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# Hardy Spaces and Oscillatory Singular Integrals

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## 1. Introduction

Consider the oscillatory singular integral operator $T$:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i(Bx,y)} K(x-y) f(y)\, dy, \tag{1}$$

where $(Bx, y)$ is a real bilinear form, and $K$ is a Calderón-Zygmund kernel, *i.e.* $K$ is $C^1$ away from the origin, has mean-value zero on each sphere centered at the origin and satisfies

$$|K(x)| \leqslant C|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \leqslant C|x|^{-n-1}.$$

It is proved by D. H. Phong and E. M. Stein in [PS], that $T$ is a bounded operator on $L^p$ spaces, with bound independent of $B$. They also introduced some variants of the $H^1$ and BMO spaces (denoted by $H^1_E$ and $\text{BMO}_E$, to avoid the confusion with the standard $H^1$ and BMO). Analogous to the fact that the classical singular integral operators are bounded from $H^1$ to $L^1$, Phong and Stein showed that $T$ extends as a bounded operator from $H^1_E$ to $L^1$. This fact was then used to prove the $L^p$ boundedness by interpolating between $L^2$ and $L^\infty$, (see [PS]).

The object of our study is a more general class of oscillatory singular integral operators. An operator in this class is obtained when the bilinear form in (1) is replaced by some real-valued polynomial in $x$ and $y$. These operators have arisen in the study of Hilbert transform along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms, etc. F. Ricci and E. M. Stein have proved in [RS] that an operator of this kind is bounded on $L^p$ spaces, with bound depending only on the total degree, not on the coefficients of the polynomial. The fact that these operators are of weak-type (1,1) was subsequently proved by S. Chanillo and M. Christ ([CC]).

It is our goal in this paper to establish a Hardy space theory for the class of oscillatory singular integral operators with polynomial phase functions. Given such an operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy, \tag{2}$$

where $P(x, y)$ is a real-valued polynomial, we will define the space $H^1_E$ as some variant of the standard $H^1$ space, and this space $H^1_E$ is closely associated with the given polynomial $P(x, y)$. First let us give the definition of the "atoms":

**Definition.** *Let $Q$ be a cube with center $x_Q$, an atom is a function $a(x)$ which is supported in $Q$, so that*

$$|a(x)| \leqslant \frac{1}{|Q|},$$

*and*

$$\int_Q e^{iP(x_Q, y)} a(y)\, dy = 0.$$

The space $H^1_E$ consists of the subspace of $L^1$ of functions $f$ which can be written as $f = \sum \lambda_j a_j$, where $a_j$ are atoms, and $\lambda_j \in \mathbb{C}$, with $\sum |\lambda_j| < \infty$. Consequently, we define $\text{BMO}_E$ as the dual space of $H^1_E$. Our main result is

**Theorem 1.** *Suppose $H^1_E$ and $T$ are defined as above. Then $T$ is a bounded operator from $H^1_E$ to $L^1$. The bound of this operator can be taken to depend only on the total degree of $P$, (not on the coefficients of $P$).*

We notice that in the paper of Phong and Stein, the fact that the phase function is a real bilinear form makes it possible to apply the Plancherel's theorem to the Fourier transform (or partial Fourier transform) associated with $B$. When $(Bx, y)$ is replaced by the polynomial $P(x, y)$, we no longer have this advantage. So we have to take a different approach, using some $L^2$ estimates of certain oscillatory integrals. This will become clear in our proof.

For $p < 1$, the Calderón-Zygmund singular integral operators are still bounded from $H^p$ to $L^p$. However, this is no longer the case for the oscillatory singular integral operators. At the end of this article, we will present a simple example which shows that this fails even in the bilinear phase function case.

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## 2. Proof of Theorem 1

PROOF. Let us assume that $a$ is a function supported in the cube $Q_0$, which is centered at the origin, and has sidelength 1, and $a$ satisfies

$$|a| \leqslant 1, \qquad \int_{Q_0} a(y)\,dy = 0.$$

First we shall prove that if $P(x,y)$ is a polynomial in $x, y$, and $P(0,y) \equiv 0$, then

$$\left\| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) a(y)\,dy \right\|_{L^1} \leqslant C, \tag{3}$$

where $C$ depends only on the total degree of $P$, and is otherwise independent of the coefficients of $P$.

