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Harnack Inequality and Green Function for a Certain Class of Degenerate Elliptic Differential Operators

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Introduction

The main purpose of this work is to obtain a Harnack inequality and estimates for the Green function for the general class of degenerate elliptic operators described below. Let

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju), \tag{0.1}$$

where $A = [a_{ij}]$ is symmetric, measurable and satisfies the following ellipticity condition

$$v(x)\sum_{i=1}^{n}\lambda_i^2(x)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le w(x)\sum_{i=1}^{n}\lambda_i^2(x)\xi_i^2, \tag{0.2}$$

for every $\xi \in \mathbf{R}^n$ and almost every x in an open bounded set Ω of $\mathbf{R}^n$. The functions λ_i are defined on $\mathbf{R}^n$ and satisfy

(0.3) $\lambda_1 \equiv 1 \quad \lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1}) \in C(\mathbf{R}^n) \bigcap C^1(\mathbf{R}^n - \Pi)$ where

$$\Pi = \left\{x \in \mathbf{R}^n : \prod_{i=1}^{n} x_i = 0\right\} \quad \text{for} \quad j = 2, \ldots, n$$

(0.4) $\lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, -x_i, \ldots, x_{j-1}),\ i = 1, \ldots, j-1$

(0.5) $0 < \lambda_j(x) \leq \Lambda$ for every $x \in \mathbf{R}^n - \Pi$, $j = 1, \ldots, n$. Moreover, there exist non-negative numbers b_{ji} such that

$$0 \leq x_i(D_i\lambda_j)(x) \leq b_{ji}\lambda_j(x)$$

for $i = 1, \ldots, j-1$, $j = 2, \ldots, n$ and for every $x \in \mathbf{R}^n - \Pi$.

A vector $(\lambda_1, \ldots, \lambda_n)$ satisfying these properties generates a distance d and a quasi-distance δ on $\mathbf{R}^n$ in such a way that $(\mathbf{R}^n, d)$ and $(\mathbf{R}^n, \delta)$ become spaces of homogeneous type with the Lebesgue measure (see [CG], [CW] and [C]) and, moreover, there exists a constant $a > 1$ such that $a^{-1}\delta < d < a\delta$ (see [FL1]). The conditions on the pair of weights (v, w) can now be stated in terms of this geometry. Given $\alpha \in (0, 1]$ and $\sigma > 1$, we introduce the class $S_{\sigma,\alpha}$ as the class of pairs (v, w) such that satisfy

(0.6) $0 < v(Q),\ w(Q) < \infty$ for every δ-ball $Q \subset \Omega$, where

$$w(Q) = \int_Q w, \quad v(Q) = \int_Q v,$$

(0.7) there exists $C > 0$ such that

$$\left(\frac{w(Q_0 \cap Q)}{w(Q_0)}\right)^{1/2\sigma} (v^{-1}(Q_0 \cap Q)\, v(Q_0))^{1/2} \leq C\,|Q_0|^{1-\alpha}\,|Q|^{\alpha}$$

for every Q_0 and Q δ-balls in Ω such that radious $(Q) \leq 8\,a^2$ radious (Q_0).

Examples of operators satisfying the preceeding conditions are the following

(0.8) $Lu = -\operatorname{div}(d(0, x)^{\beta} D_1 u,\ d(0, x)^{-\beta}\,|x_1|^{\gamma} D_2 u)$

for $x = (x_1, x_2) \in \mathbf{R}^2$, $\gamma > 0$ and $\beta > 0$. Since our results will apply when $\alpha \in (1 - (\sum_j G_j)^{-1}, 1)$, $(G_1 = 1$ and $G_j = 1 + \sum_{i=1}^{j-1} b_{ji} G_i$, $j = 2, \ldots, n)$, we get Harnack's inequality and estimates for Green's function for the operator L in (0.8) when

$$1 - \frac{1}{2+\gamma} < \frac{1}{4}\,\frac{(4+\beta)(2-\beta)}{2+\beta}.$$

We point out that our results contain as special cases those in Moser ([M]), Fabes, Kenig and Serapioni ([FKS]), Fabes, Jerison and Kenig ([FJK]), Chanillo and Wheeden ([ChW2]), Franchi and Lanconelli ([FL2]) and ([FL3]) and Franchi and Serapioni ([FS]).

In Section 1, we present a brief survey of results of the particular geometry introduced by Franchi and Lanconelli. Section 2 is devoted to the construction of a family of δ-balls which resembles the dyadic cubes. In Section 3, we prove Sobolev and Poincaré inequalities. Section 4 contains an analysis of the relations among our conditions on (v, w) and those in the work of Chanillo and Wheeden. Finally, Section 5 contains the statements of the results about Harnack's inequality and estimates of Green's function.

1.

In this section we give the definitions of the natural distance d and the quasi-distance δ and state its basic properties.

Let us start introducing the notions of λ-subunit vector and λ-subunit curve: a vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{R}^n$ is a λ-subunit vector at a point x if

$$\left(\sum_{j=1}^{n} \gamma_j \xi_j\right)^2 \leq \sum_{j=1}^{n} \lambda_j^2(x)\xi_j^2, \quad \text{for every} \quad \xi \in \mathbf{R}^n;$$

we say that $\gamma\colon [0, T] \to \mathbf{R}^n$ is a λ-subunit curve if it is an absolutely continuous curve and $\dot{\gamma}(t)$ is a λ-subunit vector at $\gamma(t)$ for a.e. $t \in [0, T]$.

Definition 1.1. *For any $x, y \in \mathbf{R}^n$ we define $d\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_0^+$ as $d(x, y) = \inf\{T \in \mathbf{R}^n :$ there exists a λ-subunit curve $\gamma\colon [0, T] \to \mathbf{R}^n$, $\gamma(0) = x, \gamma(T) = y\}$.*

Remark 1.2 ([FL1], [FL3]): d is a well defined distance. In fact our hypotheses on $\lambda = (\lambda_1, \ldots, \lambda_n)$ guarantee the existence of a λ-subunit curve joining x and y, for any pair of points x and y.

Four our purposes it is useful to introduce a quasi-distance δ, more explicitely defined and sometimes easier than d to work with.

If $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ put $H_0(x, t) = x$ and $H_{k+1}(x, t) = H_k(x, t) + t\lambda_{k+1}(H_k(x, t))e_{k+1}$ for $k = 0, \ldots, n-1$. Here $\{e_k\}_{k=1}^n$ is the usual canonical basis in $\mathbf{R}^n$. It is clear that the function $s \to F_j(x, s) = s\lambda_j(H_{j-1}(x, s))$, is strictly increasing on $(0, \infty)$ for any $x = (x_1, \ldots, x_n)$ such that $x_k \geq 0$, $k = 1, \ldots, j-1$, and for $j = 1, \ldots, n$. Hence it is possible to define the inverse function of $F_j(x, \cdot)$, that is $\phi_j(x, \cdot) = (F_j(x, \cdot))^{-1}$ for $j = 1, \ldots, n$.

Definition 1.3. *For any $x, y \in \mathbf{R}^n$ we define $\delta\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_0^+$ as*

$$\delta(x, y) = \max_{j=1,\dots,n} \phi_j(x^*, |x_j - y_j|)$$

where $x^* = (|x_1|, \dots, |x_n|)$.

The following two Lemmas contain the basic properties of the functions F_j, ϕ_j, d and δ.

Lemma 1.4. *Put* $G_1 = 1$ *and* $G_j = 1 + \sum_{i=1}^{j-1} G_i b_{ji}$ *for* $j = 2, \dots, n$. *Then*

(1.5) *for every* $x \in \mathbf{R}^n$, $s > 0$, $\theta \in (0, 1)$ *we have*

$$\theta^{G_j} \leq \frac{F_j(x^*, \theta s)}{F_j(x^*, s)} \leq \theta,$$

$$\theta \leq \frac{\phi_j(x^*, \theta s)}{\phi_j(x^*, s)} \leq \theta^{1/G_j};$$

(1.6) *if* $x = (x_1, \dots, x_n)$ *and* $y = (y_1, \dots, y_n)$ *verifies* $|y_i| + F_i(y^*, \theta s) \leq |x_i| + KF_i(x^*, s)$, $i = 1, \dots, j$, *for some* $s > 0$, $\theta \in (0, 1]$ *and* $K \geq 1$, *we have*

$$\frac{F_{j+1}(j^*, \theta s)}{F_{j+1}(x^*, s)} \leq \theta K^{G_{j+1}-1}.$$

PROOF: For (1.5) see Proposition 4.3 of [FL2]. Let us prove (1.6) from (0.5) we get that λ_{j+1} is increasing in each variable on $\{x \in \mathbf{R}^n : x_k \geq 0 \ k = 1, \dots, j\}$, then

$$\begin{aligned} F_{j+1}(y^*, \theta s) &= \theta s \lambda_{j+1}(|y_1| + F_1(y^*, \theta s), \dots, |y_j| + F_j(y^*, \theta s)) \\ &\leq \theta s \lambda_{j+1}(|x_1| + F_1(x^*, Ks), \dots, |x_j| + F_j(x^*, Ks)) \\ &\leq \theta K^{G_{j+1}-1} F_{j+1}(x^*, s), \end{aligned}$$

the last inequality follows from (1.5).

In the sequel we shall use the following notation for d-balls, δ-balls and their dilations

$$\begin{aligned} S(x, r) &= \{y \in \mathbf{R}^n : d(x, y) < r\}, \\ Q(x, r) &= \{y \in \mathbf{R}^n : \delta(x, y) < r\} \\ \alpha S(x, r) &= S(x, \alpha r), \alpha Q(x, r) = Q(x, \alpha r), \alpha > 0. \end{aligned}$$

Lemma 1.7. *There exist constants $a, b, A \in (1, \infty)$, depending only on n and the constants in* (0.5), *such that*

$$(1.8)\quad \frac{1}{a} \leq \frac{d(x, y)}{\delta(x, y)} \leq a, \textit{ for all } \; x, y;$$

$$(1.9)\quad \frac{1}{b} \; |x - y| \leq d(x, y) \leq b\,|x - y|^{\eta}$$

if $|x - y| \leq 1$, *where* $\eta = \min_j \{1/G_j\}$;

$$(1.10)\quad |2S| \leq A\,|S|, \; |2Q| \leq A\,|Q| \textit{ for any d-ball } S \textit{ and any } \delta\textit{-ball } Q.$$

PROOF: For (1.8) see Theorems 2.6 and 2.7 in [FL1]. (1.9) and (1.10) follow immediately from the above Lemma and (1.8).

