \alpha(k) r_i^k,$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \qquad 1 \leqslant k \leqslant n.$$

Let D be a bounded domain in $\mathbb{R}^n$ with $0 \in D$ and $H^{n-1}(\partial D) < +\infty$. We shall say D is a pseudo sphere if

(*a*) ∂D is homeomorphic to the unit sphere, S, in $\mathbb{R}^n$
(*b*) $g(0) = a \int_{\partial D} g \, dH^{n-1}$, whenever g is harmonic in D and continuous on $\bar{D}$.

In (*b*), a denotes a constant. The construction of pseudo spheres in $\mathbb{R}^2$, which are not circles, was first done by Keldysh and Lavrentiev to show the existence of domains not of Smirnov type (see [11, Ch. 3]). Also a completely different proof of existence has been given by Duren, Shapiro, and Shields in [3] (see also [2, Ch. 10]). Both proofs are heavily reliant on conformal mapping and $\mathbb{R}^2$ facts, such as: the logarithm of the gradient of a harmonic function is subharmonic.

In [12, p. 347], Shapiro asked whether there exists a pseudo sphere in $\mathbb{R}^n$ which is not a sphere. In this paper we answer Shapiro's question in the affirmative and even prove a little more:

Theorem 1. *There exists a pseudo sphere D in $\mathbb{R}^n$, $n \geqslant 3$, which is not a sphere. In fact D can be chosen so that there is a homeomorphism f from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f(S) = \partial D$ and*

$$c(\beta)^{-1}|x-y|^{1/\beta} \leqslant |f(x) - f(y)| \leqslant c(\beta)|x-y|^{\beta},$$

whenever $\beta \in (0, 1)$ and $|x - y| \leqslant 1/2$.

In Theorem 1, as in the sequel, $c(\beta)$ denotes a positive constant depending only on β and n. Also, c will denote a positive constant depending only on n, not necessarily the same at each occurrence. Our method of proof is inspired by the proof of Keldysh and Lavrentiev in [9]. Here though conformal mapping techniques are not available. We outline our proof with $a = 1$ in (b). Let Ω be a bounded domain with $0 \in \Omega$ and let G be Green's function for Ω with pole at 0. That is,

$$G(x) - \frac{1}{n(n-2)\alpha(n)}|x|^{2-n}, \qquad x \in \mathbb{R}^n,$$

is harmonic in Ω and G has boundary value 0 in the sense of Perron-Wiener-Brelot. It is known that if $\partial\Omega$ is sufficiently smooth, then

$$\nabla G(x) = \left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}\right)$$

extends continuously to $\bar{\Omega} - \{0\}$. Under this assumption suppose that $|\nabla G| \geqslant 1$ on $\partial\Omega$. In Section 2, given ϵ, $0 < \epsilon \leqslant \epsilon_0$, we add smooth bumps to $\partial\Omega$ by «pushing out» $\partial\Omega$ along certain small surface elements in $\{x \in \partial\Omega: |\nabla G(x)| > 1 + \epsilon\}$ of approximate side length r, $0 < r \leqslant r_0$. Let Ω', G' be the smooth domain, and Green's function with pole at 0, obtained from this process. Then $\Omega \subset \Omega'$ and we shall choose the bumps so that for $\epsilon \leqslant t \leqslant 1$,

$$H^{n-1}(\partial\Omega') \geqslant H^{n-1}(\partial\Omega) + \eta(t)H^{n-1}\{x: |\nabla G(x)| > 1 + t\}, \tag{1.1}$$

where η is a positive function on $(0, \infty)$. It turns out that η can be chosen independent of Ω, Ω'. We note from the Hopf boundary maximum principle (see [6, Lemma 3.4]) and $|\nabla G| \geqslant 1$ on $\partial\Omega$, that $|\nabla G'| \geqslant 1$ on $\partial\Omega \cap \partial\Omega'$. Also from Schauder type estimates, it will follow that $|\nabla G'| \geqslant 1$ on the bumps. Hence,

$$|\nabla G'(x)| \geqslant 1, \qquad x \in \partial\Omega'. \tag{1.2}$$

Next we modify the identity mapping slightly in a neighborhood of each bump, to get h, a homeomorphism from $\mathbb{R}^n$ into $\mathbb{R}^n$, with $h(\partial\Omega) = \partial\Omega'$. In Section 3 using a lemma of Wolff ([14, Lemma 2.7]) we will show the bumps can be chosen so that

$$\int_{\partial\Omega'} |\nabla G'| \log |\nabla G'| \, dH^{n-1} \leqslant \int_{\partial\Omega} |\nabla G| \log |\nabla G| \, dH^{n-1}. \tag{1.3}$$

The proof of (1.3) is somewhat involved, but luckily much of the hardwork has been done for us by Wolff.

In Section 4 we use (1.1)-(1.3) and induction to construct D. More specifically put $D_0 = B(0, \rho)$ and let

$$G_0(x) = \frac{1}{n(n-2)\alpha(n)} (|x|^{2-n} - \rho^{2-n}), \qquad x \in B(0, \rho),$$

be Green's function for $B(0, \rho)$, where ρ is chosen so that if $x \in \partial B(0, \rho)$, then

$$|\nabla G(x)| = \frac{1}{n\alpha(n)} \rho^{1-n} = 2. \tag{1.4}$$

We put $\Omega = D_0$ and modify Ω as above to obtain $\Omega' = D_1$, $G' = G_1$, with ϵ replaced by ϵ_1 and h by h_1. Suppose D_k has been constructed for $0 \leqslant k \leqslant m$. Again we put $\Omega = D_m$ and modify Ω as above to obtain $\Omega' = D_{m+1}$, $G' = G_{m+1}$, with ϵ replaced by $\epsilon_{m+1} = 2^{-(m+1)}\epsilon_0$, and h by h_{m+1}. By induction we get $(D_k)_0^\infty, (h_k)_1^\infty, (G_k)_0^\infty$, satisfying (1.1), (1.2), with Ω', Ω, replaced by D_{k+1}, D_k, respectively. Let $h_0(x) = \rho x$, and let $f_k = h_k \circ h_{k-1} \circ \cdots \circ h_0$, where $\circ$ denotes composition. Then it will follow from our construction for $k = 1, 2, \ldots,$ that

$$c(\beta)^{-1}|x - y|^{1/\beta} \leqslant |f_k(x) - f_k(y)| \leqslant c(\beta)|x - y|^{\beta}, \tag{1.5}$$

when $x, y \in \mathbb{R}^n$ and $|x - y| \leqslant 1/4$. Moreover, each f_k is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f_k(S) = \partial D_k$. Set $D = \bigcup_0^\infty D_k$, and note from (1.5) that there exists a subsequence (f_{n_k}) of (f_k) which converges to a homeomorphism f of $\mathbb{R}^n$, satisfying the conclusions of Theorem 1. Thus (*a*) in the definition of a pseudo sphere is valid. To prove (*b*) we first note from Green's Theorem and (1.2) that

$$1 = \int_{\partial D_k} |\nabla G_k| \, dH^{n-1} \geqslant H^{n-1}(\partial D_k), \tag{1.6}$$

for $k = 0, 1, \ldots$ Second, observe for each $\delta > 0$ that

$$\lim_{k \to \infty} H^{n-1}\{x \in \partial D_k : |\nabla G_k(x)| > 1 + \delta\} = 0, \tag{1.7}$$

since otherwise we could use (1.1) and iteration to get a contradiction to (1.6) for large k. Next from (1.2), (1.3), and iteration we deduce that for $\alpha > 1$, $k = 0, 1, \ldots$

$$(1.8)\quad \log\alpha \int_{\{|\nabla G_k| > \alpha\}} |\nabla G_k|\, dH^{n-1} \leqslant \int_{\partial D_k} |\nabla G_k| \log|\nabla G_k|\, dH^{n-1} \leqslant c < +\infty.$$

Also in Section 4 we show that as $k \to \infty$,

$$(1.9)\qquad H^{n-1}|_{\partial D_{n_k}} \to H^{n-1}|_{\partial D},$$

weakly as measures on $\mathbb{R}^n$. Let $g \geqslant 0$ be a harmonic function in D which is continuous on $\bar{D}$. Then from (1.2), (1.9), and Green's Theorem we get

$$(1.10)\qquad g(0) = \int_{\partial D_{n_k}} g|\nabla G_{n_k}|\, dH^{n-1} \geqslant \int_{\partial D_{n_k}} g\, dH^{n-1} \to \int_{\partial D} g\, dH^{n-1},$$

as $k \to \infty$. To obtain the reverse inequality for fixed $\delta < 10^{-3}$ and $\alpha > 10^3$, put

$$\begin{aligned} E_k &= \{x \in \partial D_{n_k} : 1 \leqslant |\nabla G_{n_k}(x)| \leqslant 1 + \delta\} \\ F_k &= \{x \in \partial D_{n_k} : 1 + \delta < |\nabla G_{n_k}(x)| \leqslant \alpha\} \\ L_k &= \{x \in \partial D_{n_k} : |\nabla G_{n_k}(x)| > \alpha\}, \end{aligned}$$

for $k = 0, 1, 2, \ldots$ Then

$$g(0) = \int_{\partial D_{n_k}} g|\nabla G_{n_k}|\, dH^{n-1} = \int_{E_k} \cdots + \int_{F_k} \cdots + \int_{L_k} \cdots = I_1 + I_2 + I_3.$$

Clearly,

$$|I_1| \leqslant (1 + \delta) \int_{\partial D_{n_k}} g\, dH^{n-1}.$$

Also from (1.7) we find that

$$|I_2| \leqslant \alpha \|g\|_\infty H^{n-1}\{x \in \partial D_{n_k} : 1 + \delta < |\nabla G_{n_k}|\} \to 0,$$

as $k \to \infty$. Here, $\|g\|_\infty$ denotes the maximum of g in $\bar{D}$. Using (1.8) we get

$$|I_3| \leqslant \|g\|_\infty \int_{\{|\nabla G_{n_k}| > \alpha\}} |\nabla G_{n_k}|\, dH^{n-1} \leqslant \frac{c}{\log\alpha} \|g\|_\infty.$$

Letting $k \to \infty$ we obtain from the above estimates and (1.9) that

$$g(0) \leqslant (1 + \delta) \int_{\partial D} g\, dH^{n-1} + \frac{c}{\log\alpha} \|g\|_\infty.$$

Finally letting $\delta \to 0$, $\alpha \to \infty$, we have

$$g(0) \leqslant \int_{\partial D} g\, dH^{n-1}.$$

In view of (1.10) we conclude that

$$g(0) = \int_{\partial D} g \, dH^{n-1} \tag{1.11}$$

when $g \geqslant 0$ is continuous on $\bar{D}$ and harmonic in D. From (1.11) with $g \equiv 1$ we note that, $H^{n-1}(\partial D) = 1$. If g_1 is continuous on $\bar{D}$, harmonic in D, and $g_1 - m \geqslant 0$ in $\bar{D}$, then from (1.11) and the above note we deduce

$$g_1(0) = (g_1 - m)(0) + m = \int_{\partial D} (g_1 - m) \, dH^{n-1} + m = \int_{\partial D} g_1 \, dH^{n-1}.$$

Thus, D is a pseudo sphere. The initial bumps on D_1 will be chosen to have low peaks relative to those added to form D_k, $k \geqslant 2$, in order to guarantee that D is not a ball.