To prove (3), we shall use induction on the degree $l$ of $y$ in $P(x,y)$.

If $l = 0$, then $e^{iP(x,y)}$ is only a function of $x$, therefore can be taken out of the integral sign, and (3) follows from the classical result of the standard $H^1$ theory. (See, for example [CW].)

Next we assume $l > 0$, and (3) is true for $l-1$. By the Ricci-Stein theorem on the $L^p$ boundedness of $T$, we have

$$\begin{aligned}\int_{|x|\leq 2} |T(a)(x)|\,dx &\leqslant C\Big(\int_{|x|\leq 2} |T(a)(x)|^2\,dx\Big)^{1/2} \\ &\leqslant C\Big(\int_{\mathbb{R}^n} |a|^2\,dx\Big)^{1/2} \leqslant C.\end{aligned}$$

Write

$$P(x,y) = \sum_{|\alpha|\geq 1,\, |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta + Q(x,y),$$

where $Q(x,y)$ is a polynomial with degree in $y$ less than or equal to $l-1$, and still satisfies $Q(0,y) \equiv 0$. For any $r > 0$, we have

$$\begin{aligned}\int_{2<|x|\leq r} |T(a)(x)|\,dx \leqslant & \int_{2<|x|\leq r} |(e^{iP(x,y)} - e^{iQ(x,y)}) K(x-y) a(y)\,dy|\,dx \\ &+ \int_{2<|x|\leq r} \left| \int_{\mathbb{R}^n} e^{iQ(x,y)} K(x-y) a(y)\,dy \right| dx.\end{aligned}$$

(If $r \leqslant 2$, all the above integrals are 0.)

By our inductive hypothesis, the second term is bounded. Also $|x-y| \geqslant |x|/2$, if $|x| > 2$, $|y| \leqslant 1$. So we have

$$\int_{2<|x|\le r} |T(a)(x)|\,dx \leqslant C + C\int_{|x|\le r} dx \int_{\mathbb{R}^n} \left|\exp\left(\sum_{\substack{|\alpha|\ge 1\\|\beta|=l}} a_{\alpha\beta}x^\alpha y^\beta\right) - 1\right| \times \frac{|a(y)|}{|x|^n}\,dy$$

$$\leqslant C + C\sum_{\substack{|\alpha|\ge 1\\|\beta|=l}} |a_{\alpha\beta}| \int_{|x|\le r} |x|^{|\alpha|-n}\,dx$$

$$\leqslant C + C\sum_{\substack{|\alpha|\ge 1\\|\beta|=l}} |a_{\alpha\beta}| r^{|\alpha|}.$$

Now, there exists $(\alpha_0, \beta_0)$ such that $|\alpha_0| \geqslant 1$, $|\beta_0| = l$, and

$$|a_{\alpha_0\beta_0}|^{1/|\alpha_0|} = \max_{\substack{|\alpha|\ge 1\\|\beta|=l}} |a_{\alpha\beta}|^{1/|\alpha|}.$$

Put $r = |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$, we have

$$\int_{2<|x|\le r} |T(a)(x)|\,dx \leqslant C,$$

where $C$ depends only on the total degree of $P(x, y)$. Now we turn to the estimate of the remaining part

$$\int_{|x|>2,\,|x|>r} |T(a)(x)|\,dx.$$

We shall need the following lemmas:

**Lemma 1.** *Suppose*

$$\phi(x) = \sum_{|\nu|\le k} a_\nu x^\nu$$

*is a real-valued polynomial in* $\mathbb{R}^n$ *of degree* $k$, *and* $\psi \in C_0^\infty$. *Then for any* $\nu$, $|\nu| = k$, $a_\nu \neq 0$, *we have*

$$\left|\int_{\mathbb{R}^n} e^{i\phi(x)}\psi(x)\,dx\right| \leqslant C|a_\nu|^{-1/k}(\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}) \tag{4}$$

To see this, simply let $\xi$ be an unit vector, such that

$$|(\xi\cdot\nabla_x)^k\phi(x)| \geqslant c|a_\nu|.$$

This is possible because

$$\frac{\partial^\nu \phi(x)}{\partial x^\nu} = \nu! a_\nu .$$

(See [ST], page 317.) Without loss of generality, we may assume

$$\xi = (1, 0, \ldots, 0).$$

Hence

$$\left| \frac{\partial^k \phi(y)}{\partial y_1^k} \right| \geqslant c|a_\nu|.$$

Now apply the one-dimensional Van der Corput's lemma to obtain (4). See also [ST].