2.

Here we shall construct families of δ-balls that resembles the family of dyadic cubes. Let τ be the set of all n-tuples $s = l_1 \dots l_n$ with $l_i = -1, 0, 1$; $i = 1, \dots, n$. For $k \in \mathbf{Z}$; $l_i = -1, 0, 1$; $j_i \in \mathbf{Z}$ and $i = 1, \dots, n$, define

$$T^k_{l_1}\colon \mathbf{R}^n \to \mathbf{R}^n;\; T^k_{l_1}(x) = x + l_1 2^k e_1,$$
$$T^k_{l_1 \dots l_{i-1} l_i}\colon \mathbf{R}^n \to \mathbf{R}^n;\; T^k_{l_1 \dots l_{i-1} l_i}(x) = T^k_{l_1 \dots l_{i-1}}(x) + l_i F_i(T^k_{l_1 \dots l_{i-1}}(x)^*, 2k) e_i,$$

and

$$x_{j_1} = (2j_1 - 1)\, 2^k e_1,$$
$$x_{j_1 \dots j_i} = x_{j_1} + \dots + x_{j_1 \dots j_{i-1}} + (2j_i - 1) F_i(x^*_{j_1 \dots j_{i-1}}, 2^k) e_i.$$

For $k \in \mathbf{Z}$ and $s = l_1 \dots l_n \in \tau$ given, the family of δ-balls

$$D^{k,s} = \{Q(T^k_{l_1 \dots l_n}(x_{j_1 \dots j_n}), 2^k) : j_k \in \mathbf{Z};\;\; i = 1, \dots, n\}$$

is an a.e. covering of $\mathbf{R}^n$.

The following Lemma states the main property of these families.

Lemma 2.2. *For $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \mathbf{R}^n$ and $r > 0$, there exists $s \in \tau$ and $Q_0 \in D^{k,s}$ with $2^{k-1} < 2r \leq 2^k$ such that*

(2.3) $Q(\tilde{x}, r) \subset Q_0$,

(2.4) $|Q_0| \leq C|Q(\tilde{x}, r)|$, *where C is independent of* $\tilde{x}$, r, k *and* s.

PROOF. It is obvious that there exists a center c_1 of a δ-ball belonging to D^{k,s_1}, with $s_1 = 1 \ldots 1$, and a l_1 in $\{-1, 0, 1\}$ such that

$$x_1(T^k_{l_1 1 \ldots 1}(c_1)) \in [\tilde{x}_1 - F_1(\tilde{x}^*, r), \tilde{x}_1 + F_1(\tilde{x}^*, r)]$$
$$\subseteq [x_1(T^k_{l_1}(c_1)) - F_1(T^k_{l_1}(c_1)^*, 2^k), x_1(T^k_{l_1}(c_1)) + F_1(T^k_{l_1}(c_1)^*, 2^k)],$$

where $x_1(T^k_{l_1}(c_1))$ denote the first component of $T^k_{l_1}(c_1)$. Now, let us suppose that we have determined $l_2, \ldots, l_m$ and c_i, with $Q(c_i, 2^k) \in D^{k,s_i}$, $s_i = l_1 \ldots l_{i-1} 1 \ldots 1$, $i = 1, \ldots, m$ in such a way that

$$(2.5)\quad x_j(c_i) = x_j(\mathrm{T}^k_{1\ldots 1 l_j}(c_j)),\ j = 1, \ldots, i-1,$$
$$x_i(T^k_{l_1 \ldots l_i}(c_i)) \in [\tilde{x}_i - F_i(\tilde{x}^*, r), \tilde{x}_i + F_i(\tilde{x}, r)]$$
$$\subset [x_i(T^k_{l_1 \ldots l_i}(c_i)) - F_i(T^k_{1\ldots 1 l_i}(c_i)^*, 2^k), x_i(T^k_{l_1 \ldots l_i}(c_i)) + F_i(T^k_{1\ldots 1 l_i}(c_i)^*, 2^k)].$$

Then

$$|\tilde{x}_i| + F_i(\tilde{x}^*, r) \leq |x_i(T^k_{1\ldots 1 l_i}(c_i))| + F_i(T^k_{1\ldots 1 l_i}(c_i)^*, 2^k),$$

for $i = 1, \ldots, m$. Now, by using Lemma 1.4, we get

$$F_{m+1}(\tilde{x}^*, r) \leq 2F_{m+1}(T^k_{1\ldots 1 l_m}(c_m)^*, 2^k).$$

From this inequality it follows inmediately that there exists a center c_{m+1} of a δ-ball in $D^{k,s_{m+1}}$, where $s_{m+1} = l_1 \ldots l_m 1 \ldots 1$, and a value of l_{m+1} such that (2.5) holds with $i = m + 1$. The inductive process continues to obtain (2.5) for $i = n$. Then, taking $c = T^k_{1\ldots 1 l_n}(c_n)$, $Q_0 = Q(c, 2^k)$ and $s = l_1 \ldots l_n$, we get (2.3). Finally (2.4) follows from the choice of k and the doubling property (1.10).

3.

The main results of this section are the following.

Theorem 3.1. *Let* $\beta > 0$, *fixed, let* $Q = Q(\tilde{x}, r)$ *be a* δ*-ball such that* $Q \subset \Omega$ *and let* $(v, w) \in S_{\sigma,\alpha}$ *for a given* $\alpha \in \left(1 - \left(\sum_j G_j\right)^{-1}, 1\right]$. *Then, for each* $u \in C^1(\overline{Q})$ *such that verifies*

(3.2) $\quad |\{x \in Q : u(x) = 0\}| \geq \beta |Q|,$

we get

$$\left(\frac{1}{w(Q)} \int_Q |u|^{2\sigma} w dx\right)^{1/2\sigma} \leq Cr \left(\frac{1}{v(Q)} \int_Q |\nabla_\lambda u|^2 v dx\right)^{1/2}, \tag{3.3}$$

where C depends only on n and the constants in (0.5) and (0.7), and $\nabla_\lambda u = (\lambda_1 D_1 u, \ldots, \lambda_n D_n u)$.

Theorem 3.4 (Sobolev inequality). *Let Q and (v, w) be as in the above theorem. Then, for any $u \in C_0^1\left(\frac{1}{2} Q\right)$ we get*

$$\left(\frac{1}{w(Q)} \int_{1/2Q} |u|^{2\sigma} w dx\right)^{1/2\sigma} \leq Cr \left(\frac{1}{v(Q)} \int_{1/2Q} |\nabla_\lambda u| \, v dx\right)^{1/2}, \tag{3.5}$$

where C depends only on n and the constants in (0.5) *and* (0.7).

Theorem 3.6 (Poincaré inequality): *Let Q and (v, w) be as in Theorem* 3.1. *Then, for any $u \in C^1(\overline{Q})$ we get*

$$\left(\frac{1}{w(Q)} \int_Q |u - u_Q|^{2\sigma} w dx\right)^{1/2\sigma} \leq Cr \left(\frac{1}{v(Q)} \int_Q |\nabla_\lambda u|^2 v dx\right)^{1/2}, \tag{3.7}$$

where $u_Q = \int_Q u \, w dx$ and C depends only on n and the constants in (0.5) *and* (0.7).

The proof of Theorem 3.1 is based on an estimate of u in terms of certain fractional integral operators applied to $\nabla_\lambda u$ and on a norm inequality with two weights for these operators.

Let us start with the definition of these operators.

Definition 3.8. *Let $k \in \mathbf{Z}$, $s \in \tau$ and $\mu \in (0, 1]$. We define*

$$(P_\mu^{k,s} f)(x) = \begin{cases} \dfrac{1}{|Q|^\mu} \displaystyle\int_Q |f(y)| \, dy, & \text{if } x \in Q \in D^{k,s} \\ 0 & \text{if } x \in \mathbf{R}^n - \displaystyle\bigcup_{Q \in D^{k,s}} Q \end{cases}$$

for $f \in L^1_{loc}(\mathbf{R}^n)$.

Now, with these operators, we can prove

Lemma 3.9. *Let* $\beta > 0$, *fixed, and let* $Q = Q(\tilde{x}, r)$ *be a* δ*-ball. If* $u \in C^1(\overline{Q})$ *verifies*

$$|\{x \in Q : u(x) = 0\}| \geq \beta |Q|$$

then, for each $\mu \in \left(1 - \left(\sum_j G_j\right)^{-1}, 1\right]$ *there exist a sequence* $\{a_i\}_{i=1}^{\infty} \subset \mathbf{R}^+$ *depending only on* μ *with* $\sum a_i < \infty$, *a sequence of integers* $\{k_i\}$ *depending only on* r *and a constant* C *such that*

$$|u(x)| \leq Cr|Q|^{\mu-1} \sum_{i=1}^{\infty} a_i \sum_{s \in \tau} (\mathrm{P}_{\mu}^{k_i,s}(\chi_Q |\nabla_\lambda u|))(x)$$

for all $x \in Q$.