We remark that D will be regular for the Dirichlet problem, so each continuous function on ∂D will have a harmonic extension to D which is continuous on $\bar{D}$. From (1.11) it follows that harmonic measure and H^{n-1} measure on ∂D are equal (see [7, Ch. 8] for the Dirichlet problem). Moreover, since $H^{n-1}(\partial D) = 1$, it follows (see [4, Section 5.8]) that D is of finite perimeter. Thus several other measures are equal to H^{n-1} measure on ∂D (see [5, Thm. 4.5.19, (16)] and [5, Thm. 3.2.26]). Also D will be a nontangentially accessible (NTA) domain in the sense of Kenig and Jerison [8]. Using the corkscrew condition for NTA domains ((i) in Section 3) it is easily deduced that every point in ∂D lies in the measure theoretic boundary of D (see [4, Section 5.8]). Hence D satisfies the hypotheses of Theorem 1 in [10], from which we conclude

$$\sup \{ |\nabla G^*(x)| : x \in D - B(0, \rho/2) \} = +\infty,$$

where G^* is Green's function for D with pole at 0. Next we remark that this paper leaves open the very interesting question as to whether f in Theorem 1 can also be chosen for some $K > 1$ to be a K quasiconformal mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, $n \geqslant 3$. In $\mathbb{R}^2$ it follows from a criteria of Ahlfors (see [1, Ch. 4]) and the Keldysh-Lavrentiev construction that the answer to the above question is yes, and in fact K can be chosen arbitrary near 1.

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2. Preliminary reductions

If $x \in \mathbb{R}^n$, we let $x' = (x_1, \ldots, x_{n-1})$ and shall write, $x = (x', x_n)$. We assume throughout this section that Ω is a bounded domain of class C^4 with $0 \in \Omega$.

More specifically, for each $y \in \partial\Omega$ there exists $s > 0$ such that $B(y, s) \cap \partial\Omega$ is a part of the graph of a four-times continuously differentiable function, defined on a hyperplane in $\mathbb{R}^n$, and $B(y, s) \cap \Omega$ lies above the graph. From compactness and a standard convering argument it follows for each $r > 0$ that there exists, $y^1, y^2, \ldots, y^N \in \partial\Omega$, such that

$$\partial\Omega \subset \bigcup_{i=1}^{N} B(y^i, 100r) \quad \text{and} \quad B(y^i, 10r) \cap B(y^j, 10r) = \varnothing, \qquad i \neq j.$$

Moreover, if $0 < r < r_0$, r_0 sufficiently small, and $y = (y', y_n) \in \{y^i\}_1^N$, then from the implicit function theorem we see there exists $\theta = \theta(\cdot, y)$, four-times continuously differentiable on $\mathbb{R}^{n-1}(\theta \in C^4(\mathbb{R}^{n-1}))$, with $\theta(0) = 0$, $\nabla'\theta(0) = 0$, such that after a possible rotation of axes:

$$\partial\Omega \cap B(y, 1000r^{1/2}) \subseteq \{(x' + y', \theta(x') + y_n) : x' \in \mathbb{R}^{n-1}\},$$
$$\Omega \cap B(y, 1000r^{1/2}) \subseteq \{(x' + y', x_n) : x_n - y_n > \theta(x'), x' \in \mathbb{R}^{n-1}\}$$

Here ∇' denotes the $\mathbb{R}^{n-1}$ gradient. Put

$$M_1 = \max_{y \in \{y^i\}_1^N} \left\{ \max_{x \in \partial\Omega \cap B(y, 1000r^{1/2})} \sum |\partial'_\alpha \theta(x', y)| \right\}$$

where the sum is taken over all multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ with $|\alpha| = \sum_{j=1}^{n-1} \alpha_j$, and $0 \leqslant |\alpha| \leqslant 4$. Also, ∂'_α denotes the corresponding partial derivative with respect to $(x')^\alpha$, $x' \in \mathbb{R}^{n-1}$. Given ϵ, $0 < \epsilon < \sigma_0 \leqslant 10^{-3}$, choose $r_0 > 0$ so small that for $0 < r \leqslant r_0$

$$M_1 r^{1/2} \leqslant 10^{-3} r^{1/4} < 10^{-9} \epsilon^4. \tag{2.1}$$

Again this choice is possible by compactness of $\partial\Omega$. In this section and the next section we allow r_0 to vary. At the end of this section we will fix σ_0 at a number, satisfying several conditions, which depends only on n. r_0 will depend on ϵ, M_1, n, and M_2, defined below.

As in Section 1 let G be Green's function for Ω with pole at 0 and assume $|\nabla G| \geqslant 1$ on $\partial\Omega$. Let

$$M_2 = \max_{y \in \{y^i\}_1^N} \left\{ \max_{x \in \bar{\Omega} \cap B(y, 1000r^{1/2})} \sum |\partial_\alpha G(x)| \right\},$$

where now $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $0 \leqslant |\alpha| \leqslant 4$, and ∂_α denotes the corresponding partial derivative with respect to x^α, $x \in \bar{\Omega}$. From Schauder's Theorem (see [6, Ch. 6]), it is clear that $M_2 < +\infty$. We choose r_0 still smaller, if necessary, so that in addition to the above conditions, we have

$$M_2 r^{1/2} \leqslant 10^{-3} r^{1/4} < 10^{-9} \epsilon^4. \tag{2.2}$$

Let l be the largest nonnegative integer such that $2^{-l}\sigma_0 > \epsilon$ and put $\sigma_k = 2^{-k}\sigma_0$, for $k = 0, 1, \ldots$ Set

$$E_k = \{x \in \partial\Omega : 1 + \sigma_k < |\nabla G(x)| \leqslant 1 + \sigma_{k-1}\}, \qquad 1 \leqslant k \leqslant l + 1,$$
$$E_0 = \{x \in \partial\Omega : |\nabla G(x)| > 1 + \sigma_0\}.$$

Let ψ, $0 \leqslant \psi \leqslant 1$, be a fixed C^∞ function on $\mathbb{R}^{n-1}$ with $\max_{\mathbb{R}^{n-1}} \psi = 1$ and support in the unit ball of $\mathbb{R}^{n-1}$, to be specified in Section 3. We form a domain Ω' of class C^4 by adding smooth bumps to $\partial\Omega$. More specifically, let L be the set of all $y \in \{y^i\}_1^N$ for which

$$B(y, 100r) \cap \bigcup_{k=0}^{i+1} E_k \neq \varnothing .$$

For fixed $y = (y', y_n) \in L$, let j be the smallest nonnegative integer with

$$B(y, 100r) \cap E_j \neq \varnothing . \tag{2.3}$$

Put

$$\xi(x') = \theta(x') - \sigma_j^2 r \lambda_j^{-1} \psi(\lambda_j x'/r) + y_n, \qquad x' \in \mathbb{R}^{n-1},$$

where $(\lambda_j)_0^\infty$ is an increasing sequence of positive numbers with $\lambda_j \geqslant 1/\sigma_j$, $j = 0, 1, \ldots$, which will be defined explicitly in Section 3. Also $(\lambda_j)_0^\infty$ will depend only on σ_0. Define Ω' by

(i) $\Omega - \bigcup_{z \in L} B(z, 10r) = \Omega' - \bigcup_{z \in L} B(z, 10r)$,

(ii) $\partial\Omega' \cap B(y, 10r) = \{(x' + y', \xi(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r)$,

(iii) $\Omega' \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \xi(x')\} \cap B(y, 10r)$.

Thus for each $y \in L$ and smallest j, $0 \leqslant j \leqslant l + 1$, satisfying (2.3), we add a bump to Ω under y, as defined above, to get Ω'. Clearly Ω' is of class C^4. Moreover, if r_0 is small enough, we claim as in (1.2) that

$$|\nabla G'(x)| \geqslant 1, \qquad x \in \partial\Omega'. \tag{2.4}$$

Indeed, if $x \in \partial\Omega' \cap \partial\Omega$, then it follows from the Hopf boundary maximum principle that (2.4) is true. To prove (2.4) for $x \in \partial\Omega' - \partial\Omega$, we let, $\hat{B}(t) = \{x' \in \mathbb{R}^{n-1} : |x'| < t\}$. We shall need the following lemma of Schauder type. In Lemma 1, ϕ, γ, are C^k functions on $\hat{B}(2)$, $k \geqslant 3$. Moreover, $\phi < 1/4$, and $\|\cdot\|_k$ denotes the C^k norm on $\hat{B}(2)$. Also, $c' = c'(\cdot, k)$, is an increasing function on $(0, \infty)$ which depends only on k.

Lemma 1. *Let*

$$H = \{(x', x_n): |x'| < 1 \text{ and } \phi(x') < x_n < 1\}.$$

Let u be harmonic in H, with $|u| \leqslant M_3 < +\infty$, *and suppose that* $u = \gamma$ *continuously on* $\{(x', \phi(x'))\} \cap \partial H$. *Then for* $k \geqslant 3$

$$\sum_{0 \le |\alpha| \le k} |\partial_\alpha u(x)| \leqslant c'(\|\phi\|_k)(\|\gamma\|_k + M_3), \qquad x \in B(0, 1/2) \cap \bar{H}.$$

Lemma 1 is given in [6, Corollary 6.7] for $C^{2,\alpha}$ domains with a constant depending on H. However, the proof is essentially unchanged if $C^{2,\alpha}$ is replaced by C^k, and $c'(\cdot)$ can be used for the resulting constant (see the remark following Lemma 6.5 in [6]). To prove (2.4) on a bump, we first let

$$Z(y, t) = \{(x', x_n): |x_n - y_n| < t, |x' - y'| < t\}$$

and note that since ψ has support in $\hat{B}(1)$,

$$(\partial\Omega' - \partial\Omega) \cap B(y, 10r) \subseteq Z(y, r\lambda_j^{-1}), \tag{2.5}$$

whenever $y \in L$ and j is the smallest integer satisfying (2.3). Second, observe from the Hopf boundary maximum principle and (2.5) that to prove (2.4) on a bump it suffices to show

$$|\nabla G^*(x)| \geqslant 1, \qquad x \in \bar{Z}(y, r\lambda_j^{-1}) \cap \partial\bar{\Omega}^*, \tag{2.6}$$

where Ω^* is obtained from Ω by adding just one bump at y as above, and G^* is the Green's function for Ω^* with pole at 0. To prove (2.6) let

$$F = \bar{Z}(y, r\lambda_j^{-1}) \cap \bar{\Omega}^*$$

and

$$M_4 = \max_{x \in F} |\nabla G^*(x)|.$$

Then from the mean value theorem of calculus and the fact that $G = 0$ on $\partial\Omega$, we deduce

$$0 \leqslant G^* - G \leqslant cM_4\sigma_j^2\lambda_j^{-1}r \tag{2.7}$$

on $\partial\Omega$. Since $G^* - G$ is harmonic in Ω, we see from the maximum principle for harmonic functions that (2.7) also holds in Ω. From (2.1), (2.2), (2.7), and the fact that

$$\nabla G(y) = \left(0, \ldots, \frac{\partial G(y)}{\partial y_n}\right)$$

we get for x in $\bar{\Omega} \cap \bar{B}(y, 20r\lambda_j^{-1})$,

$$
\begin{aligned}
(2.8)\ |G^*(x) - |\nabla G(y)|(x_n - y_n)| \\
&\leqslant cM_4\sigma_j^2\lambda_j^{-1}r + |G(x) - |\nabla G(y)|(x_n - y_n)| \\
&\leqslant cM_4\sigma_j^2\lambda_j^{-1}r + \int_{y_n+\theta(x'-y')}^{x_n} \left| \frac{\partial G}{\partial t_n}(x', t_n) - \frac{\partial G}{\partial t_n}(y', y_n) \right| dt_n \\
&\quad + |\nabla G(y)|\, |\theta(x' - y')| \\
&\leqslant cM_4\sigma_j^2\lambda_j^{-1}r + cM_2(\lambda_j^{-1}r)^2 + cM_2M_1(\lambda_j^{-1}r)^2 \\
&\leqslant c(M_4\sigma_j^2 + \epsilon^2)\lambda_j^{-1}r.
\end{aligned}
$$

