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Local Properties of Stationary Solutions of some Nonlinear Singular Schrödinger Equations

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Abstract

We study the local behaviour of solutions of the following type of equation $-\Delta u - V(x)u + g(u) = 0$ when V is singular at some points and g is a nondecreasing function. Emphasis is put on the case when $V(x) = c|x|^{-2}$ and g has a power-like growth.

Introduction

In this article we study the local behaviour of a solution u of the following time-independent, N-dimensional, nonlinear Schrödinger equation

$$-\Delta u - V(x)u + g(u) = 0 \tag{0.1}$$

near an isolated singularity of the potential V, g being some asymptotically nondecreasing real valued function. In many physical examples V is a Coulombian potential:

$$V(x) = \sum_{i=1}^{k} z_i |x - a_i|^{-1} \tag{0.2}$$

in the case of a nucleus in the Thomas-Fermi-Dirac-von Weizsäcker theory [3], [4]. However it is mathematically more exciting when V can be compared with $|x - a|^{-2}$ near the isolated singularity a. In that case the interference between the Laplacian, the potential and the nonlinearity is very strong. The model equation is the following

$$-\Delta u - \frac{c}{|x|^2} u + u|u|^{q-1} = 0 \tag{0.3}$$

where $q > 1$ and c is some real number. If we look for a specific solution of (0.3) under the form

$$u_s(r) = \alpha r^{\beta}, \tag{0.4}$$

then

$$\beta = -\frac{2}{q-1}$$

and

$$\alpha^{q-1} = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right).$$

Henceforth the solution u_s exists if and only if

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) > 0. \tag{0.5}$$

It is worth noticing that if (0.5) does not hold then

$$c \leqslant \left(\frac{N-2}{2}\right)^2, \tag{0.6}$$

and this condition plays a fundamental role in the description of the fundamental solutions of the equation

$$\Delta\phi + \frac{c}{|x|^2}\phi = 0. \tag{0.7}$$

If (0.6) is satisfied let β be $\sqrt{(N-2)^2 - 4c}$ and μ_i the two fundamental solutions of (0.7), that is

$$\mu_1(x) = \begin{cases} |x|^{-(N-2+\beta)/2} & \text{if } c < \left(\dfrac{N-2}{2}\right)^2, \\ |x|^{-(N-2)/2} \operatorname{Ln}(1/|x|) & \text{if } c = \left(\dfrac{N-2}{2}\right)^2, \end{cases} \tag{0.8}$$

$$\mu_2(x) = \begin{cases} |x|^{-(N-2-\beta)/2} & \text{if } c < \left(\dfrac{N-2}{2}\right)^2, \\ |x|^{-(N-2)/2} & \text{if } c = \left(\dfrac{N-2}{2}\right)^2. \end{cases} \tag{0.9}$$

It is important to notice that μ_2 is the regular solution of (0.7) in the sense that $c|\bullet|^{-2}\mu_2(\bullet)$ is locally integrable in $\mathbb{R}^N$ and

$$\Delta\mu_2 + \frac{c}{|x|^2}\mu_2 = 0 \tag{0.10}$$

holds in $D'(\mathbb{R}^N)$ (if $c \leqslant 0$, u is continuous), as the same holds for μ_1 if and only if $c > 0$; in any case $\mu_2 = o(\mu_1)$ near 0. Our first removability result deals with the meaning of the equation in the sense of distributions.

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega\setminus\{0\}$, g a continuous real valued function satisfying*

$$\begin{cases} \liminf\limits_{r\to\infty} g(r)/r^q > 0 \\ \limsup\limits_{r\to-\infty} g(r)/(-r^q) < 0 \end{cases} \tag{0.11}$$

and $V \in C^0(\Omega^*)$ is such that

$$-\infty < |x|^2 V(x) \leqslant c \tag{0.12}$$

near 0 for some constants $q > 1$ and c. If we assume either $q > N/(N-2)$, or $1 < q \leqslant N/(N-2)$ and

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) \leqslant 0, \tag{0.13}$$

any $u \in C^1(\Omega^*)$ satisfying

$$-\Delta u - Vu + g(u) = 0 \tag{0.14}$$

in $D'(\Omega^*)$ can be extended as a solution of the same equation in $D'(\Omega)$.

We must remark that if (0.5) is satisfied with $q > N/(N-2)$ there exist singular solutions of the model problem (0.3) with a rather weak singularity; this must be compared with

$$\Delta u + u^q = 0 \tag{0.15}$$

for which the same holds when $q > N/(N-2)$. Our second removability result is to compare a solution of (0.14) in Ω^* with the regular solution of (0.10).

Theorem 1.2. *Let Ω and V be as in Theorem 1.1 and let g be a continuous real valued function satisfying (0.11) for some $q > 1$. Assume also that*

$$0 = g(0) = g^{-1}(0) \tag{0.16}$$

and

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) \leqslant 0 \tag{0.17}$$

hold. Then if u is any $C^1(\Omega^*)$ function satisfying (0.14) in $D'(\Omega^*)$, u/μ_2 remains locally bounded in Ω.

It is important to notice that, as (0.17) holds, (0.6) also holds which allows us to have a comparison principle.

Our second section is devoted to the extension of Vázquez-Veron's isotropy theorems [23], [24] to the potential case. Let us introduce some notations: let S^{N-1} be the unit sphere in $\mathbb{R}^N$, $(r, \sigma) \in \mathbb{R}_*^+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$ and $\bar{\rho}(r)$ the spherical average of a function $\rho(r, \sigma)$, that is

$$\bar{\rho}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \rho(r, \sigma)\, d\sigma. \tag{0.18}$$

Theorem 2.1. *Assume Ω is an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, g is a continuous nondecreasing real valued function and $u \in C^1(\Omega^*)$ is a solution of (0.14) in $D'(\Omega^*)$ where $V \in C^0(\Omega^*)$ is a radial function such that*

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2, \quad \text{for every} \quad x \in \Omega^*. \tag{0.19}$$

If u satisfies

$$\liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0, \tag{0.20}$$

then $u(x)/\mu_1(x)$ admits a limit in $\mathbb{R} \cup \{-\infty, \infty\}$ as x tends to 0.

A similar isotropy result holds for a solution u of (0.14) in an exterior domain of $\mathbb{R}^N$. An interesting class of solutions of (0.14) in Ω^* are those which present a singular linear behaviour near 0. As the equation in the sense of distributions in Ω is not very singnificative except when $c = 0$ where the Dirac mass plays a fundamental role [6], [10], the good criterion for the behaviour of linear singularities will be the existence of a finite, not always 0, limit of $u(x)/\mu_1(x)$ as x tends to 0, as in [29].

Theorem 3.1. *Assume g is a continuous nondecreasing real valued function and*

$$c < \left(\frac{N-2}{2}\right)^2. \tag{0.21}$$

Then the equation

$$-\Delta u - \frac{c}{|x|^2} u + g(u) = 0 \tag{0.22}$$

admits solutions u in

$$B_1(0) \setminus \{0\} = \{x \in \mathbb{R}^N : 0 < |x| < 1\}$$

such that

$$\lim_{x \to 0} u(x)/\mu_1(x) = \gamma, \tag{0.23}$$

where γ is any arbitrary real number if and only if

$$\int_1^\infty (g(t) + |g(-t)|) t^{2(1-\alpha_1)/\alpha_1} \, dt < \infty, \tag{0.24}$$

where

$$\alpha_1 = -\left(N - 2 + \sqrt{(N-2)^2 - 4c}\right)/2.$$

When $c = 0$, condition (0.24) is the one introduced by Brézis and Bénilan [6] for solving equations of type

$$-\Delta u + g(u) = m \tag{0.25}$$

where m is a bounded measure. When

$$c = \left(\frac{N-2}{2}\right)^2$$

the situation is quite more complicated (see Vázquez [22] for the case $N = 2$).

We define

$$(0.26)\qquad \begin{cases} b_g^+ = \inf\left\{b > 0: \int_0^1 g(t^{-(N-2)/(N+2)}\,\mathrm{Ln}\,(1/t)/b)\,dt < \infty\right\}, \\ b_g^- = \inf\left\{b > 0: \int_0^1 g(t^{-(N-2)/(N+2)}\,\mathrm{Ln}\,t/b)\,dt > -\infty\right\}, \end{cases}$$

and we prove

Theorem 3.2. *Assume g is a continuous nondecreasing real valued function. Then the equation*

$$(0.27)\qquad -\Delta u - \left(\frac{N-2}{2|x|}\right)^2 u + g(u) = 0$$

admits solutions u in $B_1(0)\setminus\{0\}$ *such that*

$$(0.28)\qquad \lim_{x\to 0} |x|^{(N-2)/2} u(x)/\mathrm{Ln}\,(1/|x|) = \gamma,$$

where γ *is a real number, if and only if*

$$(0.29)\qquad -(N+2)/(2b_g^-) \leqslant \gamma \leqslant (N+2)/(2b_g^+).$$

The Dirichlet problems corresponding to Theorems 3.1, 3.2 *are also solved.*

In the last section we study the limit properties of the solutions u of (0.3) (as $|x|$ tends to 0 or ∞ as well). If we perform the classical transformation

$$(0.30)\qquad u(r,\sigma) = r^{-2/(q-1)} v(t,\sigma), \qquad t = \mathrm{Ln}\, r$$

and denote by $\Delta_{S^{N-1}}$ the Laplace-Beltrami operator on S^{N-1}, then

$$(0.31)\qquad v_{tt} + \left(N - 2\frac{q+1}{q-1}\right)v_t + \Delta_{S^{N-1}} v + \lambda v - v|v|^{q-1} = 0$$

holds in $(-\infty, 0)$ or $(0, \infty)$ with

$$(0.32)\qquad \lambda = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right),$$

moreover u is bounded. When $q \neq (N+2)/(N-2)$ the study of this equation is an extension of previous results of Veron [26] [27], Chen-Matano-Veron [13] and Bidaut-Veron-Veron [7]. A typical result is the following

Theorem 4.1. *Assume* $q \in (1,\infty)\setminus\{(N+2)/(N-2)\}$ *and u is a solution of* (0.3) *in* $B_1(0)\setminus\{0\}$. *Then* $r^{2/(q-1)}u(r,\cdot)$ *converges in the* $C^3(S^{N-1})$*-topology* *to some compact connected subset ξ' of the set ξ of the $C^3(S^{N-1})$-functions ω satisfying*

(0.33) $$-\Delta_{S^{N-1}}\omega + \omega|\omega|^{q-1} = \lambda\omega$$

on S^{N-1}. Moreover there exists precisely one $\omega \in \xi$ such that

(0.34) $$\lim_{r\to 0} r^{2/(q-1)}u(r, \cdot) = \omega(\cdot),$$

at least in the following cases:

(i) *u is nonnegative,*
(ii) $\lambda \leqslant N-1$,
(iii) *q is an odd integer,*
(iv) *ξ' is an hyperbolic limit manifold in the sense of Simon* [21],
(v) $N = 2$ *and* $c \leqslant 1$.

When $q = (N+2)/(N-2)$ the study is more complicated, in particular because of the conformal invariance of (0.3) and the existence of solitary waves satisfying (0.3) (see [7]). Convergence results hold at least for nonnegative solutions [17]. When $\lambda \leqslant 0$ there always holds

(0.35) $$\lim_{x\to 0} |x|^{2/(q-1)}u(x) = 0$$

for any solution u of (0.3) in $B_1(0)\setminus\{0\}$ and the exact behaviour is given by μ_2 from Theorem 1.2 except in the particular case $\lambda = 0$, $q > (N+2)/(N-2)$ and we prove

Theorem 4.2. *Assume $0 < c \leqslant ((N-2)/2)^2$, $\lambda = 0$ and $q > (N+2)/(N-2)$. If u is any solution of (0.3) in $B_1(0)\setminus\{0\}$, then the following limit exists*

(0.36) $$\lim_{x\to 0} u(x)/(\mu_2(x)\,\mathrm{Ln}\,(1/|x|)^{\alpha_2/2}) = l$$

with

$$\alpha_2 = \left(2 - N + \sqrt{(N-2)^2 - 4c}\right)/2 = -2/(q-1)$$

and

(0.37) $$l \in \{0, \pm((N(q-1) - 2(q+1))/(q-1)^2)^{1/(q-1)}\}.$$

When $\lambda > 0$ it may happen that (0.35) holds. In that case the behaviour of u near 0 is most often described by the solutions of

(0.38) $$-\Delta\zeta - \frac{c}{|x|^2}\zeta = 0,$$

satisfying (0.35) except when $-2/(q-1)$ is a solution of the algebraic equation

(0.39) $$X^2 + (N-2)X + c - k(k+N-2) = 0$$

for some integer k. Only when $N = 2$ and $c \leqslant 0$ this spectral case is understood. In some cases, when the rate of blow-up of u near 0 is of order $|x|^{-(N-2)/2}$, u may behave as a finite superposition of travelling waves near 0 (up to the damping factor $|x|^{(N-2)/2}$).

We also study the asymptotics of the solution of (0.3) in an exterior domain and end this section with some orbit connecting questions where the structure of the set of the stationnary solutions of (0.33) plays a fundamental role.

Our paper is organized as follows

(1) Removable singularities.
(2) The isotropy theorems.
(3) Solutions with linear singularities.
(4) The power case.

1. Removable Singularities

In this section we assume that $\Omega \supset \bar{B}_1(0)$, $\Omega^* = \Omega\setminus\{0\}$ and we first prove the following a priori estimate of Osserman's type [19], [10], [31].

Lemma 1.1. *Assume $u \in L^\infty_{\mathrm{loc}}(\Omega^*)$ satisfies $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega^*)$ and*

(1.1) $$-\Delta u - \frac{c}{|x|^2}u + au^q \leqslant b$$

a.e. on $\{x \in \Omega : u(x) \geqslant 0\}$, for some constants $a > 0$, b and $c \geqslant 0$ and $q > 1$. Then

(1.2) $$u(x) \leqslant A|x|^{-2/(q-1)} + B \qquad (\text{for all} \quad x \in \bar{B}_{1/2}(0)\setminus\{0\}),$$

where

(1.3) $$A = \sigma(N,q)\left(\frac{1+c}{a}\right)^{1/(q-1)}, \qquad B = \sigma(N,q)\left(\frac{b}{a}\right)^{1/q},$$

with $\sigma(N,q) > 0$.