**Lemma 2.** *Let*

$$P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$$

*denote a polynomial in* $\mathbb{R}^n$ *of degree d. Suppose* $\epsilon < 1/d$, *then*

$$\int_{|x| \leq 1} |P(x)|^{-\epsilon} dx \leqslant A_\epsilon \Bigl( \sum_{|\alpha| \leq d} |a_\alpha| \Bigr)^{-\epsilon}.$$

*The bound* $A_\epsilon$ *depends on* $\epsilon$ *(and the dimension n), but not on the coefficients* $\{a_\alpha\}$.

This is a result of Ricci and Stein. See [RS], page 182.

Now we continue our proof of Theorem 1. Let

$$R_j = \{x \in \mathbb{R}^n : 2^j \leqslant |x| < 2^{j+1}\},$$

for $j \geqslant 0$, and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy

$$\varphi(x) \equiv 1 \quad \text{for} \quad |x| \leqslant 1, \qquad \varphi(x) \equiv 0 \quad \text{for} \quad |x| \geqslant 2.$$

Define $T_j$ by

$$(T_j f)(x) = \chi_{R_j}(x) \int_{\mathbb{R}^n} e^{iP(x,y)} \varphi(y) f(y)\, dy,$$

and consider the operator $T_j T_j^*$:

$$T_j T_j^*(f)(x) = \int_{\mathbb{R}^n} L_j(x, z) f(z)\, dz,$$

where

$$L_j(x,z) = \chi_{R_j}(x)\chi_{R_j}(z)\int_{\mathbb{R}^n} e^{i(P(x,y)-P(z,y))}|\varphi(y)|^2\,dy.$$

Write

$$P(x,y) - P(z,y) = \sum_{\substack{|\alpha|\geq 1\\ |\beta|=l}} a_{\alpha\beta}y^{\beta}(x^{\alpha}-z^{\alpha}) + (Q(x,y)-Q(z,y)),$$

where the degree of $y$ in $Q(x,y) - Q(z,y)$ is less than or equal to $l-1$.

Applying Lemma 1, with $\nu = \beta_0$, we obtain

$$|L_j(x,z)| \leqslant C\left|\sum_{|\alpha|\geq 1} a_{\alpha\beta_0}(x^{\alpha}-z^{\alpha})\right|^{-1/l}\chi_{R_j}(x)\chi_{R_j}(z).$$

On the other hand, it is obvious that $|L_j(x,z)| \leqslant C$, so let $N > 0$ be a large number (to be chosen later), we have

$$|L_j(x,z)| \leqslant C\left|\sum_{|\alpha|\geq 1} a_{\alpha\beta_0}(x^{\alpha}-z^{\alpha})\right|^{-1/Nl}\chi_{R_j}(z).$$

By rescaling we would obtain the same norm if we were to replace $L_j(x,z)$ by $L_j'(x,z) = 2^{nj}L_j(2^jx, 2^jz)$, so we have

$$|L_j'(x,z)| \leqslant C2^{nj}\left|\sum_{|\alpha|\geq 1}(a_{\alpha\beta_0}2^{j|\alpha|})x^{\alpha} - \sum_{|\alpha|\geq 1} a_{\alpha\beta_0}2^{j|\alpha|}z^{\alpha}\right|^{-1/Nl}\chi_{R_0}(x)\chi_{R_0}(z).$$