Put, $\beta = 10r/\lambda_j$,

$$
\begin{aligned}
\phi(x') &= \beta^{-1}(\xi(\beta x') - y_n), && x' \in \hat{B}(2), \\
u(x) &= \beta^{-1}G^*(\beta x + y) - |\nabla G(y)|x_n, && x \in H,
\end{aligned}
$$

where H is defined relative to ϕ as in Lemma 1. Using (2.1) it is easily checked that $\|\phi\|_4 \leqslant c\sigma_j^2\|\psi\|_4 + c\epsilon^2$. Since $u = -|\nabla G(y)|\phi$ on $\partial H \cap B(1)$, we find from this inequality, (2.8), and Lemma 1 with $k = 4$ that

$$|\nabla u(x)| \leqslant c'(\|\phi\|_4)(M_4\sigma_j^2 + c\sigma_j^2|\nabla G(y)| + c\epsilon^2|\nabla G(y)| + c\epsilon^2)$$

$x \in B(0, 1/2) \cap H$, where

$$c'(\|\phi\|_4) \leqslant c'(\|\psi\|_4 + 1) = c_0.$$

From this inequality and the fact that $\epsilon \leqslant 2\sigma_j$, $|\nabla G(y)| \geqslant 1$, we deduce

$$||\nabla G^*(x)| - |\nabla G(y)|| \leqslant c_0M_4\sigma_j^2 + c_1\sigma_j^2|\nabla G(y)|, \tag{2.9}$$

for $x \in \bar{Z}(y, r\lambda_j^{-1}) \cap \bar{\Omega}^*$. Let σ_0, $0 < \sigma_0 \leqslant 10^{-3}$, be so small that

$$c_0 + c_1 < 10^{-3}\sigma_0^{-1}. \tag{2.10}$$

Choosing x so that

$$|\nabla G^*(x)| = M_4,$$

we conclude from the triangle inequality and (2.9) that

$$M_4(1 - c_0\sigma_j^2) \leqslant (1 + c_1\sigma_j^2)|\nabla G(y)|.$$

Hence,

$$M_4 \leqslant (1 + 2c_0\sigma_j^2)(1 + c_1\sigma_j^2)|\nabla G(y)|. \tag{2.11}$$

Now from (2.2), (2.3), we see that $|\nabla G(y)| \geq 1 + \sigma_j/2$. Using this fact, (2.10), and (2.11), in (2.9), we deduce

$$|\nabla G^*(x)| \geq (1 - 2(c_0 + c_1)\sigma_j^2)|\nabla G(y)| \geq 1 + \frac{1}{4}\sigma_j.$$

Hence (2.6) is valid. From our earlier remarks it now follows that (2.4) is valid.

If

$$c_2 = \int_{\mathbb{R}^{n-1}} |\nabla'\psi(x')|^2 \, dx',$$

and

$$\sigma_0 \leq c_2 \leq \alpha(n-1)\Big(\max_{\mathbb{R}^{n-1}} |\nabla'\psi|\Big)^2 \leq \sigma_0^{-1} 10^{-6}, \tag{2.12}$$

then from (2.1), it follows that

$$\begin{aligned} H^{n-1}(Z(y, r\lambda_j^{-1}) \cap \partial\Omega') &= \int_{\hat{B}(r\lambda_j^{-1})} \sqrt{1 + |\nabla'\xi|^2} \, dx' \\ &\geq \int_{\hat{B}(r\lambda_j^{-1})} \sqrt{1 + \sigma_j^4 |\nabla'\psi(\lambda_j r^{-1} x')|^2} \, dx' - \epsilon^8 \alpha(n-1)(r/\lambda_j)^{(n-1)} \\ &= \Big(\int_{\hat{B}(1)} \sqrt{1 + \sigma_j^4 |\nabla'\psi(x')|^2} \, dx'\Big)(r/\lambda_j)^{(n-1)} - \epsilon^8 \alpha(n-1)(r/\lambda_j)^{(n-1)} \\ &\geq \Big(1 + \frac{1}{4}\sigma_j^4 c_2 - \epsilon^8\Big)\alpha(n-1)(r/\lambda_j)^{(n-1)} \\ &\geq \frac{1}{8}\sigma_j^4 c_2 \alpha(n-1)(r\lambda_j^{-1})^{(n-1)} + H^{n-1}(Z(y, r\lambda_j^{-1}) \cap \partial\Omega). \end{aligned} \tag{2.13}$$

Given $t \geq \epsilon$, let k be the least nonnegative integer such that $t \geq \sigma_k$, $0 \leq k \leq l + 1$. Let $J = J(k)$, be the set of all i such that (2.3) holds with $y = y^i$ and $j \leq k$. From (2.1) it is clear that

$$\begin{aligned} H^{n-1}\{x \in \partial\Omega : |\nabla G(x)| \geq 1 + t\} &\leq H^{n-1}\Big(\bigcup_{i \in J} B(y^i, 100r) \cap \partial\Omega\Big) \\ &\leq 2 \sum_{i \in J} \alpha(n-1)(100r)^{n-1}. \end{aligned} \tag{2.14}$$

Using (2.13), (2.14), and (2.5) we deduce

$$H^{n-1}(\partial\Omega') \geq H^{n-1}(\partial\Omega) + \frac{c_3\sigma_k^4}{\lambda_k^{n-1}} H^{n-1}\{x \in \partial\Omega : |\nabla G(x)| > 1 + t\}, \tag{2.15}$$

where $c_3 > 0$ depends only on n. Let

$$\eta(t) = \begin{cases} \dfrac{c_3\sigma_o^4}{\lambda_o^{n-1}}, & \sigma_0 \leqslant t \\ \dfrac{c_3\sigma_k^4}{\lambda_k^{n-1}}, & \sigma_k \leqslant t < \sigma_{k-1}, \qquad k = 1, 2, \ldots \end{cases}$$

Clearly η does not depend on Ω or Ω'. Rewriting (2.15) in terms of η we obtain (1.1).

Next we define the homeomorphism h mentioned in Section 1. If $y \in L$ and j is the smallest positive integer for which (2.3) holds, define h on $Z(y, r)$ by $h(x) = (x', h^*(x))$, where

$$h^*(x', x_n) = \begin{cases} \dfrac{(r + y_n - \xi(x' - y'))(x_n - r - y_n)}{r - \theta(x' - y')} + r + y_n, & x \in Z(y, r) \cap \Omega \\ \dfrac{(\xi(x' - y') + r - y_n)(x_n + r - y_n)}{r + \theta(x' - y')} - r + y_n, & x \in Z(y, r) \cap (\mathbb{R}^n - \Omega) \end{cases}$$

Define $h(x) = x$ in the complement of the union of all $Z(y, r)$ for which (2.3) holds. We note that h restricted to $Z(y, r) = Z$ is simply a projection by lines parallel to the x_n axis of $Z \cap (\mathbb{R}^n - \Omega)$, $Z \cap \Omega$, respectively onto $Z \cap (\mathbb{R}^n - \Omega')$, $Z \cap \Omega'$, which keeps $\partial Z(y, r)$ fixed. Thus, h is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $h(\bar{\Omega}) = \bar{\Omega}'$. Moreover, using (2.1) it is easily checked that

$$(2.16) \qquad (1 - c_4\sigma_0^2)|x - z| \leqslant |h(x) - h(z)| \leqslant (1 + c_4\sigma_0^2)|x - z|,$$

when $x, z \in \mathbb{R}^n$ and

$$(2.17) \qquad |x - z| - c_4\sigma_0^2 r \leqslant |h(x) - h(z)| \leqslant |x - z| + c_4\sigma_0^2 r,$$

when $|x - z| > r$. Also for use in proving (1.9) we shall show for $x, z \in \partial\Omega$, that

$$(2.18) \qquad |h(x) - h(z)| \geqslant (1 - c_5 r^{1/2})|x - z|.$$

Indeed, suppose $x, z \in \partial\Omega$, $5r \leqslant |x - z| < 100r^{1/2}$, $x \in B(w, 100r)$, and $z \in B(y, 100r)$, where $w, y \in \{y^i\}_1^N$. Let θ be defined relative to y as previously and recall that $B(y, 1000r^{1/2}) \cap \partial\Omega$ can be expressed in terms of θ. Let $\nu(p)$ denote the outer unit normal to p in $\partial\Omega$ and let $\bullet$ denote inner product. Then

$$|\nu(y) \bullet \nu(w)| = (1 + |\nabla\theta(w' - y')|^2)^{-1/2} > 1 - \frac{1}{4}M_1^2|w' - y'|^2.$$

Thus if δ denotes the angle between $\nu(y)$ and $\nu(w)$, then

$$\delta < 4M_1|w' - y'| < 164M_1|x - z|.$$

Next suppose $h(x) = (v', v_n)$ and $v' \neq x'$. Then we can draw the right triangle with vertices x, $h(x)$, and $P = (v', x_n)$. Let l_1, l_2, and l_3 be the sides of this triangle connecting x to $h(x)$, $h(x)$ to P, and P to x, respectively. Then from the definition of h we see that $\nu(w)$ is parallel to l_1, and so l_1, l_2 form an angle δ at $h(x)$. Also, $|x - h(x)| < r$, so from trigonometry and the above inequality,

$$|v' - x'| < r \sin \delta < 164 M_1 r |x - z|.$$

From this inequality and (2.1) we deduce

$$\begin{aligned}|h(z) - h(x)| &> |v' - z'| \\ &\geqslant |x' - z'| - |v' - x'| \\ &> (1 - cM_1^2 r)|x - z| \\ &> (1 - c_5 r^{1/2})|x - z|.\end{aligned}$$

Hence (2.18) is valid when $5r \leqslant |x - z| < 100 r^{1/2}$. If $|x - z| < 5r$, then (2.18) remains true as follows easily from the fact that the bumps are greater than $6r$ apart. If $100 r^{1/2} \leqslant |x - z|$ then it follows from (2.17) that (2.18) is true.

Finally in this section we fix σ_0 to be the largest number for which (2.10), (2.12), hold and

$$c_4 \sigma_0^2 \leqslant \frac{1}{2}. \tag{2.19}$$

Note from (2.12) that $0 < \sigma_0 \leqslant 10^{-3}$.

3. Wolff's lemma

To prove (1.3) in Section 1 we shall need some definitions. Let Ω_1 be a bounded domain. If diam $\Omega_1 = 1$, then Ω_1 is an NTA domain with constant A if it has the following properties:

(i) (Corkscrew condition.) For each $x \in \partial\Omega_1$, $0 < r < A^{-1}$, there are points $P_r(x) \in \Omega_1$, $Q_r(x) \in \mathbb{R}^n - \Omega_1$, with $|P_r(x) - x| \leqslant Ar$, $|Q_r(x) - x| \leqslant Ar$, and $\operatorname{dist}(P_r(x), \partial\Omega_1) \geqslant A^{-1} r$, $\operatorname{dist}(Q_r(x), \partial\Omega_1) \geqslant A^{-1} r$,

(ii) (Harnack chain condition.) For each $x, y \in \Omega_1$ there is a path γ from x to y with length $(\gamma) \leqslant A|x - y|$ and $\operatorname{dist}(\gamma(t), \partial\Omega_1) \geqslant A^{-1} \min\{|\gamma(t) - x|, |\gamma(t) - y|\}$.

In general Ω_1 is an NTA domain with constant A, if a scaling of it with diameter 1 has constant A. Ω_1 is said to be Lipschitz on scale t with constant A, provided for each $z \in \partial\Omega_1$, there is a coordinate system such that $\partial\Omega_1 \cap B(z, t)$ is the graph of a Lipschitz function defined on $\mathbb{R}^{n-1}$ with Lipschitz norm less than or equal to A. Moreover, $\Omega_1 \cap B(x, t)$ lies above the graph of this function.