PROOF. Let x_0 be such that $0 < |x_0| < 1/2$. Set

$$G = \left\{ x \in \Omega \colon |x - x_0| < \frac{1}{2} |x_0| \right\}$$

and $G^+ = \{x \in G \colon u(x) \geqslant 0\}$. The function u is essentially bounded in G and

$$-\Delta u + \frac{a}{2} u^q \leqslant \beta = \max_{r > 0} \left\{ b + \frac{4c}{|x_0|^2} r - \frac{a}{2} r^q \right\} \tag{1.4}$$

a.e. in G^+. If we compute β we find

$$\beta = b + \frac{4(q-1)c}{|x_0|^2} \left(\frac{8c}{aq|x_0|^2} \right)^{1/(q-1)} \tag{1.5}$$

As in [10], [31] we consider a function v under the following form

$$v(x) = \rho \left(\frac{1}{4} |x_0|^2 - |x - x_0|^2 \right)^{-2/(q-1)} + \tau. \tag{1.6}$$

If

$$\eta = \max \left\{ \frac{2N}{q-1}, \frac{4(q+1)}{(q-1)^2} \right\}, \qquad \rho = \left(\frac{2\eta}{a} \right)^{1/(q-1)}, \qquad \tau = \left(\frac{2\beta}{a} \right)^{1/q},$$

v satisfies

$$-\Delta v + \frac{a}{2} v^q \geqslant \beta \tag{1.7}$$

in G. Using Kato's inequality as in [10], [11], we deduce $v \geqslant u$ in G, which implies $v(x_0) \geqslant u(x_0)$ and gives (1.3).

Lemma 1.2. *Assume* $1 < q \leqslant N/(N-2)$,

$$c + \frac{2}{q-1} \left(\frac{2q}{q-1} - N \right) \leqslant 0 \tag{1.8}$$

and $u \in L^\infty_{\text{loc}}(\Omega^*)$ *satisfies* $\Delta u \in L^\infty_{\text{loc}}(\Omega^*)$ *and* (1.1) *a.e. on* $\{x \in \Omega \colon u(x) \geqslant 0\}$ *for some constants* $a > 0$ *and* $b \geqslant 0$. *Then* $\mu_2 u^+ \in L^q_{\text{loc}}(\Omega)$.

PROOF. From Kato's inequality we have

$$-\Delta u^+ - \frac{c}{|x|^2} u^+ + a(u^+)^q \leqslant b \tag{1.9}$$

in $D'(\Omega^*)$, and from (1.2) and (1.8) we deduce that $c \leqslant 0$ and

$$u^+ \leqslant K_0 \mu_1 \tag{1.10}$$

near 0, for some $K_0 > 0$. Let ϕ be an element of $C_0^\infty(\Omega)$, $\phi \geqslant 0$, η_n be a C^∞-function in Ω such that $0 \leqslant \eta_n \leqslant 1$ and

$$\eta_n(x) = \begin{cases} 0 & \text{if } \quad 0 < |x| < 1/n \\ 1 & \text{if } \quad |x| \geqslant 2/n \end{cases} \tag{1.11}$$

with $|\nabla \eta_n| \leqslant Kn$, $|\Delta \eta_n| \leqslant Kn^2$.

Then we claim that

$$\int (u^+)^q \mu_2 \phi \, dx < \infty.$$

As a test function we take $\phi \eta_n \mu_2$ and get

$$\int u^+ \left(-\Delta(\phi \eta_n \mu_2) - \frac{c}{|x|^2} (\phi \eta_n \mu_2) \right) + a \int (u^+)^q \phi \eta_n \mu_2 \leqslant K(\phi). \tag{1.12}$$

But

$$\Delta(\phi \eta_n \mu_2) = \phi \eta_n \, \Delta \mu_2 + \mu_2 \, \Delta(\phi \eta_n) + 2 \, \nabla \mu_2 \, \nabla(\phi \eta_n)$$

and (1.12) becomes

$$\int u^+ (-\mu_2 \, \Delta(\phi \eta_n) - 2 \, \nabla \mu_2 \, \nabla(\phi \eta_n)) + a \int (u^+)^q \phi \eta_n \mu_2 \leqslant K(\phi). \tag{1.13}$$

Let $\Gamma_n = \{x \in \Omega: 1/n < |x| < 2/n\}$ and let χ_{Γ_n} be the characteristic function of Γ_n. There exist K_1, K_2 such that $K_i > 0$ and

$$|\Delta(\phi \eta_n)| \leqslant K_1 + K_2 n^2 \chi_{\Gamma_n}, \qquad |\nabla(\phi \eta_n)| \leqslant K_1 + K_2 n \chi_{\Gamma_n}. \tag{1.14}$$

Plugging into (1.12) implies

$$\begin{aligned} a \int (u^+)^q \phi \eta_n \mu_2 < K(\phi) &+ K_0 \int \mu_1 \mu_2 (K_1 + K_2 n^2 \chi_{\Gamma_n}) \\ &+ 2K_0 \int \mu_1 |\nabla \mu_2| (K_1 + K_2 n \chi_{\Gamma_n}). \end{aligned} \tag{1.15}$$

As $\mu_1 \mu_2 = |x|^{2-N}$ and $\mu_1 |\nabla \mu_2| = K'|x|^{1-N}$, the right-hand side of (1.15) is bounded independently of n. Letting n tend to infinity implies the claim.

Lemma 1.3. *Under the hypotheses of Theorem* 1.1, *$g(u)$ and Vu are locally integrable in Ω.*

PROOF. We shall treat separately the cases $1 < q \leqslant N/(N-2)$ and $q > N/(N-2)$ but from (0.11) and Lemma 1.1 in any case $|x|^{2/(q-1)}u(x)$ is locally bounded in Ω.

Case 1. $1 < q \leqslant N/(N-2)$. From Kato's inequality we have

$$-\Delta u^{+} - Vu^{+} + \operatorname{sign}^{+}(u)g(u) \leqslant 0 \tag{1.16}$$

in $D'(\Omega^*)$. Let ζ_ϵ be $\mu_2/(\epsilon + \mu_2)$ $(\epsilon > 0)$. As a test function we take $\phi\eta_n\zeta_\epsilon$ where ϕ and η_n are as in Lemma 1.2 with $\phi \equiv 1$ in $\bar{B}_{1/2}(0)$. We get

$$\int u^{+}((-\Delta(\phi\eta_n\zeta_\epsilon) - V\phi\eta_n\zeta_\epsilon) + \int \operatorname{sign}^{+}(u)g(u)\phi\eta_n\zeta_\epsilon \leqslant 0. \tag{1.17}$$

As

$$\Delta(\phi\eta_n\zeta_\epsilon) = \phi\eta_n\,\Delta\zeta_\epsilon + \zeta_\epsilon\,\Delta(\phi\eta_n) + 2\,\nabla\zeta_\epsilon\,\nabla(\phi\eta_n)$$

and

$$\nabla\zeta_\epsilon = \frac{\epsilon}{(\epsilon + \mu_\epsilon)^2}\nabla\mu_2,$$

$$\Delta\zeta_\epsilon = \frac{\epsilon}{(\epsilon + \mu_2)^2}\Delta\mu_2 - \frac{2\epsilon}{(\epsilon + \mu_2)^3}|\nabla\mu_2|^2.$$

As

$$\Delta\mu_2 = -\frac{c}{|x|^2}\mu_2,$$

we get

$$\begin{aligned}-\Delta(\phi\eta_n\zeta_\epsilon) - V\phi\eta_n\zeta_\epsilon = &-\left(-\frac{c\epsilon\mu_2}{|x|^2(\epsilon + \mu_2)^2} - \frac{2\epsilon}{(\epsilon + \mu_2)^3}|\nabla\mu_2|^2\right)\phi\eta_n\\ &- V\phi\eta_n\frac{\mu_2}{\epsilon + \mu_2}\\ &- \zeta_\epsilon\,\Delta(\phi\eta_n)\\ &- \frac{2\epsilon}{(\epsilon + \mu_2)^2}\nabla\mu_2\,\nabla(\phi\eta_n).\end{aligned}$$

Henceforth (1.17) implies

$$\begin{aligned}&\int \frac{\mu_2}{\epsilon + \mu_2}\phi\eta_n(-Vu^{+} + \operatorname{sign}^{+}(u)g(u))\\ &\leqslant -\int \frac{c\epsilon\mu_2 u^{+}}{|x|^2(\epsilon + \mu_2)^2}\phi\eta_n + \int u^{+}\zeta_\epsilon\,\Delta(\phi\eta_n) + 2\epsilon\int \frac{u^{+}}{(\epsilon + \mu_2)^2}\nabla\mu_2\,\nabla(\phi\eta_n).\end{aligned} \tag{1.18}$$

We also have

$$|\Delta(\phi\eta_n)| \leqslant K_1\chi_{\Gamma_2} + n^2K_2\chi_{\Gamma_n}, \tag{1.19}$$

$$|\nabla(\phi\eta_n)| \leqslant K_1\chi_{\Gamma_2} + nK_2\chi_{\Gamma_n}, \tag{1.20}$$

and

$$\left|\int u^+\zeta_\epsilon\,\Delta(\phi\eta_n)\right| \leqslant K_1\int_{\Gamma_2} u^+ + n^2K_2\int_{\Gamma_n} u^+\zeta_\epsilon. \tag{1.21}$$

As

$$\int_{\Gamma_n} u^+\zeta_\epsilon \leqslant \left(\int_{\Gamma_n}(u^+)^q\mu_\epsilon\right)^{1/q}\left(\int_{\Gamma_n}\frac{\mu_2}{(\epsilon+\mu_2)^{q/(q-1)}}\right)^{(q-1)/q},$$

and

$$\left(\int_{\Gamma_n}\frac{\mu_2}{(\epsilon+\mu_2)^{q/(q-1)}}\right)^{(q-1)/q} \leqslant \frac{c(N)}{\epsilon}n^{-(\alpha_2+N)(q-1)/q}$$

where

$$\alpha_2 = \frac{2-N+\sqrt{(N-2)^2-4c}}{2}$$

is such that $\mu_2(x) = |x|^{\alpha_2}$; as (1.8) holds $\alpha_1 \leqslant -2/(q-1)$ and as $\alpha_1+\alpha_2 = 2-N$ we deduce that

$$-(\alpha_2+N)(q-1)/q + 2 \leqslant 0. \tag{1.22}$$

From Lemma 1.2 $\mu_2(u^+)^q$ is locally integrable in Ω; henceforth

$$\lim_{n\to\infty} n^2\int_{\Gamma_n} u^+\zeta_\epsilon = 0. \tag{1.23}$$

In the same way

$$\epsilon\int\frac{u^+}{(\epsilon+\mu_2)^2}\nabla\mu_2\,\nabla(\phi\eta_n) \leqslant 2\epsilon K_1\int_{\Gamma_2}u^+\frac{|\nabla\mu_2|}{\mu_2} + 2Kn\int_{\Gamma_n}\frac{\epsilon u^+}{(\epsilon+\mu_2)^2}|\nabla\mu_2|$$

and

$$\int_{\Gamma_n}\frac{\epsilon u^+}{(\epsilon+\mu_2)^2}|\nabla\mu_2| \leqslant \int_{\Gamma_n}u^+\zeta_\epsilon\frac{\epsilon\alpha_2}{|x|(\epsilon+\mu_2)} \leqslant n\alpha_2\int_{\Gamma_n}u^+\zeta_\epsilon,$$

which yields

$$\lim_{n\to\infty} n\int_{\Gamma_n}\frac{u^+}{(\epsilon+\mu_2)^2}\nabla\mu_2 = 0. \tag{1.24}$$

For the last right-hand side term of (1.18) we have

$$-c\epsilon\int\frac{u^+\mu_2}{|x|^2(\epsilon+\mu_2)^2}\phi\eta_n \leqslant -cn^2\int_{\Gamma_n}u^+\zeta_\epsilon. \tag{1.25}$$

Using (1.23), (1.24) and the facts that V is negative and $\operatorname{sign}^+(u)g(u)$ is bounded below by some constant imply

$$\int\phi(-Vu^+ + \operatorname{sign}^+(u)g(u)) \leqslant K \tag{1.26}$$

for some $K>0$. In the same say

$$\int\phi(-Vu^- + \operatorname{sign}^-(u)g(u)) \leqslant K. \tag{1.27}$$

Henceforth Vu and $g(u)$ are locally integrable in Ω.

Case 2. $q > N/(N-2)$. From Lemma 1.1 and (0.12) uV is locally integrable in Ω. Taking $\eta_n\phi$ as a test function in (1.16) implies

$$\int u^+(-\Delta(\phi\eta_n) - V\phi\eta_n) + \int\operatorname{sign}^+(u)g(u)\phi\eta_n \leqslant 0. \tag{1.28}$$

Using (1.19) yields

$$\left|\int u^+\,\Delta(\phi\eta_n)\right| \leqslant K_1\int_{\Gamma_2}u^+ + n^2K_2\int_{\Gamma_n}u^+ \leqslant K_1\int_{\Gamma_2}u^+ + K_2K'n^{2-N+2/(q-1)} \leqslant K''.$$

Letting n tend to infinity implies

$$\int\operatorname{sign}^+(u)g(u)\phi < \infty.$$

In the same way

$$\int\operatorname{sign}^-(u)g(u)\phi < \infty$$

which ends the proof

PROOF OF THEOREM 1.1. *Case* 1. $1 < q \leqslant N/(N-2)$.

As a test function we take $\eta_n \zeta_\epsilon \phi$ where η_n and ζ_ϵ are as in Lemma 1.3 and $\phi \in C_0^\infty(\Omega)$. We have

$$\int u(-\Delta(\eta_n \phi \zeta_\epsilon)) - \int V u \eta_n \phi \zeta_\epsilon + \int g(u) \eta_n \phi \zeta_\epsilon = 0 \tag{1.29}$$

and

$$\Delta(\eta_n \phi \zeta_\epsilon) = \eta_n \phi \, \Delta \zeta_\epsilon + \eta_n \zeta_\epsilon \, \Delta \phi + \phi \zeta_\epsilon \, \Delta \eta_n + 2\eta_n \nabla \phi \, \nabla \zeta_\epsilon + 2\phi \, \nabla \eta_n \nabla \zeta_\epsilon + 2\zeta_\epsilon \nabla \phi \, \nabla \eta_n .$$

As in Lemma 1.3, Case 1 it is easy to let n tend to infinity and obtain, for $\epsilon > 0$ fixed,

$$\int u(-\phi \, \Delta \zeta_\epsilon - \zeta_\epsilon \, \Delta \phi + 2 \, \nabla \phi \, \nabla \zeta_\epsilon) + \int (g(u) - Vu) \phi \zeta_\epsilon = 0. \tag{1.30}$$

But

$$|u \, \nabla \phi \, \nabla \zeta_\epsilon| \leqslant |\nabla \phi| \, |u| \frac{\epsilon \alpha_2 \mu_2}{|x|(\epsilon + \mu_2)^2} \leqslant \frac{\alpha_2}{2|x|} |\nabla \phi| \, |u|,$$

$$\begin{aligned} |u \phi \, \Delta \zeta_\epsilon| &\leqslant |c| \, |u| \, |\phi| \frac{\epsilon \mu_2}{|x|^2 (\epsilon + \mu_2)^2} + 2\alpha_2^2 |u| \, |\phi| \frac{\epsilon \mu_2^2}{|x|^2 (\epsilon + \mu_2)^3} \\ &\leqslant \frac{|c|}{2} |\phi| \frac{|u|}{|x|^2} + \frac{8\alpha_2^2}{27} |\phi| \frac{|u|}{|x|^2}. \end{aligned}$$

If $c = 0$ the two terms $|u \, \nabla \phi \, \nabla \zeta_\epsilon|$ and $|u \phi \, \Delta \zeta_\epsilon|$ vanish; if $c < 0$ we know from Lemma 1.3 that $u/|x|^2$ is locally integrable in Ω. Henceforth, from Lebesgue's theorem, we get

$$\lim_{\epsilon \to 0} \int u(-\phi \, \Delta \zeta_\epsilon - \zeta_\epsilon \, \Delta \phi + 2 \, \nabla \phi \, \nabla \zeta_\epsilon) = \int (-u \, \Delta \phi) \tag{1.31}$$

and

$$\int (-u \, \Delta \phi) - \int V u \phi + \int g(u) \phi = 0. \tag{1.32}$$

Case 2. $q > N/(N-2)$.