PROOF. By using similar techniques than those in Lemma 4.3 in [FS] and a dyadic partition, we get

$$\begin{aligned}(3.10)\qquad |u(x)| &\leq C_0 \int_0^{2a^2r} \frac{1}{|S(x,t)|} \int_{S(x,C_0t)} |\nabla_\lambda u(y)| \chi_Q(y)\,dy\,dt \\ &\leq C_0 r \sum_{i=1}^{\infty} \left(\int_{\frac{a^2r}{2^{i-1}}}^{\frac{a^2r}{2^{i-2}}} \frac{dt}{|S(x,tr)|^{1-\mu}} \right) \\ &\qquad \frac{1}{\left|S\left(x, \frac{a^2r}{2^{i-1}}\right)\right|^{\mu}} \int_{S(x,\, C_0a^2r/2^{i-2})} |\nabla_\lambda u(y)| \chi_Q(y)\,dy,\end{aligned}$$

for all $x \in Q$. From Lemmas 1.7 and 2.2 follows that for each i there exists $k_i \in \mathbf{Z}$, $s_i \in \tau$ and $Q_i \in D^{k_i,s_i}$ such that

$$2^{k_i-1} < \frac{C_0 a^3 r}{2^{i-3}} \leq 2^{k_i},$$

$$S\left(x, \frac{C_0 a^2 r}{2^{i-1}}\right) \subset Q_i,$$

$$|Q_i| \leq C \left| S\left(x, \frac{C_0 a^2 r}{2^{i-1}}\right) \right|,$$

On the other hand, by (1.5) and (1.8), we get $|S(x, tr)| \geq Ct^{\sum G_j} |S(x, r)|$ for $t \in (0, 1]$. Then, taking $i_0 \in \mathbf{N}$ such that $a^2/2^{i_0-1} < 1 \leq a^2/2^{i_0-2}$, we obtain from (3.10)

$$|u(x)| \leq Cr \sum_{i=1}^{i_0} \left(\int_{\frac{a^2}{2^{i-1}}}^{\frac{a^2}{2^{i-2}}} \frac{dt}{|S(x, tr)|^{1-\mu}} \right) \frac{1}{|Q_i|^{\mu}} \int_{Q_i} |\nabla_\lambda u(y)| \chi_Q(y) dy$$

$$+ \frac{1}{|S(x, r)|^{1-\mu}} \sum_{i=i_0+1}^{\infty} \left(\int_{\frac{a}{2^{i-1}}}^{\frac{a^2}{2^{i-1}}} t^{-(\sum_j G_j)(1-\mu)} dt \right) \frac{1}{|Q_i|^{\mu}} \int_{Q_i} |\nabla_\lambda u(y)| \chi_Q(y) dy$$

$$\leq Cr |S(x, r)|^{\mu-1} \sum_{i=1}^{\infty} a_i \sum_{s \in \tau} (P_\mu^{k_i, s}(|\nabla_\lambda u| \nu_Q))(x)$$

where $a_i = 1$ for $i = 1, \ldots, i_0$ and $a_i = 2^{(i-1)[(\mu-1)(\sum_j G_j)+1]}$. Thus, since $|Q| \simeq |S(x, r)|$ we get the thesis.

Remark 3.11. From the proof is easy to see that $2^{k_i} \geq 8a^2 r$ then $Q \subset S(\tilde{x}, ar) \subset S(x, 2ar) \subset S(x, C_0 ar/2^{i-2})$ and thus

$$P_\mu^{k_i, s_i}(|\nabla_\lambda u| \chi_Q)(x) = \frac{1}{|Q_i|^{\mu}} \int_Q |\nabla_\lambda u| \, dy.$$

In the following Lemma we prove a two weight norm inequality for the operators $P_\mu^{x,s}$. The proof is based on techniques of E. T. Sawyer (see [S]).

Lemma 3.12. *Suppose $1 < p \leq q < \infty$. Let $E \subset \mathbf{R}^n$ be a bounded open set and let (v, w) be a pair of non negative integrable functions defined on E. Then*

$$\left(\int_E |P_\mu^{k,s} f|^q w dx \right)^{1/q} \leq C_0 \left(\int_E |f|^p v dx \right)^{1/p}, \tag{3.13}$$

for all $f \in L^p(E, vdx)$ with supp $f \subset E$, *if and only if*

$$(3.14)\quad w(E \cap Q)^{1/q} \left(\frac{v^{1/p-1}(E \cap Q)}{|Q|^{\mu}} \right)^{1-1/p} \leq C_0$$

for all $Q \in D^{k,s}$. The constant C_0 in (3.13) *and* (3.14) *is the same.*

PROOF. For sake of simplicity P_μ and D denote $P_\mu^{k,s}$ and $D^{k,s}$ respectively. Now assume that (3.13) holds. Let us first show that $v^{-1/(p-1)}\chi_{E \cap Q} \in L^p(E, vdx)$ for $Q \in D$. Suppose this is not the case, then since

$$\int_{E \cap Q} v^{-1/(p-1)} dx = \int_{E \cap Q} v^{-p/(p-1)} v dx = \infty,$$

we can find a g in $L^p(E, vdx)$ such that

$$\int_{E \cap Q} g v^{-1} v dx = \infty,$$

which is a contradiction with (3.13) taking $f = g$ on E for every $Q \in D$. We get (3.14) by taking $f = v^{-1/(p-1)}\chi_{Q_0 \cap Q}$ in (3.13). Conversely assume that (3.14) holds. Then

$$\int_E |P_\mu f|\, wdx = \sum_{Q \in D} \int_{E \cap Q} \left(\frac{1}{|Q|^{\mu}} \int_Q |f|\, dy \right)^q wdx$$

$$\leq \sum_{Q \in D} w(E \cap Q) \left(\frac{v^{-1/(p-1)}(E \cap Q)}{|f|^{\mu}} \right)^{q(p-1)/p} \left(\int_Q |f|^p v dx \right)^{q/p}.$$

Finally, from (3.14) and the fact that,

$$\sum_{Q \in D} \left(\int_Q |f|^p v dx \right)^{q/p} \leq \left(\sum_{Q \in D} \int_Q |f|^p v dx \right)^{q/p}$$

for $q \geq p$, we obtain (3.13).

PROOF OF THEOREM (3.1). From Lemma (3.9) we get

$$\left(\frac{1}{w(Q)}\int_Q |u|^{2\sigma} wdx\right)^{1/2\sigma}$$
$$\leq \frac{Cr}{|Q|^{1-\mu}} \sum_{i=1}^{\infty} a_i \sum_{s\in\tau} \left(\frac{1}{w(Q)}\int_Q |P_\mu^{k_i,s}(|\nabla_\lambda u|\chi_Q)|^{2\sigma} wdx\right)^{1/2\sigma}.$$

Now, the inequality (3.3) follows by applying (0.7), (0.8) and Lemma 3.12 for the k_i such that $2^{k_i} \leq 8a^2 r$ and the Remark 3.11, the Schwartz inequality and (0.8) for the remainders.

PROOF OF THEOREM 3.4. We need only to apply the above theorem in the ball Q, keeping in mind that, by the doubling property (1.10), it follow inmediately that $|Q - (1/2)Q| \simeq |Q|$.

PROOF OF THEOREM 3.6. With Q and u given it is always possible to find a number $b = b(Q, u)$ sich that $Q^+ = \{x \in Q : u(x) \geq b\}$ and $Q^- = \{x \in Q : u(x) \leq b\}$ verifies

$$(3.15) \quad |Q^+| \geq \frac{1}{2}|Q| \quad \text{and} \quad |Q^-| \geq \frac{1}{2}|Q|.$$

Assume this fact, then both functions $(u - b)^+$ and $(u - b)^-$ satisfy the hipotheses of Theorem (3.1) with $\beta = 1/2$. By that Theorem we get

$$\frac{1}{w(Q)}\int_{Q^\pm} |u - b|^{2\sigma} wdx \leq (Cr)^{2\sigma}\left(\frac{1}{w(Q)}\int_{Q^\pm} |\nabla_\lambda u|^2 vdx\right)^{\sigma},$$

adding these two inequalities we have

$$\frac{1}{w(Q)}\int_Q |u - b|^{2\sigma} wdx \leq 2(Cr)^{2\sigma}\left(\frac{1}{v(Q)}\int_Q |\nabla_\lambda u|^2 vdx\right)^{\sigma}.$$

Then, since

$$\left(\int_Q |u - u_Q|^{2\sigma} w dx\right)^{1/2\sigma} \leq \left(\int_Q |u - b|^{2\sigma} w dx\right)^{1/2\sigma}$$

$$+ \left(\frac{1}{w(Q)} \int_Q |u - b| \, w dx\right) w(Q)^{1/2\sigma}$$

$$\leq 2 \left(\int_Q |u - b|^{2\sigma} w dx\right)^{1/2\sigma},$$

we obtain the thesis. Let us prove (3.15). Observe that the two functions $\phi(t) = |\{x \in Q: u(x) \leq t\}|$ and $\psi(t) = |\{x \in Q: u(x) \geq t\}|$ are respectively increasing, right-continuous and decreasing, left-continuous. Define $b = \inf\{t: \phi(t) \geq 1/2|Q|\}$ then by the right-continuity $\phi(b) \geq 1/2|Q|$. Suppose now, by contradiction, that $\psi(b) < 1/2|Q|$. Then by the left-continuity there is $t < b$ such that $\psi(t) < |Q|/2$, so that $\phi(t) > |Q|/2$ and this contradicts the definition of b. Finally (3.15) holds.

4.

In [FS], B. Franchi and R. Serapioni prove inequalities of type (3.3), (3.5) and (3.7) for the case $v = Cw$. The assumption on the weight is that $w \in A_2$ respect to the d-balls, *i.e.:* $w(S) w^{-1}(S) \simeq |S|^2$ for all d-ball S. Inequalities of the same type for the euclidean case, *i.e.:* $\lambda_i = 1$ for all i, have been proved by S. Chanillo and R. Wheeden in [ChW1]. The hypotheses on the pair of weights in that work are the euclidean case of

(4.1) $w \in D_\infty$ respect to δ- balls, *i.e.:* $w(Q) \simeq w(2Q)$ for every δ-ball Q

(4.2) $v \in A_2$ respect to δ-balls, *i.e.:* $v(Q) v^{-1}(Q) \simeq |Q|^2$ for every δ-ball Q

(4.3) there exists $\sigma > 1$ and $C > 0$ such that

$$\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{1/2\sigma} \leq C \left(\frac{v(\theta Q)}{v(Q)}\right)^{1/2}, \text{ for every } \delta\text{-ball } Q, \text{ and } \theta \in (0,1],$$

We say that (v, w) belongs to C_σ if (v, w) satisfy the conditions (4.1), (4.2) and (4.3). The main purpose of this section is to find relations among the conditions of type A_2 and C_σ and the condition $S_{\sigma,\alpha}$. We begin with the following result.

Lemma 4.4. *Let* $(v, w) \in S_{\sigma,\alpha}$ *for some* $\alpha \in [1 - (\Sigma_j G_j)^{-1}, 1]$ *and* $\Omega = \mathrm{R}^n$. *The* (v, w) *verifies* (4.2) *and* (4.3).