Now suppose for some $w \in \partial\Omega_1$ and $t > 0$ that after a possible rotation of coordinates,

$$\partial\Omega_1 \cap B(w, t) = \{x : x_n = w_n\} \cap B(w, t)$$
$$\Omega_1 \cap B(w, t) = \{x : x_n > w_n\} \cap B(w, t) \tag{3.1}$$

Let $p \leqslant 0$ be a C^∞ function with support in $\hat{B}(1)$, suppose $\lambda > 2 \max_{\mathbb{R}^{n-1}} |p| + 1$, and define $\Omega_2 \supset \Omega_1$ as follows:

(*a*) $\Omega_1 - B(w, t) = \Omega_2 - B(w, t)$,

(*b*) $\partial\Omega_2 \cap B(w, t) = \{(x' + w', w_n + t\lambda^{-1}p(t^{-1}\lambda x')) : x' \in \mathbb{R}^{n-1}\} \cap B(w, t)$,

(*c*) $\Omega_2 \cap B(w, t) = \{(x' + w', x_n) : x_n > w_n + t\lambda^{-1}p(t^{-1}\lambda x')\} \cap B(w, t)$.

Let $\hat{p}$ be the continuous harmonic extension of p to $(\mathbb{R}^n)^+ = \{(x', x_n) : x_n > 0\}$ and put

$$\Lambda(p) = \int_{\mathbb{R}^{n-1}} \left(\left(\frac{\partial \hat{p}}{\partial x_n} \right)^3 - 3|\nabla' p|^2 \frac{\partial \hat{p}}{\partial x_n} \right)(x', 0)\, dx'$$

where $\nabla' p$, as in Section 2, is the $\mathbb{R}^{n-1}$ gradient. Next if $d = \operatorname{diam} \Omega_1$, we assume

$$B(0, d/A) \subseteq \Omega_1 \subseteq B(0, Ad). \tag{3.2}$$

Denote Green's functions for Ω_1, Ω_2, with pole at 0, by G_1, G_2, respectively, and let ω_1 be harmonic measure on Ω_1 with respect to 0. If $\partial\Omega_1$ is sufficiently smooth we observe that

$$\omega_1(E) = \int_{E \cap \partial\Omega_1} |\nabla G_1|\, dH^{n-1}, \qquad E \text{ Borel.}$$

Then Wolff proved [14, Lemma 2.7].

Lemma 2. *Let Ω_1 be* NTA *and Lipschitz on scale t with constant A. Suppose Ω_1 satisfies* (3.1), (3.2), *and Ω_2 is obtained by adding a bump to Ω_1 as in* (*a*)-(*c*). *If $\Lambda(p) < 0$, then there exists $\lambda^* = \lambda^*(A, p)$, $c_6 = c_6(A, p)$, such that for $\lambda \geqslant \lambda^*$,*

$$\int_{\partial\Omega_2} |\nabla G_2| \log |\nabla G_2|\, dH^{n-1} \leqslant \int_{\partial\Omega_1} |\nabla G_1| \log |\nabla G_1|\, dH^{n-1} - \frac{c_6}{\lambda^{n-1}} \omega_1(B(w, t)).$$

Actually Wolff proves this Lemma only in $\mathbb{R}^3$, but the proof for $\mathbb{R}^n$, $n \geqslant 3$, is essentially unchanged. To show the existence of $p \leqslant 0$ for which $\Lambda(p) < 0$, Wolff first shows that $\Lambda(q) < 0$ for $n = 3$ when $q(x') = -|x' + e_3|^{-1}$, $x' \in \mathbb{R}^3$,

$e_3 = (0, 0, 1)$. In view of this function, the natural function to consider for $n \geqslant 3$ is

$$q(x') = -|x' + e_n|^{2-n}, \qquad e_n = (0, \dots, 0, 1), \qquad x' \in \mathbb{R}^{n-1},$$

for which $\hat{q}(x) = -|x + e_n|^{2-n}$, $x \in (\mathbb{R}^n)^+$. Then

$$\Lambda(q) = (n-1)(n-2)^3\alpha(n-1)\int_0^\infty (r^2+1)^{-3n/2}(1-3r^2)r^{n-2}\,dr$$

$$= -\frac{(n-1)(n-2)^4\alpha(n-1)\Gamma(n-1/2)\Gamma(n/2-1/2)}{4\Gamma(3n/2)} < 0,$$

where Γ denotes the Euler gamma function and the integral was evaluated using the substitution $r = \tan\theta$, as well as, the beta function. Let Φ, $0 \leqslant \Phi \leqslant 1$, be a C^∞ function on $\mathbb{R}^{n-1}$ with support in $\hat{B}(2)$, $|\nabla'\Phi| \leqslant 1000$, and $\Phi = 1$ on $\hat{B}(1)$. Now if

$$q_m(x') = \Phi(m^{-1}x')q(x'), \qquad x' \in \mathbb{R}^{n-1},$$

then it follows easily from properties of conjugate harmonic functions (see [13, Ch. 6]) that

$$\Lambda(q_m) \to \Lambda(q) \quad \text{as} \quad m \to \infty.$$

Taking a suitable dilation of q_m for large m, we get $p \leqslant 0$ in $C^\infty(\mathbb{R}^{n-1})$ with $\operatorname{supp} p \subseteq \hat{B}(1)$, and $\Lambda(p) < 0$.

We now define ψ and $(\lambda_k)_0^\infty$ introduced in Section 2. Let ψ, $0 \leqslant \psi \leqslant 1$, be a fixed $C^\infty(\mathbb{R}^{n-1})$ function with support in $\hat{B}(1)$, $\max_{\mathbb{R}^{n-1}} \psi = 1$, and $\Lambda(\psi) > 0$. Recall that $\sigma_k = 2^{-k}\sigma_0$, $k = 0, 1, \dots$, and define λ_k as follows: let $A = 200$ in Lemma 2 and $p = -\sigma_k^2\psi$. Let $\lambda_k' = \max\{\sigma_k^{-1}, b_k^{-1}, \lambda_k^*\}$, $k = 0, 1, \dots$, where $b_k = c_6(200, -\sigma_k^2\psi)$, $\lambda_k^* = \lambda^*(200, -\sigma_k^2\psi)$. Put $\lambda_m = \max_{0 \le k \le m}\lambda_k'$, $m = 0, 1, \dots$ and note that $(\lambda_k)_0^\infty$ depends only on n since σ_0 and ψ are fixed.

Let $\Omega, \Omega', \epsilon, r, L$, and $(E_k)_0^{l+1}$, be as in Section 2 and suppose also that Ω is NTA with constant 100. Moreover, we assume $B(0, \rho) \subseteq \Omega \subseteq B(0, 2)$, where ρ is as in (1.4). From our choice of r we see that Ω is Lipschitz on scale $r^{1/2}$ with constant 2. In order to apply Lemma 2, we need to add flat bumps under each $y \in L$. For fixed $y \in L$ let j be the smallest nonnegative integer for which (2.3) holds, *i.e.*

$$B(y, 100r) \cap E_j \neq \varnothing.$$

Suppose that $L = \{z_1, z_2, \dots, z_m\}$ and put $L_k = \{z_1, \dots, z_k\}$, $1 \leqslant k \leqslant m$. For fixed $y \in L$ we assume that $B(y, 1000r^{1/2}) \cap \Omega$, $B(y, 1000r^{1/2}) \cap \partial\Omega$, can be expressed as in Section 2 relative to θ. Let

$$\hat{\xi}(x') = -100M_1 r^2 \Phi\left(\frac{x'}{r}\right) + \left(1 - \Phi\left(\frac{x'}{r}\right)\right)\theta(x') + y_n, \qquad x' \in \mathbb{R}^{n-1},$$

$$\tilde{\xi}(x') = \hat{\xi}(x') - \sigma_j^2 r \lambda_j^{-1} \psi(\lambda_j x'/r), \qquad x' \in \mathbb{R}^{n-1},$$

where Φ was defined earlier in Section 3 and M_1 is as in (2.1). Define $\hat{\Omega}_k$, $1 \leqslant k \leqslant m$, as follows:

$$(\hat{\text{I}}) \qquad \hat{\Omega}_k - \bigcup_{z \in L_k} B(z, 10r) = \Omega - \bigcup_{z \in L_k} B(z, 10r),$$

$$(\hat{\text{II}}) \qquad \partial\hat{\Omega}_k \cap B(y, 10r) = \{(x' + y', \hat{\xi}(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r),$$

$$(\hat{\text{III}}) \qquad \hat{\Omega}_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \hat{\xi}(x')\} \cap B(y, 10r),$$

for each $y \in L_k$. $\tilde{\Omega}_k \supseteq \hat{\Omega}_m$, $1 \leqslant k \leqslant m$, is defined similarly by

$$(\tilde{\text{I}}) \qquad \tilde{\Omega}_k - \bigcup_{z \in L_k} B(z, 10r) = \hat{\Omega}_m - \bigcup_{z \in L_k} B(z, 10r),$$

$$(\tilde{\text{II}}) \qquad \partial\tilde{\Omega}_k \cap B(y, 10r) = \{(x' + y', \tilde{\xi}(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r),$$

$$(\tilde{\text{III}}) \qquad \tilde{\Omega}_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \tilde{\xi}(x')\} \cap B(y, 10r),$$

for each $y \in L_k$. From (2.1) and the definition of Ω' we see that $\hat{\Omega}_m \supseteq \Omega$, $\tilde{\Omega}_m \supseteq \Omega'$. Using the fact that Ω is NTA with constant 100 and local smoothness of $\hat{\Omega}_k$, $\tilde{\Omega}_k$, it is easily checked that $\hat{\Omega}_k$, $\tilde{\Omega}_k$, $1 \leqslant k \leqslant m$, are NTA and Lipschitz on scale r with constant 200. Let $\hat{\Omega}_0 = \Omega$, $\tilde{\Omega}_0 = \hat{\Omega}_m$. We first apply Lemma 2 with $t = r$, $\Omega_1 = \tilde{\Omega}_0$, $\Omega_2 = \tilde{\Omega}_1$, after a possible rotation. We next apply Lemma 2 with $\Omega_1 = \tilde{\Omega}_1$ and $\Omega_2 = \tilde{\Omega}_2, \ldots$, etc. Let $\hat{G}_k$, $\tilde{G}_k$, $\hat{\omega}_k$, $\tilde{\omega}_k$, be the Green's functions and harmonic measures relative to 0 for $\hat{\Omega}_k$, $\tilde{\Omega}_k$. Applying the above argument m times we obtain an inequality for $\hat{G}_m = \tilde{G}_0$ and $\tilde{G}_m$. Using the definition of $(\lambda_k)_0^\infty$, we conclude

$$(3.3) \quad \int_{\partial\tilde{\Omega}_m} |\nabla\tilde{G}_m| \log |\nabla\tilde{G}_m| \, dH^{n-1}$$
$$\leqslant \int_{\partial\hat{\Omega}_m} |\nabla\hat{G}_m| \log |\nabla\hat{G}_m| \, dH^{n-1} - c(\lambda_{l+1})^{-(n-1)} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_{k+1}, 2r)).$$

Next we define a function τ on $[0, 1]$ by $\tau(s) = \min\{\lambda_k : \sigma_k \leqslant s\}$, $0 < s \leqslant 1$. Choosing r_0 still smaller, if necessary, we assume, as we may, that for $0 < r \leqslant r_0$,

$$(3.4) \qquad r^{1/16} \leqslant \tau(\epsilon)^{-(n-1)}.$$

Note that $\tau(\epsilon) = \lambda_{l+1}$.

