REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 7, N.° 3, 1991

Mean value and Harnack inequalities for a certain class of degenerate parabolic equations

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Introduction

In this paper we study the behavior of solutions of degenerate parabolic equations of the form

$$(1.1)\qquad v(x)u_t(x, t) = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x, t)D_{x_j}u(x, t)),$$

where the coefficients are measurable functions, and the coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

$$(1.2)\qquad w_1(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq w_2(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$ and $(x, t) \in \Omega \times (a, b)$, Ω a bounded open set in $\mathbf{R}^n$.

We are going to assume some conditions on the weights (non-negative functions that are locally integrable) v, w_1, w_2 and on the functions λ_j, $j = 1, \dots, n$, in order to be able to derive mean value and Harnack inequalities for solutions of (1.1). The assumptions on λ_j, which we list below, are the ones stated in [FL2].

[1] This work was supported by FAPESP —Fundação de Amparo à Pesquisa do Estado de São Paulo— Brazil.

(1.3) $\lambda_1 \equiv 1$, $\lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1})$, $j = 2, \ldots, n$, $x \in \mathbf{R}^n$.

(1.4) Let $\Pi = \{x \in \mathbf{R}^n \colon \Pi x_k = 0\}$. Then $\lambda_j \in C(\mathbf{R}^n) \bigcap C^1(\mathbf{R}^n \setminus \Pi)$ and $0 < \lambda_j(x) \leq \Lambda$, $x \in \mathbf{R}^n \setminus \Pi$, $j = 1, \ldots, n$.

(1.5) $\lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, -x_i, \ldots, x_{j-1})$, for $j = 2, \ldots, n$ and $i = 1, \ldots, j-1$.

(1.6) There is a family of $n(n-1)/2$ non-negative numbers $\varrho_{j,i}$ such that $0 \leq x_i(D_{x_i}\lambda_j)(x) \leq \varrho_{j,i}\lambda_j(x)$, for $2 \leq j \leq n$, $1 \leq i \leq j-1$ and all $x \in R^n \setminus \Pi$.

Denote $\Gamma = \Omega \times (a, b)$ and define $H = H(\Gamma)$ to be the closure of $\mathrm{Lip}(\Gamma)$ under the norm

$$(1.7)\quad \|u\|^2 = \iint_\Gamma u^2(x, t)(v(x) + w_2(x))\,dxdt + \iint_\Gamma |\nabla_\lambda u(x, t)|^2 w_2(x)\,dxdt + \iint_\Gamma u_t^2(x, t)v(x)\,dxdt,$$

where $\nabla_\lambda u = (\lambda_1 D_{x_1}u, \ldots, \lambda_n D_{x_n}u)$. Thus, $H(\Gamma)$ is the collection of all $(n+2)$-triples (u, β, B) such that there exists $u_k \in \mathrm{Lip}(\Gamma)$ with $u_k \to u$, $\nabla_\lambda u_k \to \beta$, $(u_k)_t \to B$, the convergence being in the appropriate L^2 space. We need to verify that β is uniquely determined and for this it is enough to show that for every $F \in C_0^\infty(\Gamma)$,

$$\int_\Gamma u\nabla_\lambda F = -\int_\Gamma \beta F.$$

In order to prove this, note that since $u \in H$, there exists $\{u_k\} \subset \mathrm{Lip}(\Gamma)$ such that $u_k \to u$ in H. Then, by (1.3),

$$\int_\Gamma u_k\lambda_i \frac{\partial F}{\partial x_i} = -\int_\Gamma \frac{\partial}{\partial x_i}(u_k\lambda_i)F = -\int_\Gamma \lambda_i \frac{\partial u_k}{\partial x_i} F.$$

Therefore,

$$\int_\Gamma u_k \nabla_\lambda F = -\int_\Gamma (\nabla_\lambda u_k)F.$$

By Schwarz's inequality and assuming that $w_2^{-1} \in L^1_{\mathrm{loc}}$,

$$\left| \int_\Gamma u_k \nabla_\lambda F - \int_\Gamma u \nabla_\lambda F \right| \leq \int_\Gamma | u_k - u | w_2^{1/2} | \nabla_\lambda F | w_2^{-1/2}$$
$$\leq \|u_k - u\|_{L^2_{w_2}} \left(\int_\Gamma | \nabla_\lambda F |^2 w_2^{-1} \right)^{1/2}$$
$$\leq c\|u_k - u\|_{L^2_{w_2}}.$$

Hence,
$$\int_\Gamma u_k \nabla_\lambda F \to \int_\Gamma u \nabla_\lambda F$$
and similarly we can show
$$\int_\Gamma (\nabla_\lambda u_k) F \to \int_\Omega \beta F.$$
In the same way we prove B is uniquely determined, if $\nu^{-1} \in L^1_{\text{loc}}$. We also define $H_0(\Gamma)$ to be the closure of $\text{Lip}_0(\Gamma)$, Lipschitz functions with compact support in Γ, under the norm defined in (1.7). It is easy to see that the bilinear form b on $\text{Lip}(\Gamma) \bigcap H(\Gamma)$ defined by
$$b(u, \phi) = \int\int_\Gamma \{u_t \phi \nu + \langle A \nabla u, \nabla \phi \rangle\} dx dt$$
can be continued to all of $H(\Gamma)$ (here and in the rest of the paper the vector ∇u is understood to be the vector $\left(\frac{1}{\lambda_1} \beta_1, \ldots, \frac{1}{\lambda_n} \beta_n \right)$ where $\nabla_\lambda u = (\beta_1, \ldots, \beta_n)$). We say $u \in H(\Gamma)$ is a solution of (1.1) if $b(u, \phi) = 0$ for any $\phi \in H_0$; $u \in H(\Gamma)$ is a subsolution if $b(u, \phi) \leq 0$ for any $\phi \in H_0(\Gamma)$, ϕ positive in the H-sense, *i.e.*, ϕ can be approximated in $H(\Gamma)$ by positive functions with compact support in Γ; $u \in H(\Gamma)$ is a supersolution if $b(u, \phi) \leq 0$ for any $\phi \in H_0$, ϕ positive in the H-sense.

We also define $\tilde{H} = \tilde{H}(\Omega)$ to be the closure of $\text{Lip}(\Omega)$ under the norm
$$|||u|||^2 = \int_\Gamma u^2(x)(\nu(x) + w_2(x)) dx + \int_\Gamma | \nabla_\lambda u(x) |^2 w_2(x) dx,$$
and $\tilde{H}_0(\Omega)$ to be the closure of $\text{Lip}_0(\Omega)$ under the norm defined above.

Next we will define a natural distance (associated with the functions λ_j, $j = 1, \ldots, n$) and state some of its properties. This metric was first introduced by [FL1].

A vector $\nu \in \mathbf{R}^n$ is called a λ-subunit vector at a point x if $\langle \nu, \xi \rangle^2 \leq \Sigma \lambda_j^2(x) \xi_j^2$, for every $\xi \in \mathbf{R}^n$. An absolutely continuous curve $\gamma: [0, T] \to \mathbf{R}^n$ is called a λ-subunit curve if $\dot{\gamma}(t)$ is a λ-subunit vector at $\gamma(t)$ for a.e. $t \in [0, T]$.

For any $x, y \in \mathbf{R}^n$ we define $d: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ by

$$d(x, y) = \inf\{T \in \mathbf{R}^+ : \text{there exists a } \lambda\text{-subunit curve } \gamma: [0, T] \to \mathbf{R}^n \text{ with } \gamma(0) = x, \gamma(T) = y\}.$$

One can check that this is a well-defined metric. There is a quasi-metric δ (a function $\delta: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ is called a quasi-metric if there exists $M \geq 1$ such that $\delta(x, y) \leq M\{\delta(x, z) + \delta(z, y)\}$ for all $x, y, z \in \mathbf{R}^n$) equivalent to d, and sometimes easier to work with than d (see [FL2]). If $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ put $H_0(x, t) = x$ and $H_{k+1}(x, t) = H_k(x, t) + t\lambda_{k+1}(H_k(x, t))e_{k+1}$ for $k = 0, \ldots, n-1$, where $\{e_k\}$ is the standard basis in $\mathbf{R}^n$. Define $\varphi_j(x^*, .) = (F_j(x^*, .))^{-1}$, the inverse function of $F_j(x^*, .)$, where $F_j(x, s) = s\lambda_j(H_{j-1}(x, s))$, for $j = 1, \ldots, n$ and $x^* = (|x_1|, \ldots, |x_n|)$.

We define δ: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ as

$$\delta(x, y) = \operatorname{Max}_{j=1,\ldots,n} \varphi_j(x^*, |x_j - y_j|).$$

Note that

(1.8) $\delta(x, y) < s$ is equivalent to $|x_j - y_j| < F_j(x^*, s)$, $1 \leq j \leq n$.

In (1.9), (1.10), (1.11) below we state some basic facts concerning δ and d (see also [FL2].

(1.9) There exists $a \geq 1$ such that for any $x, y \in \mathbf{R}^n$,

$$a^{-1} \leq \frac{d(x, y)}{\delta(x, y)} \leq a.$$

Consequently, δ is a quasi-metric with $\delta(x, y) \leq a^2[\delta(x, y) + \delta(z, y)]$ and $\delta(x, y) \leq a^2\delta(y, x)$.

(1.10) For any $x \in \mathbf{R}^n$, $s > 0$ and $\theta \in]0, 1[$

$$\theta^{G_j} \leq \frac{F_j(x^*, \theta s)}{F_j(x^*, s)} \leq \theta$$

where $G_1 = 1$ and $G_j = 1 + \Sigma_{l=1}^{j-1} G_l \varrho_{j,l}$, for $j = 2, \ldots, n$.

(1.11) We denote $S(x, r) = \{y \in \mathbf{R}^n : d(x, y) < r\}$ and $Q(x, r) = \{y \in \mathbf{R}^n : \delta(x, y) < r\}$ and we will call $S(x, r)$ a d-ball and $Q(x, r)$ a δ-ball. Note that there is a constant $A > 1$ such that $|S(x, 2r)| \leq A|S(x, r)|$ and $|Q(x, 2r)| \leq A|Q(x, r)|$, where $|\cdot|$ denotes Lebesgue measure. Also, by (1.8), $|Q(x, r)| = \Pi_{j=1}^n F_j(x^*, r)$. If $Q = Q(x, r)$, we write $r = r(Q)$.

In general we say that a non-negative and locally integrable function $w(x)$ is a doubling weight ($w \in D$) if there exists a constant $A > 1$ such that $w(2Q) \leq Aw(Q)$ for any δ-ball Q, where $2Q = Q(x, 2r)$, if $Q = Q(x, r)$ and

$$w(Q) = \int_Q w(x)\,dx.$$

(1.12) If $w \in D$ then there exists $\alpha > 0$ such that, for all $r > 0$, $\theta \in]0, 1]$, and $x \in R^n$, $w(Q(x, \theta r)) \geq \theta^\alpha w(Q(x, r))$.

Given $1 < p < \infty$, we say $w \in A_p$ if there is a constant $c > 0$ such that for all δ-balls Q in $\mathbf{R}^n$.

$$(1.13) \qquad \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/p-1}\,dx\right)^{p-1} \leq c.$$

Note that if we have the A_p condition with respect to δ, we have the same condition holding for the metric d, *i.e.* (1.13) holds with Q replaced by S (using doubling and the equivalence between d and δ). If v is a weight, $w \in A_p(v)$ means an analogous inequality holds with dx and $|Q|$ replaced by $v(x)dx$ and $v(Q)$, respectively. We use the notation $A_\infty(v) = \cup_{p>1} A_p(v)$. The theory of weights in homogeneous spaces was studied by A. P. Calderón in [C] and frequently we refer to this paper.

If $x, y \in \mathbf{R}^n$, we shall denote by $H(t, x, y) = (H_1(t, x, y), \ldots, H_n(t, x, y))$ the solution at time t of the Cauchy problem $\dot{H}_j(., x, y) = y_j\lambda_j(H(., x, y))$, $H_j(0, x, y) = x_j$, $j = 1, \ldots, n$.

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $0 < \epsilon_j < \alpha_j$, $j = 1, \ldots, n$, we denote $\Delta_\epsilon^\alpha = \{y \in \mathbf{R}^n\colon \epsilon_j \leq y_j \leq \alpha_j, j = 1, \ldots, n\}$. If $\sigma \in \{-1, 1\}^n$, we put $T_\sigma y = (\sigma_1 y_1, \ldots, \sigma_n y_n)$, $Q^\sigma(x, r) = \{y \in Q(x, r)\colon \sigma_j(y_j - x_j) \geq 0, j = 1, \ldots, n\}$ and $\Delta_\epsilon^\alpha(\sigma) = T_\sigma(\Delta_\epsilon^\alpha)$.

Now we can state two results proved in [FS].