PROOF. By taking $Q_0 = Q$ in (0.7), we get $v(Q_0)v^{-1}(Q_0) \le C \mid Q_0 \mid^2$ and thus v satisfy (4.2). Now assume that $\alpha = 1 - (\Sigma_j G_j)^{-1}$ then, from (0.7), we get

$$\left(\frac{w(\theta Q)}{w(Q)}\right)^{1/2\sigma} (v^{-1}(\theta Q)v(Q))^{1/2} \le C \mid Q \mid^{(\sum_j G_j)^{-1}} \mid \theta Q \mid^{1-(\sum_j G_j)^{-1}}$$

for any δ-ball Q and any $\theta \in (0, 1]$. From this and Hölder inequality follows (4.3), in fact

$$\left(\frac{\mid \theta Q \mid}{\mid Q \mid}\right)^{(\sum_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{1/2\sigma} \le C \frac{\mid \theta Q \mid}{(v^{-1}(\theta Q)v(Q))^{1/2}}$$

$$\le C \left(\frac{v(\theta Q)}{v(Q)}\right)^{1/2}.$$

Now, to complete the proof, it is sufficient to prove that if $\alpha_1, \alpha_2 \in (0, 1]$ and $\alpha_1 > \alpha_2$ then $S_{\sigma,\alpha_1} \subset S_{\sigma,\alpha_2}$. Note that only is necessary to prove that (0.7) with $\alpha = \alpha_2$ holds. This is trivial if $w(Q_0 \cap Q)$ or $v^{-1}(Q_0 \cap Q)$ is zero. Assume that both $w(Q_0 \cap Q)$ are positive. The inequality in (0.7) with $\alpha = \alpha_1$ is equivalent to

$$\left(\frac{\mid Q_0 \mid}{\mid Q \mid}\right)^{\alpha_1} \le C \frac{\mid Q_0 \mid}{(v^{-1}(Q_0 \cap Q)v(Q_0))^{1/2}} \left(\frac{w(Q_0)}{w(Q_0 \cap Q)}\right)^{1/2\sigma}. \tag{4.5}$$

On the other hand, since $Q_0 \cap Q \neq \emptyset$ and radious $(Q) \le 8a^2$ radious (Q_0), the doubling property (1.10) allow us to write

$$\left(\frac{|Q_0|}{|Q|}\right)^{\alpha_1} \leq C\left(\frac{|Q_0|}{|Q|}\right)^{\alpha_1}$$

with C independent of Q_0 and Q. From this and (4.5) follows that $(v, w) \in S_{\sigma,\alpha_2}$.

We shall next show that condition C_σ implies a condition $S_{\sigma,\alpha}$. In the proof of this fact we shall use the following result.

Lemma 4.6. *Let* (v, w) *be a pair of non negative weights satisfying* (4.1) *and* (4.3). *Then there exist* $\eta \in (0, 1)$ *and* $\sigma' \in (1, \sigma)$ *such that the inequality* (4.3) *holds with* $(\Sigma_j G_j)/\eta$ instead of $\Sigma\, G_j$ and σ' instead of σ.

PROOF. Since $w \in D_\infty$ we get

$$\begin{aligned} w(Q) &= w\left(\frac{1}{2}\,Q\right) + w\left(Q - \frac{1}{2}\,Q\right) \\ &\geq (1 + C)\, w\left(\frac{1}{2}\,Q\right) \end{aligned}$$

for every δ-ball Q. By iteration we have a $\beta \geq 1$ such that

$$(4.7)\qquad \frac{w(\theta Q)}{w(Q)} \geq C\theta^\beta \quad \text{for every} \quad \theta \in (0, 1], \quad \text{and every} \quad \delta\text{-ball } Q.$$

On the other hand, from (1.5) it follows that

$$|\theta Q| \geq \theta^{\Sigma_j G_j} |Q| \quad \text{for every} \quad \theta \in (0, 1], \quad \text{and every} \quad \delta\text{-ball } Q.$$

Then, this inequality (4.7) and (4.4) allow us to obtain the inequality

$$\begin{aligned} \left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}(1-\epsilon\beta)} \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}+\epsilon} &\leq C\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}(1-\epsilon\beta)} \theta^{\epsilon\beta}\left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}} \\ &\leq C\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}} \\ &\leq C\left(\frac{v(\theta Q)}{v(Q)}\right)^{1/2}, \end{aligned}$$

for all $\theta \in (0, 1]$, $\epsilon > 0$, and all δ-ball Q. Finally, by taking ϵ in $(0, \min\{1/\beta, (1 - 1/\sigma)/2\sigma\})$, we get the thesis with $\eta = 1 - \epsilon\beta$ and $\sigma' = \dfrac{\sigma}{1 + 2\epsilon\sigma}$.

Lemma 4.8. *Let $(v, w) \in C_\sigma$. Then $(v, w) \in S_{\sigma',\alpha}$ for some $\alpha \in ((\Sigma_j G_j)^{-1}, 1]$ and some $\sigma' \in (1, \sigma)$.*

PROOF. We only need to prove (0.7). Let $Q_0 = Q(x_1, r_1)$ and $Q = Q(x_2, r_2)$ be two δ-balls such that $r_2 \le 8a^2 r_1$. If $Q_0 \cap Q = \emptyset$ there is nothing to prove. Assume $Q_0 \cap Q \neq \emptyset$, then there exists $C_1 = C_1(a) > 1$ and $C_2 = C_2(a)$ such that $Q_0 \subset \tilde{Q} = (C_1 r_1/r_2)\, Q \subset C_2 Q_0$. Now, let σ' and η be as in the above Lemma and $\theta = r_2/C_1 r_1$. Then, from (4.1), (4.2), (1.10) and Lemma 4.6 it follows that

$$\left(\frac{w(Q_0 \cap Q)}{w(Q_0)}\right)^{1/2\sigma'} (v^{-1}(Q_0 \cap Q)v(Q_0))^{1/2}$$

$$\le C\left(\frac{w(\theta\tilde{Q})}{w(\tilde{Q})}\right)^{1/2\sigma'} (v^{-1}(\theta\tilde{Q})v(\tilde{Q}))^{1/2}$$

$$\le C\left(\frac{|\tilde{Q}|}{|\theta\tilde{Q}|}\right)^{(\Sigma_j G_j)^{-1}\eta} (v^{-1}(\theta\tilde{Q})v(\theta\tilde{Q}))^{1/2}$$

$$\le C|\tilde{Q}|^{(\Sigma_j G_j)^{-1}\eta}\, |\theta\tilde{Q}|^{1-(\Sigma_j G_j)^{-1}\eta}$$

$$\le C|Q_0|^{(\Sigma_j G_j)^{-1}\eta}\, |Q|^{1-(\Sigma_j G_j)^{-1}\eta}.$$

Thus, $(v, w) \in S_{\sigma',\alpha}$ with $\alpha = 1 - (\Sigma_j G_j)^{-1}\eta$.

Now, by using the above result, we get

Lemma 4.9. *Let $w \in A_2$ with respect to d or δ-balls. Then $(w, w) \in S_{\sigma,\alpha}$ for some $\sigma > 1$ and some $\alpha \in (1 - (\Sigma_j G_j)^{-1}, 1]$.*

PROOF. From the previous Lemma, we only need to prove that $(w, w) \in C_\sigma$ for some $\sigma > 1$. We know that

$$\frac{w(\theta Q)}{w(Q)} \geq C \frac{w(\theta Q) w^{-1}(Q)}{|Q|^2} \geq C \frac{w(\theta Q) w^{-1}(\theta Q)}{|Q|^2} \geq C \left(\frac{|\theta Q|}{|Q|}\right)^2$$

for all $\theta \in (0, 1]$ and all δ-ball Q. Then, by taking $\sigma > 1$ such that $1/\sigma > 1 - (\Sigma_j G_j)^{-1}$, we get

$$\left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}} = \left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}+\frac{1}{2}-\frac{1}{2}}$$

$$\leq C \left(\frac{w(\theta Q)}{W(Q)}\right)^{\frac{1}{2}} \left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}+\frac{1}{\sigma}-1}$$

$$\leq C \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2}}.$$

Thus $(w, w) \in C_\sigma$.

Let us describe some examples of pairs (v, w) that satisfy the hypotheses (0.6) and (0.7) for any $\lambda_1, \ldots, \lambda_n$ in the conditions (0.3) to (0.5).

EXAMPLE 4.10. In [FS], Franchi and Serapioni prove that $d(0, x)^\beta$ for $\beta \in (-n, n)$ is a weight in A_2 with respect to d-balls. In particular they prove the following inequalities

$$(4.11) \quad \int_{S(y,r)} d(0, x)^\beta dx \leq [d(0, y) + r\beta |\beta|]^\beta |S(y, r)|$$

if $d(0, y) \geq 2r$,

$$\int_{S(y,r)} d(0, x)^\beta dx \leq Cr^\beta |S(y, r)|,$$

if $d(0, y) \leq 2r$.

These facts allows us to prove that there exists values of β in $(0, n)$ such that $v(x) = d(0, x)^{\beta}$ and $w(x) = v(x)^{-1}$ belong to a class $S_{\sigma,\alpha}$ for some $\sigma > 1$ and some α in $(1 - (\Sigma_j G_j)^{-1}, 1]$. Since both v and w belong to A_2 we only need to show that there exists σ and α such that (0.7) holds with S instead of Q_0 and θS instead of Q for any θ in $(0.1]$, where S is any d-ball. Let us now prove this fact. First, note that, by the A_2 condition, we get

$$(4.12)\quad \left(\frac{w(\theta S)}{w(S)}\right)^{\frac{1}{2\sigma}} (v^{-1}(\theta S)v(S))^{1/2} \leq C \frac{(w(\theta S)w^{-1}(S))^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/2}}$$

for all $\sigma > 1$. Let $S = S(y, r)$. If $d(0, y) \geq 2r$, from (4.11) we get

$$\frac{(w(\theta S)w^{-1}(S))^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/\sigma}} \leq 3^{\beta} |S|^{1-\frac{1}{2}\left(1+\frac{1}{\sigma}\right)} |\theta S|^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}.$$

On the other hand, if $d(0, y) \leq 2\theta r$, from the same inequalities and Lemmas 1.4 and 1.7 it follows that

$$\frac{(w(\theta S)w^{-1}(S))^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/\sigma}} \leq C \frac{(\theta^{-\beta}|S|\,|\theta S|)^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/\sigma}}$$

$$\leq C|S|^{1-\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)} |\theta S|^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)}.$$

The case $2\theta r < d(0, y) < 2r$ follows in a similar way, so we get that (v, w) belongs $S_{\sigma,\alpha}$ for

$$\alpha = \frac{1}{2}\left(1 + \frac{1}{\sigma}\right)\left(1 - \frac{\beta}{n}\right).$$

Then, by taking $\sigma = 1 + \beta/n$, we can choose $\beta_0 \in (0, n)$ such that $\alpha \in (1 - (\Sigma_j G_j)^{-1}, 1]$ for all $\beta \in (0, \beta_0]$.