To prove (1.3) we must show that $\hat{G}_m$, $\tilde{G}_m$, in (3.3) can be replaced by G, G', with an error term at most,

$$c\tau(\epsilon)^{-(n-1)} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_k, 2r)).$$

To do so we introduce Ω'_k, $0 \leqslant k \leqslant m$, defined by, $\Omega'_0 = \Omega'$, and for $1 \leqslant k \leqslant m$,

$$\text{(I}'\text{)} \qquad \Omega'_k - \bigcup_{z \in L_k} B(z, 10r) = \Omega' - \bigcup_{z \in L_k} B(z, 10r),$$

$$\text{(II}'\text{)} \qquad \partial\Omega'_k \cap B(y, 10r) = \{(x' + y', \xi(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r),$$

$$\text{(III}'\text{)} \qquad \Omega'_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \xi(x')\} \cap B(y, 10r),$$

for each $y \in L_k$. Denote the corresponding Green's functions and harmonic measures relative to 0, by G'_k, ω'_k, $0 \leqslant k \leqslant m$. We shall also need the following facts about the NTA domain Ω_1 with constant A satisfying (3.2). If $z \in \partial\Omega_1$, then

$$\text{(3.5)} \qquad \begin{aligned} c(A)^{-1}\omega_1(B(z,t)) &\leqslant t^{n-2} \max_{B(z,t)\cap\Omega_1} G_1 \\ &\leqslant c(A)t^{n-2}G_1(P_t) \\ &\leqslant c(A)\omega_1(B(z,t)), \end{aligned}$$

for $0 < t < A^{-1}$, where $P_t = P_t(z)$. Moreover,

$$\text{(3.6)} \qquad \omega_1(B(z, 2t)) \leqslant c(A)\omega_1(B(z, t)).$$

(3.6) is called the doubling inequality for harmonic measure. If $z \in \partial\Omega_1$ and u, v are two positive harmonic functions in Ω_1 which vanish continuously on $\partial\Omega_1 - B(z, t)$, and $P_t = P_t(z)$, then for $x \in \Omega_1 - B(z, 2t)$

$$\text{(3.7)} \qquad c(A)^{-1}u(P_t)/v(P_t) \leqslant u(x)/v(x) \leqslant c(A)u(P_t)/v(P_t).$$

Moreover, (3.7) is valid when u and v vanish on $\partial\Omega_1 \cap B(z, 2t)$, and $x \in B(z, t) \cap \Omega_1$. (3.7) is called the rate inequality. Finally there exists $\mu = \mu(A) > 0$ so that for z and P_t as above, and $x \in B(z, t) \cap \Omega_1$,

$$\text{(3.8)} \qquad G_1(x) \leqslant c(|x - z|/t)^{\mu} G_1(P_t).$$

For the proof of (3.5)-(3.8) see [8, Sections 4 and 5].

From (3.5), (3.6), (3.8) with $t = A^{-1}$, and the fact that $\omega_1(B(z, A^{-1})) \geqslant c(A)^{-1}$, when $z \in \partial\Omega_1$, we see there exists $\nu(A)$, $0 < \nu < 1$, with

$$\text{(3.9)} \qquad c(A)^{-1}t^{1/\nu} \leqslant \omega_1(B(z, t)) \leqslant c(A)t^{\mu+n-2}, \qquad 0 < t < A^{-1}.$$

We claim that

$$\sum_{k=0}^{m-1} \omega_k^*(B(z_{k+1}, 6r)) \leqslant c \sum_{k=0}^{m-1} \omega_k^+(B(z_{k+1}, 6r)), \tag{3.10}$$

whenever $*$ and $+$ are elements of $\{\wedge, \sim, '\}$. Indeed from our construction and the maximum principle for harmonic functions we have,

$$\begin{aligned}\hat{\omega}_0(B(z_{k+1}, 6r) - B(z_{k+1}, 2r)) &\leqslant \omega_j^*(B(z_{k+1}, 6r) - B(z_{k+1}, 2r)) \\ &\leqslant \tilde{\omega}_m(B(z_{k+1}, 6r) - B(z_{k+1}, 2r)),\end{aligned}$$

when $0 \leqslant j \leqslant m$, $0 \leqslant k \leqslant m-1$, and $* \in \{\wedge, \sim, '\}$. Summing and using the doubling inequality it follows that

$$c^{-1} \sum_{k=0}^{m-1} \hat{\omega}_0(B(z_{k+1}, 6r)) \leqslant \sum_{k=0}^{m-1} \omega_k^*(B(z_{k+1}, 6r)) \leqslant c \sum_{k=0}^{m-1} \tilde{\omega}_m(B(z_{k+1}, 6r)).$$

On the other hand, from the maximum principle we deduce

$$\sum_{k=0}^{m-1} \tilde{\omega}_m(B(z_{k+1}, 6r)) \leqslant \sum_{k=0}^{m-1} \hat{\omega}_0(B(z_{k+1}, 6r)).$$

Hence our claim is true. We shall show for $0 \leqslant k \leqslant m-1$ that

$$\begin{aligned}\int_{\partial\Omega'_k} |\nabla G'_k| \log |\nabla G'_k| \, dH^{n-1} \\ \leqslant \int_{\partial\Omega'_{k+1}} |\nabla G'_{k+1}| \log |\nabla G'_{k+1}| \, dH^{n-1} + c r^{1/2} \omega'_k(B(z_{k+1}, 6r)),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\int_{\partial\hat{\Omega}_{k+1}} |\nabla \hat{G}_{k+1}| \log |\nabla \hat{G}_{k+1}| \, dH^{n-1} \\ \leqslant \int_{\partial\hat{\Omega}_k} |\nabla \hat{G}_k| \log |\nabla \hat{G}_k| \, dH^{n-1} + c r^{1/2} \hat{\omega}_k(B(z_{k+1}, 6r)).\end{aligned} \tag{3.12}$$

Summing (3.11) and using (3.10), it then follows that

$$\begin{aligned}\int_{\partial\Omega'} |\nabla G'| \log |\nabla G'| \, dH^{n-1} \\ \leqslant \int_{\partial\tilde{\Omega}_m} |\nabla \tilde{G}_m| \log |\nabla \tilde{G}_m| \, dH^{n-1} + c r^{1/2} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_{k+1}, 6r)),\end{aligned} \tag{3.13}$$

where we have used the fact that $\Omega'_0 = \Omega_0$, $\Omega'_m = \tilde{\Omega}_m$.

Summing (3.12) and using (3.10), we find

$$(3.14)\quad \int_{\partial\hat{\Omega}_m} |\nabla\hat{G}_m| \log|\nabla\hat{G}_m|\, dH^{n-1} \leqslant \int_{\partial\Omega} |\nabla G| \log|\nabla G|\, dH^{n-1} + cr^{1/2} \sum_{k=0}^{m-1} \hat{\omega}_k(B(z_{k+1}, 6r)),$$

since $\hat{\Omega}_0 = \Omega$. Putting (3.13), (3.14), into (3.3) and using (3.6) we get (1.3) provided r_0 is small enough, thanks to (3.4). Thus (1.3) is true once we prove (3.11)-(3.12).

We prove only (3.11), (3.12), for $k = 0$, since the proof of all the other inequalities is the same. To prove (3.12) for $k = 0$ we first observe from (3.5) that

$$(3.15)\quad \max_{B(z_1, 6r)\cap\hat{\Omega}_1} \hat{G}_1 \leqslant cr^{2-n}\hat{\omega}_1(B(z_1, 6r)).$$

Using (3.15), (2.1), and applying Lemma 1 with $k = 4$ after scaling $B(x_1, 6r) \cap \hat{\Omega}_1$, we find for x, y in the closure of $B(z_1, 3r)\cap\hat{\Omega}_1$,

$$(3.16)\quad |\nabla\hat{G}_1(x) - \nabla\hat{G}_1(y)| \leqslant c|x - y|r^{-n}\hat{\omega}_1(B(z_1, 6r)),$$

while from (3.15), a barrier argument, (3.5)-(3.6) and (ii), we have

$$(3.17)\quad c^{-1}r^{1-n}\hat{\omega}_1(B(z_1, 6r)) \leqslant |\nabla\hat{G}_1(x)| \leqslant cr^{1-n}\hat{\omega}_1(B(z_1, 6r)).$$

Clearly (3.17) and (3.9) imply

$$(3.18)\quad |\log|\nabla\hat{G}_1(x)|| \leqslant -c\log r,$$

when x is in the closure of $B(z_1, 3r)\cap\hat{\Omega}_1$. Using (3.16)-(3.18), (3.6), (2.1), and parametrizing $\partial\Omega$ and $\partial\hat{\Omega}_1$ in terms of θ and ξ, for $y = z_1$, we obtain with $z_1 = (y', y_n)$, $\hat{x} = (x' + y', \xi(x'))$, $x = (x' + y', \theta(x') + y_n)$,

(3.19)

$$\left| \int_{\partial\Omega\cap B(z_1, 3r)} |\nabla\hat{G}_1| \log|\nabla\hat{G}_1|\, dH^{n-1} - \int_{\partial\hat{\Omega}_1\cap B(z_1, 3r)} |\nabla\hat{G}_1| \log|\nabla\hat{G}_1|\, dH^{n-1} \right|$$

$$\leqslant \int_{\hat{B}(3r)} ||\nabla\hat{G}_1| \log|\nabla\hat{G}_1||(x) \left|\sqrt{1 + |\nabla'\theta(x')|^2} - \sqrt{1 + |\nabla'\xi(x')|^2}\right| dx'$$

$$+ \int_{\hat{B}(3r)} ||\nabla\hat{G}_1|(x) - |\nabla\hat{G}_1|(\hat{x})||\log|\nabla\hat{G}_1(x)|| \sqrt{1 + |\nabla'\xi(x')|^2}\, dx'$$

$$+ \int_{\hat{B}(3r)} |\nabla \hat{G}_1|(\hat{x})|\log|\nabla \hat{G}_1(x)| - \log|\nabla \hat{G}_1(\hat{x})||\sqrt{1 + |\nabla' \xi(x')|^2}\, dx'$$
$$\leqslant (-cM_1^2 r^2 \log r - cM_1 r \log r + \log(1 + M_1 r))\hat{\omega}_1(B(z_1, 6r))$$
$$\leqslant cr^{1/2}\hat{\omega}_1(B(z_1, 6r)).$$

Next from (3.17), (2.1) and the fact that each point of $B(z_1, 6r) \cap \partial\hat{\Omega}_1$ lies within 200 $M_1 r^2$ of a point of $B(z_1, 6r) \cap \partial\Omega$, we get

$$(\hat{G}_1 - G)(x) \leqslant cM_1 r^{3-n}\hat{\omega}_1(B(z_1, 6r)) \tag{3.20}$$

for $x \in \partial\Omega$. From the maximum principle for harmonic functions and the fact that $\Omega \subseteq \hat{\Omega}_1$, we conclude this inequality holds in Ω. Let $\phi(x') = \theta(6rx')/6r$, and define H relative to ϕ as in Lemma 1. Put

$$u(x) = \frac{1}{6r}(\hat{G}_1(6rx + z_1) - G(6rx + z_1)), \qquad x \in \bar{H},$$
$$\phi_1(x') = \frac{1}{6r}(\xi(6rx') - y_n),$$
$$H_1 = \{x\colon |x'| < 8, \phi_1(x') < x_n < 2\},$$
$$u_1(x) = \frac{1}{6r}\hat{G}_1(6rx + z_1), \qquad x \in \bar{H}_1.$$

We note from (2.1) that

$$\max\{\|\phi\|_4, \|\phi_1\|_4\} \leqslant cM_1 r. \tag{3.21}$$

Using (3.20), (3.21), we first apply Lemma 1 with u, H, replaced by u_1, H_1. As in (3.16) we get

$$\sum_{0 \le |\alpha| \le 4} |\partial_\alpha u_1(x)| \leqslant cr^{1-n}\hat{\omega}_1(B(z_1, 6r)), \qquad x \in H. \tag{3.22}$$