Let $\gamma \in]0, 1[$ and $\sigma \in \{-1, 1\}^n$ be fixed. Then there exists $\epsilon, \alpha \in \mathbf{R}^n$ as above such that, for all $r > 0$ and $x \in \mathbf{R}^n$,

$$(1.14) \qquad |H(r, x, \Delta_\epsilon^\alpha(\sigma)) \cap Q^\sigma(x, r)| \geq (1 - \gamma)\,|Q^\sigma(x, r)|,$$

where $H(r, x, \Delta_\epsilon^\alpha(\sigma)) = \{H(x, r, y)\colon y \in \Delta_\epsilon^\alpha(\sigma)\}$.

Also, there are positive constants c_1, c_2 depending only on ϵ, α and $\varrho_{j,i}$ such that

$$(1.15)\qquad c_1\,|\,S(x,r)\,| \;\le\; \prod \int_0^r \lambda_j(H(t,x,y))dt \;\le\; c_2\,|\,S(x,r)\,|$$

for each $x \in \mathbf{R}^n$, $r > 0$ and $y \in \Delta^\alpha_\epsilon(\sigma)$.

If $q \geq 2$, we say that Sobolev inequality holds for w_1, w_2 for any $u \in \tilde{H}_0(Q)$, Q a δ-ball in $\mathbf{R}^n$,

$$(1.16)\qquad \left(\frac{1}{w_2(Q)}\int_Q |\,u\,|^q w_2 dx\right)^{1/q} \le cr(Q)\left(\frac{1}{w_1(Q)}\int_Q |\,\nabla_\lambda u\,|^2 w_1 dx\right)^{1/2}.$$

Given $q \geq 2$, we say the Poincaré inequality holds for w_1, w_2 and μ if there are constant $c > 0$ and $a > 0$ (see (1.9)) such that for any δ ball Q and every $u \in \tilde{H}(a^2Q)$ we have

$$(1.17)\qquad \left(\frac{1}{w_2(Q)}\int_Q |\,u - av_{\mu,Q}u\,|^q w_2 dx\right)^{1/q} \le$$

$$\le cr(Q)\left(\frac{1}{w_1(Q)}\int_{a^2Q} |\,\nabla_\lambda u\,|^2 w_1 dx\right)^{1/2},$$

where $av_{\mu,Q}u = \dfrac{1}{\mu(Q)}\displaystyle\int_Q u d\mu$ and $a^2Q = Q(x, a^2r)$ if $Q = Q(x, r)$.

The reason that we have a^2Q on the right side of (1.17) is that we do not have a Kohn type argument (see also [J]) for the quasi-metric δ. In the d-metric, we can state (1.17) with equal balls on both sides. For the metric δ, however, we have convenient cut-off functions (see [FL1]) that are important in order to get Caccioppoli estimates for solutions of (1.1) (see C.1), (C.2) and (C.3)). This explains the reason that we work with δ instead of d.

We can now state our main results.

Theorem A (Harnack's inequality).

Suppose that:

(i) v, w_1, $w_2 \in A_2$,

(ii) *the Poincaré inequality holds for* w_1, w_2 *and* w_1, v *with* $\mu = 1$ *and some* $q > 2$,

(iii) $w_2 v^{-1} \in A_\infty(v)$.

If u *is a non-negative solution of* (1.1) *in the cylinder* $R = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta)$, *then*

$$\text{ess sup}_{R^-} u \le c_1 \exp\{c_2[\alpha^{-2}\beta\Lambda(Q(x_0,\alpha)) + \alpha^2\beta^{-1}(\lambda(Q(x_0,\alpha)))^{-1}]\}\,\text{ess inf}_{R^+} u,$$

where $R^- = Q(x_0, \alpha/2) \times (t_0 - 3\beta/4, t_0 - \beta/4)$, $R^+ = Q(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta)$, $\Lambda(Q) = w_2(Q)/v(Q)$, $\lambda(Q) = w_1(Q)/v(Q)$, *for a* δ*-ball* Q. *Here the constants* c_1, c_2 *depend only on the constants which arise in* (i), (ii), (iii).

We write

$$\int\int_R f(x,t)m(x,t)dxdt = \int\int_R f(x,t)m(x,t)dxdt \Big/ \int\int_R m(x,t)dxdt.$$

Theorem B *(Mean value inequality). Assume that hypotheses* (i), (ii), (iii) *of Theorem A hold. Let* $0 < p < \infty$, $\alpha, \beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$ *and let* $Q = Q(x_0, \alpha)$, $Q' = Q(x_0, \alpha')$ *and* $R = Q \times (t_0 - \beta, t_0 + \beta)$, $R'_+ = Q' \times (t_0 - \beta', t_0 + \beta)$. *If* u *is a solution of* (1.1) *in* R, *then* u *is bounded in* R'_+ *and*

$$\text{ess sup}_{R'_+} |u|^p \le D(\alpha^2\beta^{-1}\lambda(Q)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q) + 1)^{h/(h-1)} \int\int_R |u|^p(\alpha^{-2}\beta w_2 + v)dxdt,$$

where $D \le C^{1/(h-1)}$ *if* $p \ge 2$, *and* $D \le c^{\log(3/p)}C^c$ *if* $0 < p < 2$, *and* $C = c\dfrac{\alpha^{2+b}\beta}{(\alpha - \alpha')^{2+b}(\beta - \beta')}$. *Here* $h > 1$, $c > 0$ *and* $b > 0$ *are constants which are independent of* $u, p, \alpha, \alpha', \beta, \beta'$.

The organization of the paper is as follows. In Section 2 we prove the following Sobolev interpolation inequality:

Theorem D: *Let* w_1, w_2 *be doubling weights,* $v \in A_2$ *and suppose* (1.17) *holds with* $w_1, w_2, \mu = 1$ *and some* $q > 2$. *If* $w_2 v^{-1} \in A_\infty(v)$ *then there exists* $h > 1$ *and constants* $c > 0$, $b > 0$ *such that for every* ϵ *satisfying* $0 < \epsilon \le 1$,

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx \le c\epsilon^{-b}\left(\frac{1}{v(Q)}\int_{(1+\epsilon)Q} u^2 v dx\right)^{h-1} \times \left(\frac{r(Q)^2}{w_1(Q)}\int_{(1+\epsilon)Q} |\nabla_\lambda u|^2 w_1 dx + \frac{1}{v(Q)}\int_{(1+\epsilon)Q} u^2 v dx\right)$$

for all $u \in \tilde{H}((1 + \epsilon)Q)$.

In section 3 we prove Theorem B. First we show, for $p \geq 2$, the following mean value inequality for subsolutions of (1.1):

(*) ess $\sup_{R'_+} u_+^p \leq$

$$(p^2C)^{\frac{h}{h-1}}(1+\alpha^2\beta^{-1}\lambda(Q)^{-1})^{1/(h-1)}(1+\alpha^{-2}\beta\Lambda(Q))^{h/(h-1)}\int\int_R u_+^p(\alpha^{-2}\beta w_2+v)dxdt,$$

where C is as in Theorem B and $u_+ = \max\{u, 0\}$. This inequality is less precise than the one we will eventually obtain because of the presence of the factor p^2 on the right. In order to prove the above inequality we apply Theorem D to the function $H_M(u(., r))$ where

$$H_M(s) = \begin{cases} s^{p/2} \text{ if } s \in [0, M] \\ M^{p/2} + \frac{p}{2} M^{(p-2)/2}(s - M) \text{ if } s \geq M \\ 0 \text{ if } s < 0, \end{cases}$$

and therefore $H_M(u(., \tau))$ is an element of $\tilde{H}(Q(x_0, \alpha)$ for a.e. $\tau \in (t_0 - \beta', t_0 + \beta)$. The first idea would be to apply Theorem D to the function $u_+^{p/2}(., \tau)$ but at this point we do not know if $u_+^{p/2}(., \tau)$ belongs to $\tilde{H}(Q(x_0, \alpha)$. Hence we have to work with $H_M(u)$, and in order to proceed with the proof of (*) we show the following Caccioppolli inequality for $H_M(u)$.

(C.1) Let $2 \leq p < \infty$ and u be a subsolution of (1.1) in R. Let $w_2 \in A_2$ and $\alpha, \alpha', \beta, \beta'$ satisfy $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$. Then

$$\text{ess}\sup_{\tau\in(t_0-\beta',t_0+\beta)}\int_Q H_M(u(x, r))^2 v(x)dx$$

$$+\int\int_{R'_+} |\nabla_\lambda(H_M(u))|^2 w_1(x)dxdt$$

$$\leq c\int\int_R u^2 H'_M(u)^2\left(\frac{w_2}{(\alpha-\alpha')^2}+\frac{v}{\beta-\beta'}\right)dxdt$$

with c independent of all parameters.

The next step is to apply (*) for $p = 2$ to deduce that u_+ is locally bounded. This fact allow us to apply Theorem D to the function $u_+^{p/2}(., \tau)$ for a.e. $\tau \in (t_0 - \beta', t_0 + \beta)$. The Caccioppoli inequality we can deduce from (C.1) for the function $u_+^{p/2}$ is not precise enough since it will have a factor p^2 in the right hand side (note that $uH_M'(u) \le pu_+^{p/2}/2$) and this is the term we want to eliminate from (*). But with a different test function from the one used in the proof of (C.1), namely, $\phi(x, t) = \eta^2 g(u)\chi(t, \tau_1, \tau_2)$ where

$$g(s) = \begin{cases} s^{p-1} & \text{if } s \in [0, M], \\ M^{p-2}s & \text{if } s \ge M, \\ 0 & \text{if } s < 0, \end{cases}$$

and η is a convenient C^∞ function with compact support, we can deduce the following Caccioppoli inequality for subsolutions of (1.1):

(C.2) Let $2 \le p < \infty$ and u be a subsolution of (1.1) in R. Let $w_2 \in A_2$ and $\alpha, \alpha', \beta, \beta'$ satisfy $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$. Then

$$\operatorname{ess\,sup}_{\tau\in(t_0-\beta', t_0+\beta)} \int_Q u_+(x, \tau)^p v(x)dx + \int\int_{R'_+} |\nabla_\lambda u_+^{p/2}|^2 w_1(x)dxdt$$

$$\le c \int\int_R u_+^p \left(\frac{w_2}{(\alpha-\alpha')^2} + \frac{v}{\beta-\beta'}\right) dxdt,$$

with c independent of all parameters.

Now following the steps of the proof of (*) using (C.2) instead of (C.1) we can prove that for $p \ge 2$

(**) $\operatorname{ess\,sup}_{R'_+} u_+^p \le$

$$C^{\frac{h}{h-1}}(\alpha^2\beta^{-1}\lambda(Q)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q) + 1)^{h/(h-1)} \int\int_R u_+^p(\alpha^{-2}\beta w_2 + v)dxdt,$$

and Theorem B will follow from (**) and an iteration argument like the one given in Lemma 3.4 of [GW2]. Finally we conclude Section 3 by making some comments about the proof of mean value inequalities for u^p, when $p < 0$, where u is a positive solution of (1.1). These inequalities will be necessary in the proof of Theorem A and in order to show them we need the following generalization of (C.2):

(C.3) Let $-\infty < p < +\infty, p \neq 0, 1$, u satisfy $0 < m < u(x, t) < M < \infty$ in R, $w_2 \in A_2$. Then if $p > 1$ and u is a subsolution in R, or if $p < 0$ and u is a supersolution in R,

$$\operatorname{ess\,sup}_{\tau \in (t_0-\beta', t_0+\beta)} \int_{Q'} u(x, \tau)^p v(x)dx + \frac{p-1}{p} \int\int_{R'_+} |\nabla_\lambda u^{p/2}|^2 w_1(x)dxdt$$
$$\leq c \int\int_R u^p \left(\frac{p}{p-1} \cdot \frac{w_2(x)}{(\alpha-\alpha')^2} + \frac{v(x)}{\beta-\beta'}\right) dxdt.$$

Moreover, if $0 < p < 1$ and u is a supersolution in R, then

$$\operatorname{ess\,sup}_{\tau \in (t_0-\beta, t_0+\beta')} \int_{Q'} u(x, \tau)^p v(x)dx + \left|\frac{p-1}{p}\right| \int\int_{R'_-} |\nabla_\lambda u^{p/2}|^2 w_1 dxdt$$
$$\leq c \int\int_R u^p \left(\left|\frac{p}{p-1}\right| \frac{w_2}{(\alpha-\alpha')^2} + \frac{v}{\beta-\beta'}\right) dxdt.$$

In this paper we do not present the proofs of (C.2) and (C.3) since their proofs are similar to the ones given in Section 2 of [GW2].