As a test function we just take $\phi \eta_n$ and we have from Lemma 1.3 and Hölder's inequality

$$\lim_{n \to \infty} \int -u \, \Delta(\phi \eta_n) - V u \phi \eta_n + g(u) \phi \eta_n = \int -u \, \Delta \phi - V u \phi + g(u) \phi, \tag{1.33}$$

which ends the proof.

Lemma 1.4. *Assume Ω is as above and V is continuous in Ω^* and satisfies*

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2 \tag{1.34}$$

near 0. *If $w \in C^0(\Omega^*)$ is a nonnegative function satisfying*

$$\Delta w + Vw \geqslant 0 \tag{1.35}$$

in $D'(\Omega^*)$ and $w = o(\mu_1)$ near 0, *then u/μ_2 remains locally bounded in Ω.*

PROOF. Let M be the supremum of w on $\{x: |x| = 1\}$ and, for $\epsilon > 0$, $\Phi_\epsilon = M\mu_2 + \epsilon\mu_1$. We write (1.35) in spherical coordinates and get

$$w_{rr} + \frac{N-1}{r^2} w_r + \frac{c}{r^2} w + \frac{1}{r^2} \Delta_{S^{N-1}} w \geqslant 0 \tag{1.36}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} (we have used (1.35)). We shall distinguish two cases:

Case 1. $c < \left(\frac{N-2}{2}\right)^2 \cdot$ We write

$$v(s, \sigma) = w(r, \sigma)/\mu_1(r), \qquad s = r^\beta, \qquad \beta = \sqrt{(N-2)^2 - 4c} \tag{1.37}$$

and get

$$s^2 v_{ss} + \frac{1}{\beta^2} \Delta_{S^{N-1}} v \geqslant 0. \tag{1.38}$$

We write $\phi_\epsilon(s, \sigma) = \Phi_\epsilon(r, \sigma)/\mu_1(r)$, $t = \operatorname{Ln}(1/s)$ and $\psi(t, \sigma) = (v - \phi_\epsilon)(s, \sigma)$. The following relation holds in $D'(\mathbb{R}_*^+ \times S^{N-1})$

$$\psi_{tt} + \psi_t + \frac{1}{\beta^2} \Delta_{S^{N-1}} \psi \geqslant 0. \tag{1.39}$$

By convolution on t we may assume that $\psi \in C^\infty(\mathbb{R}_*^+, C^0(S^{N-1}))$ and if we approximate ψ by the solution χ_η ($\eta > 0$) of

$$-\eta \Delta_{S^{N-1}} \chi_\eta + \chi_\eta = \psi, \tag{1.40}$$

which converges to ψ in $L^2(S^{N-1})$ as η tends to 0, we deduce that

$$\frac{d^2}{dt^2} \|\psi^+(t, \cdot)\|_{L^2(S^{N-1})} + \frac{d}{dt} \|\psi^+(t, \cdot)\|_{L^2} \geqslant 0, \tag{1.41}$$

which implies that the function $s \to \|(v - \phi_\epsilon)^+(s, \cdot)\|_{L^2(S^{N-1})}$ is convex on $(0, 1)$. As it vanishes at 0 and 1, it is always 0. Letting ϵ tend to 0 implies the claim.

Case 2. $c = \left(\frac{N-2}{2}\right)^2$. We just write

$$v(t, \sigma) = w(r, \sigma)/r^{-(N-2)/2}, \qquad t = \operatorname{Ln}(1/r) \tag{1.42}$$

and v satisfies

$$v_{tt} + \Delta_{S^{N-1}} v \geqslant 0 \tag{1.43}$$

in $D'(\mathbb{R}_*^+ \times S^{N-1})$. By the same approximation we see that the convexity and the fact that $v = o(t)$ at infinity imply the estimate $w \leqslant K\mu_2$.

PROOF OF THEOREM 1.2. *Case* 1. We assume that

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) < 0. \tag{1.44}$$

In that case $2/(q-1) < (N - 2 + \beta)/2$ and $u = o(\mu_1)$ near 0. As we have

$$\Delta u^+ + \frac{c}{|x|^2} u^+ \geqslant \operatorname{sign}^+(u) g(u) \geqslant 0 \tag{1.45}$$

in $D'(\Omega^*)$, we deduce $u^+ \leqslant k\mu_2$. We do the same with u^-.

Case 2. We assume that $c \leqslant 0$.
In that case u^+ satisfies

$$\Delta u^+ \geqslant \operatorname{sign}^+(u) g(u) \geqslant a(u^+)^q - b. \tag{1.46}$$

From Brézis-Veron's result [11] u^+ is locally bounded in Ω; henceforth $u^+ \leqslant k\mu_2$. The same with u^-.

Case 3. We assume

$$0 < c \leqslant \left(\frac{N-2}{2}\right)^2, \tag{1.47}$$

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) = 0. \tag{1.48}$$

For $n \in \mathbb{N}^*$ let ϕ_n be the solution of

$$
(1.49)\qquad \begin{cases} \Delta\phi_n + \dfrac{c}{|x|^2}\phi_n - a\phi_n^q = 0 & \text{in} \quad B_1(0)\backslash B_{1/n}(0) \\ \phi_n = \max\{u^+(x): |x| = 1\} & \text{on} \quad \partial B_1(0) \\ \phi_n = \max\{u^+(x): |x| = 1/n\} & \text{on} \quad \partial B_{1/n}(0) \end{cases}
$$

(ϕ_n exists by minimization techniques and it is positive and unique) where a is defined as in (1.46) or Lemma 1.1. Let σ be $b/(c + 2N)$. Then $\psi_n = \phi_n + \sigma|x|^2$ satisfies

$$
(1.50)\qquad \Delta\phi_n + \frac{c}{|x|^2}\psi_n - a\psi_n^q \leqslant b.
$$

We then deduce, as in Lemma 1.4, that $u^+ \leqslant \psi_n$ in $B_1(0)\backslash B_{1/n}(0)$. As ϕ_n remains locally bounded in $B_1(0)\backslash B_{1/n}(0)$, independently of n (Lemma 1.1), we deduce that (up to a subsequence) it converges in the $C^1_{\text{loc}}(\bar{B}_1(0)\backslash\{0\})$-topology to a function ϕ which is radial and satisfies

$$
(1.51)\qquad \begin{cases} \Delta\phi + \dfrac{c}{|x|^2}\phi - a\phi^q = 0 & \text{in} \quad B_1(0)\backslash\{0\}, \\ \phi = \max\{u^+(x): |x| = 1\} & \text{on} \quad \partial B_1(0), \end{cases}
$$

and

$$
(1.52)\qquad u(x) \leqslant \phi(x) + \sigma|x|^2
$$

in $\bar{B}_1(0)\backslash\{0\}$. Moreover, in the range (1.48), we have

$$
(1.53)\qquad \phi(x) \leqslant c\mu_1(x) = c|x|^{\alpha_1} = c|x|^{-2/(q-1)}.
$$

If we set $\phi(x) = \tilde{\phi}(r)$ and

$$
(1.54)\qquad \eta(t) = r^{2/(q-1)}\tilde{\phi}(r), \qquad t = \operatorname{Ln}(1/r)
$$

then we get

$$
(1.55)\qquad \eta_{tt} - \left(N - 2\frac{q+1}{q-1}\right)\eta_t - a\eta^q = 0
$$

in $(0, +\infty)$.

(i) If $q = (N+2)/(N-2)$ the first order coefficient is 0 and

$$
(1.56)\qquad W(\eta, \eta_t) = \frac{1}{2}\eta_t^2 - \frac{a}{q+1}\eta^{q+1}
$$

is constant. As η is nonnegative and bounded, the only admissible constant is 0 and $\eta(t)$ tends to 0 as t tends to infinity.

(ii) If $q \neq (N+2)/(N-2)$ then

$$\frac{d}{dt} W(\eta, \eta_t) = \left(N - 2\frac{q+1}{q-1}\right)\eta_t^2. \tag{1.57}$$

From La Salle invariance principle $\lim_{t \to \infty} \eta(t) = 0$.

Henceforth

$$\lim_{x \to 0} u^+(x)/\mu_1(x) = 0. \tag{1.58}$$

As the same holds for u^- we deduce the claim from Lemma 1.4.

Remark 1.1. Using Theorems 2.1 and 3.1 it is possible to extend Theorem 1.2 to the case where g satisfies

$$\begin{cases} \liminf_{r \to \infty} \dfrac{g(r) \operatorname{Ln} r}{r^q} > 0 \\ \limsup_{r \to -\infty} \dfrac{g(r) \operatorname{Ln} r}{|r|^q} < 0 \end{cases} \tag{1.59}$$

and (0.17) (see [24] for the zero potential case and [16]).

2. The isotropy Theorems

In this section Ω is an open subset of $\mathbb{R}^N$ containing $\bar{B}_1(0)$, $\Omega^* = \Omega \setminus \{0\}$, g is a continuous nondecreasing real valued function and $V \in C(\Omega^*)$ is a radial potential such that

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2, \quad \text{for all} \quad x \in \Omega^*. \tag{2.1}$$

We are interested in solutions $u \in C^1(\Omega^*)$ of

$$-\Delta u - Vu + g(u) = 0. \tag{2.2}$$

Lemma 2.1. *Assume $u \in C^1(\Omega^*)$ satisfies* (2.2) *in $D'(\Omega^*)$ and*

$$\liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0. \tag{2.3}$$

Then there exists a constant $K \geqslant 0$ *such that*

$$\|u(r, \cdot) - \bar{u}(r)\|_{L^2(S^{N-1})} \leqslant K r^{(2-N+\sqrt{N^2-4c})/2} \tag{2.4}$$

for $0 < r \leqslant 1$.

PROOF. *Case* 1. $c < \left(\frac{N-2}{2}\right)^2$. In radial coordinates we have

$$u_{rr} + \frac{N-1}{r} u_r + V(r)u + \frac{1}{r^2} \Delta_{S^{N-1}} u = g(u). \tag{2.5}$$

We write u as in (1.37), that is

$$v(r, \sigma) = u(r, \sigma)/\mu_1(r), \qquad s = r^\beta, \qquad \beta = \sqrt{(N-2)^2 - 4c}, \tag{2.6}$$

and get

$$s^2 v_{ss} + \frac{1}{\beta^2} \Delta_{S^{N-1}} v + \frac{1}{\beta^2} (s^{2/\beta} V(s^{1/\beta}) - c) v = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} v) \tag{2.7}$$

with $\alpha_1 = (2 - N - \beta)/2$.

Let $\bar{\rho}(s)$ be the spherical average of a function $\rho(s, \sigma)$; then

$$s^2 \bar{v}_{ss} + \frac{1}{\beta^2} (V(s^{1/\beta}) - c) \bar{v} = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} \overline{g(s^{\alpha_1/\beta} v)}.$$

As

$$\int_{S^{N-1}} (-\Delta v(v - \bar{v}))\, d\sigma \geqslant (N-1) \int_{S^{N-1}} (v - \bar{v})^2\, d\sigma,$$

$$\int_{S^{N-1}} (g(s^{\alpha_1/\beta} v) - \overline{g(s^{\bar{\alpha}_1/\beta} v)})(v - \bar{v})\, d\sigma \geqslant 0,$$

and (2.1) we get

$$s^2 \int_{S^{N-1}} (v_{ss} - \bar{v}_{ss})(v - \bar{v})\, d\sigma - \frac{N-1}{\beta^2} \int_{S^{N-1}} (v - \bar{v})^2\, d\sigma \geqslant 0. \tag{2.8}$$

Setting

$$X(s) = \left(\int_{S^{N-1}} (v - \bar{v})^2(s)\, d\sigma \right)^{1/2},$$

we obtain

$$s^2 X_{ss} - \frac{N-1}{\beta^2} X \geqslant 0 \tag{2.9}$$

in $D'(0, 1)$. As $X(s_n) = o(s_n^{(\beta - \sqrt{N^2 - 4c})/2\beta})$ for some sequence $\{s_n\}$ converging to 0 we deduce that

$$X(s) \leqslant K s^{(\beta + \sqrt{N^2 - 4c})/2\beta}, \tag{2.10}$$

which is (2.4).

Case 2. $c = \left(\frac{N-2}{2}\right)^2$ · We write u as in (1.42):

$$w(t, \sigma) = r^{(N-2)/2} u(r, \sigma), \qquad t = \operatorname{Ln}(1/r); \tag{2.11}$$

then

$$w_{tt} + \Delta_{S^{N+1}} w + (\tilde{V}(t) - c) w = e^{(2-N)t/2} g(e^{(N-2)t/2} w) \tag{2.12}$$

where

$$\tilde{V}(t) = e^{(2-N)t} V(e^{(N-2)t/2}).$$

If we set

$$X(t) = \left(\int_{S^{N-1}} (w - \bar{w})^2(t)\, d\sigma \right)^{1/2},$$

then

$$X_{tt} - (N-1) X \geqslant 0 \tag{2.13}$$

in $D'(\mathbb{R}_*^+)$, and $X(t_n) = o(e^{\sqrt{N-1}\, t_n})$ for some sequence $\{t_n\}$ tending to ∞. The maximum principle implies that

$$X(t) \leqslant K e^{-\sqrt{N-1}\, t} \tag{2.14}$$

which is (2.4).

In order to have a L^∞ estimate we need the following result the proof of which is essentially contained in [24].