We note that $u_1 = 0$ on $\partial H_1 \cap \{(x', \phi_1(x'))\}$ and $u = u_1 = \gamma$ on $\partial H \cap \{(x', \phi(x'))\}$. Using these notes and (3.21)-(3.22) we deduce

$$\sum_{|\alpha|=0}^{3} |\partial'_\alpha \gamma(x', \phi(x'))| = \sum_{|\alpha|=0}^{3} |\partial'_\alpha(u_1(x', \phi(x')) - u_1(x', \phi_1(x')))| \tag{3.23}$$
$$\leqslant cM_1 r^{2-n}\hat{\omega}_1(B(z_1, 6r)).$$

Applying Lemma 1 to u and H, with $k = 3$ we find from (3.20)-(3.23)

$$\sum_{|\alpha|=0}^{3} |\partial_\alpha u(x)| \leqslant cM_1 r^{2-n}\hat{\omega}_1(B(z_1, 6r)),$$

for $x \in B(0, 1/2) \cap H$. Hence if $x \in B(z_1, 3r) \cap \bar{\Omega}$, then

$$|\nabla \hat{G}_1 - \nabla G|(x) \leqslant cM_1 r^{2-n} \hat{\omega}_1(B(z_1, 6r)) \leqslant c|\nabla \hat{G}_1(x)| M_1 r, \tag{3.24}$$

where the last inequality is just (3.17). From (3.24) and (2.1) we obtain

$$\begin{aligned}
&\left| \int_{\partial\Omega \cap B(z_1, 3r)} |\nabla G| \log |\nabla G| \, dH^{n-1} - \int_{\partial\Omega \cap B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| \, dH^{n-1} \right| \\
&\quad \leqslant \int_{\partial\Omega \cap B(z_1, 3r)} ||\nabla G| - |\nabla \hat{G}_1|| \, |\log |\nabla G|| \, dH^{n-1} + \int_{\partial\Omega \cap B(z_1, 3r)} |\nabla \hat{G}_1| \\
&\quad\quad \times \left| \log \left(\frac{|\nabla G|}{|\nabla \hat{G}_1|} \right) \right| dH^{n-1} \\
&\quad \leqslant -cM_1 r \log r \, \hat{\omega}_1(B(z_1, 6r)) + \hat{\omega}_1(B(z_1, 6r)) \log (1 + cM_1 r) \\
&\quad \leqslant c r^{1/2} \hat{\omega}_1(B(z_1, 6r)).
\end{aligned} \tag{3.25}$$

Let $P = P_{3r}(z_1)$ and let $G(\bullet, Y)$ denote Green's function with pole at $Y \in \Omega$. Following Wolff (see [14, (2.7)]) we first note fom (3.20) and the rate inequality (3.7) with $u = \hat{G}_1 - G$, $v = G(\bullet, P)$, $t = 2r$, that

$$G(x, P)^{-1}(\hat{G}_1 - G)(x) \leqslant cM_1 r \, \hat{\omega}_1(B(z_1, 6r)), \qquad x \in \Omega - B(z_1, 3r).$$

Second, given w in $\partial\Omega - B(z_1, 3r)$, we apply the rate inequality with $u = G(\bullet, P)$, $v = G(\bullet, P_t(w))$, $t = 2|w - z_1|$ in $\Omega - B(z_1, t)$, provided $0 \in \Omega - B(z_1, 2t)$. We get for $x = 0$,

$$t^{n-2} G(P_t(w), P) \leqslant cG(0, P)/G(0, P_t(w)).$$

If $0 \in B(z_1, 2t)$, then it follows easily from Harnack's inequality and $t \geqslant \rho/2$ (since $B(0, \rho) \subseteq \Omega$) that

$$G(P_t(w), P) \leqslant ct^{2-n} G(0, P).$$

From the above inequalities, (3.8) and Harnack's inequality, we find for $P_t = P_t(w)$,

$$G(P_t, P) \leqslant ct^{2-n}(r/t)^{\mu}.$$

Third, we use the rate inequality in $B(w, 10^{-3}t) \cap \Omega$ with $u = \hat{G}(\bullet, P)$, $v = \hat{G}_1(\bullet, 0)$; the above inequalities, (3.5) and (3.6), to obtain

$$\begin{aligned}
r^{-1}(\hat{\omega}_1(B(z_1, 6r)))^{-1} M_1^{-1} (\hat{G}_1 - G)(x) \hat{G}_1(x, 0)^{-1} &\leqslant cG(x, P) \hat{G}_1(x, 0)^{-1} \\
&\leqslant c(r/t)^{\mu} (\hat{\omega}_1(B(z_1, t)))^{-1},
\end{aligned}$$

for $x \in B(w, 10^{-3}t) \cap \Omega$. Letting $x \to w$ and using (2.1) we conclude from this inequality that

$$(3.26)\quad (|\nabla \hat{G}_1|^{-1} |\nabla \hat{G}_1 - \nabla G|)(w) \leqslant c r^{3/4+\mu} \hat{\omega}_1(B(z_1, 6r))(\hat{\omega}_1(B(z_1, t)))^{-1} |z_1 - w|^{-\mu}.$$

Now

$$(3.27)\quad \left| \int_{\partial\Omega - B(z_1, 3r)} |\nabla G| \log |\nabla G| \, dH^{n-1} - \int_{\partial\hat{\Omega}_1 - B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| \, dH^{n-1} \right|$$

$$\leqslant \int_{\partial\Omega - B(z_1, 3r)} \big||\nabla G| - |\nabla \hat{G}_1|\big| \, |\log |\nabla G|| \, dH^{n-1}$$

$$+ \int_{\partial\Omega - B(z_1, 3r)} |\nabla \hat{G}_1| \, |\log(|\nabla G| / |\nabla \hat{G}_1|)| \, dH^{n-1}$$

$$= I_1 + I_2.$$

If $F_k = B(z_1, 3^{k+1}r) - B(z_1, 3^k r)$, $k = 1, 2, \ldots$ then from (3.26) we have

$$I_1 \leqslant \sum_{k=1}^{\infty} \int_{F_k \cap \partial\Omega} \big||\nabla G| - |\nabla \hat{G}_1|\big| \, |\log |\nabla G|| \, dH^{n-1}$$

$$\leqslant -c r^{3/4+\mu} \log r \, \hat{\omega}_1(B(z_1, 6r)) \left(\sum_{k=1}^{\infty} k 3^{-k\mu} \right) r^{-\mu}$$

$$\leqslant c r^{1/2} \hat{\omega}_1(B(z_1, 6r)).$$

A similar estimate holds for I_2. Using these estimates in (3.27) we get

$$(3.28)\quad \left| \int_{\partial\Omega - B(z_1, 3r)} |\nabla G| \log |\nabla G| \, dH^{n-1} - \int_{\partial\hat{\Omega}_1 - B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| \, dH^{n-1} \right|$$

$$\leqslant c r^{1/2} \hat{\omega}_1(B(z_1, 6r)).$$

Next, since

$$\hat{\omega}_1(B(z_1, 6r)) \leqslant \hat{\omega}_0(B(z_1, 6r)),$$

we can replace $\hat{\omega}_1$ by $\hat{\omega}_0$ in (3.28), (3.25), and (3.19). Doing this and combining (3.28), (3.25), (3.19), we conclude that (3.12) is true for $k = 0$.

To prove (3.11) for $k = 0$, let j be the smallest positive integer such that $E_j \cap B(z_1, 10r) \neq \varnothing$. Put $r' = 10r/\lambda_j$ and let $z \in B(z_1, 6r) \cap \partial\Omega_1'$. Then it is easily checked that (3.16)-(3.18) hold with $\hat{G}_1, \hat{\omega}_1, r, z_1$, replaced by G_1', ω_1', r', z, respectively, when $x, y \in B(z, 3r')$. Now from (3.4) we have

$$(3.29)\quad \frac{r}{10} \geqslant \frac{r}{\lambda_j} \geqslant \frac{r}{\lambda_{l+1}} = \frac{r}{\tau(\epsilon)} \geqslant r^{\gamma}$$

where

$$\gamma = 1 + \frac{1}{16(n-1)} \leqslant \frac{33}{32}.$$

Let z^* be the point in $\partial\Omega'$ obtained by projecting z in the rotated x_n direction onto $\partial\Omega'$. Then from the new version of (3.16)-(3.18), and the fact that

$$|z - z^*| < 200 M_1 r^2 < r',$$

thanks to (2.1), (3.29) we find

$$\begin{aligned}|(|\nabla G_1'| \log |\nabla G_1'|)(z) - (|\nabla G_1'| \log |\nabla G_1'|)(z^*)| \\ \leqslant ||\nabla G_1'|(z) - |\nabla G_1'|(z^*)| \log r' + |\nabla G_1'(z)| \log (|\nabla G_1'|(z)/|\nabla G_1'|(z^*))| \\ \leqslant -cM_1 r^2 \log (r')(|\nabla G_1'(z)|/r').\end{aligned}$$

Using this inequality, (3.29), and parametrizing $\partial\Omega'$, $\partial\Omega_1'$, we get as in (3.19)

(3.30)

$$\left| \int_{\partial\Omega' \cap B(z_1, 3r)} |\nabla G_1'| \log |\nabla G_1'| \, dH^{n-1} - \int_{\partial\Omega_1' \cap B(z_1, 3r)} |\nabla G_1'| \log |\nabla G_1'| \, dH^{n-1} \right| \leqslant c r^{1/2} \omega_1'(B(z_1, 6r)).$$

Next suppose $z \in \partial\Omega'$ and observe as in (3.20) that

(3.31) $(G_1' - G')(z) \leqslant cM_1 r^2 (r')^{1-n} \omega_1'(B(z_1, 6r')) \leqslant cM_1 r^2 (r')^{1-n} \omega_1'(B(z_1, 6r)).$

It follows from the maximum principle for harmonic functions that (3.31) holds in Ω'. If $z = (\bar{z} + y', \xi(\bar{z})) \in \partial\Omega'$, put

$$\phi'(x') = \frac{1}{6r'}(\xi(6r'x' + \bar{z}) - \xi(\bar{z})),$$

$$H' = \{x: |x'| < 1, \phi'(x') < x_n < 1\},$$

$$u'(x) = \frac{1}{6r'}(G_1'(6r'x + z) - G(6r'x + z)), \qquad x \in \bar{H}',$$

$$\phi_1'(x') = \frac{1}{6r'}(\tilde{\xi}(6r'x' + \bar{z}) - \xi(\bar{z})),$$

$$H_1' = \{x: |x'| < 8, \phi_1'(x') < x_n < 2\},$$

$$u_1' = \frac{1}{6r'} G_1'(6r'x + z), \qquad x \in \bar{H}_1'.$$

We note that

$$\|\phi'\|_4 + \|\phi_1'\|_4 \leqslant c,$$
$$\|\phi' - \phi_1'\|_4 \leqslant cM_1 r.$$

Using these inequalities in place of (3.21) and Lemma 1 we get

$$\sum_{0 \le |\alpha| \le 4} |\partial_\alpha u_1'(x)| \leqslant c(r')^{1-n}\omega_1'(B(z, 6r')) \leqslant c(r')^{1-n}\omega_1'(B(z_1, 6r))$$

in H'. Also, as in (3.23), we see for $u' = \gamma'$ on $\partial H' \cap \{(x', \phi'(x'))\}$, that

$$\sum_{|\alpha|=0}^{3} |\partial_\alpha' \gamma'|(x') \leqslant cM_1 r(r')^{1-n}\omega_1'(B(z_1, 6r)).$$

From this inequality, (3.31) and Lemma 1 it follows as in (3.24) that

$$\begin{aligned} (3.32)\quad |\nabla G_1' - \nabla G'|(x) &\leqslant cM_1 r^2 (r')^{-n}\omega_1'(B(z_1, 6r)) \\ &\leqslant cM_1(r^2/r')\omega_1'(B(z_1, 6r))(\omega_1'(B(z_1, 6r')))^{-1}|\nabla G_1'(x)|, \end{aligned}$$

$x \in B(z, 3r') \cap \bar{\Omega}'$. We cover $\partial\Omega' \cap B(z_1, 3r)$ by at most $c(r/r')^{n-1}$ balls, $B(z, 3r')$, $z \in \partial\Omega' \cap B(z_1, 3r)$. Using (3.32) in each ball and arguing as in (3.25) we have

(3.33)

$$\begin{aligned} \left| \int_{\partial\Omega' \cap B(z_1', 3r)} |\nabla G'| \log |\nabla G'| \, dH^{n-1} - \int_{\partial\Omega' \cap B(z_1, 3r)} |\nabla G_1'| \log |\nabla G_1'| \, dH^{n-1} \right| \\ \leqslant -cM_1 r(r/r')^n \log r \, \omega_1'(B(z_1, 6r)) \\ \leqslant cr^{1/2}\omega_1'(B(z_1, 6r)), \end{aligned}$$

thanks to (3.29) and (2.1).