In Section 4, we prove

Theorem E: *Let v and w_1 be weights such that there exists $s > 1$ with*

$$\left(\frac{r(I)}{r(B)}\right)^2 \left(\frac{1}{|I|}\int_I \left(\frac{v}{v(B)}\right)^s dx\right)^{1/s} \left(\frac{1}{|I|}\int_I \left(\frac{w_1}{w_1(B)}\right)^{-s} dx\right)^{1/s} \leq c \tag{1.18}$$

for all δ-balls I, B with $I \subset 2a^2 B$ (a as in (1.9)), where c is a constant independent of the balls. Let $Q = Q(\xi, r)$ and φ be a C^1 function such that $\varphi \equiv 1$ in $Q(\xi, kr)$, $1/2 \leq k < 1$, $0 \leq \varphi \leq 1$, supp $\varphi \subset Q$ and

$$\varphi(x)\varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y))$$

for all x, y, t, t_0 with $0 \leq t \leq t_0$. Then, if $u \in \operatorname{Lip}(Q)$,

$$\int_Q |u(x) - A_Q|^2 \varphi(x)v(x)dx \leq c \frac{v(Q)}{w_1(Q)} r(Q)^2 \int_Q |\nabla_\lambda u(x)|^2 \varphi(x) w_1(x)dx,$$

where $A_Q = \dfrac{1}{\varphi(Q)} \displaystyle\int_Q u(x)\varphi(x)dx.$

Finally, in Section 5, we prove Theorem A. This theorem follows as an application of Bombieri's lemma ([GW2]). In order to verify the hypotheses of Bombieri's lemma we need Theorem B and Theorem F, which we state next. We write

$$(\nu \otimes 1)(A) = \int\int_A \nu(x)dxdt,$$

where $\nu = \nu(x)$, $x \in \mathbf{R}^n$, and $A \subset \mathbf{R}^{n+1} = \{(x, t): x \in \mathbf{R}^n, t \in R\}$.

Theorem F: *Suppose ν is a doubling weight, $w_2 \in A_2$,* (1.18) *holds and $w_2\nu^{-1} \in A_\infty(\nu)$. Let Q_R be a δ-ball of radius R, $t_0 \in (a, b)$ and $\tilde{w}_2 = w_2/w_2(Q_R)$ and $\tilde{\nu} = \nu/\nu(Q_R)$. If u is a solution of* (1.1) *in $Q_{3R/2} \times (a, b)$ which is bounded below by a positive constant, then there are constants c_1, M_2, $\varkappa$ and V such that if for $s > 0$ we define*

$$E^+ = \{(x, t) \in Q_R \times (t_0, b): \log u < -s - M_2(b - t_0) - V\}$$
$$E^- = \{(x, t) \in Q_R \times (a, t_0): \log u > s - M_2(a - t_0) - V\},$$

then

$$((\tilde{\nu} + \tilde{w}_2) \otimes 1)(E^+) \leq c_1 \left(\frac{1}{s} \frac{\nu(Q_R)}{w_1(Q_R)} \frac{R^2}{b - t_0}\right)^\varkappa (b - t_0)$$

and

$$((\tilde{\nu} + \tilde{w}_2) \otimes 1)(E^-) \leq c_1 \left(\frac{1}{s} \frac{V(Q_R)}{w_1(Q_R)} \frac{R^2}{t_0 - a}\right)^\varkappa (t_0 - a).$$

Here c_1 and $\varkappa$ depend only on the constants in the conditions on ν and w_2, $M_2 \approx \dfrac{w_2(Q_R)}{R^2\nu(Q_R)}$, and V is a constant which depends on u.

In order to prove this theorem, if we follow the steps of Lemma 4.9 of [GW2], we just have to verify that a certain test function (see [FL1]) satisfies the conditions of Theorem E. This will be done in Lemma 5.4.

2. Interpolation Inequality

In this section we prove Theorem D. We start with

Theorem 2.1. *Let w_1, w_2, and μ be doubling weights and suppose* (1.17) *holds for w_1, w_2 with any μ, and for some $q > 2$. If $Q = Q(\xi, r)$ and $w_2\nu^{-1} \in A_\infty(\nu)$ then there exist $h > 1$ and a constant $c > 0$, independent of Q and u, such that*

$$\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx$$

$$\leq c \left(\frac{1}{v(Q)} \int_Q u^2 v dx \right)^{h-1} \left(\frac{r^2}{w_1(Q)} \int_{Q(\xi, a^2 r)} |\nabla_\lambda u|^2 w_1 dx + (av_{\mu,Q} |u|)^2 \right)$$

for all $u \in \tilde{H}(a^2 Q)$, *and* a *as in* (1.9). *Also if* (1.17) *is replaced by* (1.16), *then*

$$\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx \leq c \left(\frac{1}{v(Q)} \int_Q u^2 v dx \right)^{h-1} \left(\frac{r^2}{w_1(Q)} \int_Q |\nabla_\lambda u|^2 w_1 dx \right)$$

for all $u \in \tilde{H}_0(Q)$.

PROOF: The proof follows as in [GW1], Theorem 3; the only differences are that we obtain $Q(\xi, a^2 r)$ in the second integral on the right when we apply Poincaré's inequality and in the end we use the results of Calderón for weights in homogeneous spaces (see [C]).

Corollary 2.2. *Let* w_1, w_2 *be doubling weights and suppose* (1.17) *holds with* w_1, w_2, $\mu = 1$ *and some* $q > 2$. *If* $w_2 v^{-1} \in A_\infty(v)$, *then there exists* $h > 1$ *and a constant* $c > 0$ *such that*

$$\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx$$

$$\leq c \left(\frac{1}{v(Q)} \int_Q u^2 v dx \right)^{h-1} \left(\frac{r^2}{w_1(Q)} \int_{a^2 Q} |\nabla_\lambda u|^2 w_1 dx + \frac{1}{v(Q)} \int_Q u^2 v dx \right)$$

for all $u \in \tilde{H}(a^2 Q)$, $Q = Q(\xi, r)$.

PROOF: The conclusion of Theorem 2.1 holds for $\mu = 1$. But, by Schwarz's inequality,

$$\begin{aligned} av_Q |u| &= \frac{1}{|Q|} \int_Q |u| \, dx \\ &= \frac{1}{|Q|} \int_Q u v^{1/2} v^{-1/2} dx \leq \frac{1}{|Q|} \left(\int_Q u^2 v dx \right)^{1/2} \left(\int_Q \frac{1}{v} dx \right)^{1/2} \\ &\leq \left(\frac{1}{v(Q)} \int_Q u^2 v dx \right)^{1/2}, \end{aligned}$$

where in the last inequality we used the fact that $\nu \in A_2$.

In the next section we prove mean value inequalities. In order to be able to iterate a certain inequality as was done in [GW2] we need a refinement of the above corollary. This refinement is Theorem D and to prove it we need the following lemmas.

Lemma 2.3. *Given* $Q = Q(\xi, s)$ *and* $0 < r < s$, *there exists* $x_1, \ldots, x_{m(r,s)}$ *in* Q, *and* $k \geq 1$ independent of ξ, r, s, *such that*

(i) $$Q(x_j, r/k) \bigcap Q(x_h, r/k) = \emptyset, \; h \neq j$$

(ii) $$Q(\xi, s) \subset \cup_{j=1}^{m(r,s)} Q(x_j, r).$$

Moreover, $m(r, s) \leq c \left(\frac{s}{r}\right)^{\nu'}$ *for some constant* ν' *depending only on the dimension.*

PROOF: If we apply Theorem 1.2, page 69, of [CoW] to the open covering of Q given by $(S(x, r/4a))_{x \in Q}$, there exist $x_1, \ldots, x_{m(r,s)}$ in Q such that: $S(x_h, r/4a) \bigcap S(x_j, r/4a) = \emptyset$ if $j \neq h$ and $Q(\xi, s) \subset \cup_{j=1}^{m(r,r)} S(x_j, r/a)$. By (1.9), $S(x_j, r/4a) \supset Q(x_j, r/4a^2)$ and $S(x_j, r/a) \subset Q(x_j, r)$. Therefore, if we choose $k = 4a^2$, (i) and (ii) follow. It remains to find an upper bound for $m(r, s)$. First, we note that $Q(x_j, r/k) \subset Q(\xi, a^2(k + 1)s/k)$. But

$$\frac{r}{k} = \frac{2a^4(k + 1)s}{k} \, \frac{r}{2a^4(k + 1)s},$$

and so by (1.10), there exists $\nu' > 0$, such that

$$\left| Q\left(x_j, \frac{r}{k}\right) \right| \geq \left(\frac{r}{2a^4(k + 1)s}\right)^{\nu'} \left| Q\left(x_j, \frac{2a^4(k + 1)s}{k}\right) \right|,$$

and since the $Q(x_j, r/k)$ are disjoint,

$$\left| Q\left(\xi, \frac{a^2(k+1)s}{k}\right) \right| \geq \sum_j \left| Q\left(x_j, \frac{r}{k}\right) \right|$$

$$\geq c\left(\frac{r}{s}\right)^{\nu} \sum_j \left| Q\left(x_j, \frac{2a^4(k+1)s}{k}\right) \right|.$$

But,

$$Q\left(x_j, \frac{2a^4(k+1)s}{k}\right) \supset Q\left(\xi, \frac{a^2(k+1)s}{k}\right)$$

and so

$$\left| Q\left(\xi, \frac{a^2(k+1)s}{k}\right) \right| \geq c\left(\frac{r}{s}\right)^{\nu} m(r, s) \left| Q\left(\xi, \frac{a^2(k+1)s}{k}\right) \right|.$$

Therefore, $m(r, s) \leq c\,(s/r)^{\nu}$.

Lemma 2.4. *If* $\delta(y, z) < s$ *then* $F_j(z^*, s) \leq (2a^2)^{G_j} F_j(y^*, s)$, G_j *as in* (1.10).

Proof: Since $Q(z, s) \subset Q(y, 2a^2 s)$, $F_j(z^*, s) \leq F_j(y^*, 2a^2 s)$. By (1.10), it follows that

$$F_j(z^*, s) \leq F_j(y^*, 2a^2 s) \leq (2a^2)^{G_j} F_j(y^*, s).$$

Lemma 2.5. *If* $0 < \epsilon < 1$ *and* $\eta \in Q = Q(\xi, s)$, *then* $Q(\eta, \epsilon s/(2a^2)^{\zeta}) \subset Q(\xi, (1+\epsilon)s)$, *where* $\zeta = \max_{j=1,\ldots,n} G_j$.

Proof: If $y \in Q(\eta, \epsilon s/(2a^2)^{\zeta})$ then by (1.8), $|y_j - \eta_j| \leq F_j(\eta^*, \epsilon s/(2a^2)^{\zeta})$ and by (1.10) and Lemma 2.4

$$F_j\left(\eta^*, \frac{\epsilon s}{(2a^2)^{\zeta}}\right) \leq \frac{\epsilon}{(2a^2)^{\zeta}} F_j(\eta^*, s) \leq \epsilon F_j(\xi^*, s).$$

Therefore

$$\begin{aligned} |y_j - \xi_j| &\leq |y_j - \eta_j| + |\eta_j - \xi_j| \leq \epsilon F_j(\xi^*, s) + F_j(\xi^*, s) \\ &= (1+\epsilon) F_j(\xi^*, s) \\ &\leq F_j(\xi^*, (1+\epsilon)s), \end{aligned}$$

where in the last inequality we used (1.10).

Proof of Theorem D.

Let $Q = Q(\xi, s)$. By Lemma 2.5, $\delta(Q, \partial(1+\epsilon)Q) \geq \epsilon s/(2a^2)^{\zeta}$. Apply Lemma 2.3 to $r = \dfrac{\epsilon s}{(2a^2)^{\zeta} a^2}$ to find $x_1, \ldots, x_{m(r,s)} \in Q$ such that: $Q(x_j, r/k) \bigcap Q(x_h, r/k) = \varnothing$ if $j \neq h$, $Q(\xi, s) \subset \bigcup_{j=1}^{m(r,s)} Q(x_j, r)$ and $m(r, s) \leq c(s/r)^{\nu}$.

Note that, by (2.5), $Q(x_j, a^2 r) = Q\left(x_j, \frac{\epsilon s}{(2a^2)^{\zeta}}\right) \subset Q(\xi, (1+\epsilon)s) = (1+\epsilon)Q$.