EXAMPLE 4.13. Let $w(x) = d(0, x)^{-\beta}$ for $\beta \in (0, n)$. From (4.11) and Lemmas 1.4 and 1.7 we get

$$(4.14)\quad \frac{w(\theta S)}{w(S)} \leq C\left(\frac{|\theta S|}{|S|}\right)^{1-\frac{\beta}{n}} \text{ for every } d\text{-ball } S \text{ and all } \theta \in (0, 1].$$

The for $\nu \equiv 1$ we get that for any $\sigma > 1$ and $\epsilon \in (0, 1)$, it holds

$$\left(\frac{w(\theta S)}{w(S)}\right)^{\frac{1}{2\sigma}} \leq C\left(\frac{|\theta S|}{|S|}\right)^{\left(1-\frac{\beta}{n}\right)\frac{1}{2\sigma}}\left(\frac{|\theta S|}{\nu^{-1}(\theta S)}\right)^{\frac{1}{2}}$$

$$\leq C\frac{|S|^{1-\left(\frac{\epsilon}{2}+\left(1-\frac{\beta}{n}\right)\frac{1}{2\sigma}\right)}|\theta S|^{\frac{\epsilon}{2}+\left(1-\frac{\beta}{n}\right)\frac{1}{2\sigma}}}{(\nu^{-1}(\theta S)\nu(S))^{1/2}}.$$

Thus, $(\nu, w) \in S_{\sigma,\alpha}$ for $\alpha = \epsilon/2 + (1 - \beta/n)/2\sigma$. By taking $\epsilon = 1 - \mathrm{b}/n$ and $\sigma = 1 + \beta/n$, we can choose $\beta_0 \in (0, n)$ such that $\alpha \in (1 - \Sigma_j G_j)^{-1}, 1]$ for every $\beta \in (0, \beta_0]$.

EXAMPLE 4.15. Let $\nu(x) = d(0, x)^\beta$, $\beta \in (0, n)$, and $w(x) \equiv 1$. Then, from (4.14), we get

$$\left(\frac{w(\theta S)}{w(S)}\right)^{\frac{1}{2\sigma}}(\nu^{-1}(\theta S)\nu(S))^{1/2} \leq C|S|^{1-\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)}|\theta S|^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)}$$

for all $\sigma > 1$. Thus, by reasoning as in the preceding examples, we get that there exists $\beta_0 \in (0, n)$ such that $(\nu, w) \in S_{\sigma,\alpha}$, for some $\sigma > 1$ and some $\alpha \in (1 - (\Sigma G_j)^{-1}, 1]$, for all $\beta \in (0, \beta_0]$.

5.

Let S be a d-ball such that $2a^2S \subset \Omega$. For ϕ and ψ in Lip $(\bar{S})$ we define

$$(5.1)\quad a_0(\phi, \psi) = \int_S \langle A\nabla\phi, \nabla\psi\rangle$$

$$(5.2)\quad a(\phi, \psi) = a_0(\phi, \psi) + \int_S \phi\psi w.$$

It is easy to prove that (5.1) defines a scalar product in $\mathrm{Lip}_0(S)$ and that (5.2) defines a scalar product in $\mathrm{Lip}(\bar{S})$.

Definition 5.3. *We denote with $H_0(S)$ and $H(S)$ to the completion of* $\mathrm{Lip}_0(S)$ *and* $\mathrm{Lip}(\bar{S})$ *respect to the norms* $\|\cdot\|_0 = a_0(\cdot\,,\,\cdot)^{1/2}$ *and* $\|\cdot\| = a(\cdot\,,\,\cdot)^{1/2}$, *respectively.*

Remark 5.4. From Sobolev inequality (Theorem 3.4) we get $H_0(S) \subset H(S)$.

Remark 5.5. It is possible to associate a function in $L^2(S, wdx)$ to each element in $H(S)$ and define its derivative as functions in $L^2(S, vdx)$.

Definition 5.6. *Let f be such that $f/w \in L^{2\sigma/(2\sigma-1)}(S, wdx)$ and let $\psi \in H(S)$. We say that $u \in H(S)$ is a solution of*

$$\begin{aligned} Lu &= f \quad in \quad S \\ u &= \psi \quad in \quad \partial S \end{aligned}$$

if

$$a_0(u, \phi) = \int_S u\phi \quad for\ all \quad \phi \in H_0(S),$$

and $u - \psi \in H_0(S)$.

Definition 5.7. *Let $F = (f_1, \ldots, f_n)$ be such that $|F|/v \in L^2(S, vdx)$ and let $\psi \in H(S)$. We say that $u \in H(S)$ is solution of*

$$\begin{aligned} Lu &= -\mathrm{div}_\lambda F \quad in \quad S \\ u &= \psi \qquad\qquad in \quad \partial S \end{aligned}$$

if

$$a_0(u, \phi) = \int_S \langle F, \nabla_\lambda \phi\rangle \quad for\ all \quad \phi \in H_0(S),$$

and $u - \psi \in H_0(S)$.

Remark 5.8. We can prove, by the representation theorem for continuous linear functional on Hilbert spaces, the existence and uniqueness of solutions for the above Dirichlet problems.

Remark 5.9. The above definitions and remarks hold if we change d-balls by δ-balls.

By using the results of Sections 3 and 4, and the technique in [ChW2], we get

Theorem 5.10 (Harnack inequality). *Let $Q_0 = Q(\tilde{x}, 4R)$ be a δ-ball in Ω. If $u \in H(Q_0)$ is a non-negative solution of $Lu = 0$ and $Q = (1/4)Q_0$ then*

$$\sup_Q u \leq \exp\left\{C\left(\frac{w(2Q)}{w((1/2)Q)}\right)^{\gamma}\left(\frac{w(2Q)}{v(2Q)}\right)^{1/2}\right\} \inf_Q u,$$

where $\gamma = (3\sigma^2 - 2\sigma + 1)/(\sigma - 1)$ and C depends only on the constants in (0.5) *and* (0.7).

Let S be a d-ball such that $2aS \subset \Omega$. For $y \in S$ and $\varrho > 0$ fixed such that $Q_\varrho = Q(y, \varrho) \subset S$, we define the mapping

$$\psi \rightarrow \frac{1}{w(Q_\varrho)} \int_{Q_\varrho} \psi w, \quad \psi \in H_0(S).$$

From Sobolev inequality (3.4) follows that the above mapping is a continuous linear functional on $H_0(S)$. Then, there is a unique $G_y^\varrho \in H_0(S)$ such that

$$a_0(G_y^p, \psi) = \frac{1}{w(Q_\varrho)} \int_{Q_\varrho} \psi w, \quad \text{for all} \quad \psi \in H_0(S).$$

In the next, $G_y^\varrho = G^\varrho(\cdot, y)$ will be called the «ϱ-approximate Green function for S with pole y». For the sake of simplicity we often will use the notation G^ϱ.

Lemma 5.11. *G^ϱ is non negative on S.*

PROOF. Follows the line of the euclidean case. (See Section 3 of [ChW3].)

Lemma 5.12. *There exists a constant C such that*

$$w(\{G_y^\varrho > t\}) \leq C\left(\frac{R^2}{v(2Q)}\right)^{\sigma} \frac{w(2Q)}{t^\sigma} \quad \text{for all } y \in S,\ \varrho > 0 \text{ with } Q(y, \varrho) \subset S$$

where R is the radious of S and Q is the δ-ball with the same centre that S and radious aR.

PROOF. The technique is the same that Chanillo and Wheeden have used for the euclidean case (see Section 3 of [ChW3]) but with Sobolev inequality for our particular geometry (Theorem 3.4).

Then, with the above result we have

Lemma 5.13. *For each $p \in (0, \sigma)$ there exists C such that*

$$\sup_{r/2<d(y,x)<r} G_y^{\varrho} \leq C\left(\frac{w(S(y,4r/3))}{v(S(y,r)}\right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{r^2}{v(S(y,r))}\left(\frac{w(S(y,8a^2r/3))}{\inf\limits_{r/2<d(y,z)<r} w(S(z,r/(4a^2))}\right)^{1/p}$$

for all $\varrho \in (0, r/4a)$ and for all y and r such that $S(y, 3a^4r) \subset \Omega$.

PROOF. Let $x \in \{x \in S : r/2 < d(y, x) < 3r/4\}$ and $\varrho \in (0, r/4a)$, then $S(x, r/4) \subset S(y, r) \setminus S(y, a\varrho)$. Note that G^{ϱ} satisfies $Lu = 0$, then for each $p \in (0, \sigma)$, we have

$$\text{(5.14)}\quad \sup_{Q(x,r/8a)} G^{\varrho} \leq C\left(\frac{w(Q(x, r/4a))}{v(Q(x, r/4a))}\right)^{\frac{\sigma^2}{p(\sigma-1)}}\left(\frac{1}{w(Q(x, r/4a))}\int_{Q(x,r/4a)} (G^{\varrho})^p w\right)^{1/p},$$

where C depends only on the constants of (0.5) and (0.7) (see Lemmas 3.1 and 3.11 of [ChW2]). On the other hand, from Lemma 6.2 follows

$$\int_{Q(x,r/4a)} (G^{\varrho})^p w \leq Cw(S(y, 2a^2r))\left(\frac{r^2}{v(S(y, 2r))}\right)^p$$

for each $p \in (0, \sigma)$. From this and (6.4) we have

$$\sup_{Q(x,r/8a)} G^{\varrho} \leq C\left(\frac{w(Q(x, r/4a))}{v(Q(x, r/4a))}\right)^{\frac{\sigma^2}{p(\sigma-1)}}\left(\frac{w(s(y, 28^2r))}{w(Q(x, r/4a))}\right)^{1/p}\frac{r^2}{v(S(y, 2r))}$$

for all p, σ and x. Then, for each $p \in (0, \sigma)$ the inequality

$$(5.15) \qquad \sup_{r/2<d(y,x)<3r/4} G^{\varrho} \leq C\left(\frac{w(S(y,\, r))}{v(S(y,\, r))}\right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{r^2}{v(S(y,\, r))} \frac{w(S(y,\, 2a^2 r))}{\inf\limits_{r/2<d(y,z)<3r/4} w(S(x,\, r/4a^2))}$$

holds for all $\varrho \in (0, r/4a)$, and all y and r such that $S(y, 3a^4 r) \subset \Omega$.