At this point we can use (3.31) in place of (3.20) and repeat the argument following (3.25) in the proof of (3.12) (for $k = 0$), since only NTA estimates were used. From (3.28) with G, $\hat{G}_1$, $\hat{\omega}_1$, replaced by G', G_1', ω_0' and (3.30), (3.33), with ω_1' replaced by ω_0', we conclude that (3.11) holds when $k = 0$. From our earlier remarks we now deduce that (1.3) is true.

4. Proof of Theorem 1

Recall that ψ, $0 \leqslant \psi \leqslant 1$, is a fixed C^∞ function with support in $\hat{B}(1)$, $\max_{\mathbb{R}^{n-1}} \psi = 1$, and $\Lambda(\psi) > 0$. Also σ_0, $0 < \sigma_0 \leqslant 10^{-3}$, was chosen to be the largest number for which (2.10), (2.12), and (2.19) are true. Finally, given ϵ, $0 < \epsilon \leqslant \sigma_0$, we note that $r_0 = r_0(\epsilon, M_1, M_2)$, was chosen so small that the inequalities in Sections 2 and 3 are true for $0 < r \leqslant r_0$.

We elaborate on the induction argument for the construction of D which was outlined in Section 1. Let $D_0 = B(0, \rho)$, where ρ satisfies (1.4). Put $\epsilon_0 = \sigma_0$ and $\epsilon_k = 2^{-k}\epsilon_0$, $k = 0, 1, 2, \ldots$ Choose a covering, $L_1 = \{B(z_{0i}, t_{0i})\}$, $1 \leqslant i \leqslant k_0$ of ∂D_0 such that $t_{0i} \leqslant 1/2$, $i = 1, 2, \ldots, k_0$, and

$$\alpha(n-1)\sum_{i=1}^{k_0} t_{0i}^{n-1} \leqslant H^{n-1}(\partial D_0) - \frac{1}{2}.$$

By compactness of D_0 we may assume $k_0 < \infty$. Let $2r_1' > 0$ denote the distance from ∂D_0 to $\mathbb{R}^n - \cup_1^{k_0} B(z_{0i}, t_{0i})$. We set $\Omega = D_0$, $\epsilon = \epsilon_1$, and apply the results in Section 2 with $r = r_1$, where r_1 is the smaller of $10^{-9}\rho, r_1'$, and $r_0 = r_0(\epsilon_1, M_1, M_2)$. Here M_1, M_2, are defined relative to D_0, G_0. Let $D_1 = \Omega'$ be the domain obtained by adding smooth bumps to D_0 and $h_1 = h$ the homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, which satisfies (2.16)-(2.18) with $r = r_1$. Moreover, $h_1(\partial D_0) = \partial D_1$. By induction, suppose for some $m \geqslant 1$ we have defined sequences: $(D_k)_0^m, (L_k)_1^m, (r_k')_1^m, (r_k)_1^m, (h_k)_1^m$. Let $L_{m+1} = \{B(z_{mi}, t_{mi})\}_1^{k_m}$, be a covering of ∂D_m such that $t_{mi} \leqslant 2^{-(m+1)}$, $1 \leqslant i \leqslant k_m$, and

$$\alpha(n-1)\sum_{1}^{k_m} t_{mi}^{n-1} \leqslant H^{n-1}(\partial D_m) - 2^{-(m+1)} \tag{4.1}$$

Let $2r_{m+1}' > 0$ be the distance from ∂D_m to $\mathbb{R}^n - \cup_1^{k_m} B(z_{mi}, t_{mi})$. Let $\Omega = D_m$, $\epsilon = \epsilon_m$, and $r = r_{m+1}$, where r_{m+1} is the smaller of $10^{-4m} r_m \rho$, r_{m+1}', and $r_0(\epsilon_{m+1}, M_1, M_2)$. Here M_1, M_2, are defined relative to D_m, G_m. Adding smooth bumps to Ω as in Section 2 we obtain $D_{m+1} = \Omega' \supseteq D_m$ and h_{m+1} a homeomorhism from $\mathbb{R}^n$ to $\mathbb{R}^n$ which satisfies (2.16)-(2.18) with $r = r_{m+1}$. Moreover, $h_{m+1}(\partial D_m) = \partial D_{m+1}$. By induction we get, $(D_k)_0^\infty$, $(H_k)_0^\infty$, $(r_k')_1^\infty$, $(r_k)_1^\infty$, and $(h_k)_1^\infty$. From our work in Section 2 we see that (1.1), (1.2), are true with Ω, Ω', G, G', replaced by $D_k, D_{k+1}, G_k, G_{k+1}$, respectively, $k = 0, 1, \ldots$

We claim that D_k, $k = 1, 2, \ldots$ is NTA with constant 100. Indeed, since $0 \leqslant \psi \leqslant 1$ and $r_k \leqslant 10^{-4k}\rho$, $k = 1, 2, \ldots$, it follows from the definition of D_k, by way of the triangle inequality, that

$$B(0, \rho) \subseteq D_k \subseteq B(0, 2\rho), \qquad k = 1, 2, \ldots \tag{4.2}$$

To prove D_k satisfies the corkscrew condition (i) in the definition of an NTA domain, we proceed by induction. If $0 < s < \rho$, and $z \in \partial D_0$, note that $B(z, s) \cap D_0$, $B(z, s) \cap (\mathbb{R}^n - D_0)$, each contain a ball of radius $s/4$. From this note and the fact that ∂D_1 lies within r_1 distance of ∂D_0, we deduce for $4r_1^{1/2} \leqslant s < \rho$, and $z \in \partial D_1$ that $B(z, s) \cap D_0$, $B(z, s) \cap (\mathbb{R}^n - D_0)$, each contain a ball of radius,

$$(1 - r_1)\frac{s}{4} - r_1 \geqslant \frac{1}{4}s(1 - 2r_1^{1/2}) = s_1.$$

If $0 < s \leqslant 4r_1^{1/2}$, then from our choice of $r_1 = r$, we have $z \in B(y, 100r_1)$, for some $y \in \{y^i\}_1^N$. Moreover, $B(y, 1000r_1^{1/2}) \cap D_1$, $B(y, 1000r_1^{1/2}) \cap \partial D_1$, can be expressed as in Section 2 relative to ξ. From (2.12) and (2.1) we observe that $|\nabla \xi| \leqslant 10^{-3}$. Using these facts and a little geometry it is easily seen that the above inequality remains valid when $0 < s \leqslant 4r_1^{1/2}$. By induction, suppose we have shown for some $m \geqslant 1$, that if $z \in \partial D_m$ and $0 < s < \rho$, then $B(z, s) \cap D_m$, $B(z, s) \cap (\mathbb{R}^n - D_m)$, each contain a ball of radius

$$\frac{1}{4} s\left(1 - 2 \sum_{k=1}^{m} r_k^{1/2}\right) = s_m. \tag{4.3}$$

If $4r_{m+1}^{1/2} \leqslant s < \rho$, and $z \in \partial D_{m+1}$, then since ∂D_{m+1} lies within r_{m+1} of ∂D_m, we deduce from (4.3) that $B(z, s) \cap D_{m+1}$, $B(z, s) \cap (\mathbb{R}^n - D_{m+1})$, each contain a ball of radius

$$\frac{1}{4}(s - r_{m+1})\left(1 - \sum_{k=1}^{m} r_k^{1/2}\right) - r_{m+1} \geqslant \frac{1}{4} s\left(1 - 2 \sum_{k=1}^{m+1} r_k^{1/2}\right) = s_{m+1}.$$

If $0 < s < 4r_{m+1}^{1/2}$, it follows from local smoothness of D_{m+1} that $B(z, s) \cap D_{m+1}$, $B(z, s) \cap (\mathbb{R}^n - D_{m+1})$, each contain a ball of radius s_{m+1}. Thus by induction we have shown for $z \in \partial D_k$, $k = 0, 1, \ldots,$ that $B(z, s) \cap D_k$, $B(z, s) \cap (\mathbb{R}^n - D_k)$, both contain a ball of radius

$$s_k \geqslant \frac{1}{4} s\left(1 - 2 \sum_{m=1}^{\infty} r_m^{1/2}\right) \geqslant \frac{1}{8} s,$$

when $0 < s < \rho$. Scalling D_k to have diameter 1, we see that (i) in Section 3 holds with $A = 16$.

To prove (ii), we proceed similarly. Suppose by induction, we have shown for some nonnegative integer m that whenever $x, z \in D_m$, we can join x to z by a curve γ with parameter interval, $[0, 1]$, in such a way that $\gamma(0) = x$, $\gamma(1) = z$, and

$$(a)\quad \operatorname{dist}(\gamma(t), \partial D_m) \geqslant \frac{1}{16}\left(1 - 2 \sum_{k=1}^{m} r_k^{1/4}\right) \min\{|\gamma(t) - x|, |\gamma(t) - z|\}, \tag{4.4}$$

$$(b)\quad \operatorname{length} \gamma \leqslant 3\left(1 + \sum_{k=1}^{m} r_k^{1/4}\right)|x - z|. \tag{4.5}$$

In case $m = 0$, replace the sums in (4.4), (4.5) by 0. From inspection we see that (4.4), (4.5) hold when $m = 0$, since $D_0 = B(0, \rho)$. Next suppose $x, z \in D_{m+1}$ and $4r_{m+1}^{1/2} \leqslant |x - z|$. Since $D_m \subseteq D_{m+1}$, we note that (4.4) and (4.5) hold trivially unless either $x \notin D_m$ or $z \notin D_m$. If $x \notin D_m$, then $x \in B(y, r_{m+1}) \cap D_{m+1}$

for some $y \in \{y_j\}_1^N$, $y \in \partial D_m$, and $x = (x', x_n)$ in the corresponding rotated coordinate system. Put $x^* = (x', x_n + r_{m+1})$ and observe that $x^* \in D_m$. If $x \in D_m$, we also let $x^* = x$. Applying the same argument to z we get $x^*, z^* \in D_m$. Let γ^* be the curve joining x^* to z^* which satisfies (4.4), (4.5). If $x \neq x^*$, we modify γ^* as follows. Let t_0, $0 < t_0 < 1$, be the largest t with $\gamma^*(t) \in \bar{B}(y, r_{m+1}^{3/4})$. If $\gamma^*(t_0) = w = (w', w_n)$, we join x, w, to $\bar{x} = (x', y_n + r_{m+1}^{3/4})$, $\bar{w} = (w', y_n + r_{m+1}^{3/4})$, respectively by line segments, l_1, l_2. We then join $\bar{x}$ to $\bar{w}$ by a line segment l_3. Let $l_1 + l_2 + l_3$ denote the resulting curve from x to w with parameter interval $[0, t_0]$. If $z \notin D_m$, we see there exists $\hat{y} \in \{y^i\}_1^N$ and largest t_1, $0 < t_0 < t_1 < 1$, such that $z \in B(\hat{y}, r_{m+1})$, and

$$\{\gamma^*(t): 0 \leqslant t < t_1\} \cap \bar{B}(\hat{y}, r_{m+1}^{3/4}) = \varnothing.$$