Then using Corollary 2.2, doubling for w_2, doubling for v and w_1 and the fact that $Q(x_j, 2a^2 s) \supset Q(\xi, s)$ and $Q(\xi, 2a^2 s) \supset Q(x_j, s)$,

$$\int_Q |u|^{2h} w_2 dx \le \sum_{j=1}^{m(r,s)} \int_{Q(x_j,r)} |u|^{2h} w_2 dx$$

$$\le c \sum_{j=1}^{m(r,s)} w_2(Q(x_j, r)) \left(\frac{1}{v(Q(x_j, r)} \int_{Q(x_j,r)} u^2 v dx\right)^{h-1}$$

$$\cdot \left\{\frac{r^2}{w_1(Q(x_j, r))} \int_{Q(x_j,a^2 r)} |\nabla_\lambda u|^2 w_1 dx + \frac{1}{v(Q(x_j, r))} \int_{Q(x_j,r)} u^2 v dx\right\}$$

$$\le c \left(\frac{s}{r}\right)^{\nu'} w_2(Q(\xi, s)) \left[\left(\frac{r}{2a^2 s}\right)^{-\alpha} \frac{1}{v(Q(\xi, s))} \int_{(1+\epsilon)Q} u^2 v dx\right]^{h-1}$$

$$\cdot \left\{\frac{s^2}{w_1(Q(\xi, s))} \left(\frac{r}{2a^2 s}\right)^{-\alpha} \int_{(1+\epsilon)Q} |\nabla_\lambda u|^2 w_1 dx\right.$$

$$\left. + \left(\frac{r}{2a^2 s}\right)^{-\alpha} \frac{1}{v(Q(\xi, s))} \int_{(1+\epsilon)Q} u^2 v dx\right\}.$$

The theorem follows if we choose $b = \nu + 2\alpha$, since $s/r = c\epsilon^{-1}$.

3. Mean value inequalities.

In this section we prove Theorem B and some other mean value inequalities. Since the proofs are similar to the ones given by [GW2], we just point out the differences. Basically, we have to be a little more careful in the iteration argument since there is a factor ϵ in Theorem D.

We asume throughout this section that:

(3.1)
- (a) $w_1, w_2, v \in A_2$
- (b) Poincaré's inequality, (1.17), holds for both of the pairs w_1, w_2 and w_1, v with some $q > 2$ and $\mu = 1$
- (c) $w_2 v^{-1} \in A_\infty(v)$.

Denote $R_{r,s} = Q(x_0, r) \times (t_0 - s, t_0 + s)$ and let $R = R_{r,s}$, $R' = R_{\varrho,\sigma}$ with $r/2 < \varrho < r$ and $s/2 < \sigma < s$ and define

$$C = c\frac{r^{2+b}s}{(r-\varrho)^{2+b}(s-\sigma)} \tag{3.2}$$

where b is given by Theorem D and c is a constant that may vary, but which only depends on the weights and on h, where $h > 1$ is the index for which Theorem D holds for both w_2 and v on the left hand side.

We also write $\lambda(Q) = w_1(Q)/v(Q)$ and $\Lambda(Q) = w_2(Q)/v(Q)$. We start this section with the proof of (C.1). This estimate will be important in deducing a mean value inequality for subsolutions of (1.1).

PROOF OF (C.1): If $u \in H$ define

$$\varphi(x, t) = \eta^2(x, t)\left[\int_0^{u(x,t)} H_M'(s)^2 ds + u(x, t)H_M'(u(x, t))^2\right]\chi(t, \tau_1, \tau_2),$$

where $\eta \in C_0^\infty(R)$ will be specified later, $t_0 - s < \tau_1 < \tau_2 < t_0 + s$ and $\chi(t, \tau_1, \tau_2)$ denotes the characteristic function of (τ_1, τ_2). The fact that the function φ is in H_0 follows as a consequence of the following result: if f is a piecewise smooth function on the real line with $f' \in L^\infty(-\infty, \infty)$ and if $u \in H$, then $f \circ u \in H$. Here we use the convention that $f'(u) = 0$ if $u \in L$ where L denotes the set of corner points of f (the proof follows the steps of Theorem 7.8 of [GT] and it also shows that $\nabla_\lambda(f \circ u) = f'(u)\nabla_\lambda u$ and $(f(u))_t = f'(u)u_t$). The proof of the above fact also verifies that in our case $\varphi \geq 0$ in the H_0-sense since $H_M(s) = 0$ for $s < 0$.

Since u is a subsolution, we have

$$\int\int_R (\langle A\nabla u, \nabla\varphi\rangle + u_t\varphi v)dxdt \leq 0. \tag{3.3}$$

Note that by another limiting argument

$$u_t\left[\eta^2\int_0^u H_M'(s)^2ds\right] = \left[u\eta^2\int_0^u H_M'(s)^2ds\right]_t - u(\eta^2)_t\int_0^u H_M'(s)^2ds - \eta^2 H_M'(u)^2u_tu,$$

and then by definition of φ, for $\tau_1 < t < \tau_2$,

$$u_t\varphi = \left[u\eta^2\int_0^u H_M'(s)^2ds\right]_t - (\eta^2)_tu\int_0^u H_M'(s)^2ds$$

and

$$\nabla \varphi = 2\eta \nabla \eta \left[\int_0^u H_M'(s)^2 ds + u H_M'(u)^2\right] + \eta^2 [H_M'(u)^2 \nabla u + f_M'(u) \nabla u],$$

where $f_M(s) = sH_M'(s)^2$ (note that $\nabla (f_M(u)) = f_M'(u) \nabla u$, since f_M is piecewise smooth with $f_M' \in L^\infty$). If we substitute the two last equations in (3.3) we get, with $Q = Q(x_0, r)$,

$$\int_Q \int_{\tau_1}^{\tau_2} \left[u\eta^2 \int_0^u H_M'(s)^2 ds\right]_t v dx dt + \int_Q \int_{\tau_1}^{\tau_2} \eta^2 H_M'(u)^2 \langle A \nabla u, \nabla u\rangle dx dt$$

$$\leq \int_Q \int_{\tau_1}^{\tau_2} \left[(\eta^2)_t u \int_0^u H_M'(s)^2 ds\right] v dx dt$$

$$-2 \int_Q \int_{\tau_1}^{\tau_2} \eta \langle A \nabla u, \nabla \eta\rangle \left[\int_0^u H_M'(s)^2 ds + u H_M'(u)^2\right] dx dt$$

$$- \int_Q \int_{\tau_1}^{\tau_2} \eta^2 \langle A \nabla u, \nabla u\rangle f_M'(u) dx dt.$$

We can drop the last term on the right since the integrand is non-negative. The second term on the right is majorized in absolute value by

$$4 \int_Q \int_{\tau_1}^{\tau_2} |\langle A \nabla u, \nabla \eta\rangle| \, \eta H_M'(u)^2 u dx dt$$

$$= 4 \int_Q \int_{\tau_1}^{\tau_2} |\langle A H_M'(u) \eta \nabla u, u H_M'(u) \nabla \eta\rangle| \, dx dt$$

$$\leq 2\epsilon \int_Q \int_{\tau_1}^{\tau_2} \langle A \nabla (H_M(u)), \nabla (H_M(u))\rangle, \eta^2 dx dt$$

$$+ \frac{2}{\epsilon} \int_Q \int_{\tau_1}^{\tau_2} \langle A \nabla \eta, \nabla \eta\rangle u^2 H_M'(u)^2 dx dt$$

where we used the fact that $|\langle Ax, y\rangle| \leq \langle Ax, x\rangle^{1/2} \langle Ay, y\rangle^{1/2} \leq \frac{\epsilon}{2} \langle Ax, x\rangle + \frac{1}{2\epsilon} \langle Ay, y\rangle$. If we pick $\epsilon = \frac{1}{4}$ we get

$$(3.4)\quad \int_Q \int_{\tau_1}^{\tau_2} \left[u\eta^2 \int_0^u H_M'(s)^2 ds \right]_t v\,dxdt$$

$$+ \frac{1}{2} \int_Q \int_{\tau_1}^{\tau_2} \eta^2 \langle A\, \nabla (H_M(u)),\ \nabla (H_M(u)) \rangle dxdt$$

$$\leq 8 \int_Q \int_{\tau_1}^{\tau_2} \langle A\, \nabla \eta,\ \nabla \eta \rangle u^2 H_M'(u)^2 dxdt + \int_Q \int_{\tau_1}^{\tau_2} \left[(\eta^2)_t u \int_0^u H_M'(s)^2 ds \right] v\,dxdt.$$

Choosen η to be zero in a neighborhood of $\{\partial Q \times (t_0 - s,\ t_0 + s)\} \cup \{Q \times (t = t_0 - s)\}$, $\eta \equiv 1$ in R'_+, $0 \leq \eta \leq 1$, $|\nabla_\lambda \eta| \leq c/(r - \varrho)$, $|\eta_t| \leq c/(s - \sigma)$ (see page 537 of [FL1]). If we pick τ_1 so close to $t_0 - s$ that $\eta(x, \tau_1) = 0$ for all $x \in Q$, drop the second term on the left of (3.4) (which is non-negative) and use Lemma 5 of [AS] it follows that

$$(3.5)\quad \text{ess sup}_{\tau_2 \in (t_0 - \sigma, t_0 + s)} \int_{Q'} u(x, \tau_2) \int_0^{u(x,\tau_2)} H_M'(s)^2 ds\ v\,dx$$

$$\leq c \int\int_R u^2 H_M'(u)^2 \left[\frac{w_2}{(r - \varrho)^2} + \frac{v}{s - \sigma} \right] dxdt.$$

If we fix $\tau_2 \in (t_0 - \sigma,\ t_0 + s)$ and τ_1 as before and if we drop the first term on the left of (3.4) (which we can see is non-negative after performing the integration) we obtain

$$(3.6)\quad \int_Q \int_{\tau_1}^{\tau_2} \eta^2 \langle A\, \nabla (H_M(u)),\ \nabla (H_M(u)) \rangle dxdt$$

$$\leq c \int\int_R u^2 H_M'(u)^2 \left[\frac{w_2}{(r - \varrho)^2} + \frac{v}{s - \sigma} \right] dxdt.$$

Letting $\tau_2 \rightarrow t_0 + s$ and using (1.2) we get

$$(3.7)\quad \int\int_{R'_+} |\nabla_\lambda (H_M(u))|^2 w_1 dxdt \leq c \int\int_R u^2 H_M'(u)^2 \left[\frac{w_2}{(r - \varrho)^2} + \frac{v}{s - \sigma} \right] dxdt.$$

Finally note that

$$H_M(u)^2 = \int_0^u (H_M(s)^2)'\,ds = \int_0^u 2H_M(s)H_M'(s)\,ds$$

$$\leq 2\int_0^u sH_M'(s)^2\,ds \leq 2u\int_0^u H_M'(s)^2\,ds,$$

since $H_M(s) \leq sH_M'(s)$. Combining this with (3.5) and (3.7), (C.1) follows with α, β, α', β' taken there to be r, s, ϱ, σ.

Lemma 3.8. *Let* $p \geq 2$, R, R' *be as defined above and assume* (3.1) *holds. If* u *is a subsolution of* (1.1) *in* R, *then* u_+ *is bounded in* $R'_+ = Q(x_0, \varrho) \times (t_0 - \sigma, t_0 + s)$ *and*

$$\operatorname{ess\,sup}_{R'_+} u_+^p$$

$$\leq (p^2C)^{\frac{h}{h-1}}\left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q)}\right)^{\frac{1}{h-1}}\left(1 + \frac{s}{r^2}\Lambda(Q)\right)^{\frac{h}{h-1}}\int\int_R u_+^p\left(\frac{s}{r^2}w_2 + v\right)dxdt,$$

with C *as in* (3.2).

PROOF: $H_M(u)$ is a function in H since $u \in H$ and H_M is a C^1 function with bounded derivative. Then by Fubini's theorem we have that $H_M(u(.,\tau)) \in \tilde{H}$ for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$. If we apply Theorem D to the function $F(x) = H_M(u(x,\tau))$, $Q = Q_\varrho$ and $\epsilon > 0$ such that $(1 + \epsilon)\varrho < r$ and combine this with (C.1) we obtain

$$\frac{1}{w_2(Q_\varrho)}\int_{Q_\varrho} H_M(u(x,\tau)^{2h}w_2(x)dx$$

$$\leq c\epsilon^{-b}\left\{\frac{1}{v(Q_\varrho)}\int\int_R u^2H_M'(u)^2\left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right)dxdt\right\}^{h-1}$$

$$\cdot\left\{\frac{\varrho^2}{w_1(Q_\varrho)}\int_{Q_{(1+\epsilon)\varrho}} |\nabla_\lambda(H_M(u(x,\tau)))|^2 w_1(x)dx\right.$$

$$\left. + \frac{1}{v(Q_\varrho)}\int\int_R u^2H_M'(u)^2\left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right)dxdt\right\}$$

for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$.