Lemma 2.2. *Assume γ, a and b are positive numbers such that $a < b$ and ϕ, $\psi \in L^2(S^{N-1})$. Then there exists a unique function $\Phi \in C([a, b]; L^2(S^{N-1}))$*

$\cap C^\infty((a,b) \times S^{N-1})$ *such that*

$$\text{(2.15)} \qquad \begin{cases} s^2\Phi_{ss} + \dfrac{1}{\gamma}\Delta_{S^{N-1}}\Phi = 0 & \textit{in} \quad (a,b) \times S^{N-1}, \\ \Phi(a,\bullet) = \phi(\bullet); \qquad \Phi(b,\bullet) = \psi(\bullet) & \textit{in} \quad S^{N-1}. \end{cases}$$

Moreover there exists a constant $C_1 > 0$ such that

$$\text{(2.16)} \qquad \|\Phi(s,\bullet)\|_{L^\infty(S^{N-1})} \leqslant C_1 \left\{ \left(1 + \frac{1}{\operatorname{Ln}(s/a)}\right)^{(N-1)/2} \|\Phi\|_{L^2(S^{N-1})} + \left(1 + \frac{1}{\operatorname{Ln}(b/s)}\right)^{(N-1)/2} \|\psi\|_{L^2(S^{N-1})} \right\}.$$

This result is, up to change of variable and unknown, essentially an estimate concerning harmonic functions in an annulus.

Lemma 2.3. *Assume the hypotheses of Lemma* 2.1 *hold with $c < (N-2)^2/4$. Then the function v introduced in* (2.6) *satisfies*

$$\text{(2.17)} \qquad \|v(s,\bullet) - \bar{v}(s)\|_{L^\infty(S^{N-1})} \leqslant \tilde{c} s^{(\beta + \sqrt{N^2 - 4c})/2\beta}$$

for some $\tilde{c} > 0$ and any $s \in (0, 1/2]$.

PROOF. Let y be the solution of

$$\text{(2.18)} \qquad \begin{cases} s^2 y_{ss} = \dfrac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} y) & \text{in} \quad (a,b), \\ y(a) = \rho, \qquad y(b) = \tau, \end{cases}$$

with $0 < a < b < 1$, ρ and τ real numbers. Let w be $v - y$, ϕ be $(v(a,\bullet) - \rho)^+$, ψ be $(v(b,\bullet) - \tau)^+$ and Φ be the solution of (2.15). Then $\Phi \geqslant 0$. If we define h as

$$h = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} (g(s^{\alpha_1/\beta} v) - g(s^{\alpha_1/\beta} y))/w,$$

then $h \geqslant 0$ and

$$\text{(2.19)} \qquad s^2 w_{ss} + \frac{1}{\beta^2}\Delta_{S^{N-1}} w = hw.$$

Henceforth Φ is a super-solution for (2.19) and $\Phi \geqslant w$. Using (2.16) with $\rho = \bar{v}(a)$, $\tau = \bar{v}(b)$ we get

$$(2.20)\quad v(s,\sigma)-y(s)\leqslant C_1\left\{\left(1+\frac{1}{\mathrm{Ln}\,(s/a)}\right)^{(N-1)/2}\|(v(a,\cdot)-\bar v(a))^+\|_{L^2(S^{N-1})}\right.$$
$$\left.+\left(1+\frac{1}{\mathrm{Ln}\,(b/s)}\right)^{(N-1)/2}\|(v(b,\cdot)-\bar v(b))^+\|_{L^2(S^{N-1})}\right\}$$

for any $\sigma\in S^{N-1}$ and any $a<s<b$.

If we take $\dfrac{s}{a}=\dfrac{b}{s}=2$ and use estimate (2.4) in the s variable, we get

$$(2.21)\qquad v(s,\sigma)-y(s)\leqslant C_2 s^{(\beta+\sqrt{N^2-4c})/2\beta}.$$

In the same way we have

$$(2.22)\qquad y(s)-v(s,\sigma)\leqslant C_2 s^{(\beta+\sqrt{N^2-4c})/2\beta},$$

which implies the claim.

Lemma 2.4. *Assume the hypotheses of Lemma* 2.1. *hold and* $c=(N-2)^2/4$. *Then the function* w *introduced in* (2.12) *satisfies*

$$(2.23)\qquad \|w(t,\cdot)-\bar w(t)\|_{L^\infty(S^{N-1})}\leqslant \tilde c e^{-\sqrt{N-1}\,t}$$

for some $\tilde c>0$ *and any* $t>0$.

PROOF. It is essentially the same at the one of Lemma 2.3 except that Lemma 2.2 is replaced by the following estimate: for $a>0$ and $\phi\in L^2(S^{N-1})$ the unique bounded solution Φ of

$$(2.24)\qquad \begin{cases}\Phi_{tt}+\Delta_{S^{N-1}}\Phi=0 & \text{in}\quad (a,+\infty)\times S^{N-1},\\ \Phi(a,\cdot)=\phi(\cdot) & \text{on}\quad S^{N-1}.\end{cases}$$

satisfies

$$(2.25)\qquad \|\Phi(t,\cdot)\|_{L^\infty(S^{N-1})}\leqslant \tilde c\left(1+\frac{1}{t-a}\right)^{(N-1)/2}\|\phi\|_{L^2(S^{N-1})}.$$

This is essentially Poisson's formula.

PROOF OF THEOREM 2.1. *Case* 1. $c<\left(\dfrac{N-2}{2}\right)^2$.

The proof follows the ideas of [24] and we have to distinguish according $\bar v$ is bounded or not near 0

Case 1.1. $\bar{v}$ is bounded near 0. There exist a sequence $\{s_n\}$ tending to 0 and some z such that $\bar{v}(s_n)$ converges to z as $n \to \infty$. Assuming $z > 0$, we write $\tilde{g}(r) = g(r) - g(0)$ and call $\tilde{v}$ the solution of

$$\text{(2.26)} \quad \begin{cases} s^2\tilde{v}_{ss} + \dfrac{1}{\beta^2}(s^{2/\beta}V(s^{1/\beta}) - c)\tilde{v} = \dfrac{1}{\beta^2}s^{(2-\alpha_1)/\beta}\tilde{g}(s^{\alpha_1/\beta}\tilde{v}) \quad \text{on} \quad (s_n, s_{n_0}), \\ \tilde{v}(s_n) = \tilde{v}(s_{n_0}) = z/2, \end{cases}$$

where n_0 is such that $v(s_n, \sigma) > z/2$ for all $n \geqslant n_0$, and $\sigma \in S^{N-1}$. It is clear that $\tilde{v} \geqslant 0$. If Λ is the solution of

$$\text{(2.27)} \quad \begin{cases} s^2\Lambda_{ss} + \dfrac{1}{\beta^2}(s^{2/\beta}V(s^{1/\beta}) - c)\Lambda + \dfrac{1}{\beta^2}s^{(2-\alpha_1)/\beta}|g(0)| = 0 \quad \text{on} \quad (s_n, s_{n_0}), \\ \Lambda(s_n) = \Lambda(s_{n_0}) = 0, \end{cases}$$

then $\Lambda \geqslant 0$ and $\Lambda(s) \leqslant Ks$ for some constant independent of n. If we set $v^* = \tilde{v} - \Lambda$, then v^* is a sub-solution for (2.7) which implies

$$\text{(2.28)} \qquad v(s, \sigma) \geqslant v^*(s) \geqslant -Ks \qquad (\text{for every} \quad (s, \sigma) \in [s_n, s_{n_0}] \times S^{N-1}),$$

and $v_k(s, \sigma) = v(s, \sigma) + Ks$ is nonnegative in $(0, s_{n_0}] \times S^{N-1}$. As the spherical average $\bar{v}_K$ of v_K satisfies

$$\text{(2.29)} \quad s^2(\bar{v}_K)_{ss} + \frac{1}{\beta^2}(s^{2/\beta}V(s^{1/\beta}) - c)\bar{v}_K \geqslant \frac{1}{\beta^2}s^{(2-\alpha_1)/\beta}g(-K) + \frac{K}{\beta^2}(s^{2/\beta}V(s^{1/\beta}) - c)s,$$

there exist two constants M and $N \geqslant 0$ such that the function $E(s) = \bar{v}_K(s) + Ms^{(2-\alpha_1)/\beta} + N(s\,\mathrm{Ln}\,s - s)$ is convex. As $E(s_n)$ tends to z we deduce $\lim_{s\to 0} E(s) = z$, which yields

$$\text{(2.30)} \qquad \lim_{s\to 0} \bar{v}(s) = z = \lim_{s\to 0} v(s, \cdot),$$

uniformly on S^{N-1}.

If $z < 0$ we proceed similarly. If $z = 0$, then it is clear by using the technique above that

$$\text{(2.31)} \qquad \lim_{s\to 0} \bar{v}(s) = 0 = \lim_{s\to 0} \|v(s, \cdot)\|_{L^\infty(S^{N-1})}.$$

Case 1.2. $\bar{v}$ is unbounded near 0. Then there exists a sequence $\{s_n\}$ tending to 0 such that $\lim_{s \to 0} \bar{v}(s_n) = \infty$ ($-\infty$ in the same way). We conclude by the same convexity argument as in Case 1.1 that

$$\lim_{s \to 0} v(s, \sigma) = \infty \tag{2.32}$$

uniformly on S^{N-1}.

Case 2. $c = \left(\frac{N-2}{2}\right)^2$. We essentially follow the ideas of Case 1 but use the t variable ($t > 0$) and Lemma 2.4. If $\bar{w}$ is bounded in $\mathbb{R}^+$ then

$$\lim_{|x| \to 0} u(x)/\mu_1(x) = 0. \tag{2.33}$$

If $\bar{w}$ is not bounded we deduce from convexity arguments that

$$\text{either} \quad \lim_{t \to \infty} w(t, \cdot) = +\infty, \quad \text{or} \quad \lim_{t \to \infty} w(t, \cdot) = -\infty, \tag{2.34}$$

uniformly on S^{N-1}. Assuming the first case we also have from convexity the fact that $\bar{v}(t)/t$ admits a limit in $\mathbb{R}^+ \cup \{+\infty\}$. This limit is the same as the one of $u(x)/\mu_1(x)$ as x tends to zero and this ends the proof.

Remark 2.1. It is interesting to notice that (2.3) is automatically satisfied as soon as g has a fast enough growth, that is

$$\begin{cases} \liminf_{r \to \infty} g(r)/r^q = \infty, \\ \limsup_{r \to -\infty} g(r)/(-r)^q = -\infty, \end{cases} \tag{2.35}$$

for some $q > 1$ such that

$$c + \frac{q+1}{q-1}\left(\frac{q+1}{q-1} - N\right) = 0. \tag{2.36}$$

In that case we have

$$\frac{2}{q-1} = \frac{N - 2 + \sqrt{N^2 - 4c}}{2}$$

and

$$u(x) = o(|x|^{-2/(q-1)})$$

from (2.35) and Lemma 1.1. In the zero potential case the limit exponent q is $(N+1)/(N-1)$.

As we have proved Theorem 2.1 we can prove a similar result for the solution of (2.2) in an exterior domain G

Theorem 2.2. *Assume* $G \supset \{x \in \mathbb{R}^N \colon |x| \geqslant 1\}$ *and* $u \in C^1(G)$ *satisfies* (2.2) *in* G *where* V *is a radial potential defined in* G *and satisfying*

$$(2.37)\qquad -\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2, \quad \textit{for every} \quad x \in G.$$

If $g(0) = 0$ *and* u *satisfies*

$$(2.38)\qquad \liminf_{r \to \infty} r^{(N-2-\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0,$$

then $u(x)/\mu_2(x)$ *admits a limit in* $\mathbb{R} \cup \{\infty, -\infty\}$ *as* $|x|$ *tends to infinity. Moreover, if* $\lim_{|x| \to \infty} u(x)/\mu_2(x) = 0$, *there exists* $\gamma \in \mathbb{R}$ *such that*

$$(2.39)\qquad \lim_{|x| \to \infty} u(x)/\mu_1(x) = \gamma.$$

The zero potential case of this result can be found in [31]. We can apply this type of methods to symmetry problems as in [28].

Corollary 2.1. *Assume* V *is a radial potential defined in* $\mathbb{R}^N \setminus \{0\}$ *and satisfying*

$$(2.40)\qquad -\infty < |x|^2 V(x) \leqslant c \leqslant \frac{N^2}{4}, \quad \textit{for every} \quad x \in \mathbb{R}^N \setminus \{0\},$$

and g *is a nondecreasing real valued function. If* $u \in C^1(\mathbb{R}^N \setminus \{0\})$ *satisfies*

$$(2.41)\qquad \liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0,$$

$$(2.42)\qquad \liminf_{r \to \infty} r^{(N-2-\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0;$$

then u *is a radial function.*

It is important to notice that the hypothesis on V is weaker as the proof essentially deals with the study of the following differential inequality

$$(2.42)\qquad X_{rr} + \frac{N-1}{r} X_r + \left(V - \frac{N-1}{r^2} \right) X \geqslant 0.$$

Other symmetry results for Schrödinger operator with singular radial potentials can be found in [28], [5].

3. Solution with Linear Singularities

We assume that Ω and Ω^* are as in Section 1 and g is a continuous nondecreasing real valued function.

PROOF OF THEOREM 3.1. We recall that we assume $c < (N-2)^2/4$.