Choosing $N$ sufficiently large and applying Lemma 2, we get

$$\begin{aligned}\sup_z \int_{\mathbb{R}^n}|L_j'(x,z)|\,dx &\leqslant C2^{nj}\sup_z\left(\sum_{|\alpha|\geq 1}|a_{\alpha\beta_0}|2^{j|\alpha|} + \left|\sum_{|\alpha|\geq 1} a_{\alpha\beta_0}2^{j|\alpha|}z^{\alpha}\right|\right)^{-1/Nl}\\ &\leqslant C2^{nj}|a_{\alpha_0\beta_0}|^{-1/Nl}2^{-j|\alpha_0|/Nl}.\end{aligned}$$

Similar estimate holds for $\sup_x \int_{\mathbb{R}^n}|L'(x,z)|\,dz$, therefore we obtain

$$\|T_jT_j^*\| \leqslant C2^{nj}|a_{\alpha_0\beta_0}|^{-1/Nl}2^{-j|\alpha_0|/Nl},$$

so

$$\|T_j\|_{L^2\to L^2} \leqslant C2^{nj/2}|a_{\alpha_0\beta_0}|^{-1/2Nl}2^{-j|\alpha_0|/2Nl}$$

Now we have

$$\begin{aligned}\int_{|x|>2,\,|x|>r}|T(a)(x)|\,dx &\leqslant \int_{|x|>2,\,|x|>r}dx\int_{\mathbb{R}^n}|K(x-y)-K(x)|\,|a(y)|\,dy\\ &\quad + \int_{|x|>2,\,|x|>r}|K(x)|\,dx\left|\int_{\mathbb{R}^n}e^{iP(x,y)}a(y)\,dy\right| = I_1 + I_2.\end{aligned}$$

The estimate for $I_1$ is easy

$$I_1 \leqslant \int_{|x|>2,\,|x|>r} dx \int_{\mathbb{R}^n} \frac{|y|\,|a(y)|}{|x|^{n+1}}\,dy$$
$$\leqslant C \int_{|x|>2} \frac{dx}{|x|^{n+1}} < C.$$

As for $I_2$, using our estimate on $T_j$ and assuming $2^{j_0} \leqslant r < 2^{j_0+1}$, for some $j_0$, we have

$$I_2 \leqslant C \int_{|x|>2,\,|x|>r} \frac{1}{|x|^n} \left| \int_{\mathbb{R}^n} e^{iP(x,y)} a(y)\,dy \right| dx$$
$$\leqslant C \sum_{j \geq j_0} \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^n} |T_j(a)(x)|\,dx$$
$$\leqslant C \sum_{j \geq j_0} \left( \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^{2n}}\,dx \right)^{1/2} \| T_j(a) \|_{L^2}$$
$$\leqslant C \sum_{j \geq j_0} 2^{-nj/2} 2^{nj/2} |a_{\alpha_0 \beta_0}|^{-1/2Nl} 2^{-j|\alpha_0|/2Nl} \leqslant C,$$

because $2^{j_0} \geqslant (1/2)|a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$, and (3) is proved.

To prove the theorem, we only need to prove that $\| T(a) \|_{L^1} \leqslant C$, for all atoms $a$, and $C$ is a constant which depends only on the total degree of $P(x, y)$.

Let $a$ be an atom associated to the cube $Q$, and the center and sidelength of $Q$ are $x_Q$ and $\delta$ respectively. We observe that

$$\delta^n(T(a))(\delta x + x_Q) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(\delta x + x_Q,\, \delta y + x_Q)} K(x-y) \delta^n a(\delta y + x_Q)\,dy.$$

Write

$$P(\delta x + x_Q, \delta y + x_Q) = R(x, y) + P(x_Q, \delta y + x_Q),$$

where $R(x, y)$ is a polynomial which satisfies $R(0, y) \equiv 0$, and the total degree of $R$ is not greater than that of $P$. Let

$$b(y) = e^{iP(x_Q,\, \delta y + x_Q)} \delta^n a(\delta y + a_Q),$$

by the definition of the atom, we have

$$\operatorname{supp}(b) \subset Q_0 \quad \text{and} \quad |b(y)| \leqslant 1,$$

also

$$\int_{Q_0} b(y)\,dy = \int_Q e^{iP(x_Q, y)} a(y)\,dy = 0.$$

Now invoking (3), we have

$$\|T(a)\|_{L^1} = \left\| \text{p.v.} \int_{\mathbb{R}^n} e^{iR(x,y)} K(x-y) b(y)\, dy \right\|_{L^1} \leqslant C.$$

This completes the proof of Theorem 5.