Integrate with respect to τ over $(t_0 - \sigma, t_0 + s)$ and apply (C.1) to get

$$\frac{1}{w_2(Q_\varrho)} \int\int_{R'_+} H_M(u(x, t))^{2h} w_2(x) dxdt$$

$$\leq c \frac{\epsilon^{-b}}{v(Q_\varrho)^{h-1}} \left(\frac{\varrho^2}{w_1(Q_\varrho)} + \frac{s+\sigma}{v(Q_\varrho)}\right) \left(\int\int_R u^2 H'_M(u)^2 \frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right) dxdt\Bigg)^h$$

Since $(r/2) < \varrho < r$ and $(s/2) < \sigma < s$, by the doubling property of the weights and the definitions of λ and Λ, it follows that

$$\frac{1}{w_2(Q_r)} \int\int_{R'_+} H_M(u(x, t))^{2h} w_2(x) dxdt$$

$$\leq c \frac{\epsilon^{-b}}{v(Q_r)^h} \left(\frac{r^2}{\lambda(Q_r)} + s\right) \left(\int\int_R u^2 H'_M(u)^2 \left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right) dxdt\right)^h.$$

A similar inequality holds with w_2 replaced by v on the left, and if we add the two inequalities, we obtain

$$\int\int_{R'_+} H_M(u)^{2h} \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right) dxdt \tag{3.9}$$

$$\leq c \frac{\epsilon^{-b}}{v(Q_r)^h} \left(\frac{r^2}{\lambda(Q_r)} + s\right) \left(\int\int_R u^2 H'_M(u)^2 \left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right) dxdt\right)^h$$

for any ϵ such that $(1 + \epsilon)\varrho < r$.

Now note that

$$\frac{w_2}{(r - (1 + \epsilon)\varrho)^2} + \frac{v}{s-\sigma} \leq \frac{r^2}{(r-(1+\epsilon)\varrho)^2 (s-\sigma)} \left\{\frac{s}{r^2} w_2 + v\right\},$$

$$\int\int_{R'_+} \left\{\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right\} dxdt \approx s,$$

$$\int\int_R \left\{\frac{s}{r^2} w_2 + v\right\} dxdt \approx s \left\{\frac{s}{r^2} w_2(Q_r) + v(Q_r)\right\} \approx sv(Q_r) \left\{\frac{s}{r^2} \Lambda(Q_r) + 1\right\},$$

$$\frac{sr^{-2} w_2(x) + v(x)}{sr^{-2} w_2(Q_r) + v(Q_r)} \leq \frac{w_2(x)}{w_2(Q_r)} + \frac{v(x)}{v(Q_r)}.$$

Thus, by raising both sides of (3.9) to the power $1/h$, normalizing and using the fact that $\epsilon^{-b/h} \leq \epsilon^{-b}$, we obtain

$$(3.10) \quad \left(\int\int_{R'_+} H_M(h)^{2h}\left(\frac{s}{r^2}w_2+v\right)dxdt\right)^{1/h}$$

$$\leq c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left(1+\frac{s}{r^2}\Lambda(Q_r)\right)\left(1+\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot\int\int_R u^2 H'_M(u)^2\left(\frac{s}{r^2}w_2+v\right)dxdt$$

for any ϵ such that $(1+\epsilon)\varrho < r$. Since $u_+^{p/2}\chi_{\{0<u<M\}} \leq H_M(u)$ and $uH'_M(u) \leq pu_+^{p/2}/2$, if we let $M \to \infty$ it follows by Fatou's lemma that

$$(3.11) \quad \left(\int\int_{R'_+} u_+^{ph}\left(\frac{s}{r^2}w_2+v\right)dxdt\right)^{1/h}$$

$$\leq cp^2\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left(1+\frac{s}{r^2}\Lambda(Q_r)\right)\left(1+\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot\int\int_R u_+^p\left(\frac{s}{r^2}w_2+v\right)dxdt.$$

Now, we have to iterate (3.11). Fix r, s, ϱ, σ with $r/2 < \varrho < r$ and $s/2 < \sigma < s$. For $k = 1, 2, \ldots$ define sequences $\{s_k\}_{k\in\mathbf{N}}$ and $\{r_k\}_{k\in\mathbf{N}}$ and $\{\epsilon_k\}_{k\in\mathbf{N}}$ by $s_1 = s$, $s_k - s_{k+1} = \dfrac{s-\sigma}{2^k}$ for $k \geq 1$, $r_1 = r$, $r_k - r_{k+1} = (r-\varrho)/2^k$ for $k \geq 1$, and $\epsilon_k = \dfrac{r-\varrho}{2^k r_k} = \dfrac{r_k - r_{k+1}}{r_k}$ for $k \geq 1$. Also, define $R_k = Q_k \times (t_0 - s_k, t_0 + s)$ for $k \geq 1$, where $Q_k = Q(x, r_k)$. Note that $R_1 = R$ and $\bigcap_{k=1}^{\infty} R_k = R'_+$. Since

$$\frac{1}{2}sr^{-2} \leq s_k r_k^{-2} \leq 4sr^{-2},$$

if we apply (3.11) with p replaced by ph^{k-1}, $p \geq 2$, and $r = r_k$, $\varrho = r_{k+1}$ and $\epsilon = \epsilon_{k+1}$ (note that $(1+\epsilon_{k+1})r_{k+1} < r_k$), we obtain

$$\left(\iint_{R_{k+1}} u_+^{ph^k}\left(\frac{s}{r^2}\, w_2 + v\right) dxdt\right)^{1/h^k}$$

$$\leq \left\{c(ph^{k-1})^2 \epsilon_{k+1}^{-b} \frac{r_k^2 s_k}{(r_k - (1+\epsilon_{k+1})r_{k+1})^2 (s_k - s_{k+1})} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right)\right.$$

$$\left.\cdot \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}\right\}^{1/(h^{k-1})} \cdot \left\{\iint_{R_k} u_+^{ph^{k-1}}\left(\frac{s}{r^2}\, w_2 + v\right) dxdt\right\}^{1/(h^{k-1})}.$$

But note that

$$\epsilon_{k+1}^{-b} \frac{r_k^2 s_k}{[r_k - (1+\epsilon_{k+1})r_{k+1}]^2 (s_k - s_{k+1})}$$

$$= 2^{(k+1)b} \frac{r_{k+1}^b}{(r-\varrho)^b} \frac{r_k^2 s_k}{\left(\frac{r-\varrho}{2^k} - \frac{r-\varrho}{2^{k+1}}\right)^2 \left(\frac{s-\sigma}{2^k}\right)}$$

$$\leq c2^{(3+b)k} \frac{r^{2+b} s}{(r-\varrho)^{2+b}(s-\sigma)}$$

$$\leq C2^{(3+b)k},$$

where C is given by (3.2). Thus,

$$\text{(3.12)} \quad \left(\iint_{R_{k+1}} u_+^{ph^k}\left(\frac{s}{r^2}\, w_2 + v\right) dxdt\right)^{1/h^k}$$

$$\leq \left\{C(ph^{k-1})^2 2^{(3+b)k}\left(1 + \frac{s}{r^2}\Lambda(Q_r)\right)\left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}\right\}^{1/h^{k-1}}$$

$$\cdot \left\{\iint_{R_k} u_+^{ph^{k-1}}\left(\frac{s}{r^2}\, w_2 + v\right) dxdt\right\}^{1/h^{k-1}}.$$

If we iterate (3.12), we obtain

ess $\sup_{R'_+} u_+^p$

$$\leq \prod_{k=1}^{\infty} \left\{ C(ph^{k-1})^2 2^{(3+b)k} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right) \cdot \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h} \right\}^{1/h^{k-1}} \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt.$$

Since $\sum_{k=1}^{\infty} \frac{1}{h^{k-1}} = \frac{h}{h-1}$ and $\sum_{k=1}^{\infty} \frac{k}{h^{k-1}} = \left(\frac{h}{h-1}\right)^2$, it follows that

ess $\sup_{R'_+} u_+^p$

$$\leq (p^2C)^{\frac{h}{h-1}} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right)^{\frac{h}{h-1}} \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{\frac{1}{h-1}} \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt,$$

and this proves the lemma. Note that if we apply the above result for $p = 2$, it follows that u_+ is bounded on R'_+.

PROOF OF THEOREM B: By Lemma 3.8 we know that u_+ is bounded in $Q_{(1+\epsilon)\varrho} \times (t_0 - \sigma, t_0 + s)$ for all ϵ such that $(1 + \epsilon)\varrho < r$. If we define $F(x) = u_+^{p/2}(x, \tau)$ then $F \in \tilde{H}(Q_{(1+\epsilon)\varrho})$ for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$ and if we follow the proof of Lemma 3.8 using (C.2) instead of (C.1), we get (see the comments in the introduction)

ess $\sup_{R'_+} u_+^p$

$$\leq C^{\frac{h}{h-1}} \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q)}\right)^{\frac{1}{h-1}} \left(1 + \frac{s}{r^2}\Lambda(Q)\right)^{\frac{h}{h-1}} \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt$$

for $p \geq 2$. For $0 < p < 2$, define I_p and I_∞ as in Lemma 3.4 of [GW2]. The only difference in our case is that

$$I_\infty(\alpha', \beta')^2 \leq c \left[\frac{1}{(\alpha - \alpha')^{2+b}(\beta - \beta')}\right]^{\frac{h}{h-1}} I_2(\alpha, \beta)^2$$

if $1/2 < \alpha' < \alpha < 1$ and $1/2 < \beta' < \beta < 1$. Thus, arguing as in Lemma 3.4 of [GW2] we prove that if u is a solution of (1.1) and $p > 0$ then

(3.13) $\operatorname{ess\,sup}_{R'_+} u_+^p \leq$

$$D\left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q)}\right)^{\frac{1}{h-1}}\left(1 + \frac{s}{r^2}\Lambda(Q)\right)^{\frac{h}{h-1}}\int\int_R u_+^p\left(\frac{s}{r^2}w_2 + v\right)dxdt,$$

where D is as in Theorem B.

If we apply (3.13) to both u and $-u$, we obtain Theorem B of the introduction, with α, β, α', β' taken there to be r, s, ϱ, σ.

In order to prove Harnack's inequality we need a mean value inequality for u^p when $-\infty < p < \infty$ and u is a non-negative solution.

We begin by noting that if we use (C.3) instead of (C.1) we can prove the following analogue of (3.11):

Lemma 3.14. *Suppose* (3.1) *holds,* $0 < m < u(x, t) \leq M < \infty$ *in* $R = R_{r,s}$, $r/2 < \varrho < r$, $s/2 < \sigma < s$ *and* $\epsilon > 0$, $(1 + \epsilon)\varrho < r$. *Then, if* $p > 1$ *and* u *is a subsolution in* R, *or if* $p < 0$ *and* u *is a supersolution in* R,

$$\left(\int\int_{R'_+} u^{ph}\left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right)dxdt\right)^{1/h}$$

$$\leq c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left(1 + \frac{p}{p-1}\frac{s}{r^2}\Lambda(Q_r)\right)\left(1 + \frac{p}{p-1}\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot\int\int_R u^p\left(\frac{p}{p-1}\frac{s}{r^2}w_2 + v\right)dxdt.$$

Moreover, if $0 < p < 1$ *and* u *is a supersolution in* R, *then*

$$\left(\int\int_{R'_-} u^{ph}\left(\frac{w_1}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right)dxdt\right)^{1/h}$$

$$\leq c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left(1 + \frac{p}{|p-1|}\frac{s}{r^2}\Lambda(Q_r)\right)\left(1 + \frac{p}{|p-1|}\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot\int\int_R u^p\left(\frac{p}{|p-1|}\frac{s}{r^2}w_2 + v\right)dxdt.$$

Both inequalities are still true if we replace the integral averages on the right by the larger integral average

$$\iint_R u^p \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)} \right) dxdt.$$

Theorem 3.15. *Assume* (3.1) *holds,* $r, s > 0$, $r/2 < \varrho < r$, $s/2 < \sigma < s$. *If u is a non negative solution of* (1.1) *in R, then for* $p > 0$

$$\text{ess sup}_{R'}\, u^p$$

$$\leq C^c \left(1 + p\frac{s}{r^2}\Lambda(Q_r)\right)^{\frac{h}{h-1}} \left(1 + p\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{\frac{1}{h-1}} \iint_R u_+^p \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right) dxdt,$$

and for $p < 0$

$$\text{ess sup}_{R'_+}\, u^p \leq C^{\frac{h}{h-1}} \left(1 + |p| \frac{s}{r^2} \Lambda(Q_r)\right)^{\frac{h}{h-1}}$$

$$\cdot \left(1 + |p| \frac{r^2}{s} \frac{1}{\lambda(Q_r)}\right)^{\frac{h}{h-1}} \iint_R u^p \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right) dxdt,$$

where C is given by (3.2).

PROOF: In Lemma 3.17 of [GW2] we replace (3.20) by the result given here in Lemma 3.14 and then argue as in Lemma 3.17 of [GW2].

4. Proof of Theorem E

We start with the following lemma.