Step 1. Suppose (0.24) is satisfied, that is

$$\int_1^\infty (g(t) + |g(-t)|)t^{2(1-\alpha_1)/\alpha_1}\,dt < \infty, \tag{3.1}$$

then we claim that for any $\gamma \in \mathbb{R}$ there exists $u \in C^1(\Omega^*)$ satisfying

$$-\Delta u - \frac{c}{|x|^2}u + g(u) = 0 \tag{3.2}$$

in Ω^* and

$$\lim_{x\to 0} u(x)/\mu_1(x) = \gamma. \tag{3.3}$$

We take $\gamma \geqslant 0$ and for $\epsilon > 0$ let y_ϵ be the solution of

$$\begin{cases} s^2(y_\epsilon)_{ss} = \dfrac{1}{\beta^2}(s+\epsilon)^{(2-\alpha_1)/\beta}g((s+\epsilon)^{\alpha_1/\beta}y_\epsilon) & \text{in} \quad (0,1), \\ y_\epsilon(0) = \gamma, \qquad y_\epsilon(1) = 0. \end{cases} \tag{3.4}$$

In order to avoid technical difficulties we suppose $g(0) = 0$. Henceforth y_ϵ is positive, convex, nondecreasing and

$$y_\epsilon(s) \leqslant \gamma + s(1-\gamma), \quad \text{for all} \quad 0 < s < 1. \tag{3.5}$$

From (3.4) we get

$$(y_\epsilon)_s(s) = (y_\epsilon)_s(1) - \frac{1}{\beta^2}\int_s^1 (\tau+\epsilon)^{(2-\alpha_1)/\beta - 2}g((\tau+\epsilon)^{\alpha_1/\beta}y_\epsilon)\,d\tau \tag{3.6}$$

for $0 < s \leqslant 1$, and $(y_\epsilon)_s(1)$ is bounded from (3.4)-(3.5). From (3.6) we deduce

$$|y_\epsilon(s_1) - y_\epsilon(s_2)| \leqslant a(s_2 - s_1) + \frac{1}{\beta^2}\int_{s_1}^{s_2}\int_s^1 (\tau+\epsilon)^{(2-\alpha_1)/\beta-2}g((\tau+\epsilon)^{\alpha_1/\beta}y_\epsilon)\,d\tau \tag{3.7}$$

for some constant $a > 0$ and $0 < s_1 < s_2 \leqslant 1$. But

(3.8) $$\int_{s_1}^{s_2}\int_s^1 (\tau+\epsilon)^{(2-\alpha_1)/\beta-2} g((\tau+\epsilon)^{\alpha_1/\beta} y_\epsilon)\, d\tau\, ds \leqslant \int_{s_1}^{s_2+\epsilon}\int_s^2 t^{(2-\alpha_1)/\beta-2} g(\gamma t^{\alpha_1/\beta})\, dt\, ds$$

(we assume $\epsilon < 1$). Set

$$\phi(x) = \int_x^2\int_s^2 t^{(2-\alpha_1)/\beta-2} g(\gamma t^{\alpha_1/\beta})\, dt\, ds,$$

then

(3.9) $$\lim_{x\to 0}\phi(x) = \int_0^2\int_s^2 t^{(2-\alpha_1)/\beta-2} g(\gamma t^{\alpha_1/\beta})\, dt\, ds = l\int_{\gamma 2^{\alpha_1/\beta}}^{\infty} t^{2(1-\alpha_1)/\alpha_1} g(t)\, dt < \infty$$

from hypothesis ($l = l(\alpha_1, \beta) > 0$). Henceforth ϕ is extendable to $[0, 2]$ as a uniformly continuous function $\tilde{\phi}$ and (3.7) reads as

(3.10) $$|y_\epsilon(s_1) - y_\epsilon(s_2)| \leqslant a|s_2 - s_1| + \frac{1}{\beta^2}(\tilde{\phi}(s_2+\epsilon) - \tilde{\phi}(s_1+\epsilon))$$

which implies the equicontinuity of $\{y_\epsilon\}_{0<\epsilon<1}$ in $C([0, 1])$ and the existence of a $y \in C([0, 1])$ satisfying

(3.11) $$\begin{cases} s^2 y_{ss} = \dfrac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} y) & \text{in} \quad (0, 1], \\ y(0) = \gamma, \qquad y(1) = 0. \end{cases}$$

The function $u_\gamma(x) = |x|^{\alpha_1} y(|x|^\beta)$ is a solution of (3.2) satisfying (3.3).

Step 2. We assume that there exists $\gamma > 0$ such that (3.3) holds for some $u \in C^1(\Omega^*)$ and that

(3.11) $$\int_1^\infty t^{2(1-\alpha_1)/\alpha_1} g(t)\, dt = \infty.$$

As $\lim_{r\to 0} \bar{u}(r)/\mu_1 = \gamma$ and

(3.12) $$\bar{u}_{rr} + \frac{N-1}{r}\bar{u}_r + \frac{c}{r^2}\bar{u} = \overline{g(u(r))} \quad \text{in} \quad (0, 1]$$

we deduce, from the monotonicity of g, that

$$\bar{u}_{rr} + \frac{N-1}{r}\bar{u}_r + \frac{c}{r^2}\bar{u} \geqslant g(\gamma r^{\alpha_1}/2). \tag{3.13}$$

Defining $\psi(s)$ by $\bar{u}(r)/r^{\alpha_1}$ with $s = r^\beta$, then

$$s^2\psi_{ss} \geqslant \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(\gamma s^{\alpha_1/\beta}/2). \tag{3.14}$$

Integrating (3.14) twice yields

$$\psi(s) \geqslant \psi(1) + \psi_s(1)(s-1) + \frac{1}{\beta^2}\int_s^1\int_t^1 \sigma^{(2-\alpha_1)/\beta - 2} g(\gamma\sigma^{\alpha_1/\beta})\, d\sigma\, dt. \tag{3.15}$$

As

$$\lim_{s\downarrow 0}\int_s^1\int_t^1 \sigma^{(2-\alpha_1)/\beta-2} g(\gamma\sigma^{\alpha_1/\beta})\, d\sigma\, dt = \infty$$

we derive

$$\lim_{s\to 0}\psi(s) = \lim_{r\to 0}\bar{u}(r)/r^{\alpha_1} = \infty, \tag{3.16}$$

a contradiction.

Remark 3.1. With the above techniques it is easy to show that if Ω is bounded with a regular boundary $\partial\Omega$, for any $\phi \in C(\partial\Omega)$ and any $\gamma \in \mathbb{R}$ there exists a unique $u_\gamma \in C(\bar{\Omega}\setminus\{0\}) \cap C^1(\Omega^*)$ satisfying

$$\begin{cases} -\Delta u_\gamma - \dfrac{c}{|x|^2}u_\gamma + g(u_\gamma) = 0 \quad \text{in} \quad \Omega^*, \\ u_\gamma = \phi \quad \text{on} \quad \partial\Omega, \qquad \lim_{x\to 0} u_\gamma(x)/\mu_1(x) = \gamma. \end{cases} \tag{3.17}$$

PROOF OF THEOREM 3.2. Here we assume that $c = (N-2)^2/4$. We recall the definition of b_g^+, b_g^-:

$$\begin{cases} b_g^+ = \inf\left\{b > 0 : \displaystyle\int_0^1 g(t^{-(N-2)/(N+2)}\,\mathrm{Ln}\,(1/t)/b)\, dt < \infty\right\}, \\ b_g^- = \inf\left\{b > 0 : \displaystyle\int_0^1 g(t^{-(N-2)/(N+2)}\,\mathrm{Ln}\, t/b)\, dt > -\infty\right\}. \end{cases}$$

Step 1. *Existence result*. It is clear that if $[-(N+2)/2b_g^-, (N+2)/2b_g^+] = \{0\}$ there exists u satisfying the equation (0.27) with a zero limit in (0.28); so we shall assume $b_g^+ < \infty$, and consider any $\gamma \in (0, (N+2)/2b_g^+]$.

Case 1. $\gamma < (N+2)/2b_g^+$. For $\epsilon > 0$ let y_ϵ be the solution of

$$(3.19)\quad \begin{cases} (y_\epsilon)_{ss} = (s+\epsilon)^{-3}\exp\left(-\dfrac{N+2}{2(s+\epsilon)}\right) g\left(\dfrac{y_\epsilon}{s+\epsilon}\right)\exp\left(\dfrac{N-2}{2(s+\epsilon)}\right) \text{ on } [0,1], \\ y_\epsilon(0) = \gamma, \qquad y_\epsilon(1) = 0. \end{cases}$$

We assume again that $g(0) = 0$; y_ϵ is decreasing, positive and convex; therefore

$$(3.20)\quad |y_\epsilon(s_1) - y_\epsilon(s_2)| \leqslant a|s_1 - s_2| + \int_{s_1+\epsilon}^{s_2+\epsilon}\int_s^2 t^{-3}e^{-(N+2)/2t} g(\gamma e^{-(N-2)/2t}t^{-1})\,dt\,ds,$$

for $0 < s_1 < s_2 < 1$ $(\epsilon < 1)$. Let ϕ be defined by

$$(3.21)\qquad \phi(x) = \int_x^2 \int_s t^{-3}e^{-(N+2)/2t} g(\gamma e^{-(N-2)/2t}t^{-1})\,dt\,ds,$$

then

$$(3.22)\quad \lim_{x\to 0}\phi(x) = l\int_0^{e^{-(N+2)/4}} g((2\gamma/(N+2))t^{-(N-2)/(N+2)}\operatorname{Ln}(1/t))\,dt < \infty.$$

As in the proof of Theorem 3.1, $\{y_\epsilon\}$ is equicontinuous in [0, 1] and there exists $y \in C([0,1])$ such that

$$(3.23)\qquad \begin{cases} y_{ss} = s^{-3}\exp(-(N+2)/2s) g(ye^{-(N-2)/2s}s^{-1}) \quad \text{on} \quad (0,1), \\ y(0) = \gamma, \qquad y(1) = 0. \end{cases}$$

If we set

$$u_\gamma(x) = |x|^{-(N-2)/2}\operatorname{Ln}(1/|x|)y(-1/\operatorname{Ln}|x|),$$

then

$$(3.24)\qquad \begin{cases} -\Delta u_\gamma - \left(\dfrac{N-2}{2|x|}\right)^2 u_\gamma + g(u_\gamma) = 0 \quad \text{in} \quad B_{e^{-1}}(0)\backslash\{0\}, \\ u_\gamma(x) = 0 \quad \text{on} \quad \partial B_{e^{-1}}(0), \\ \lim\limits_{x\to 0} u_\gamma(x)/\mu_1(x) = \gamma. \end{cases}$$

Case 2. $\gamma = (N+2)/2b_g^+$. Let y_n be the solution of

$$(3.25)\qquad \begin{cases} (y_n)_{ss} = s^{-3}\exp(-(N+2)/2s)g(y_n e^{-(N-2)/2s}s^{-1}) & \text{on} \quad (0,1), \\ y_n(0) = (N+2)/(2b_g^+) - \dfrac{1}{n}, \qquad y_n(1) = 0. \end{cases}$$

The function y_n is again decreasing, positive and convex and the sequence $\{y_n\}$ is increasing and bounded. From Dini's Theorem it is uniformly convergent on (0, 1] and its limit y is continuous on [0, 1] and satisfies (3.23) with $\gamma = (N+2)/2b_g^+$.

Step 2. Assume there exists γ and a solution u of (0.27) such that

$$\lim_{x\to 0}\frac{u(x)}{\mu_1(x)} = \gamma$$

and we assume for example that

$$(3.26)\qquad \gamma > (N+2)/2b_g^+.$$

(we proceed similarly if $\gamma < -(N+2)/2b_g^-$). We define

$$(3.27)\qquad u(r,\sigma) = \frac{1}{t}\exp((N-2)/2t)v(t,\sigma), \qquad t = -1/\mathrm{Ln}\, r,$$

and v satisfies

$$(3.28)\qquad \begin{cases} v_{ss} + \Delta_{S^{N-1}}v = s^{-3}\exp(-(N+2)/2s)g(ve^{-(N-2)/2s}s^{-1}) & \text{in} \quad \Omega^*, \\ \lim_{s\to 0} v(s,\cdot) = \gamma \quad \text{uniformly on} \quad S^{N-1}. \end{cases}$$

We consider $\epsilon_0 \in (0, \gamma - (N+2)/2b_g^+)$ and set $\lambda = \gamma - \epsilon_0 > (N+2)/2b_g^+$. For s small enough we have $v(s,\sigma) \geqslant \lambda$ (for every $\sigma \in S^{N-1}$) and it is the same with the spherical average $\bar{v}(s)$. Therefore

$$(3.29)\qquad \bar{v}_{ss} \geqslant s^{-3}\exp(-(N+2)/2s)g(\lambda s^{-1}\exp((N-2)/2s)).$$

Integrating (3.29) twice as in step 1 and using the definition of b_g^+ implies

$$\lim_{s\to 0}\bar{v}(s) = +\infty,$$

a contradiction.

Remark 3.2. The Dirichlet problem is also solvable in the case $c = (N-2)^2/4$.

Remark 3.3. If $g(r)$ behaves like r^q $(q > 1)$ at infinity (and $-|r|^q$ at $-\infty$), (3.1) means that

$$0 < -\alpha_1 < \frac{2}{q-1} \quad \text{or} \quad 1 < q < \frac{N+2+\beta}{N-2+\beta} \tag{3.30}$$

and

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) < 0.$$

In the critical case $c = \left(\frac{N-2}{2}\right)^2$, the role of the Sobolev exponent $\frac{N+2}{N-2}$ is enlightened:

$$b_g^+, b_g^- = \begin{cases} \infty & \text{if} \quad q \geqslant (N+2)/(N-2), \\ 0 & \text{if} \quad 1 < q < (N+2)/(N-2). \end{cases} \tag{3.31}$$

Remark 3.4. Let u_γ be the solution on (3.17), for any γ if $c < (N-2)^2/4$ or if $c = (N-2)^2/4$ or if $c = (N-2)^2/4$ and $b_g^+ = 0$. Then the mapping $\gamma \mapsto u_\gamma$ is increasing. If we assume that

$$\int_A^\infty \frac{ds}{\sqrt{sg(s)}} < \infty \tag{3.32}$$

for some $A > 0$, u_γ is bounded above in $\bar{\Omega}\backslash\{0\}$ by a continuous function in $\bar{\Omega}\backslash\{0\}$. Then $u_\infty = \lim_{\gamma\to\infty} u_\gamma$ exists. In the case $g(r) = |r|^{q-1}r$ we shall prove in Section 4 that

$$\lim_{x\to 0} |x|^{2/(q-1)} u_\infty(x) = \left(c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)^{1/(q-1)} \tag{3.34}$$

Moreover u_∞ is the unique solution of (3.17) with $\gamma = \infty$ (see [31] for example).

4. The Power Case

In this Section we study the solutions of (0.3), that is

$$-\Delta u - \frac{c}{|x|^2} u + |u|^{q-1} u = 0 \tag{4.1}$$

in $B_1(0)\backslash\{0\}$ or in $\complement B_1(0)$ or in $\mathbb{R}^N\backslash\{0\}$. As some of the results are direct extensions of [13] and [7], we shall abreviate their proof.

PROOF OF THEOREM 4.1. It is clear from the classical energy method that in the case $q \neq (N+2)/(N-2)$, $r^{2/(q-1)}u(r, \bullet)$ converges in the $C^3(S^{N-1})$-topology to a compact connected subset of the set ξ of solutions of the following equation on S^{N-1}:

$$-\Delta_{S^{N-1}}\omega + |\omega|^{q-1}\omega = \left(c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)\omega, \tag{4.2}$$

(see [2], [7], [13], [26], [27]). Set

$$\lambda = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right),$$

then

(i) if $u \geqslant 0$, $\xi \cap C^+(S^{N-1})$ is reduced to 0 and $\lambda^{1/(q-1)}$ $(\lambda > 0)$,
(ii) if $0 < \lambda \leqslant N-1$, ξ is reduced to 0, $\lambda^{1/(q-1)}$ and $-\lambda^{1/(q-1)}$,
(iii) if q is an odd integer $r \mapsto r^q$ is a real analytic function and we can apply Bidaut-Veron-Veron and Simon's theorem [7], [8], [21].
(v) if ξ' is an hyperbolic limit manifold, that is for any $\omega \in \xi'$ and any $\psi \in C^2(S^{N-1})$ satisfying

$$-\Delta\psi + q|\omega|^{q-1}\psi - \left(c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)\psi = 0 \tag{4.3}$$

there exists a one-parameter family $\{\omega_s\}_{0 \leq s \leq 1}$ of elements of ξ' such that

$$\lim_{s \to 0} s^{-1}(\omega_s - \omega) = \psi \tag{4.4}$$

in $C^2(S^{N-1})$. We can use Simon's result [21, Theorem 6.6]. Henceforth we are left with (v): $N = 2$, $c \leqslant 1$.