As above, we get line segments $\tilde{l}_1, \tilde{l}_2, \tilde{l}_3$, with $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3$ joining $\gamma^*(t_1)$ to z. Moreover, $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3$ has parameter interval $[t_1, 1]$. Let $\hat{\gamma} = \gamma^*$ on $[t_0, t_1]$ and if $x \notin D_m$, then $\hat{\gamma} = l_1 + l_2 + l_3$ on $[0, t_0]$. Otherwise, $\hat{\gamma} = \gamma^*$ on $[0, t_0]$. If $z \notin D_m$, then $\hat{\gamma} = \tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3$ on $[t_1, 1]$, while if $z \in D_m$, then $\hat{\gamma} = \gamma^*$ on $[t_1, 1]$. From (4.5) we deduce

$$\begin{aligned}
\text{(4.6)} \qquad \operatorname{length} \hat{\gamma} &\leqslant \operatorname{length} \gamma^* + 10 r_{m+1}^{3/4} \\
&\leqslant 3\left(1 + \sum_{k=1}^{m} r_k^{1/4}\right)|x^* - z^*| + 10 r_{m+1}^{3/4} \\
&\leqslant 3\left(1 + \sum_{k=1}^{m} r_k^{1/4}\right)|x - z| + 12 r_{m+1}^{3/4} \\
&\leqslant 3\left(1 + \sum_{k=1}^{m+1} r_k^{1/4}\right)|x - z|.
\end{aligned}$$

Moreover, from local smoothness of ∂D_{m+1} it is easily checked for $t \in [0, t_0] \cup [t_1, 1]$, that

$$\operatorname{dist}(\hat{\gamma}(t), \partial D_{m+1}) \geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m+1} r_k^{1/4}\right) \min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\}.$$

If $t \in [t_0, t_1]$, then by construction

$$\begin{aligned}
\min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\} &\geqslant r_{m+1}^{3/4} - r_{m+1} \\
&\geqslant \frac{1}{2} r_{m+1}^{3/4}.
\end{aligned}$$

Using this inequality, (4.4), and the fact that $\gamma^* = \hat{\gamma}$ on $[t_0, t_1]$ we get for $t \in [t_0, t_1]$,

$$(4.7)\ \operatorname{dist}(\hat{\gamma}(t), \partial D_{m+1}) \geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m} r_k^{1/4}\right)\min\{|\hat{\gamma}(t) - x^*|, |\hat{\gamma}(t) - z^*|\}$$

$$\geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m} r_k^{1/4}\right)\min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\} - \frac{r_{m+1}}{16}$$

$$\geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m+1} r_k^{1/4}\right)\min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\}.$$

If $|x - z| < 4r_{m+1}^{1/2}$, then from local smoothness of ∂D_{m+1}, we see there exists $\hat{\gamma}$ for which (4.6) and (4.7) hold. Thus by induction, we obtain (4.4), (4.5), for $m = 0, 1, 2, \ldots$ Since $\sum_1^\infty r_k^{1/4} \leqslant 1/10$, we conclude that D_m, $m = 0, 1, \ldots,$ is NTA with constant 100. From this fact, (4.2), and our work in Section 3 we now find that (1.3) holds with $\Omega = D_k$, $\Omega' = D_{k+1}$, $k = 0, 1, \ldots$

Next let $h_0(x) = \rho x$, and $f_k = h_k \circ h_{k-1} \circ \cdots \circ h_0$. Then f_k is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f_k(S) = \partial D_k$. From (2.16), (2.19), and iteration, we find

$$(4.8)\qquad \begin{aligned} 2^{-k}\rho|x - z| &\leqslant \rho(1 - c_4\sigma_0^2)^k|x - z| \\ &\leqslant |f_k(x) - f_k(z)| \\ &\leqslant \rho(1 + c_4\sigma_0^2)^k|x - z| \\ &\leqslant \rho 2^k|x - z|, \end{aligned}$$

for $x, z \in \mathbb{R}^n$. If $r_j < |x - z|$ for some $j \geqslant 1$, then from (4.8) and the fact that $r_{k+1} \leqslant 10^{-4k} r_k \rho$, we deduce for $l \geqslant j$,

$$r_{l+1} < 2^{-l}\rho|x - z| \leqslant |f_l(x) - f_l(z)|.$$

From this inequality, (2.17), (2.19) and iteration we find for $k > j$,

$$|f_j(x) - f_j(z)| - \frac{1}{2}\sum_{m=j+1}^{k} r_m \leqslant |f_k(x) - f_k(z)| \leqslant |f_j(x) - f_j(z)| + \frac{1}{2}\sum_{m=j+1}^{k} r_m.$$

Using the above inequality, (4.8) with $j = k$, and the fact that

$$\sum_{m=j+1}^{\infty} r_m \leqslant \rho 10^{-j} r_j \leqslant \rho 10^{-j}|x - z|,$$

we get

$$(4.9)\qquad 2^{-(j+1)}\rho|x - z| \leqslant |f_k(x) - f_k(z)| \leqslant \rho 2^{j+1}|x - z|.$$

Given $\beta \in (0, 1)$, we have

$$2^{j+1} \leqslant c(\beta)|x - z|^{\beta - 1},$$

when $r_j \leqslant |x - z| \leqslant r_{j-1}$, $j = 2, 3, \ldots$ for some $c(\beta)$, independent of j. Here we have used, $r_m \leqslant c10^{-m^2}$, $m = 1, 2, \ldots$, which follows easily from our choice of $(r_m)_1^\infty$. Using the above inequality in (4.9), we obtain

$$c(\beta)^{-1}|x - z|^{1/\beta} \leqslant |f_k(x) - f_k(z)| \leqslant c(\beta)|x - z|^\beta,$$

for $|x - z| \leqslant 1/4$. Hence (1.5) is true. As in Section 1 we put $D = \bigcup_0^\infty D_k$ and choose a subsequence (f_{n_k}) of (f_k) such that (f_{n_k}) converges uniformly to f on compact subsets of $\mathbb{R}^n$. We claim that D is not a sphere. Indeed, since $\max_{\mathbb{R}^{n-1}} \psi = 1$, and (2.1), (3.4) hold for r_1, ϵ_0, D_1, we see that if $\rho_1 = \rho + (2\lambda_0)^{-1}\sigma_0^2 r_1$, then $D_1 \cap (\mathbb{R}^n - B(0, \rho_1)) \neq \varnothing$. Also, by construction, there exists $x_0 \in \partial D_1$ with $|x_0| = \rho$. Using the definition of $(r_m)_1^\infty$ and the triangle inequality we see that $f(x_0) \in \partial D$ and $|f(x_0)| < \rho_1$. Therefore, D is not a sphere.

It remains only to prove (1.9) in order to obtain Theorem 1 from the remarks in Section 1. To this end let

$$p_j(x) = f \circ f_j^{-1}(x) = \lim_{k \to \infty} h_{n_k} \circ \cdots \circ h_{j+1}(x),$$

when $x \in \partial D_j$ and $j = 1, 2, \ldots$ Iterating (2.18) we deduce that if

$$e_j = \prod_{m=j+1}^{\infty} (1 - c_5 r_m^{1/2}),$$

then

$$e_j|x - y| \leqslant |p_j(x) - p_j(y)|, \qquad x, y \in \partial D_j.$$

If q_j denotes the inverse of p_j, it follows that

$$|q_j(x) - q_j(y)| \leqslant e_j^{-1}|x - y|, \tag{4.10}$$

when $x, y \in \partial D$. Next we use Kirsbraun's Theorem ([5, 2.10.43]) to extend q_j to $\mathbb{R}^n$ (also denoted q_j) in such a way that (4.10) holds whenever $x, y \in \mathbb{R}^n$.

From (4.10) it is easily seen by comparing coverings of each set that

$$H^{n-1}(q_j(F)) \leqslant e_j^{1-n} H^{n-1}(F), \qquad F \subseteq \mathbb{R}^n. \tag{4.11}$$

$j = 1, 2, \ldots$ Let $g \geqslant 0$ be a continuous function on $\mathbb{R}^n$, and put $\nu(E) = H^{n-1}(q_j^{-1}(E) \cap \partial D)$. Then from (4.11) with $F = q_j^{-1}(E) \cap \partial D$, we have

$$H^{n-1}(E \cap \partial D_j) \leqslant e_j^{1-n}\nu(E).$$

Also from the usual change of variables formula [5, Thm. 2.4.18] and the above inequality we get

(4.12) $$e_j^{n-1}\int_{\partial D_j} g\,dH^{n-1} \leqslant \int_{\mathbb{R}^n} g\,d\nu = \int_{\partial D} g\circ q_j\,dH^{n-1}.$$

Letting $j\to\infty$, $j\in(n_k)_1^\infty$, we obtain from the definition of $(r_k)_1^\infty$ that $e_j\to 1$, while

$$\int_{\partial D} g\circ q_j\,dH^{n-1}\to\int_{\partial D} g\,dH^{n-1},$$

since $q_{n_k}(x)\to x$, uniformly on compact subsets of $\mathbb{R}^n$. Hence from (4.12) we have

(4.13) $$\limsup_{k\to\infty}\int_{\partial D_{n_k}} g\,dH^{n-1}\leqslant\int_{\partial D} g\,dH^{n-1}.$$

On the other hand from our choice of $(r_k)_1^\infty$ we see that L_m, $m=1,2,\ldots,$ is a covering for D. Thus if ϕ_δ^{n-1} is as in Section 1, then

$$\phi_{2^{-m}}^{n-1}(\partial D)\leqslant H^{n-1}(\partial D_m)-2^{-m}.$$

Letting $m\to\infty$, we find

(4.14) $$H^{n-1}(\partial D)\leqslant\liminf_{m\to\infty}H^{n-1}(\partial D_m).$$

From (4.13), (4.14), it follows that if $0\leqslant g\leqslant 1$ on $\bar{D}$, then

$$\begin{aligned}
H^{n-1}(\partial D)&\leqslant\liminf_{k\to\infty}H^{n-1}(\partial D_{n_k})\\
&\leqslant\liminf_{k\to\infty}\int_{\partial D_{n_k}} g\,dH^{n-1}+\limsup_{k\to\infty}\int_{\partial D_{n_k}}(1-g)\,dH^{n-1}\\
&\leqslant\limsup_{k\to\infty}\int_{\partial D_{n_k}} g\,dH^{n-1}+\int_{\partial D}(1-g)\,dH^{n-1}\\
&\leqslant\int_{\partial D} g\,dH^{n-1}+\int_{\partial D}(1-g)\,dH^{n-1}\\
&=H^{n-1}(\partial D).
\end{aligned}$$

Thus equality holds everywhere and so

(4.15) $$\lim_{k\to\infty}\int_{\partial D_{n_k}} g\,dH^{n-1}=\int_{\partial D} g\,dH^{n-1}$$

when $0\leqslant g\leqslant 1$. In general we can write, $g=ag_1+b$, where $0\leqslant g_1\leqslant 1$ on D, for properly chosen $a,b\in\mathbb{R}$. Applying (4.15) to g_1, 1 we find that (4.15) holds when g is continuous on $\mathbb{R}^n$. Hence, (1.9) is true.

The proof of Theorem 1 is now complete.

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John L. Lewis
Department of Mathematics
University of Kentucky
Lexington, KY 40506
U.S.A.

Andrew Vogel*
Department of Mathematics
University of Synacuse
New York, NY 13244
U.S.A.

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