Lemma 4.1. *Suppose* $Q = Q(\xi, r)$ *and* φ *is a* C^1 *function such that* $\varphi \equiv 1$ *in* $kQ = Q(\xi, kr)$, $0 < k < 1$, $0 \leq \varphi \leq 1$, $\text{supp}\, \varphi \subset Q$ *and*

$$\varphi(x)\varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y)) \tag{4.2}$$

for all x, y, t, t_0 *with* $0 \leq t \leq t_0$. *If u is a Lipschitz function,*

$E = \{x \in Q(\xi, kr): u(x) = 0\}$ *and* $|E| \geq \beta |Q|$ *for some* $0 < \beta < 1$, *then if* $x \in Q$,

$$(4.3) \qquad |u(x)| \sqrt{\varphi(x)} \leq c \int_Q |\nabla_\lambda u(z)| \sqrt{\varphi(z)} \frac{\delta(x, z)}{|Q(x, \delta(x, z))|} dz,$$

where c is independent of Q, u, x.

PROOF: (The general outline of this proof follows the steps of the proof of Lemma 4.3 in [FS].) If $x \in Q = Q(\xi, r)$ then $Q(\xi, r) \subset Q(x, 2a^2 r)$ and $Q(x, r) \subset Q(\xi, 2a^2 r)$. Therefore, by doubling, $|Q(x, r)| \simeq |Q|$. Now, we note that there exists $\sigma \in \{-1, 1\}^n$ such that $|E \cap Q^\sigma(x, 2a^2 r)| \geq c\beta |Q^\sigma(x, 2a^2 r)|$. In fact, $E = \cup_\sigma (Q^\sigma(x, 2a^2 r) \cap E)$ and so there exists σ such that

$$(4.4) \qquad |Q^\sigma(x, 2a^2 r) \cap E| \geq \beta 2^{-n} |Q| \geq c\beta |Q^\sigma(x, 2a^2 r)| .$$

We also claim that there exist $\alpha, \epsilon \in \mathbf{R}^n$, independent of x and r, $0 < \epsilon_j < \alpha_j$, $j = 1, \ldots, n$, such that

$$(4.5) \qquad |E \cap Q^\sigma(x, 2a^2 r) \cap H(2a^2 r, x, \Delta_\epsilon^\alpha(\sigma))| \geq \frac{c\beta}{2} |Q^\sigma(x, 2a^2 r)| .$$

To prove this fact, apply (1.14) to $\gamma = \dfrac{c\beta}{2}$ and find $\alpha, \epsilon \in \mathbf{R}^n$, $0 < \epsilon_j < \alpha_j$, $j = 1, \ldots, n$, such that

$$|H(2a^2 r, x, \Delta_\epsilon^\alpha(\sigma)) \cap Q^\sigma(x, 2a^2 r)| \geq \left(1 - \frac{c\beta}{2}\right) |Q^\sigma(x, 2a^2 r)| .$$

Then,

$$|Q^\sigma(x, 2a^2 r)| \geq |(Q^\sigma(x, 2a^2 r) \cap E) \cup (Q^\sigma(x, 2a^2 r) \cap H(\ldots))| =$$
$$|Q^\sigma(x, 2a^2 r) \cap E| + |Q^\sigma(x, 2a^2 r) \cap H(\ldots)| - |E \cap Q^\sigma(x, 2a^2 r) \cap H(\ldots)|$$
$$\geq |Q^\sigma(x, 2a^2 r)| \left(c\beta + 1 - \frac{C\beta}{2}\right) - |E \cap Q^\sigma(x, 2a^2 r) \cap H(\ldots)|$$

and therefore the claim follows.

We can assume $x \notin E$ and define $\Sigma = \{y \in \Delta_\epsilon^\alpha(\sigma): H(2a^2 r, x, y) \in E\}$. Let K be a smooth function supported in $\Delta_{\epsilon/2}^{2\alpha}(\sigma)$, $0 \leq K \leq 1$, $K \equiv 1$ on $\Delta^\alpha_\epsilon(\sigma)$. Suppose $u \in \mathrm{Lip}(Q)$. If $y \in \Sigma$ then

$$|u(x)| \sqrt{\varphi(x)} = |u(x) - u(H(2a^2 r, x, y))| K(y) \sqrt{\varphi(x)},$$

and if we integrate on Σ, we obtain

$$|u(x)|\ \sqrt{\varphi(x)}\ |\Sigma| = \int_{\Sigma} |u(x) - u(H(2a^2r, x, y))|\, K(y)\sqrt{\varphi(x)}dy.$$

Now we note that $\varphi(H(2a^2r, x, y)) = 1$ if $y \in \Sigma$ and using (4.2) we get $\varphi(x) \leq \varphi(H(t, x, y))$ for any $0 \leq t \leq 2a^2r$. Therefore,

$$|u(x)|\ \sqrt{\varphi(x)}\ |\Sigma| \leq \int_{\mathrm{supp}K} \left| \int_0^{2a^2r} \frac{d}{dt}(u(H(t, x, y)))dt \right| \sqrt{\varphi(H(t, x, y))}dy$$

$$\leq \int_{\mathrm{supp}K} \left| \int_0^{2a^2r} \langle \nabla u(H(t, x, y)), \dot{H}(t, x, y)\rangle dt \right| \sqrt{\varphi(H(t, x, y))}dy$$

$$\leq \int_0^{2a^2r} \int_{\mathrm{supp}K} |\nabla_\lambda u(H(t, x, y))|\ |y|\ \sqrt{\varphi(H(t, x, y))}dydt.$$

If we make change of variables $z = H(t, x, y)$ in $\Delta^{2\alpha}_{\epsilon/2}(\sigma)$, then

$$\left|\det \frac{\partial z}{\partial y}(t, x, y)\right| = \Pi_{j=1}^n \int_0^t \lambda_j(H(s, x, y))ds.$$

For $y \in \Delta^{2\alpha}_{\epsilon/2}(\sigma)$, the last product is equivalent to $|Q(x, t)|$ by (1.15). Hence

$$(4.6)\quad |u(x)|\ \sqrt{\varphi(x)} \leq \frac{c}{|\Sigma|}\int_0^{2a^2r} \frac{1}{|Q(x, t)|}\int_{H(t,x,\Delta^{2\alpha}_{\epsilon/2}(\sigma))} |\nabla_\lambda u(z)|\ \sqrt{\varphi(z)}dzdt.$$

Note that there exists $c > 0$ such that $H(t, x, \Delta^{2\alpha}_{\epsilon/2}(\sigma)) \subset Q(x, ct)$. In fact, if we define $\gamma(s) = H(s/|y|, x, y)$ then

$$\langle \dot{\gamma}(s), \xi\rangle^2 = \left\{\sum_{j=1}^n \lambda_j\left(H\left(\frac{s}{|y|}, x, y\right)\right) y_j\xi_j\right\}^2 \frac{1}{|y|^2}$$

$$\leq \sum_{j=1}^n \lambda_j^2\left(H\left(\frac{s}{|y|}, x, y\right)\right)\xi_j^2$$

$$= \sum_{j=1}^n \lambda_j(\gamma(s))\,\xi_j^2$$

for every $\xi \in R^n$. So, γ is a λ-subunit curve starting from x and attaining $H(t, x, y)$ at the time $s = t|y|$. Therefore by (1.9),

$$\delta(x, H(t, x, y)) \leq ad(x, H(t, x, y)) \leq at|y| \leq ct$$

where $c = 2\alpha a$

Thus, from (4.6), we obtain

$$|u(x)|\sqrt{\varphi(x)} \le \frac{c}{|\Sigma|}\int_0^{2a^2r}\frac{1}{|Q(x,t)|}\int_{Q(x,ct)}|\nabla_\lambda u(z)|\sqrt{\varphi(z)}\,dzdt$$

and, interchanging the order of integration and using the fact that supp $\varphi \subset Q$ (the argument we are going to present next is due to Chanillo, Sawyer and Wheeden), we get

$$(4.7)\qquad |u(x)|\sqrt{\varphi(x)} \le \frac{c}{|\Sigma|}\int_Q |\nabla_\lambda u(z)|\sqrt{\varphi(z)}\left(\int_{c\delta(x,z)}^{\infty}\frac{dt}{|Q(x,t)|}\right)dz.$$

We claim that $\displaystyle\int_{ch}^{\infty}\frac{dt}{|Q(x,t)|} \le c\frac{ch}{|Q(x,h)|}$. To prove this we note that, by (1.8),

$$\frac{|Q(x,t)|}{t} = \Pi_{j=2}^{n}F_j(x^*,t),$$

and consequently by (1.10), there exists $\epsilon > 0$ such that if $t > \tau$ then

$$\frac{|Q(x,t)|}{t} \ge c\left(\frac{t}{\tau}\right)^{\epsilon}\frac{|Q(x,\tau)|}{\tau}.$$

Hence,

$$\int_{ch}^{\infty}\frac{dt}{|Q(x,t)|} = \int_{ch}^{\infty}\frac{t}{|Q(x,t)|}\frac{dt}{t} \le \int_{ch}^{\infty}\frac{h}{|Q(x,h)|}\left(\frac{h}{t}\right)^{\epsilon}\frac{dt}{t} = c\frac{h}{|Q(x,h)|}.$$

Finally, we note that $|\Sigma| \ge c > 0$, with c independent of x, since, by the change of variables $z = H(2a^2r, x, y)$,

$$\begin{aligned}|\Sigma| &= \int_\Sigma dy \simeq \int_{H(2a^2r,x,\Sigma)}\frac{1}{|Q(x,2a^2r)|}\,dz\\ &= \frac{|H(2a^2r,x,\Sigma)|}{|Q(x,2a^2r)|} = \frac{|E \cap H(2a^2r,x,\Delta_\epsilon^\alpha(\sigma))|}{|Q(x,2a^2r)|}\\ &\ge c\beta\frac{|Q^\sigma(x,2a^2r)|}{Q(x,2a^2r)|} \ge c > 0.\end{aligned}$$

The lemma follows by combining the last two last estimates with (4.7).

PROOF OF THEOREM E.

Define $Tf(x) = \int_{\mathbf{R}^n} f(y)K(x, y)dy$, where $K(x, y) = \dfrac{\delta(x, y)}{|Q(x, \delta(x, y))|}$.

Fix S a d-ball. In order to show that for a pair of weights $\tilde{v}$, $\tilde{w}$ we have $\|Tf\|_{L^2(S,\tilde{v})} \leq \|f\|_{L^2(S,\tilde{w})}$ (where $\|f\|_{L^2(S,\tilde{v})} = \left(\int_S f^2\tilde{v}\right)^{1/2}$) for all $f \geq 0$, $\operatorname{supp} f \subset S$, according to [SW], we need to verify that the following conditions hold:

(a) there exists $s > 1$ such that

$$\varphi(I)\,|I| \left(\frac{1}{|I|}\int_I \tilde{v}^s dx\right)^{\frac{1}{2s}} \left(\frac{1}{|I|}\int_I \tilde{w}^{-s} dx\right)^{\frac{1}{2s}} \leq c$$

for all d-balls $I \subset 2S$, where $\varphi(I)$ is defined to be

$$\varphi(I) = \sup\left\{K(x, y) : x, y \in I,\ d(x, y) \geq \frac{1}{2} r(I)\right\};$$

(b) there is $\epsilon > 0$ such that

$$\frac{|I|}{|I|} \leq c_\epsilon \frac{\varphi(I)}{\varphi(I')} \left(\frac{r(I')}{r(I)}\right)^\epsilon$$

for all pairs of d-balls $I' \subset I$.

Note that it is convenient to work with d since the results of [SW] hold for pseudo-metrics (a pseudo-metric d is a quasi-metric satisfying $d(x, y) = d(y, x)$ for all $x, y \in \mathbf{R}^n$).

Define $\tilde{v} = \dfrac{v}{v(S)}$ and $\tilde{w} = \dfrac{w_1}{w_1(S)} r(S)^2$. Note that if $x, y \in I$ and $d(x, y) \geq \dfrac{1}{2} r(I)$, then by (1.9)

$$K(x, y) = \frac{\delta(x, y)}{|Q(x, \delta(x, y))|} \leq \frac{2ar(I)}{\left|Q\left(x, \frac{1}{2a} r(I)\right)\right|} \leq c \frac{r(I)}{|Q(x, r(I))|},$$

and since $x \in I$, $|Q(x, r(I))| \simeq |I|$. Therefore,

$$\varphi(I) \leq c \frac{r(I)}{|I|}.$$

So, the expression in (a) is bounded by

$$c \frac{r(I)}{|I|} |I| \left(\frac{1}{|I|} \int_I \left(\frac{v}{v(S)}\right)^s dx\right)^{\frac{1}{2s}} \left(\frac{1}{|I|} \int_I \left(\frac{w_1}{w_1(S)} r(S)^2\right)^{-s} dx\right)^{\frac{1}{2s}}$$

$$\leq c \frac{r(I)}{r(S)} \left(\frac{1}{|I|} \int_I \left(\frac{v}{v(S)}\right)^s dx\right)^{\frac{1}{2s}} \left(\frac{1}{|I|} \int_I \left(\frac{w_1}{w_1(S)}\right)^{-s} dx\right)^{\frac{1}{2s}},$$

which is equivalent to the expression in condition (1.18) (if we use doubling and (1.9)). This proves (a).