Lemma 4.1. *Assume A is an open subset strictly included into*

$$\bar{B}_1^+(0)\backslash\{0\} = \{(r, \theta) \in \mathbb{R}^2 : 0 < r \leqslant 1, 0 \leqslant \theta \leqslant r\}$$

and assume $c \leqslant 1$. Then

$$\lambda_1(A) = \inf\left\{\frac{1}{2}\int_A \left(|\nabla\phi|^2 - \frac{c}{|x|^2}f^2\right)dx : \phi \in W_0^{1,2}(A)\right\} > 0.$$

PROOF. Let $A = \{(t, \theta) : (e^t, \theta) \in A\}$ and $\psi(t, \theta) = \phi(r, \theta)$ with $r = e^t$. Then

$$\int_A \left(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2 - \frac{c}{r^2}\phi^2\right) r\,dr\,d\theta = \int_A (\psi_t^2 + \psi_\theta^2 - c\psi^2)\,dt\,d\theta.$$

As $A \subset (\operatorname{Ln} a, \operatorname{Ln} b) \times (0, \pi)$ for some $0 < a < b < 1$ and as the first eigenvalue of $-\Delta$ in $W_0^{1,2}((\operatorname{Ln} a, \operatorname{Ln} b) \times (0, \pi))$ is $\pi^2\left(\frac{1}{\pi^2} + \frac{1}{(\operatorname{Ln} b/a)^2}\right)$ we deduce by the monotonicity property that

$$\lambda_1(A) \geqslant 1 - c + \frac{\pi^2}{(\operatorname{Ln} b/a)^2} > 0,$$

which is the claim.

We define v by (0.30); as $N = 2$, it satisfies

$$v_{tt} - \frac{4}{q-1} v_t + v_{\theta\theta} + \left(c + \left(\frac{2}{q-1}\right)^2\right) v - |v|^{q-1} v = 0 \tag{4.5}$$

in $(-\infty, 0) \times S^1$. As in [13] we are left with the situation where the α-limit set of the negative trajectory of $v(t, \cdot)$ defined by

$$\Gamma^- = \bigcap_{t<0} \overline{\bigcup_{\tau \leq t} v(t, \cdot)}^{C^3(S^1)} \tag{4.6}$$

is included into one of the non trivial-S^1-action connected component of the set of solutions of

$$\omega_{\theta\theta} + \left(c + \left(\frac{2}{q-1}\right)^2\right)\omega - |\omega|^{q-1}\omega = 0 \quad \text{on} \quad S^1, \tag{4.7}$$

that is

$$\Gamma^- \subset \{\omega(\cdot + \alpha) : \alpha \in S^1\}, \tag{4.8}$$

where ω is a solution of (4.7) with anti-period π/k ($k \in \mathbb{N}^*$). The following result is then an extension of [13 Lemma 1.6].

Lemma 4.2. *Let ω be an element of Γ^-. If $\omega_\theta(\theta_0) > 0$ (resp. < 0) at some $\theta_0 \in S^1$, then there exists $t^* \leqslant 0$ such that*

$$v_\theta(t, \theta_0) \geqslant 0 \qquad (\textit{resp.} \leqslant 0) \tag{4.9}$$

for any $t \leqslant t^*$.

PROOF. For proving it we may assume $\theta_0 = 0$ and define

$$\tilde{u}(r, \theta) = u(r, \theta) - u(r, -\theta); \tag{4.10}$$

then $\tilde{u}$ satisfies

$$(4.11) \qquad -\Delta\tilde{u} - \frac{c}{|x|^2}\tilde{u} + d(x)\tilde{u} = 0$$

in $B_1^+(0)\setminus\{0\}$ where $d(x) \geqslant 0$. Let us set

$$(4.12) \quad \theta^+ = \{x \in \bar{B}_1^+\setminus\{0\} : \tilde{u}(x) > 0\}; \qquad \theta^- = \{x \in \bar{B}_1^-\setminus\{0\} : \tilde{u}(x) < 0\}.$$

If C is a connected component of θ^+ or θ^-, we claim that

$$(4.13) \qquad 0 \in \partial C \quad \text{or} \quad C \cap \partial B_1(0) \neq \varnothing.$$

Assume the contrary; if C is such a component, there exists a, b such that $0 < a < b < 1$ and

$$(4.14) \qquad \bar{C} \subset \{(r,\theta) : a < r < b, \quad 0 \leqslant \theta \leqslant \pi\} = \Gamma_{a,b}^+.$$

Extending $\tilde{u}$ by 0 in $\Gamma_{a,b}^+\setminus C$ then the new function $\tilde{u}^e$ belongs to $W_0^{1,2}(\Gamma_{a,b}^+)$ and

$$(4.15) \qquad \int_{\Gamma_{a,b}^+}\left(|\nabla\tilde{u}^e|^2 - \frac{c}{|x|^2}(\tilde{u}^e)^2 + d(x)(\tilde{u}^e)^2\right) = 0.$$

Then $\tilde{u} = 0$ in C from Lemma 4.1, contradiction. The remaining of the proof of Lemma 4.2 goes as in [10 Lemma 1.6 (i)].

Remark 4.1. Using Lemma 4.2 and comparison principles implies that if $\omega_\theta(\theta) > 0$ for $\theta \in [\theta_0, \theta_1] \subset S^1$, then there exists $t^* \leqslant 0$ such that

$$(4.16) \qquad v_\theta(t,\theta) \geqslant 0, \qquad \int_{\theta_0}^{\theta_1} v_\theta(t,\theta)\,d\theta > 0$$

for any $t \leqslant t^*$, $\theta \in [\theta_0, \theta_1]$. However it is interesting to notice that the other assertions of [13, Lemma 1.6] do not hold for $0 < x \leqslant 1$ as they involve Neuman boundary data.

The remaining of the proof of Theorem 4.1 goes exactly as in [13, Theorem 1.1].

Remark 4.2. The potential $c|x|^{-2}$ of Theorem 4.1 can be replaced by a more general potential V such that $V \in C^{1+\epsilon}(\bar{B}_1(0)\setminus\{0\})$ and $r^2V(r,\bullet)$ converges to c as r tends to 0 in the $C^{1+\epsilon}(S^{N-1})$-topology. In the case (iv) we have also to assume: either $c < 1$ or $|x|^2V(x) \leqslant 1$ in some punctured neighborhood of 0.

It is clear that if

$$\lambda = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)$$

is nonpositive, the set ξ of the solutions of (4.2) is reduced to $\{0\}$ and from Theorem 1.2 u is described by μ_2. However, if $q > (N+2)/(N-2)$ and if $\lambda = 0$ we have

$$\alpha_1 < \alpha_2 = -2/(q-1). \tag{4.17}$$

The superposition of the linear and the nonlinear effect gives rise to the phenomena described in Theorem 4.2.

PROOF OF THEOREM 4.2. *Step* 1. *A priori estimate.* We claim that for any $\epsilon \in (0,1)$ there exists $K_\epsilon > 0$ such that

$$|u(x)| \leqslant L(N,q)(|x|^2 \operatorname{Ln}(1/|x|))^{-1/(q-1)}(1 + K_\epsilon(\operatorname{Ln}(1/|x|))^{\epsilon-1}) \tag{4.18}$$

where

$$L(N,q) = \left(\left(\frac{1}{q-1}\right)\left(N - 2\frac{q+1}{q-1}\right)\right)^{1/(q-1)}. \tag{4.19}$$

We use the function $v(t,\sigma)$ defined in (0.30) and v satisfies

$$v_{tt} + \left(N - 2\frac{q+1}{q-1}\right)v_t + \Delta_{S^{N-1}}v - v|v|^{q-1} = 0 \tag{4.20}$$

in $(-\infty, 0) \times S^{N-1}$ and $\lim_{t\to-\infty} v(t,\sigma) = 0$ uniformly on S^{N-1}.

Let $\psi(t) = L(N,q)(-t)^{-1/(q-1)} + M(-t)^{-\rho}$, $M, \rho > 0$, then

$$\begin{aligned} \psi_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\psi_t - \psi^q \\ &= L(N,q)\frac{q}{(q-1)^2}(-t)^{-2-1/(q-1)} \\ &\quad + \rho(\rho+1)M(-t)^{-\rho-2} + L^q(N,q)(-t)^{-q/(q-1)} \\ &\quad + M\rho\left(N - 2\frac{q+1}{q-1}\right)(-t)^{-\rho-1} \\ &\quad - L^{q-1}(N,q)(-t)^{-q/(q-1)} - qML^{q-1}(N,q)t^{-\rho-1} \\ &\quad + o(t^{-\rho-1}). \end{aligned} \tag{4.21}$$

If we choose $\dfrac{1}{q-1} < \rho < \dfrac{q}{q-1}$ then $-2 - \dfrac{1}{q-1} < -\rho - 1$ and

$$M_\rho\left(N - 2\frac{q+1}{q-1}\right) < qML^{q-1}(N, q). \tag{4.22}$$

Henceforth, there exists $T < 0$ such that

$$\psi_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\psi_t - \psi^q \leqslant 0 \tag{4.23}$$

in $(-\infty, T) \times S^{N-1}$. Choosing M large enough we conclude that $v \leqslant \psi$. Arguing similarly for the negative part of v yields (4.18), (4.19).

Step 2. *End of the proof.* Let us define

$$\zeta(t, \sigma) = (-t)^{1/(q-1)} v(t, \sigma). \tag{4.24}$$

ζ is bounded in $(-\infty, -1] \times S^{N-1}$ where it satisfies

$$\zeta_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\zeta_t + \Delta_{S^{N-1}}\zeta - \frac{2}{t(q-1)}\zeta_t + \frac{q}{t^2(q-1)}\zeta + \frac{1}{t}(|\zeta|^{q-1}\zeta - L^{q-1}(N, q)\zeta) = 0. \tag{4.25}$$

From Agmon-Douglis-Niremberg [15] and Schauder theory all the derivatives $(\partial^\alpha/\partial t^\alpha)\nabla_\beta\zeta$ up to the order 3 are uniformly bounded in $(-\infty, -1] \times S^{N-1}$. Henceforth the α-limit set Γ^- of the trajectory of $\zeta(t, \cdot)$, $t \leqslant -1$, is a non-empty compact subset of $C^2(S^{N-1})$. Multiplying (4.25) by ζ_t and integrating over $(-\infty, -1] \times S^{N-1}$ yields

$$\int_{-\infty}^{-1}\int_{S^{N-1}} \zeta_t^2 \, d\sigma \, dt < \infty, \tag{4.26}$$

after some easy integrations by parts. This immediately implies

$$\int_{-\infty}^{-1}\int_{S^{N-1}} \zeta_{tt}^2 \, d\sigma \, dt < \infty.$$

The uniform continuity of ζ_t and ζ_{tt} yields

$$\lim_{t \to -\infty} \|\zeta_t(t, \cdot)\|_{L^2(S^{N-1})} = \lim_{t \to -\infty} \|\zeta_{tt}(t, \cdot)\|_{L^2(S^{N-1})} = 0. \tag{4.27}$$

If we multiply (4.25) by $\phi \in C^{\infty}(S^{N-1})$ and take some element $l \in \Gamma^{-}$ we get

$$\int_{S^{N-1}} l \, \Delta\phi \, d\sigma = 0. \tag{4.28}$$

Henforth Γ^{-} is reduced to a constant l which satisfies

$$l \in [-L(N,q), L(N,q)], \tag{4.29}$$

from (4.18). If we integrate (4.25) on $(t, -1) \times S^{N-1}$ we get

$$\int_{t}^{-1} \int_{S^{N-1}} \frac{1}{T} (L^{q-1}(N,q) - \zeta^{q-1})\zeta \, d\sigma \, dt = \Phi(t) \tag{4.30}$$

where $\Phi(t)$ admits a limit as t tends to $-\infty$. Henceforth it is the same with the left-hand side of (4.30) and l must satisfy

$$(L^{q-1}(N,q) - l^{q-1})l = 0 \tag{4.31}$$

which ends the proof.

Remark 4.3. A similar argument can be found in [2] for the study of the isolated singularities of the solutions of

$$-\Delta u = u^{N/(N-2)} \qquad (u > 0) \tag{4.32}$$

or in [27] for the study of the long range behaviour of the solutions of

$$-\Delta u + |u|^{2/(N-2)}u = 0 \tag{4.33}$$

in an exterior domain.

When

$$\lim_{x \to 0} |x|^{2/(q-1)} u(x) = 0$$

we are usually in the situation where the behaviour of u near 0 is essentially of linear type. If we look for solutions of

$$\Delta\zeta + \frac{c}{|x|^2}\zeta = 0 \tag{4.34}$$

in $B_1(0)\setminus\{0\}$ under the form

$$\zeta(r,\sigma) = y(r)\phi(\sigma), \tag{4.35}$$

we find out that ϕ must be an eigenfunction of $-\Delta_{S^{N-1}}$ with corresponding eigenvalue $\lambda_k = k(k+N-2)$ and y must satisfy

$$r^2 y_{rr} + r(N-1)y_r + (c-\lambda_k)y = 0 \tag{4.36}$$

($k \in \mathbb{N}$); the corresponding characteristic equation is

$$X^2 + (N-2)X + c - \lambda_k = 0, \tag{4.37}$$

with discriminant $\delta_k = (N-2+2k)^2 - 4c$. If $\delta_k \geqslant 0$ (4.36) admits two fundamental solutions with constant sign

$$\mu_1^k(x) = \begin{cases} |x|^{-(N-2+\beta_k)/2} & \text{if} \quad \delta_k > 0, \\ |x|^{-(N-2)/2} \operatorname{Ln}(1/|x|) & \text{if} \quad \delta_k = 0, \end{cases} \tag{4.38}$$

$$\mu_2^k(x) = \begin{cases} |x|^{-(N-2-\beta_k)/2} & \text{if} \quad \delta_k > 0, \\ |x|^{-(N-2)/2} & \text{if} \quad \delta_k = 0, \end{cases} \tag{4.39}$$

with $\beta_k = \sqrt{\delta_k}$. If $\delta_k < 0$ the space of solutions of (4.36) is generated by

$$\begin{cases} \nu_1^k(x) = |x|^{(2-N)/2} \cos\left(\sqrt{-\delta_k} \operatorname{Ln} \sqrt{r}\right), \\ \nu_2^k(x) = |x|^{(2-N)/2} \sin\left(\sqrt{-\delta_k} \operatorname{Ln} \sqrt{r}\right). \end{cases} \tag{4.40}$$

Surprisingly the case $q > (N+2)/(N-2)$ is simpler than the case $1 < q < (N+2)/(N-2)$.