The above inequality allows us to obtain a similar one but on $S \setminus (1/2)\, S$. Indeed, if G_0^{ϱ} denotes the ϱ-approximate Green function for $S_0 = S(y, 4r/3)$ with pole y then, by the weak maximum principle (the proof is similar that the Lemma 2.6 of [ChW3]), se have $G^{\varrho} \leq G_0^{\varrho}$ on S, and from this and (5.15)

$$\sup_{2r/3<d(y,x)<r} G^{\varrho} \leq C\left(\frac{w(S(y,\, 4/3r))}{v(S(y,\, 4r/3))}\right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{r^2}{v(S(y,\, 4r/3))} \left(\frac{w(S(y,\, 8a^2 r/3))}{\inf\limits_{2r/a<d(y,z)<r} w(S(2,\, r/3a^2))}\right)^{1/p}$$

Then, the thesis follows from (5.15) and the last inequality.

Lemma 5.16. *Let $S(x_0, R)$ be a d-ball such that $S(x_0, 13a^4 R) \subset \Omega$. Then, for each $p \in (0, \sigma)$, there exists a constant C, independent of x_0 and R such that*

$$(5.17) \qquad \sup_{r/2<d(y,x)<r} G_y^{\varrho}(x) \leq C \int_r^R \frac{t^2}{v(S(y,\, t))} (F_1(y,\, t)^{\frac{\sigma^2}{\sigma-1}} F_2(y,\, t))^{1/p} \frac{dt}{t},$$

for all $y \in S(x_0, R/2)$, $r \in (0, R/2)$ and $\varrho \in (0, r/4a)$, where

$$F_1(y,\, t) = \frac{w(S(y,\, 12a^2 t))}{v(S(y,\, t))}$$

and

$$F_2(y,\, t) = \frac{w(S(y,\, 12a^2 t))}{\inf\limits_{d(y,z)<12a^2 t} w\left(S\left(z,\, \frac{t}{128a^5}\right)\right)}.$$

PROOF. First, let us consider the case $y = x_0$. For $s > 0$. Let us denote with S_s to $S(y, s)$ and with G_s^ϱ to the approximate Green function for S_s with pole y. Now, for $(t_1, t_2) \in \mathbf{R}^+ \times \mathbf{R}^+$, we define

$$g(t_1, t_2) = \left(\frac{w(S_{a^2 t_1})}{v(S_{t_2})}\right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{t_1^2}{v(S_{t_2})} \left(\frac{w(S_{a^2 t_1})}{\inf\limits_{d(y,z)<t_1} w(S(z, t_2))}\right)^{1/p}$$

Note that this function depend on p, is increasing on t_1 and decreasing on t_2. Let $r < R/2$ and $m \in \mathbf{N}$ such that $(3/2)^{m-1} r \leq R < (3/2)^m r$. Then on $S_r - S_{r/2}$, holds

$$(5.18) \qquad G_R \leq G^\varrho_{(3/2)^m r} = G_r^\varrho + \sum_{j=1}^{m} \left(G^\varrho_{(3/2)^j r} - G^\varrho_{(3/2)^{j-1} r}\right).$$

From the above lemma follows

$$(5.19) \quad \sup_{S_r - S_{r/2}} G_r^\varrho \leq Cg\left(\frac{8a^2 r}{3}, \frac{r}{4a^2}\right), \quad \text{for all} \quad \varrho \in \left(0, \frac{r}{4a}\right),$$

for each $p \in (0, \sigma)$. By the other hand, by using a similar argument to the Lemma 2.7 of [ChW3] and the above lemma, we get

$$\sup_{S_s} \left(G^\varrho_{(3/2)s} - G_s^\varrho\right) \leq Cg\left(4a^2 s, \frac{3s}{8a^2}\right), \quad \text{for all} \quad \varrho \in \left(0, \frac{s}{4a}\right),$$

for each $p \in (0, \sigma)$. Then, from this inequality, (5.19) and (5.18) follows

$$\begin{aligned}
(5.20) \quad \sup_{S_r - S_{r/2}} G_R^\varrho &\leq \sum_{j=1}^{m-1} g\left(6a^2 \left(\frac{3}{2}\right)^{j-1} r, \left(\frac{3}{2}\right)^{j-1} \frac{r}{4a^2}\right) \\
&\leq C \sum_{j=1}^{m-1} \int_{(3/2)^{j-1} r}^{(3/2)^j r} g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t} \\
&\leq C \int_r^R g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}, \quad \text{for all} \quad \varrho \in \left(0, \frac{r}{4a}\right)
\end{aligned}$$

for each $p \in (0, \sigma)$. Finally, since any $y \in S(x_0, R/2)$ verifies $S(x_0, R) \subset S(y, 2R)$, we get $G^{\varrho} \leq G^{\varrho}_{2R}$ on $S(x_0, R)$ and then, from (5.20), follows the thesis

$$\sup_{r/2<d(y,x)<r} G^{\varrho} \leq C \int_r^{2R} g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}$$

$$= C \int_r^{R} g\left(12a^2 t, \frac{t}{3a^2}\right) \frac{dt}{t} + \int_r^{2r} g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}$$

$$\leq C \int_r^{R} g\left(12a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}, \quad \text{for all } \varrho \in \left(0, \frac{r}{4a}\right).$$

Corollary 5.21. *With the same hypothesis as in Lemma* 5.16 *for a.e.* $y \in S(x_0, R/2)$ *there is a constant* $C = C(y, x_0, R, w, v) > 0$ *such that*

$$\sup_{r/2<d(y,x)<r} G^{\varrho}_y(x) \leq C \min\left\{\int_r^R \frac{t^{\,2}}{|S(y,t)|}\frac{dt}{t}, \int_r^R \frac{1}{|S(y,t)|}\frac{dt}{t}\right\}, \tag{5.22}$$

for all $r \in (0, R/2)$ and all $\varrho \in (0, r/4a)$.

PROOF. We set $C_1(y) = \sup_{t\leq R} \dfrac{w(S(y, 12a^2 t))}{v(S(y,t))}$, $C_2(y) = \sup_{t\leq R} \dfrac{S(y,t)}{v(S(y,t))}$ and $C_3(y) = \sup_{t\leq R} \dfrac{S(y,t)}{w(S(y,t))}$. Now from (5.17) follows

$$\sup_{r/2<d(y,x)<r} G^{\varrho}_y(x) \leq C C_1(y)^{\left(\frac{\sigma^2}{\sigma-1}+1\right)\frac{1}{p}} C_2(y) \int_r^R \frac{t^2}{|S(y,t)|}\frac{dt}{t}, \tag{5.23}$$

for each $y \in S(x_0, R/2)$ and for all $r \in (0, R/2)$ and $\varrho \in (0, r/4a)$. On the other hand, from (1.5), (1.8) and (1.10), we get

$$\left(\frac{t}{a^2 R}\right)^{\sum_j G_j} \leq \frac{|S(y,t)|}{|S(y,R)|} \qquad 0 < t \leq R,$$

then from Lemma 4.4, it follows that

$$\frac{t^2}{v(S(y,\,t))} \le CR^2\left(\frac{w(S(y,\,aR))}{w(S(y,\,t/a))}\right)^{1/\sigma}\frac{1}{v(S(y,\,R))}\,, \quad 0 < r \le R.$$

This inequality and (5.17) allows us to obtain

$$\sup_{r/2<d(y,x)<r} G_y^\varrho(x) \le CC_1(y)^{\left(\frac{\sigma^2}{\sigma-1}+1\right)\frac{1}{p}}\, C_3(y)^{1/\sigma} R^2\left(\frac{w(S(y,\,aR))}{v(S(y,\,R))}\right)^{1/\sigma},$$
$$\int_r^R \frac{1}{|(S(y,\,t)|^{1/\sigma}}\frac{dt}{t}.$$

Now, from (5.23) and the above inequality, by taking infinum on p, (5.22) follows.

The next Lemma gives us an estimate of $\nabla_\lambda G^\varrho$ in terms of G^ϱ.

Lemma 5.24. *Let $S = S(x_0,\,R)$ be a d-ball such that $2S \subset \Omega$. Then there exists a constant C such that*

$$\int_{S\setminus Q(y,r)} \langle \Delta\,\nabla G_y^\varrho,\ \nabla G_y^\varrho\rangle \le \frac{C}{r^2}\int_{Q(y,r)\setminus Q(y,r/2)} (G_y^\varrho)^2 w,$$

for all $y \in (1/2)\,S$, $r \in (0,\,R/2a)$ and $\varrho \in (0,\,r/2)$.

PROOF. The proof is very similar to that of Lemma 4.2 of [ChW3] by keeping in mind that exists $\eta \in C^\infty(\mathbf{R}^n)$ such that $\eta \equiv 0$ on $Q(y,\,r/2)$, $\eta \equiv 1$ outside $Q(y,\,r)$ and $|\nabla_\lambda \eta| \le C/r$ (see [FL2], p. 537).

Now, the above lemma and Corollary 5.21 allows us to obtain the following result.

Lemma 5.25. *Let $S = S(x_0,\,R)$ be a d-ball such that $13a^4S \subset \Omega$. If $n > 2$, then, for a.e. $y \in (1/2)\,S$ there exists a constant $C = C(y,\,x_0,\,R,\,w,\,v) > 0$ such that*

$$\int_{S\setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v \le C \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t},$$

for all $r \in (0, R/2a)$ *and for* $\varrho \in (0, R/2a)$.