To show (b) we note that if $x, y \in I$ and $d(x, y) \geq \frac{1}{2} r(I)$ then

$$K(x, y) \geq \frac{(2a)^{-1} r(I)}{|Q(x, 2ar(I))|} \geq c \frac{r(I)}{|I|}.$$

Thus $\varphi(I) \simeq \frac{r(I)}{|I|}$. Then, if $I' \subset I$, $\frac{\varphi(I)}{\varphi(I')} \simeq \frac{r(I)}{r(I')} \frac{|I'|}{|I|}$ and we obtain (b) with $\epsilon = 1$.

By doubling and (1.9), it follows that

$$\| Tf \|_{L^2(Q, \tilde{v})} \leq c \| f \|_{L^2(Q, \tilde{w})}$$

for all $f \geq 0$, supp $f \subset Q$, where $\tilde{v} = \frac{v}{v(Q)}$ and $\tilde{w} = \frac{w_1}{w_1(Q)} r(Q)^2$.

Suppose u is a Lipschitz function in Q and $|E| = |\{x \in Q(\xi, kr): u(x) = 0\}| \geq \beta |Q|$, $1/2 < k < 1$. If we combine Lemma 4.1 and the fact that $\| Tf \|_{L^2(Q, \tilde{v})} \leq c \| f \|_{L^2(Q, \tilde{w})}$ we obtain

$$\left(\frac{1}{v(Q)} \int_Q |u(x)|^2 \varphi(x) v(x) dx\right)^{1/2} \leq cr(Q) \left(\frac{1}{w_1(Q)} \int_Q |\nabla_\lambda u(z)|^2 \varphi(z) w_1(z) dz\right)^{1/2} \tag{4.8}$$

Given Q and a general Lipschitz function u, there is a number $\mu = \mu(u, Q)$, the media of u in Q, such that if $Q^+ = \{x \in Q: u(x) \geq \mu\}$ and $Q^- = \{x \in Q: u(x) \leq \mu\}$ then $|Q^+| \geq |Q|/2$ and $|Q^-| \geq |Q|/2$. Hence, $u_1 = \max\{u - \mu(u, kQ), 0\}$ and $u_2 = \max\{\mu(u, kQ) - u, 0\}$ satisfy the hypohesis of Lemma (4.1) for some β depending on k and so if we apply (4.8) to u_1 and u_2 and add both inequalities, we get

$$\text{(4.9)} \quad \int_Q |u(x) - \mu|^2 \varphi(x) v(x) dx \leq c r(Q)^2 \frac{v(Q)}{w_1(Q)} \int_Q |\nabla_\lambda u(z)^2 \varphi(z) w_1(z) dz.$$

Finally, it is easy to see that in (4.9) μ can be replaced by the average A_Q of u defined in Theorem E. In fact,

$$\text{(4.10)} \quad \begin{aligned} &\int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx \\ &\leq 2 \int_Q |u(x) - \mu|^2 \varphi(x) v(x) dx \\ &+ 2 \int_Q |\mu - A_Q|^2 \varphi(x) v(x) dx, \end{aligned}$$

and

$$\begin{aligned} \int_Q |\mu - A_Q|^2 \varphi(x) v(x) dx &= (\varphi v)(Q) |\mu - A_Q|^2 \\ &= (\varphi v)(Q) \left| \mu - \frac{1}{\varphi(Q)} \int_Q u(x) \varphi(x) dx \right|^2 \\ &\leq (\varphi v)(Q) \left(\frac{1}{\varphi(Q)} \int_Q |u(x) - \mu| \varphi(x) dx \right)^2 \\ &\leq \frac{(\varphi v)(Q)}{(\varphi(Q))^2} \int_Q |u(x) - \mu|^2 \varphi^2(x) v(x) dx \int_Q \frac{1}{v(x)} dx, \end{aligned}$$

where in the last inequality we used Schwarz's inequality. Since $v \in A_2$ and $0 \leq \varphi \leq 1$, it follows from (4.9) and (4.10) that

$$\begin{aligned} &\int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx \\ &\leq c r(Q)^2 \left[1 + \left(\frac{|Q|}{\varphi(Q)} \right)^2 \right] \frac{v(Q)}{w_1(Q)} \int_Q |\nabla_\lambda u(z)|^2 \varphi(z) w_1(z) dz. \end{aligned}$$

This finishes the proof of Theorem E if we note that $\varphi(Q) \simeq |Q|$ since $1/2 \leq k \leq 1$.

The next corollary is also helpful.

Corollary 4.11. *Theorem E is also true with* $A_Q = \frac{1}{(\varphi v)(Q)} \int_Q u\varphi v dx$.

Just note that

$$\begin{aligned}\int_Q |\mu - A_Q|^2 \varphi v dx &= (\varphi v)(Q) |\mu - A_Q|^2 \\ &\leq (\varphi v)(Q) \left(\frac{1}{(\varphi v)(Q)} \int_Q |\mu - u| \varphi v dx \right)^2 \\ &\leq \int_Q |\mu - u|^2 \varphi v dx,\end{aligned}$$

where the last inequality follows by Schwarz's inequality.

5. Harnack's inequality

The proof of Theorem A follows as an application of Bombieri's lemma which we state next. For its proof see Section 5 of [GW2].

Lemma 5.1. *Let $R(\varrho)$ be a one parameter family of rectangles in $\mathbf{R}^{n+1}$, $R(\sigma) \subset R(\varrho)$, $1/2 \leq \sigma \leq \varrho \leq 1$ and let v be a doubling measure in $\mathbf{R}^{n+1}$. Let A, μ, M, m, θ and $\varkappa$ be positive constants such that $M \geq 1/\mu$ and suppose that f is a positive measurable function defined in a neighborhood of $R(1)$ satisfying*

$$\text{ess sup}_{R(\sigma)} f^p \leq \frac{A}{(\varrho - \sigma)^m} \iint_{R(\varrho)} f^p v(x) dx dt \tag{5.2}$$

for all σ, ϱ, p, $1/2 \leq \theta \leq \sigma < \varrho < 1$, $0 < p < M$ and

$$v(\{(x, t) \in R(1) : \log f > s\}) \leq \left(\frac{\mu}{s}\right)^{\varkappa} v(R(1)) \tag{5.3}$$

for all $s > 0$. *Then there is a constant* $\gamma = \gamma(A, m, \varkappa) > 0$ *such that*

$$\log(\text{ess sup}_{R(\theta)} u) \leq \frac{\gamma}{(1-\theta)^{2m}} \mu.$$

Hence, in order to prove Theorem A, we need a mean value inequality (that we proved in Section 3) and a logarithm estimate which is given by Theorem F (some steps of its proof we will present in this section). The next lemma shows that the test function described on page 537 of [FL1] satisfies the conditions of Theorem E. Then, as we said before, the proof of Theorem F follows as Lemma 4.9 of [GW2].

Lemma 5.4. *Given* $Q = Q(\xi, r)$ *and* $0 < k < 1$, *there exists* $\varphi \in C^1$ *such that* $\varphi \equiv 1$ *in* kQ, $0 \leq \varphi \leq 1$, $\operatorname{supp}\varphi \subset Q$, $|\nabla_\lambda \varphi| \leq \dfrac{c}{r(1-k)}$ *and* $\varphi(x) \cdot \varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y))$ *for all* x, y, t, t_0 *with* $0 \leq t \leq t_0$.

PROOF: Consider the function φ given by [FL1], page 537:

$$\varphi(x) = \prod_{j=1}^{n} \psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right),$$

where $\psi \in C^\infty(\mathbf{R})$, $0 \leq \psi \leq 1$, $\psi(t) = \psi(-t)$, $\psi \equiv 1$ on $[-k, k]$, $\psi = 0$ outside $]-1, 1[$, $|\psi'(t)| \leq 2(1-k)^{-1}$, for all $t \in \mathbf{R}$. Here, we show that φ satisfies the last condition since all the others are proved in [FL1], page 537.

Fix t, $0 < t < t_0$, x and y. Define $z = H(t, x, y)$. Then,

$$z_j = x_j + y_j \int_0^t \lambda_j(H(s, x, y))ds.$$

Suppose $z_j - \xi_j \geq 0$. If $y_j \geq 0$ then

$$|z_j - \xi_j| \leq x_j - \xi_j + y_j \int_0^{t_0} \lambda_j(H(s, x, y))ds = H_j(t_0, x, y) - \xi_j.$$

On the other hand, if $y_j < 0$,

$$|z_j - \xi_j| \leq |x_j - \xi_j|.$$

Thus, if $z_j - \xi_j \geq 0$ then $|z_j - \xi_j| \leq |H_j(t_0, x, y) - \xi_j|$ or $|z_j - \xi_j| \leq |x_j - \xi_j|$. The same holds if $z_j - \xi_j < 0$. Since $\psi(t)$ can be chosen to be non-increasing for positive t, then $\varphi(z) \geq a_1 \ldots a_n$, where

$$a_j = \psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right)$$

or

$$a_j = \psi\left(\frac{|H_j(t_0, x, y) - \xi_j|}{F_j(\xi^*, r)}\right).$$

Since $0 \leq \psi \leq 1$,

$$a_j \geq \psi\left(\frac{|H_j(t_0, x, y) - \xi_j|}{F_j(\xi^*, r)}\right) \psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right)$$

for $1 \leq j \leq n$. Therefore,

$$\varphi(z) \geq \varphi(x)\varphi(H(t_0, x, y)).$$

The next three lemmas are needed in order to show that the hypothesis in Theorem A imply those in Theorems D and E.

Lemma 5.5. *Assume that Poincaré's inequality holds for w_1, w_2 with $q = 2$ and $\mu = 1$. Then*

$$\left(\frac{r(I)}{r(B)}\right)^2 \frac{w_2(I)}{w_2(B)} \leq c\,\frac{w_1(I)}{w_1(B)}$$

for any pair of δ-balls I, B, with $I \subset 2B$.

PROOF: Suppose $I = Q(u_0, r(I))$ and $B = Q(x, r(B))$ and define

$$F(u) = \sum_{j=1}^{n} \frac{|u_j - (u_0)_j|}{F_j(u_0^*, r(I))}\, r(I)\varphi(u)$$

where φ is the function described in lemma (5.4) associated with I (as opposed to B) and $k = 1/2$. If $u \in I$, by (1.8)

$$\left|\frac{\partial F}{\partial u_k}(u)\right| \leq \frac{r(I)}{F_k(u_0^*, r(I))} + \frac{\partial \varphi}{\partial u_k}(u) n r(I),$$

for $k \in \{1, \dots n\}$, and using the fact that $\lambda_k(u) = \lambda_k(u^*) \leq \lambda_k(H(u^*, r(I)))$ if $u \in I$ we get

$$\left|\lambda_k(u)\frac{\partial F}{\partial u_k}(u)\right| \leq \frac{F_k(u^*, r(I))}{F_k(u_0^*, r(I))} + n r(I)\lambda_k(u)\frac{\partial \varphi}{\partial u_k}(u)$$

and by Lemma 2.4 and the fact that $|\nabla_\lambda \varphi| \le c/r(I)$ we have $|\nabla_\lambda F(u)| \le c\chi_I$.
We have Poincaré's inequality for F, *i.e.*,

$$(5.6)\qquad \left(\frac{1}{w_2(B)}\int_{n4^{\eta+1}B} |F(u) - av_{n4^{\eta+1}B}F|^2 w_2(u)du\right)^{1/2}$$
$$\le cr(B)\left(\frac{1}{w_1(B)}\int_{na^2 4^{\eta+1}B} |\nabla_\lambda F(u)|^2 w_1(u)du\right)^{1/2},$$

where $\eta = \max_{j=1,\ldots,n}\{G_j\}$. The right side of (5.6) is bounded by $cr(B)\left(\frac{w_1(I)}{w_1(B)}\right)^{1/2}$ by doubling and the fact that $|\nabla_\lambda F| \le c\chi_I$. Now, if $u \notin \frac{1}{4}I$ there exists $k \in \{1, \ldots, n\}$ such that

$$|u_k - (u_0)_k| \ge F_k\left(u_0^*, \frac{1}{4}r(I)\right)$$

and then if $u \in \frac{1}{2}I \setminus \frac{1}{4}I$ (note that $\varphi(u) = 1$)

$$(5.7)\qquad F(u) \ge \frac{F_k\left(u_0^*, \frac{1}{4}r(I)\right)}{F_k(u_0^*, r(I))} r(I) \ge \left(\frac{1}{4}\right)^{G_k} r(I) \ge \frac{1}{4^\eta} r(I).$$

Also, if $u \in I$, $F(u) \le nr(I)$ and therefore

$$av_{n4^{\eta+1}B}F \le \frac{|I|}{|n4^{\eta+1}B|} nr(I).$$

But, by (1.10), $F_j(x_B^*, n4^{\eta+1}r(B)) \ge 2n4^\eta F_j(x_B^*, 2r(B))$, and by (1.11),

$$|n4^{\eta+1}B| \ge (2n4^\eta)^n |2B| \ge 2n4^\eta |2B|.$$

Hence, since $I \subset 2B$, $av_{n4^{\eta+1}B}F \le r(I)/2\cdot 4^\eta$ and then if $u \in \frac{1}{2}I \setminus \frac{1}{4}I$ (using also 5.7)),

$$|F(u) - av_{n4^{\eta+1}B}F| \ge cr(I).$$

Therefore, the left hand side of (5.6) is larger than a constant times

$$\left[\frac{(r(I))^2}{w_2(B)} w_2\left(\frac{1}{2} I \setminus \frac{1}{4} I\right)\right]^{1/2} \geq cr(I)\left(\frac{w_2(I)}{w_2(B)}\right)^{1/2},$$

where in the last inequality we used the fact that $w_2\left(\frac{1}{2} I \setminus \frac{1}{4} I\right) \simeq w_2(I)$, which is shown in the next lemma.