Theorem 4.3. *Assume $q > (N+2)/(N-2)$ and that $-2/(q-1)$ is not a solution of* (4.37) *for some $k \in \mathbb{N}$. If u is a solution of* (4.1) *in $B_1(0) \setminus \{0\}$ such that*

$$\lim_{x \to 0} |x|^{2/(q-1)} u(x) = 0; \tag{4.41}$$

then we have the following alternative.

(i) *either there exists $l \in \mathbb{N}$ satisfying $\delta_l > 0$ and $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_l I)$, $\psi \neq 0$, such that*

$$\lim_{r \to 0} r^{(N-2-\beta_l)/2} u(r, \cdot) = \psi(\cdot) \tag{4.42}$$

in the $C^2(S^{N-1})$-topology,

(ii) *or $u \equiv 0$.*

PROOF. As $-2/(q-1)$ is not a root of (4.37) we can apply [13, Lemma 2.1]. Henceforth there exists $\epsilon < 0$ such that

$$|u(x)| \leqslant M|x|^{-2/(q-1)+\epsilon} \tag{4.43}$$

near 0. Let k_0 be the smallest integer such that $\delta_{k_0} > 0$; then, as in [13], we derive the estimate

$$|u(x)| \leqslant M|x|^{-(N-2-\beta_{k_0})/2} \tag{4.44}$$

near 0 and this estimate yields easily that there exists $\psi_0 \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_{k_0} I)$ such that

$$\lim_{x \to 0} r^{(N-2-\beta_{k_0})/2} u(r, \sigma) = \psi_0. \tag{4.45}$$

If $\psi_0 = 0$, then, as in [13], we obtain

$$|u(x)| \leqslant M|x|^{-(N-2-\beta_{k_0+1})/2}, \tag{4.46}$$

etc., and we carry on as above. If we assume that

$$\lim_{x \to 0} |x|^{(N-2-\beta_k)/2} u(x) = 0 \tag{4.47}$$

for any $k \in \mathbb{N}$, we conclude that $u \equiv 0$ from Aronszajn's unique continuation theorem [1].

If $1 < q < (N+2)/(N-2)$ we have $(N-2)/2 < 2/(q-1)$ and the properties of u will depend on the sign of $(N-2)^2 - 4c$.

Theorem 4.4. *Assume* $1 < q < (N+2)/(N-2)$, *that* $-2/(q-1)$ *is not a root of* (4.37) *for some* $k \in \mathbb{N}$ *and* $(N-2)^2 \geqslant 4c$. *If* u *is a solution of* (4.1) *in* $B_1(0)\setminus\{0\}$ *satisfying* (4.41); *then let* k_0 *be the largest integer such that*

$$(N - 2 + \beta_{k_0})/2 < 2/(q-1); \tag{4.48}$$

(i) *either there exist an integer* $k \in [0, k_0]$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that*

$$\lim_{r \to 0} u(r, \cdot)/\mu_1^k(r) = \psi(\cdot) \tag{4.49}$$

in the $C^2(S^{N-1})$*-topology,*

(ii) *or there exist an integer* $k \geqslant 0$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that* (4.49) *holds with* μ_2^k *instead of* μ_1^k,

(iii) *or* $u \equiv 0$.

The proof is the same as the one of Theorem 4.3.

Theorem 4.5. *Assume* $1 < q < (N+2)/(N-2)$, *that* $-2/(q-1)$ *is not a root of* (4.37) *for some* $k \in \mathbb{N}$ *and* $(N-2)^2 < 4c$. *If* u *is a solution of* (4.1) *in* $B_1(0)\setminus\{0\}$ *satisfying* (4.41), *let* $k_0 \geqslant 1$ *be the smallest integer such that* $\delta_{k_0} \geqslant 0$.

Case I. $2/(q-1) \geqslant (N-2+\beta_{k_0})/2$. *Let* $k_1 \geqslant k_0$ *be the largest integer such that*

$$(N-2+\beta_{k_1})/2 < 2/(q-1). \tag{4.50}$$

Then

(i) *either there exist an integer* $k \in [k_0, k_1]$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that* (4.49) *holds.*

(ii) *or there exists* k_0 *couples of functions* (ϕ_k, ψ_k) *both belonging to* $\operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *for* $k \in \mathbb{N} \cap [0, k_0 - 1]$, *one of the above functions at least being nonzero, such that*

$$\lim_{r\to 0}\Big\{ r^{(N-2)/2}u(r,\bullet) - \sum_{k=0}^{k_0-1}\Big(\cos\big(\sqrt{-\delta_k}\,\operatorname{Ln}\sqrt{r}\,\big)\phi_k + \sin\big(\sqrt{-\delta_k}\,\operatorname{Ln}\sqrt{r}\,\big)\psi_k\Big)\Big\} = 0 \tag{4.51}$$

in the $C^2(S^{N-1})$*-topology,*

(iii) *or there exist an integer* $k \geqslant k_0$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that* (4.49) *holds with* μ_2^k *instead of* μ_1^k,

(iv) *or* $u \equiv 0$.

Case II. $(N-2-\beta_{k_0})/2 > 2/(q-1)$. *Only the parts* (ii), (iii) *and* (iv) *of Case I hold.*

PROOF. As in the proof of Theorem 4.2 the (i) of Case I is clear as $-2/(q-1)$ is not a root of (4.37). Henceforth we may assume that

$$|u(x)| \leqslant M|x|^{(2-N)/2} \tag{4.52}$$

and define

$$w(t,\sigma) = r^{(N-2)/2}u(r,\sigma), \qquad t = \operatorname{Ln} r. \tag{4.53}$$

Therefore w satisfies

$$w_{tt} + \Delta_{S^{N-1}}w + \left(c - \frac{(N-2)^2}{4}\right)w + e^{(q-1)(N-2)t/2}w|w|^{q-1} = 0 \tag{4.54}$$

in $(-\infty, 0] \times S^{N-1}$ where it stays bounded. Let w^k be the projection of w onto $\operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ for $0 \leqslant k \leqslant k_0 - 1$. Then w_k satisfies

$$w_{tt}^k + \left(c - \frac{(N-2-2k)^2}{4}\right)w^k + e^{(q-1)(N-2)t/2} f_k = 0 \tag{4.55}$$

where f_k is bounded, and it is easy to check that

$$\lim_{t \to \infty} \left(w^k(t) - \cos\left(\sqrt{-\delta_k}\, t/2\right)\phi_k - \sin\left(\sqrt{-\delta_k}\, t/2\right)\psi_k\right) = 0 \tag{4.56}$$

for some ϕ_k, ψ_k in $\operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$. As the α-limit set of the trajectory of $(w(t, \cdot))_{t \leq 0}$ is included into the direct sum of the $\operatorname{Ker}(\Delta_{S^{n-1}} + \lambda_k I)$ for $k = 0, \ldots, k_0 - 1$, we get (ii). The remaining of the proof is as in Theorem 4.2.

Remark 4.4. If $N = 2$ and $c \leqslant 0$, Theorem 4.4 holds for any $q > 1$.

Similar types of results (with some times many cases to examine) hold for the exterior problem. We just give the basic ones corresponding to Theorem 4.1-4.2.

Theorem 4.6. *Assume* $q \in (1, \infty) \setminus \{(N+2)/(N-2)\}$ *and* u *is a solution of* (4.1) *in* $G \supset \{x: |x| \geqslant 1\}$. *Then* $r^{2/(q-1)}u(r, \cdot)$ *converges in the* $C^3(S^{N-1})$-*topology to some compact connected subset of the set* ξ *of the* $C^3(S^{N-1})$-*functions satisfying* (4.2). *Moreover there exists precisely one* $\omega \in \xi$ *such that*

$$\lim_{r \to 0} r^{2/(q-1)} u'r, \cdot) = \omega(\cdot), \tag{4.57}$$

at least if one of the conditions (i)-(v) *of Theorem* 4.1 *is fulfilled.*

Theorem 4.7. *Assume* $1 < q < (N+2)/(N-2)$, $0 < c \leqslant (N-2)/2)^2$ *and* $\lambda = 0$. *If* u *is any solution of* (4.1) *in* $G \supset \{x: |x| \geqslant 1\}$, *then the following limit exists*

$$\lim_{|x| \to \infty} u(x)/(\mu_1(x)(\operatorname{Ln}|x|)^{\alpha_1/2}) = \tilde{l} \tag{4.58}$$

with

$$\alpha_1 = \frac{2 - N + \sqrt{(N-2)^2 - 4c}}{2} = -\frac{2}{q-1}$$

and

$$\tilde{l} \in \{0, \pm((2(q+1) - N(q-1))/(q-1)^2)^{1/(q-1)}\}. \tag{4.59}$$

From Theorems 4.1 and 4.6 we know that a global solution of (4.1) in $\mathbb{R}^N \setminus \{0\}$ satisfies

$$\lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) \in \xi_-, \qquad \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) \in \xi_+ \tag{4.60}$$

where ξ_- and ξ_+ are two compact connected subsets of ξ. If we define

$$E(\eta) = \int_{S^{N-1}} \left(\frac{1}{2} |\nabla \eta|^2 + \frac{1}{q+1} |\eta|^{q+1} - \frac{\lambda}{2} \eta^2 \right) d\sigma, \tag{4.61}$$

then $E_{|\xi_-} = E_-$, $E_{|\xi_+} = E_+$ and

$$\left(N - 2\frac{q+1}{q-1} \right) \int_{-\infty}^{\infty} \int_{S^{N-1}} v_t^2 d\sigma\, \delta\tau = E_+ - E_-, \tag{4.62}$$

where we have used the notations of (0.30). This relation tells us what are the set of elements of ξ for which a connecting orbit may exist. The way to constructing connecting orbits is to go through a semiflow as in [13] and to constructing such a semiflow we need an existence and uniqueness result for some initial boundary value problem.

Theorem 4.8. *Assume $1 < q$. Then for any $\phi \in C(\partial B_1(o))$ there exists a unique $u \in C(\complement B_1(0)) \cap C^3(\complement \bar{B}_1(0))$ satisfying*

$$-\Delta u - \frac{c}{|x|^2} u + u|u|^{q-1} = 0 \tag{4.63}$$

in $\complement(\bar{B}_1(0))$ and $u = \phi$ on $\partial B_1(0)$ if one of the two following conditions is fulfilled

(I) *$\lambda > 0$, u and ϕ are nonnegative, and either $1 < q \leqslant (N+2)/(N-2)$, or $q > (N+2)/(N-2)$ and $c > (N-2)^2/4$,*
(II) *either $c \leqslant 0$, or $0 < c \leqslant (N-2)^2/4$ and $1 < q \leqslant (N+2)/(N-2)$.*

PROOF. *Case I-Step* 1. *Uniqueness.* If $\lambda > 0$ and $u \geqslant 0$, we know, from Theorem 4.1 if $q \neq (N+2)/(N-2)$ or from [17] when $q = (N+2)/(N-2)$, that

$$\lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) = L \in \{0, \lambda^{1/(q-1)}\}. \tag{4.64}$$

If $\phi \neq 0$, $u > 0$ in $\bar{B}_1(0)$ from the strong maximum principle. If $L = 0$ we get

$$\bar{v}_{tt} + \left(N - 2\frac{q+1}{q-1} \right) \bar{v}_t + (\lambda - \epsilon(t))\bar{v} = 0, \tag{4.65}$$

where $\epsilon(t)$ is a positive function tending to 0 at infinity. Let $\tilde{\delta}$ be the discriminant of the equation with constant coefficients associated to (4.65)

$$\tilde{\delta} = \left(N - 2\frac{q+1}{q-1}\right)^2 - 4\lambda = (N-2)^2 - 4c. \tag{4.66}$$

If $1 < q < (N+2)/(N-2)$, 0 is a source and $\bar{v}$ cannot tend to 0 except if it is identically 0; if $q \geqslant (N+2)/(N-2)$, then $\tilde{\delta} < 0$ and any solution of (4.65) tending to 0 at infinity must oscillate around 0, a contradiction. Henceforth $L = \lambda^{1/(q-1)}$. If we linearize (0.31) at $\lambda^{1/(q-1)}$ we obtain the following equation

$$\phi_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\phi_t + \Delta_{S^{N-1}}\phi - (q-1)\lambda\phi = 0. \tag{4.67}$$

As this equation satisfies the maximum principle we deduce that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\| v(t, \cdot) - \lambda^{1/(q-1)} \|_{L^\infty(S^{N-1})} \leqslant C_\epsilon e^{-(\tau - \epsilon)t}, \tag{4.68}$$

where

$$\tau = \frac{1}{2}\left\{\left(N - 2\frac{q+1}{q-1}\right) + \sqrt{\left(N - 2\frac{q+1}{q-1}\right)^2 + 4\lambda(q-1)}\right\}, \tag{4.69}$$

which implies

$$\| u(r, \cdot) - \lambda^{1/(q-1)} r^{-2/(q-1)} \|_{L^\infty(S^{N-1})} \leqslant C_\epsilon r^{-2/(q-1) - \tau + \epsilon}. \tag{4.70}$$

Assume now u and $\hat{u}$ are two solutions of (4.63) with same initial data $\phi \geqslant 0$, u and $\hat{u} \geqslant 0$. Then for any $R > 1$ we have

$$-\int_{B_R(0)} \left(\frac{\Delta\hat{u}}{u} - \frac{\Delta\hat{u}}{\hat{u}}\right)(u^2 - \hat{u}^2) + \int_{B_R(0)} (|u|^{q-1} - |\hat{u}|^{q-1})(u^2 - v^2) = 0, \tag{4.71}$$

and

$$\int_{B_R(0)} \left(\frac{\Delta u}{u} - \frac{\Delta\hat{u}}{\hat{u}}\right)(u^2 - \hat{u}^2) = -\int_{\partial B_R(0)} \left(\frac{u_\nu}{u} - \frac{\hat{u}_\nu}{\hat{u}}\right)(u^2 - \hat{u}^2) + \int_{B_R(0)} \left(\left| \nabla u - \frac{u}{\hat{u}}\nabla\hat{u}\right|^2 + \left|\nabla\hat{u} - \frac{\hat{u}}{u}\nabla u\right|^2\right).$$

But from (4.70) we have

$$\int_{\partial B_R(0)} \left(\frac{u_\nu}{u} - \frac{\hat{u}_\nu}{\hat{u}}\right)(u^2 - \hat{u}^2) = O(R^{(N-2-4/(q-1)-\tau+\epsilon)}) \tag{4.72}$$

and

$$N-2-\frac{4}{q-1}-\tau=\frac{1}{2}\left\{N-2\frac{q+1}{q-1}-\sqrt{\left(N-2\frac{q+1}{q-1}\right)^2+4\lambda(q-1)}\right\}$$

which is negative. For ϵ small enough we deduce

$$(4.73)\quad \int_{B_R(0)}\left(\left|\nabla u-\frac{u}{\hat{u}}\nabla\hat{u}\right|^2+\left|\nabla\hat{u}-\frac{\hat{u}}{u}\nabla u\right|^2\right) + \int_{B_R(0)}(|u|^{q-1}-|\hat{u}|^{q-1})(u^2-v^2)=0,$$

which implies the uniqueness.