PROOF. Let $\varrho \in \left(0, \frac{r}{8a^2}\right)$, then

$$\int_{S\setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v \le \frac{C}{r^2} w(Q(y,r)) \left(\sup_{r/2a<d(y,x)<ar} G_y^\varrho(x)\right)^2 \tag{5.26}$$

$$\le \frac{C}{r^2} w(Q(y,r)) \left(\sum_{j=1}^{j\le \log_2 a^2+1} \sup_{\frac{ar}{2^{j+1}}<d(y,x)<\frac{ar}{2^j}} G_y^\varrho(x)\right)^2$$

$$\le \frac{Cw(Q(y,r))}{r^2} \left(\min \int_{r/2a}^{R} \frac{t^2}{|S(y,t)|} \frac{dt}{t}, \int_{r/2a}^{R} \frac{1}{|S(y,r)|^{1/\sigma}} \frac{dt}{t}\right)^2$$

$$\le \frac{C}{r^2} w(Q(y,r)) \left(\int_{r/2a}^{R} \frac{t^2}{|S(y,t)|} \frac{dt}{t}\right) \left(\int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t}\right).$$

On the other hand, from Lemmas 1.4 and 1.7 it follows that

$$\int_{r/2a}^{R} \frac{t^2}{|S(y,t)|}\frac{dt}{t} \le \sum_{j=1}^{\infty} \int_{\frac{2^j r}{2a}}^{\frac{2^{j+1} r}{2a}} \frac{t^2}{|S(y,t)|}\frac{dt}{t}$$

$$\le \sum_{j=0}^{\infty} \left(\frac{2^j r}{2a}\right) \frac{1}{\left|S\left(y, \frac{2^j r}{a}\right)\right|}$$

$$\le \frac{C\, r^2}{|S(y,r)|} \sum_{j=0}^{\infty} 2^{j(2-n)}$$

$$= \frac{C\, r^2}{|S(y,r)|}\,.$$

Then, from this and (5.26), we get

$$(5.27)\qquad \int_{S \setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v \le C \frac{w(Q(y,r))}{|Q(y,r)|} \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t}\,.$$

Now, if $\varrho \in [r/8a^2, R/2a)$, by applying Sobolev inequality in $Q = Q(x_0, aR)$ we have

$$a_0(G_j^\varrho, G_j^\varrho) = \frac{1}{w(Q(y,\varrho)} \int_{Q(y,\varrho)} G_y^\varrho w$$

$$\le CR \left(\frac{w(2Q)}{w(Q(y,\varrho))}\right)^{1/2\sigma} \left(\frac{1}{w(2Q)} \int_S |\nabla_\lambda G_y^\varrho|^2 v\right)^{1/2}$$

$$\le \frac{1}{w(Q(y,\varrho))}\, a_0(G_y, G_y)^{1/2}.$$

Then

$$\int_S |\nabla_\lambda G_y^\varrho|^2 v \leq a_0(G_y^\varrho, G_y^\varrho) \leq \frac{C}{w(Q(y, \varrho))^{1/\sigma}}$$

$$\leq \left(\frac{|S(y, r)|}{w(Q(y, r/8a^2))}\right)^{1/\sigma} \int_{r/2a}^{R} \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t}.$$

Finally, from (5.27) and the above inequality, we get our thesis.

With Lemma 5.25 we can prove the following integrability property of $|\nabla_\lambda G^\varrho|$.

Lemma 5.28. *Let $S = S(x_0, R)$ be a d-ball such that $13a^4 S \subset \Omega \subset \mathbf{R}^n$ with $n > 2$. Then for each $q \in (0, 2\sigma/(\sigma + 1))$ and a.e. $y \in (1/2)S$ there exists $C = C(q, y, x_0, R, w, v) > 0$ such that*

$$(5.29) \quad \int_S |\nabla_\lambda G_y^\varrho|^q \leq C \quad \textit{for all} \quad \varrho \in \left(0, \frac{R}{2a}\right).$$

PROOF. Let $y \in \frac{1}{2} S$ and $\varrho \in \left(0, \frac{R}{2a}\right)$. From Lemma 5.25 we get

$$v(\{|\nabla_\lambda G_y^\varrho| > s\}) \leq \frac{1}{s^2} \int_{S \setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v + v(Q(y, r))$$

$$\leq \frac{C}{s^2} \int_{r/2a}^{R} \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t} + |Q(y, r)| \sup_{t \leq R} \frac{v(Q(y, t))}{|Q(y, t)|}$$

$$\leq C\left(\frac{1}{s^2} \int_{r/2a}^{R} \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t} + |Q(y, r)|\right)$$

for a.e. $y \in (1/2)\, S$ and for all $r \in (0, R/2a)$ and $s > 0$, where $C = C(y, S, w, v)$. Then, by using the properties of d and δ, it follows that

$$\nu(\{|\nabla_\lambda G_y^\varrho| > s\}) \leq C\left(|Q(y, r)| + \frac{1}{s^2}\sum_{j=0}^{\infty}\int_{\frac{2^j r}{2a}}^{\frac{2^{j+1} r}{2a}} \frac{1}{|S(y, t)|^{1/\sigma}}\frac{dt}{t}\right)$$

$$\leq C\left(|Q(y, r)| + \frac{1}{s^2 |S(y, r)|^{1/\sigma}}\sum_{j=0}^{\infty} 2^{-nj/\sigma}\right)$$

$$\leq C\left(|Q(y, r)| + \frac{1}{s^2 |Q(y, r)|^{1/\sigma}}\right).$$

Now, by choosing r such that $|Q(y, r)| = s^{-\frac{2\sigma}{\sigma+1}}$, holds

$$\nu(\{|\nabla_\lambda G_y^\varrho| > s\}) \leq C\, s^{-\frac{2\sigma}{\sigma+1}}$$

for all $s > |Q(y, R/2a)|^{-(\sigma+1)/2\sigma}$ for all $\varrho \in (0, R/2a)$ and for a.e. $y \in (1/2)\, S$. From this inequality (5.29) follows inmediately.

Let S be a d-ball such that $2aS \subset \Omega \subset \mathbf{R}^n$ with $n \geq 3$ and $(p, q) \in [1, \sigma) \times [1, 2\sigma/(\sigma + 1))$. Let $X_{p,q}$ be the completion of $\mathrm{Lip}_0(S)$ with the norm

$$\|\phi\|_{p,q} = \left(\int_S |\phi|^p w\right)^{1/p} + \left(\int_S |\nabla_\lambda \phi|^q v\, dx\right)^{1/q}.$$

Now the above results about G_y^ϱ and $\nabla_\lambda G_y^\varrho$, allow to prove existence of the Green functions $G_y(\cdot) = G(\cdot, y)$ for S with pole y.

Theorem 5.30. *Let $S = S(x_0, R)$ be a d-ball such that $13a^4 S \subset \Omega$. For a.e. $y \in (1/2)\, S$ exists a sequence $\{\varrho_k\} \subset (0, R/2a)$ and a function G_y such that*

(5.31) $\varrho_k \downarrow 0$ *and* $\{G_y^{\varrho_k}\}$ *converges weakly to* G_y *on* $X_{p,q}$

for all $(p, q) \in [1, \sigma) \times \left[1, \frac{2\sigma}{\sigma + 1}\right)$;

(5.32) $\|G_y\|_{p,q} \leq C$

for every $(p, q) \in [1, \sigma) \times \left[1, \frac{2\sigma}{\sigma + 1}\right)$, *where* C *is independent of* y;

(5.33) *if* (v, w) *satisfies* $\left(\frac{w}{v}\right)^{\frac{q_0}{q_0-1}} v \in L^1(S, dx)$ *for some* $q_0 \in \left(1, \frac{2\sigma}{\sigma + 1}\right)$

then

$$\int_S \langle A \nabla G_y, \nabla \phi \rangle = \phi(y)$$

for all $\phi \in \mathrm{Lip}_0(S)$, *and a.e.* $y \in \frac{1}{2} S$;

(5.34) *if* u *is solution of* (5.6) *with* $\psi = 0$, *then*

$$u(y) = \int_S G_y(x) f(x) dx \quad a.e. \quad y \in \frac{1}{2} s;$$

(5.35) *if* u *is solution of* (5.7) *with* $\psi = 0$, *then*

$$u(y) = \int_S \langle \nabla_\lambda G_y(x), F(x) > dx \quad a.e. \quad y \in \frac{1}{2} S.$$

PROOF. Follows from the above results about G_y^ϱ and $\nabla_\lambda G_y^\varrho$ with the arguments in sections 5 and 6 of [ChW3].

Now, we prove another estimate for G_y^ϱ. Then we shall to use this and the above to prove some estimates of size for the Green function.

Lemma 5.37. *Let* $S(x_0, R)$ *be a d-ball such that* $13a^4 S \subset \Omega$. *Then exists a constant* C *depending only on the constants of* (0.5) *and* (0.7) *such that*

$$(5.37) \qquad \inf_{r/2 < d(y,x) < r} G_y^\varrho(x) \geq C \int_r^R \frac{t^2}{w(S(y, 2at))} e^{-CF_2(y,t)^{\gamma} F_1(y,t)^{1/2}} \frac{dt}{t}$$

for all $y \in S(x_0, R/2)$, $r \in (0, R/8a^2)$ *and* $\varrho \in 0, r/4a^2)$, *where* $F_1(y, t)$ *and* $F_2(y, t)$ *are the functions of* (5.17) *and* $\gamma = \dfrac{3\sigma^2 - 2\sigma + 1}{\sigma - 1}$.