Lemma 5.8. *If w is a doubling weight then $W(Q(u, 2s) \setminus Q(u, s))$ is equivalent to $w(Q(u, s))$.*

PROOF: Choose $\eta \in Q(u, 2s)$ such that $\delta(u, \eta) = \frac{3s}{2}$. By Lemma 2.5,

$$Q\left(\eta, \frac{3\epsilon s}{2(2a^2)^{\zeta}}\right) \subset Q\left(u, (1+\epsilon)\frac{3s}{2}\right)$$

for any $0 < \epsilon < 1$.

Choose j such that $\delta(u, \eta) = \varphi_j(u^*, |\eta_j - u_j|)$. Then, if $y \in Q\left(\eta, \frac{3\epsilon s}{2(2a^2)^{\zeta}}\right)$,

$$F_j\left(u^*, \frac{3s}{2}\right) = |\eta_j - u_j| \leq |\eta_j - y_j| + |y_j - u_j|$$
$$\leq F_j\left(\eta^*, \frac{3\epsilon s}{2(2a^2)^{\zeta}}\right) + |y_j - u_j|,$$

By (1.10) and Lemma 2.4,

$$F_j\left(u^*, \frac{3s}{2}\right) \leq \epsilon F_j\left(u^*, \frac{3s}{2}\right) + |y_j - u_j|.$$

Thus,

$$|y_j - u_j| \geq (1-\epsilon) F_j\left(u^*, \frac{3s}{2}\right) \geq F_j\left(u^*, (1-\epsilon)\frac{3s}{2}\right).$$

If we choose $\epsilon = 1/3$ we have proved that

$$Q\left(\eta, \frac{s}{2(2a^2)^{\zeta}}\right) \subset Q(u, 2s) \setminus Q(u, s).$$

The lemma follows by doubling.

Lemma 5.9. *If $w_1 \in A_2$, $v \in A_\infty$ and Poincaré's inequality holds for w_1, v with $q = 2$ and $\mu = 1$, then condition* (1.21) *holds.*

PROOF: If $v \in A_\infty$ there exists $s > 1$ such that

$$\left(\frac{1}{|I|}\int_I \left(\frac{v}{v(B)}\right)^s dx\right)^{1/s} \le \frac{1}{|I|}\frac{v(I)}{v(B)}.$$

So, since Poincaré's inequality holds for w_1, v with $q = 2$, by Lemma 5.5

$$\left(\frac{r(I)}{r(B)}\right)^2 \left(\frac{1}{|I|}\int_I \left(\frac{v}{v(B)}\right)^s dx\right)^{1/s} \le c\,\frac{1}{|I|}\frac{w_1(I)}{w_1(B)},$$

and the above condition is equivalent to condition (1.18) since $w_1 \in A_2$.

Now we are ready to prove Theorem A.

PROOF OF THEOREM A

Let u be a non-negative solution of (1.1) in the cylinder $R_{\alpha,\beta} = R_{\alpha,\beta}(x_0, t_0) = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta)$. If we define $T(x, t) = (x, \beta t + t_0)$ and $\bar{u}(x, t) = u(T(x, t))$ then u is a solution in $R_{\alpha,1}(x_0, 0)$ of the equation

$$v(x)\bar{u}_t = \operatorname{div}(\bar{A}(x, t)\nabla \bar{u}),$$

where the coefficients matrix $\bar{A} = (\bar{a}_{ij})$ are defined by $\bar{a}_{ij}(x, t) = \beta a_{ij}(x, \beta t + t_0)$ and satisfies the degeneracy condition

$$\bar{w}_1(x)\sum_{j=1}^n \lambda_j^2(x)\xi_j^2 \le \sum_{j=1}^n \bar{a}_{ij}(x, t)\xi_i\xi_j \le \bar{w}_2(x)\sum_{j=1}^n \lambda_j^2(x)\xi_j^2,$$

if we put $\bar{w}_i = \beta w_i$, for $i = 1, 2$.

Suppose $|p| < [\alpha^{-2}\bar{\Lambda}(Q(x_0, \alpha)) + \alpha^2/\bar{\lambda}(Q(x_0, \alpha))]^{-1}$, where $\bar{\Lambda}(Q) = \bar{w}_2(Q)/v(Q)$, $\bar{\lambda}(Q) = \bar{w}_1(Q)/v(Q)$. Write

$$R^-(\varrho) = Q\left(x_0, \frac{(\varrho + 1)\alpha}{3}\right) \times \left(-\frac{1}{2} - \frac{\varrho}{2}, -\frac{1}{2} + \frac{\varrho}{2}\right)$$

$$R^+(\varrho) = Q\left(x_0, \frac{(\varrho + 1)\alpha}{3}\right) \times \left(\frac{1}{2} - \frac{\varrho}{2}, 1\right)$$

If we take $1/2 < \varrho < r \leq 1$ then the mean value inequalities in Theorem 3.15 applied to u give

$$(5.10)\quad \text{ess sup}_{R^-(\varrho)}\bar{u}^p \leq c\,\frac{1}{(r-\varrho)^m}\int\!\!\int_{R^-(r)}\bar{u}^p\left(\frac{\bar{w}_2}{\bar{w}_2(Q_\alpha)}+\frac{v}{v(Q_\alpha)}\right)dxdt,$$

for some $m > 0$, if $p > 0$, where $Q_\alpha = Q(x_0, \alpha)$, and

$$(5.11)\quad \text{ess sup}_{R^+(\varrho)}\bar{u}^p \leq c\,\frac{1}{(r-\varrho)^m}\int\!\!\int_{R^+(r)}\bar{u}^p\left(\frac{\bar{w}_2}{\bar{w}_2(Q_\alpha)}+\frac{v}{v(Q_\alpha)}\right)dxdt,$$

if $p < 0$. Moreover, by Theorem B, $\bar{u}$ is locally bounded and by adding $\epsilon > 0$, we may assume by letting $\epsilon \to 0$ at the end of the proof that $\bar{u}$ is bounded below in $R_{\alpha,1}(x_0, 0)$ by a positive constant.

Now, by Theorem F, we have

$$(5.12)\quad \left[\left(\frac{v}{v(Q_\alpha)}+\frac{\bar{w}}{\bar{w}_2(Q_\alpha)}\right)\otimes 1\right](E^+)$$
$$\leq \left\{\frac{1}{s}\,\frac{v(Q_\alpha)}{\bar{w}_1(Q_\alpha)}\,\alpha^2\right\}^\varkappa$$
$$\leq c\left\{\frac{1}{s}\left[\alpha^{-2}\bar{\Lambda}(Q_\alpha)+\alpha^2\frac{1}{\bar{\lambda}(Q_\alpha)}\right]\right\}^\varkappa,$$

and the same inequality holds for E^-, where E^+, E^- are defined in Theorem F with $u = \bar{u}$, $R = 2/3\alpha$, $a = -1$, $b = 1$, $t_0 = 0$, $M_2 \simeq \bar{\Lambda}(Q_\alpha)/\alpha^2$.

By (5.10) and (5.12), we can apply Bombieri's lemma to the family of rectangles $R^-(\varrho)$ with $\mu = \alpha^{-2}\bar{\Lambda}(Q_\alpha(x_0)) + \alpha^2/\bar{\lambda}(Q_\alpha(x_0))$, $M = 1/\mu$ and $f = e^{-M_2+V(0)}\bar{u}$, obtaining

$$\text{ess sup}_{R^-(1/2)}f \leq C\exp\{\,c[\alpha^{-2}\bar{\Lambda}(Q_\alpha) + \alpha^2/\bar{\lambda}(Q_\alpha)]\},$$

and this implies that

$$(5.13)\quad \text{ess sup}_{R^-(1/2)}\bar{u} \leq C\exp\{c[\alpha^{-2}\bar{\Lambda}(Q(x_0,\alpha)) + \alpha^2/\bar{\lambda}(Q(x_0,\alpha))] - V(0)\}.$$

Also, by (5.11) and (5.12), we can apply Bombieri's lemma to the family of rectangles $R^+(\varrho)$, $f = e^{-M_2-V(0)}\bar{u}^{-1}$, with μ, M, M_2 and $V(0)$ as before, and we obtain

$$\text{ess sup}_{R^+(1/2)} f \leq C \exp\{c[\alpha^{-2}\overline{\Lambda}(Q_\alpha) + \alpha^2/\overline{\lambda}(Q_\alpha)]\},$$

which implies that

$$e^{-V(0)} \leq C\, e^{c[\alpha^{-2}\overline{\Lambda}(Q(x_0,\alpha)) + \alpha^2/\overline{\lambda}(Q(x_0,\alpha))]} \text{ess inf}_{R^+(1/2)} \overline{u}. \tag{5.14}$$

Combining (5.13) and (5.14) it follows that

$$\text{ess sup}_{R^-(1/2)} \overline{u} \leq c_1 e^{c[\alpha^{-2}\overline{\Lambda}(Q(x_0,\alpha)) + \alpha^2/\overline{\lambda}(Q(x_0,\alpha))]} \text{ess inf}_{R^+(1/2)} \overline{u}.$$

Since, $T(R^-(1/2)) = R^-$, $T(R^+(1/2)) = R^+$ and $\alpha^{-2}\overline{\Lambda}(Q_\alpha) + \alpha^2/\overline{\lambda}(Q_\alpha) = \alpha^{-2}\beta\Lambda(Q_\alpha) + \alpha^2\beta^{-1}/\lambda(Q_\alpha)$, Theorem A follows.

REMARK: Using the equivalence between d and δ we can prove the following analogues of Theorem A and B for the metric d.

Theorem A′: *Assume* (i), (ii), (iii) *of Theorem* A. *If u is a non-negative solution of* (1.1) *in the cylinder* $R = S(x_0, \alpha a^2) \times (t_0 - \beta, t_0 + \beta)$, *then*

$$\text{ess sup}_{R^-} u \leq c_1 \exp\{c_2[\alpha^{-2}\beta\, \Lambda(S(x_0,\alpha)) + \alpha^2\beta^{-1}\lambda(S(x_0,\alpha))^{-1}]\}\, \text{ess inf}_{R^+} u$$

where $R^- = S(x_0, \alpha/2) \times (t_0 - 3\beta/4, t_0 - \beta/4)$, $R^+ = S(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta)$, $\Lambda(S) = w_2(S)/v(S)$ *and* $\lambda(S) = w_1(S)/v(S)$ *for a d-ball S. Here the constants* c_1, c_2 *depend only on the constants which arise in* (i), (ii), (iii).

Theorem B′: *Assume hypothesis* (i), (ii), (iii) *of Theorem* A *hold. Let* $0 < p < \infty$, $\alpha, \beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$ *and let* $S(x_0, \alpha) = S$, $S(x_0, \alpha') = S'$ *and* $R(\alpha, \beta) = S \times (t_0 - \beta, t_0 + \beta)$, $R'_+(\alpha, \beta) = S' \times (t_0 - \beta', t_0 + \beta)$. *If u is a solution of* (1.1) *in* $R(a^2\alpha, \beta)$, *then u is bounded in* $R'_+(\alpha, \beta)$ *and*

$$\text{ess sup}_{R'_+(\alpha,\beta)} |u|^p \leq$$

$$D(\alpha^2\beta^{-1}\lambda(S)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(S) + 1)^{h/(h-1)} \int\!\!\int_{R(a^2\alpha,\beta)} |u|^p(\alpha^{-2}\beta w_2 + v)\,dx\,dt$$

where D is as in Theorem B, and $C = c\dfrac{\alpha^{2+b}\beta}{(\alpha - \alpha')^{2+b}(\beta - \beta')}$. *Here* $h > 1$, *constants which are independent of* u, p, α, α', β, β'.

Acknowledgments

I wish to thank Prof. Richard Wheeden for having suggested this problem, as well as for his generous advice throughout the work.

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Recibido: 20 de julio de 1990.

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