## 3. An Extension

In [RS], Ricci and Stein pointed out that the $L^p$ boundedness still holds, if the Calderón-Zygmund kernel in the operator is replaced by some more general distribution. For $H^1_E$, the same thing is true, *i.e.*

**Theorem 2.** *If $K(x,y)$ is a distribution and $C^1$ away from the diagonal $\{x=y\}$, and satisfies*:

(i) $|K(x,y)| \leqslant C|x-y|^{-n}$ *and* $|\nabla K(x,y)| \leqslant C|x-y|^{-n-1}$.

(ii) *The operator*

$$f \to \int K(x,y) f(y)\, dy$$

*extends as a bounded operator on $L^2(\mathbb{R}^n)$.*

*Then the operator*

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y)\, dy \tag{5}$$

*is bounded from $H^1_E$ to $L^1$.*

The proof of Theorem 2 is essentially the same as Theorem 1.

## 4. The Dual Space $\mathrm{BMO}_E$

We define the sharp function $f^{\#}_E$ to be

$$(f^{\#}_E)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x) - f^E_Q(x)|\, dx,$$

where

$$f^E_Q(x) = e^{-iP(x_Q,x)} \left( \frac{1}{|Q|} \int_Q e^{-iP(x_Q,y)} f(y)\, dy \right)$$

and as the dual space of $H^1_E$, $\mathrm{BMO}_E$ is given by

$$\mathrm{BMO}_E = \{f \in L^1_{\mathrm{loc}} : f^{\#}_E \in L^\infty\}$$

and

$$\|f\|_{\mathrm{BMO}_E} = \|f^{\#}_E\|_{L^\infty}.$$

The dual statement of Theorem 2 is

**Theorem 3.** *The operator $T^*$ ($T$ given by (5)) extends as a bounded operator from $L^\infty$ to* $\mathrm{BMO}_E$.

## 5. A Counterexample

In this section, we shall give a simple example to show that the $H^1$ theory on the oscillatory singular integral operators cannot be extended to the $H^p$ case, if $p < 1$.

Let $T$ be defined as

$$(Ta)(x) = \text{p.v.} \int_{\mathbb{R}^1} e^{ixy} \frac{1}{x-y} a(y)\, dy.$$

Take $\delta > 0$, $\delta$ is very small, and $a$ is a function supported on $I_\delta = [-\delta, \delta]$, given by

$$a(y) = \begin{cases} (2\delta)^{-1/p} & \text{if } y \in [\delta/2, \delta], \\ -(2\delta)^{-1/p} & \text{if } y \in [-\delta, -\delta/2], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $a$ satisfies

$$|a| \leqslant |I_\delta|^{-1/p}, \qquad \int_{I_\delta} a(y)\, dy = 0.$$

Therefore, we have

$$\operatorname{Im}(Ta)(x) = (2\delta)^{-1/p}\left(\int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x-y}\, dy + \int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x+y}\, dy\right).$$

Let $x \in (\pi/4\delta,\ \pi/3\delta)$, then $x - y > 0$, $x + y > 0$ for $y \in [\delta/2, \delta]$. Also $\pi/8 < xy < \pi/3$.

Hence

$$\operatorname{Im}(Ta)(x) > c_0(2\delta)^{-1/p}\left(\int_{\delta/2}^{\delta} \frac{1}{x-y}\,dy + \int_{\delta/2}^{\delta} \frac{1}{x+y}\,dy\right)$$
$$= c_0(2\delta)^{-1/p}\log\left(1 + \frac{\delta x}{(x^2 - \delta x/2 - \delta^2/2)}\right)$$
$$> c_0'\delta^{1-1/p}x^{-1},$$

for some constant $c_0' > 0$. Then, we have

$$\int_{\mathbb{R}^1} |Ta(x)|^p\,dx \geqslant c_0'^p \int_{\pi/4\delta}^{\pi/3\delta} (\delta^{1-1/p})^p x^{-p}\,dx = c\delta^{2(p-1)}.$$

This is unbounded as $\delta \to 0$ and $p < 1$.

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