PROOF. First, let us consider the case $y = x_0$. Let $r \in (0, R/4a^2)$. For $s > 0$, let us denote with G_y^ϱ the approximate Green function for $S(y, s)$ with pole y. From Lemma 5.24, we get

$$(5.38)\quad \int_{S(y,2ar)\setminus Q(y,r)} \langle A \nabla G_{2ar}^\varrho, \nabla G_{2ar}^\varrho \rangle \leq \frac{C}{r^2} w(Q(y,r)) \left(\sup_{S(y,ar)-S(y,r/2a)} G_{2ar}^\varrho \right)^2$$

On the other hand, since $LG_{2ar}^\varrho = 0$ on $S(y, 2ar) \setminus S(y, r/4a)$, the Harnack inequality (Theorem 5.10) allow us to obtain

$$\sup_{Q(x,r/8a^2)} G_{2ar}^\varrho \leq \exp\left\{ C \left(\frac{w(S(y,2ar))}{\inf_{d(y,z)<ar} w(S(z, r/16a^3))} \right)^\gamma \left(\frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\} \inf_{Q(x,r/8a^2)} G_{2ar}^\varrho$$

for all $\varrho \in (0, r/4a^2)$ and all $x \in S(y, ar) \setminus S(y, r/2a)$. From this inequality and (5.38) follows that

$$(5.39)\quad \left(\inf_{S(y,ar)\setminus S(y,r/2a)} G_{2ar}^\varrho \right)^2$$

$$\leq \frac{Cr^2}{w(Q(y,r))} \exp\left\{ -C \left(\frac{w(S(y, 2ar))}{\inf_{d(y,z)<ar} w(S(z, r/16a^3))} \right)^\gamma \left(\frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\}$$

$$\int_{S(y,2ar)\setminus Q(y,r)} \langle A \nabla G_{2ar}^\varrho, \nabla G_{2ar}^\varrho \rangle .$$

Now, by taking $\eta \in C_0^\infty(S(y, 2ar))$ such that $\eta \equiv 1$ on $Q(y, r)$ and $|\nabla_\lambda \eta| \leq \dfrac{C}{r}$ and applying the ellipticity condition, we get

$$1 = a_0(G^{\varrho}_{2ar}, \eta)$$

$$\leq \frac{C}{r}\, w(y,\, 2ar))^{1/2} \left(\int_{S(y,2ar)\setminus Q(y,ar)} \langle A \nabla G^{\varrho}_{2ar},\ \nabla G^{\varrho}_{2ar} \rangle \right)^{1/2}.$$

Then, from (5.39), holds

$$(5.40)\qquad \inf_{S(y,ar)\setminus S(y,r/2a)} G^{\varrho}_{2ar}$$

$$\geq \frac{Cr^2}{w(S(y,2ar))} \exp\left\{ -C \left(\frac{w(S(y,\, 2ar))}{\inf\limits_{d(y,z)<ar} w(S(z,\, r/16a^3))} \right)^{\gamma} \left(\frac{w(S(y,\, 2ar))}{v(S(y,\, 2ar))} \right)^{1/2} \right\}$$

for all $\varrho \in (0,\, r/4a^2)$. Now, by applying the weak maximum principle, it follows that

$$(5.41)\qquad G^{\varrho}_{2ar} - G^{\varrho}_{r}$$

$$\geq \frac{Cr^2}{w(S(y,\, 2ar))} \exp\left\{ -C \left(\frac{w(S(y,\, 2ar))}{\inf\limits_{d(y,z)<ar} w(S(z,\, r/16a^3))} \right)^{\gamma} \left(\frac{w(S(y,\, 2ar))}{v(S(y,\, 2ar))} \right)^{1/2} \right\}$$

a.e. in $S(y,\, r)$ and for all $\varrho \in (0,\, r/4a^2)$. On the other hand, if $m \in \mathbf{N}$ is such that $(2a)^m r \leq R < (2a)^{m+1} r$, we get

$$(5.42)\qquad G^{\varrho}_{R} \geq G^{\varrho}_{(2a)^m r} = G^{\varrho}_{2ar} + \sum_{j=1}^{m-1} (G^{\varrho}_{(2a)^{j+1} r} - G_{(2a)^j r}),\ \text{a.e. in } S(y,\, 2ar).$$

Then, with $S_t = S(y,\, t)$ and

$$g(t_1,\, t_2)$$

$$= \frac{t_1^2}{w(S_{t_2})} \exp\left\{ -C \left(\frac{w(S_{t_2})}{\inf\limits_{d(y,z)<ar} w(S(z,\, t_1/16a^3))} \right)^{\gamma} \left(\frac{w(S_{t_2}))}{v(S_{t_1}))} \right)^{1/2} \right\},\ t_1,\ t_2 \in \mathbf{R}^+.$$

from (5.40) and (5.41) the inequality

$$(5.43)\quad G_R^{\varrho} \geq C \sum_{j=0}^{m-1} g((2a))^j r, (2a)^{j+1} r)$$

$$\geq C \int_{2ar}^{R} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t}$$

$$= C \left[\int_{4a^2 r}^{R} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t} + \int_{2ar}^{4a^2 r} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t} \right]$$

$$\geq C \left[\int_{2ar}^{R} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t} + \frac{1}{2} \int_{r}^{2ar} g\left(\frac{t}{2a}, 2at\right) \frac{dt}{t} \right]$$

$$\geq C \int_{r}^{R} g\left(\frac{t}{4a^2}, 2at\right) \frac{dt}{t},$$

holds for a.e. in $S(y, ar) \setminus S(y, r/2a)$ and for all $\varrho \in (0, r/4a^2)$. Finally, the result for $y \neq x_0$ follows easily from the fact that $G_y^{\varrho} \geq G_{R/2}^{\varrho}$ a.e. in $S(y, R/2)$ and (5.43).

In the following we prove a result of functional analysis which shall be used to prove the size estimates.

Lemma 5.44. *Let $E \subset \mathbf{R}^n$ be a measurable set, $h \in L_{\mathrm{loc}}(E, dx)$ be positive a.e. in E and $p \in (1, \infty)$. If $\{f_k\} \subset L^p(E, dx)$ converges weakly to a function f and satisfies* $\sup_k f_k \leq C_0$ *a.e. in a bounded set $F \subset E$, for some constant C_0, then $f \leq C_0$ a.e. in F.*

PROOF. From the hypothesis we get

$$\int_E (C_0 - f_k)\, g h dx \geq 0,$$

for every k *and for all* $g \in L^p(E, hdx)$ such that supp $g \subset F$ and $g \geq 0$ a.e. in F. Then, by letting $k \to \infty$, the inequality

$$\int_E (C_0 - f)\, ghdx \geq 0,$$

holds. Now, by taking $g = \chi_{F \cap \{f > C_0\}}$, we get the thesis.

Finally, we are in position to prove

Theorem 5.45. *Let* $S = S(x_0, R)$ *be a d-ball such that* $13a^4 S \subset \Omega$. *Then for a.e.* $y \in (1/2)\, S$. G_y *is non-negative a.e. in* S *and satisfies*

$$\text{(5.46)} \qquad \sup_{r/2 < d(y,x) < r} G_y(x) \leq C \int_r^R \frac{t^2}{v(S(y, t))} ((F_1(y, t))^{\gamma_1} F_2(y, t))^{1/p} \frac{dt}{t}$$

for each $p \in (0, \sigma)$

$$\text{(5.47)} \qquad \inf_{r/2 < d(y,x) < r} G_y(x) \geq C \int_r^R \frac{t^2}{w(S(y, 2at))} e^{-CF_1(y,t)^{1/2} F_2(y,t)^{\gamma_2}} \frac{dt}{t}$$

for all $r \in (0, R/8a^2)$, where $\gamma_1 = \dfrac{\sigma^2}{\sigma - 1}$, $\gamma_2 = \dfrac{3\sigma^2 - 2\sigma + 1}{\sigma - 1}$ and F_1 and F_2 are define as in (5.17).

PROOF. It follow from Lemmas 5.11 and 5.36 and Theorem 5.30 by using the above lemma.

Finally, we shall prove a result about the integrability without weights of the Green function.

Theorem 5.48. *Let* $S = S(x_0, R)$ *be a d-ball such that* $17a^4 S \subset \mathbf{R}^n$ *with* $n \geq 3$. *Then* $G_y \in L^p(S, dx)$ *for a.e.* $y \in (1/2)\, S$ *and for all*

$$p \in \left(1, \max\left\{\frac{\sum G_j}{\sum G_j - 2}, 1 + \frac{n}{\sum G_j} (\sigma - 1)\right\}\right).$$

PROOF. From (5.46) by applying a similar argument to that Corollary 5.21 follows

$$\sup_{r/2<d(y,x)<r} G_y(x) \le C \min\left\{\int_r^R \frac{t^2}{|S(y,t)|}\frac{dt}{t}, \int_r^R \frac{1}{|S(y,t)|^{1/\sigma}}\frac{dt}{t}\right\} \tag{5.49}$$

for a.e. $y \in (1/2)S$ and all $r \in (0, R/8a^2)$ where $C = C(y, x_0, R, w, v)$. By denoting with G_{16a^2R} the Green function for $S(y, 16a^2R)$ with pole y, the weak maximum principle, (5.49) and the properties of d allow us to get

$$\begin{aligned}
\int_{S(x_0,R)} (G_y)^p\,dx &\le \int_{S(x_0,R)} (G_{16a^2R})^p\,dx \\
&\le \sum_{i=0}^{\infty} \int_{S(y,2R/2^i)-S(y,2R/2^{i+1})} (G_{16a^2R})^p\,dx \\
&\le C \sum_{i=0}^{\infty} \left(\int_{\frac{R}{2^{i-1}}}^{16a^2R} \frac{t^2}{|S(y,t)|}\frac{dt}{t}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \int_{\frac{R}{2^{i-1}}2^j}^{\frac{R}{2^{i-1}}2^{j+1}} \frac{t^2}{|S(y,t)|}\frac{dt}{t}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{(R\,2^{1-i}2^{j+1})^2}{\left|S\left(y, \frac{R}{2^{i-1}}2^j\right)\right|}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \left(\frac{(R\,2^{1-i})^2}{\left|S\left(y, \frac{R}{2^{i-1}}\right)\right|} \sum_{j=0}^{\infty} 2^{j(2-n)}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \frac{(R\,2^{i-1})^{2p}}{\left|S\left(y, \frac{R}{2^{i-1}}\right)\right|^{p-1}}
\end{aligned} \tag{5.50}$$

$$\leq C\left(\frac{(2R)^p}{|S(y,2R)|^{p-1}}+\frac{R^{2p}}{|S(y,R)|^{p-1}}\sum_{j=1}^{\infty}2^{i((\Sigma_j G_j-2)p-\Sigma_j G_j)}\right).$$

Note that the right hand side is finite if and only if $p < (\Sigma_j G_j)/(\Sigma_j G_j - 2)$. On the other hand, by applying (5.46) again and a similar argument, we get

$$\int_{S(x_0,R)}(G_j)^p dx \leq \frac{C}{|S(y,R)|^{p/\sigma-1}}\left(1+\sum_{i=1}^{\infty}2^{i1/\sigma((p-1)\Sigma_j G_j+n(1-\sigma))}\right),$$

where the right member is finite iff $p < 1 + \dfrac{n}{\Sigma_j G_j}(\sigma - 1)$. Finally, from this and (5.50), we obtain the thesis.

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