Step 2. *Existence.* We consider the following iterative scheme

$$(4.74)\quad \begin{cases} -\Delta u_n+u_n^q=\dfrac{c}{|x|^2}u_{n-1} & \text{in } \complement\bar{B}_1(0) \qquad (n\geqslant 1),\\ u_n=\phi & \text{on } \partial B_1(0)\\ u_0=0. \end{cases}$$

$\{u_n\}$ is increasing. For $\Lambda>\lambda^{1/(q-1)}$ the function $\psi_\Lambda(r)=\Lambda r^{-2/(q-1)}$ satisfies

$$(4.75)\quad -\Delta\psi_\Lambda+\psi_\Lambda^q\geqslant\frac{c}{|x|^2}\psi_\Lambda .$$

If we choose $\Lambda\geqslant\|\phi\|_{L^\infty(\partial B_1(0))}$ we deduce $\psi_\Lambda\geqslant u_1$ and finally $0<u_1<u_2<\cdots<u_n<\psi_\Lambda$. Clearly u_n converges to a solution u of (4.63) with initial value ϕ.

Case II. Step 1. *Uniqueness.* Let u and $\hat{u}$ be two solutions, $w=u-\hat{u}$, $v(s,\sigma)=w(r,\sigma)/\mu_1(r)$, $s=r^\beta$ (we assume $c<(N-2)^2/4$, the case $c=(N-2)^2/4$ is treated by the same technique, see Lemma 1.4); then

$$(4.76)\quad s^2(\|w(s,\cdot)\|_{L^2(S^{N-1})})_s\geqslant 0$$

in $D'(0,+\infty)$. As $\|w(s,\cdot)\|_{L^2(S^{N-1})}=o(s)$ at infinity, $w\equiv 0$.

Step 2. Existence. We approximate u by the solution of the following problem in $B_n(0)\backslash B_1(0)$

$$(4.77)\quad \begin{cases} -\Delta u_n-\dfrac{c}{|x|^2}u_n+|u_n|^{q-1}u_n=0 & \text{in } B_n(0)\backslash B_1(0),\quad n\geqslant 2,\\ u_n=\phi & \text{on } \partial B_1(0),\\ u_n=0 & \text{on } \partial B_n(0). \end{cases}$$

u_n is unique (see Step 1), uniformly bounded, therefore it is convergent to the desired u.

Remark 4.5. Using a phase plane analysis for the radial solutions of (0.31) we can see that Theorem 4.8 is optimal. It is also of some interest to notice that if $\lambda \leqslant 0$ there exists no positive solution of (4.63) in $\complement B_1(0)$: if u were such a solution, then $\lim_{r\to\infty} r^{2/(q-1)}u(r, \cdot) = 0$ from Theorem 4.1 if $q < (N+2)/(N-2)$ and [17] if $q = (N+2)/(N-2)$ and then

$$(4.78) \qquad \left(N - 2\frac{q+1}{q-1}\right)\int_0^\infty \int_{S^{N-1}} v_t^2 d\sigma\, dt = -E(\phi) + \frac{1}{2}\int_{S^{N-1}} v_t^2(0, \cdot)\, d\sigma$$

where we use the notations of (0.30) and (4.61), and $u(x) = \phi$ on $\partial B_1(0)$. If $q = (N+2)/(N-2)$ we deduce $E(\phi) = 0 \Rightarrow \phi = 0$. But we can replace $\phi = v(0, \cdot)$ by $v(T, \cdot)$ for any $T \geqslant 0$.

Remark 4.6. Thanks to Theorem 4.8 we can define a semiflow Φ on $X = C^+(S^{N-1})$ in Case I or on $C(S^{N-1})$ in Case II by the formula

$$(4.79) \qquad \Phi_t(\phi)(\cdot) = u(t, \cdot) \qquad (t \geqslant 0)$$

if u satisfies (4.63) in $\complement \bar{B}_1(0)$ and $u = \phi$ on $\partial B_1(0)$. Clearly Φ satisfies

(i) $\Phi_0 = I$,
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$,
(iii) $(t, \phi) \mapsto \Phi_t(\phi)$ is continuous in (t, ϕ).

The proof of those assertions is the same as the one of [13, Proposition 3.2]. Moreover Φ is strongly order preserving in the sense that being given ϕ_1 and ϕ_2 on X, $\phi_1 \geqslant \phi_2$, $\phi_1 \not\equiv \phi_2$, then for any $t > 0$ there exists $\delta > 0$ such that for any η_1, $\eta_2 \in X$, satisfying

$$\|\phi_1 - \eta_1\|_{C^0(S^{N-1})} \leqslant \delta,$$
$$\|\phi_2 - \eta_2\|_{C^0(S^{N-1})} \leqslant \delta$$

we have

$$(4.80) \qquad \begin{aligned} &\Phi_t(\eta_1) \geqslant \Phi_t(\eta_2), \\ &\Phi_t(\eta_1) \not\equiv \Phi_t(\eta_2). \end{aligned}$$

Finally, if B is a bounded subset of X and $t > 0$, $\Phi_t(B)$ is relatively compact in X. Those results are what we need to apply Matano's Theorem concerning heteroclinic orbits of Φ connecting two equilibria ω_1 and ω_2 such that $[\omega_1, \omega_2]$ contains no other equilibria than ω_1 and ω_2 [18].

Remark 4.7. In order to apply Matano's method we need to know what is the structure of the set ξ of the solutions of

$$-\Delta_{S^{N-1}}\omega + \omega|\omega|^{q-1} = \lambda\omega \tag{4.81}$$

on S^{N-1}. The complete structure is far out of reach, but using the geometric technique we have introduced in [30] one can describe some of the solutions of (4.81) associted to a tesselation of S^{N-1}. We first recall that if G is a subgroup of $O(N)$ generated by reflections through hyperplanes containing 0 and if G is finite, then G contains a finite number of reflections through hyperplanes $(H_k)_{k\in K}$ containing 0 and those hyperplanes divide $\mathbb{R}^N$ into a finite number of angular polyhedra $(P_i)_{i\in I}$, each of them being limited by at most N faces [12], [9]. Moreover those polyedra are all equal and G acts transitively on them. The intersections of those angular polyedra with S^{N-1} are spherical simplexes $(S_i)_{i\in I}$ on which G also acts transitively. Henceforth we can consider only model simplex S as *a fundamental domain for G*. The complete description of those finite groups generated by reflections can be found in [9] but on $\mathbb{R}^3$ there exists only five types of subgroups: type I is generated by reflections through two hyperplanes with angle π/n; type II is generated by the reflections through two hyperplanes with angle π/n and a reflection through an hyperplane orthogonal to them; type III, IV and V are associated to Plato's polyedra [14] and have respectively 24, 48 and 120 elements. In order to construct a solution of (4.81) with a high degree of complexity we consider a finite subgroup of reflections G with fundamental simplicial domain S on S^{N-1} and we call $\lambda(S)$ the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S)$. It is clear that $\lambda(S)$ is an eigenvalue of $\Delta_{S^{N-1}}$ on S^{N-1}. If $\lambda > \lambda(S)$ we call ω_S the unique positive solution of the following equation on S

$$-\Delta_{S^{N-1}}\omega_S + \omega_S^q = \lambda\omega_S, \tag{4.82}$$

ω_S vanishing on ∂S (ω_S is a minimizer). We then extend ω_S by reflection to whole S^{N-1} according the formula

$$\omega_{G|S_i} = \det(g_i)\omega_S \circ g_i^{-1} \tag{4.83}$$

if $S_i = g_i(S)$ for some $g_i \in G$. As the vertices have codimension 2 in S^{N-1} and ω_G is bounded, ω_G belongs to ξ [20]. For $\lambda > 0$ let ξ^* be the subset of ξ of solutions of (4.81) containing the three constants and the solutions which are of type ω_G for some finite subgroup of reflections G (as ω_G is constructed, $\omega_G \circ \tau$, for any $\tau \in O(N)$, is of the same type). If ω_G and $\omega_{G'}$ are two nonconstant elements of ξ^* associated to G and G' with fundamental simplicial domains S and S' and if S is a disjoint union of a finite number k of $g'_j(S')$, $1 \leqslant j \leqslant k$, $g'_j \in G'$ we shall say that *the frequency of* $\omega_{G'}$ *is a multiple of the frequency of* ω_G. As $\omega_{G|S}$ is energy minimizing we clearly have

$$(4.84)\quad \int_S \left(\frac{1}{2}|\nabla\omega_G|^2 + \frac{1}{q+1}|\omega_G|^{q+1} - \frac{\lambda}{2}\omega_G^2\right)$$
$$< \int_S \left(\frac{1}{2}|\nabla\omega_{G'}|^2 + \frac{1}{q-1}|\omega_{G'}|^{q+1} - \frac{\lambda}{2}\omega_{G'}^2\right)$$

which clearly implies

$$(4.85)\qquad E(\omega_G) < E(\omega_{G'}) < 0.$$

Using also the techniques of Theorem 4.8, for example the increasing iterative scheme

$$(4.86)\qquad \begin{cases} -\Delta_{S^{N-1}}\omega_n + \omega_n^q = \lambda\omega_{n-1} & \text{in } S \\ \omega_n = 0 & \text{on } \partial S \end{cases}$$

with $\omega_0 = \omega^+_{G'|S'}$ $(S' \subset S)$, we deduce

$$(4.87)\qquad \omega_G > \omega_{G'} \quad \text{in } S.$$

The following result the proof of which is an extension of Theorem 4.8 will be useful in the sequel for constructing a semiflow.

Theorem 4.9. *Assume Ω is an open subset of S^{N-1}, K_Ω is the piece of cone defined by*

$$(4.88)\qquad K_\Omega = \{\tau\sigma : \tau > 1, \sigma \in \Omega\},$$

$\lambda(\Omega)$ is the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(\Omega)$ and $\tilde{\partial} K_\Omega$ is the lateral boundary of K_Ω; let q be bigger than 1. Then for any $\phi \in C_0(\Omega)$ there exists a unique $u \in C(\bar{K}_\Omega) \cap C^3(K_\Omega)$ satisfying

$$(4.89)\qquad -\Delta u - \frac{c}{|x|^2}u + u|u|^{q-1} = 0$$

in K_Ω, $u = 0$ on $\tilde{\partial} K_\Omega$, $u = \phi$ on Ω if one of the following two conditions is fulfilled

(I) $\lambda > \lambda_1(\Omega)$, *u and ϕ are nonnegative, and either* $1 < q \leqslant (N+2)/(N-2)$ *or* $q > (N+2)/(N-2)$ *and* $c > \lambda_1(\Omega) + (N-2)^2/4$,

(II) *either* $c \leqslant \lambda_1(\Omega)$, *or* $\lambda_1(\Omega) < c \leqslant \lambda_1(\Omega) + (N-2)^2/4$ *and* $1 < q \leqslant (N+2)/(N-2)$.

With this result we can define a semiflow Φ^G on $X = C_0^+(\Omega)$ in Case I or on $C_0(\Omega)$ in Case II and, if $\partial\Omega$ is Lipschitz, Φ^G is a strongly order preserving semiflow on X mapping bounded subsets of X into relatively compact subsets of X for $t > 0$. We are now able to study the solutions u of (4.63) in $\mathbb{R}^N \setminus \{0\}$ such that

$$\begin{cases} \lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) = \omega_1, & \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) = \omega_2 \quad \text{or} \\ \lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) = \omega_2, & \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) = \omega_1 \end{cases} \tag{4.90}$$

and by extension of the notations of Remark 4.7 we shall say that the two constants $\pm\lambda^{1/(q-1)}$ are of type ω_G with $G = \{I_d\}$ and $S = S^{N-1}$, $\lambda(S) = 0$.

Theorem 4.10. *Assume* $q > 1$, ω_1 *and* $\omega_2 \in \xi^*$. *Then there exists a solution* u *of* (4.63) *in* $\mathbb{R}^N \setminus \{0\}$ *satisfying* (4.90) *if*

(*A*) $\omega_1 = 0$, $\omega_2 = \omega_G$ *for some* G *and one of the following two conditions is fulfilled*

(i) $1 < q \leqslant (N+2)/(N-2)$,
(ii) $q > (N+2)/(N-2)$ *and* $c > \lambda(S) + (N-2)^2/4$;

(*B*) $\omega_1 = \lambda^{1/(q-1)}$, $\omega_2 = \omega_G$ *for some non trivial* G *and either* $c \leqslant 0$, *or* $0 < c \leqslant (N-2)^2/4$ *and* $1 < q \leqslant (N+2)/(N-2)$;

(*C*) $\omega_1 = \omega_G$, $\omega_2 = \omega_{G'}$ *the frequency of* $\omega_{G'}$ *is a multiple of the frequency of* ω_G *and either* $c \leqslant \lambda(S)$ *or* $\lambda(S) < c \leqslant \lambda(S) + (N-2)^2/4$ *and* $1 < q \leqslant (N+2)/(N-2)$.

The proof of the Theorem is essentially a consequence of the construction of Remark 4.7 and of Theorem 4.8 for B and C and Theorem 4.9 applied in K_S for A; in that last case the positive solution constructed in the cone with basis 0 and vertex 0 is extended by reflection to be a solution of (4.63) in $\mathbb{R}^N \setminus \{0\}$. It must also be noticed that the case A, B and C imply $\lambda > \lambda(S)$, $\lambda > \lambda(S)$ and $\lambda > \lambda(S') > \lambda(S)$ respectively.

Remark 4.8. The complete set of the critical values of E is not known, in particular is it true that all the connected components of ξ have different energy value (the energy is constant on each connected component from Sard's theorem)? Such an exclusion principle is valid on ξ^*.

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