

# Hardy Space Estimates for Multilinear Operators, I

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**Abstract.** In this article, we study bilinear operators given by inner products of finite vectors of Calderón-Zygmund operators. We find that necessary and sufficient condition for these operators to map products of Hardy spaces into Hardy spaces is to have a certain number of moments vanishing and under these assumptions we prove a Hölder-type inequality in the  $H^p$  space context.

## Introduction

Probably, the most important example of a multilinear operator is the determinant of the Jacobian of a map  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ . In two dimensions the determinant of the Jacobian of a pair of functions  $(f, g)$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  is the bilinear map

$$J(f, g) = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1}$$

which has very interesting properties. A theorem of P. L. Lions and Y. Meyer, says that  $J$  maps the product of Sobolev spaces  $L_1^2 \times L_1^2$  into the Hardy space  $H^1$ . This theorem has been extended for  $L^p$  Sobolev spaces by [CMLS] as follows: The determinant of the Jacobian maps  $L_1^p \times L_1^q$  into  $H^\gamma$  as long as  $1 \geq \gamma > 2/3$ ;  $p, q > 1$  and  $p^{-1} + q^{-1} = \gamma^{-1}$ . The spaces  $H^\gamma(\mathbb{R}^n)$ ,  $\gamma \leq 1$  are the usual real variable Hardy spaces as defined in [S] or [SW]. The result of [CMLS] is false when  $\gamma = 2/3$  and leads naturally to the following question: Why can't the Jacobian-determinant map into some  $H^\gamma$  space for  $\gamma \leq 2/3$ ?

In these articles we prove that a more general class of bilinear operators map into  $H^r$  for arbitrarily small  $r > 0$  *only when* they have a certain number of moments vanishing. The determinant of the Jacobian has always integral zero but it does not have higher moments vanishing and this is the reason it cannot map into  $H^\gamma$  for  $\gamma \leq 2/3$ . Other bilinear operator on  $\mathbb{R}^n$  have higher moments vanishing and they map into  $H^r$  for  $r < n/(n+1)$ . (The index  $2/3$  corresponds to the case  $n = 2$ .)

A good example of an operator with integral and first moments vanishing in  $\mathbb{R}^2$  is the determinant of the Hessian of a map  $(f, g): \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$H(f, g) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 g}{\partial y \partial x} - \frac{\partial^2 f}{\partial y \partial x} \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 g}{\partial x^2}.$$

By introducing the Riesz transforms,  $\widehat{R}_j f(\xi) = i\xi_j \widehat{f}(\xi)/|\xi|$  the Jacobian-determinant  $J$  and Hessian-determinant  $H$  can be studied through the bilinear maps  $\widetilde{J}$  and  $\widetilde{H}$  given by

$$\begin{aligned} \widetilde{J}(f, g) &= (R_1 f)(R_2 g) - (R_2 f)(R_1 g) \\ \widetilde{H}(f, g) &= (R_1^2 f)(R_2^2 g) - 2(R_1 R_2 f)(R_2 R_1 g) + (R_2^2 f)(R_1^2 g) \end{aligned}$$

which are of the form

$$(0.1) \quad \sum_{j=1}^N (T_j^1 f)(T_j^2 g)$$

for some Calderón-Zygmund operators  $\{T_j^1\}$ ,  $\{T_j^2\}$ .

In part II of this work, we prove that  $\widetilde{H}$  maps  $L^p \times L^q$  into  $H^r$  for  $p, q > 1$ ,  $1 \geq r > 1/2$  and  $r^{-1} = p^{-1} + q^{-1}$ . We conclude that  $H$  maps functions with Laplacean in  $L^p \times L^q$  into  $H^r$  for the same  $p, q, r$  as above. The exponent  $r = 1/2$  is a natural lower bound in this case since the assumptions  $p, q > 1$  imply that  $r = (p^{-1} + q^{-1})^{-1} > 1/2$ .

The question investigated in this article is under what conditions can we have boundedness into  $H^r$  for  $r \leq 1/2$ . Since  $r = (p^{-1} + q^{-1})^{-1}$  we must have  $p \leq 1$  or  $q \leq 1$  and obviously the  $L^p$  spaces are not a suitable starting point. If we replace the  $L^p$  spaces with  $H^p$  for  $p \leq 1$  however, we get boundedness into  $H^r$  for arbitrarily small  $r$ . We treat general bilinear operators of the form (0.1) and we assume that these operators have for a given  $r > 0$  a required number of moments vanishing to map into  $H^r$ .

## 1. Preliminaries

We are given two families of tempered distributions  $\{K_i^1\}_{i=1}^N$ ,  $\{K_i^2\}_{i=1}^N$ , homogeneous of degree 0 and we are assuming that:

- (1) The Fourier transforms of  $\{K_i^1\}$ ,  $\{K_i^2\}$  are bounded functions.
- (2)  $\{K_i^1\}$ ,  $\{K_i^2\}$  are sufficiently smooth away from the origin and

$$\left| \frac{\partial^\gamma}{\partial x^\gamma} K_i^j \right| \leq C|x|^{-n-|\gamma|}$$

for all  $|\gamma| \leq M$  for some sufficiently large  $M$ , ( $j = 0$  or  $1$ ).

- (3) For all multi-indices  $\alpha$  and  $\gamma$  of norm less than  $M$  for some large  $M$ , the partial derivatives of  $\{K_i^1\}$  and  $\{K_i^2\}$  satisfy:

$$\left| \frac{\partial^\gamma}{\partial y^\gamma} \frac{\partial^\alpha}{\partial x^\alpha} (K_i^j(x-y) - K_i^j(x)) \right| \leq A \frac{|y|}{|x|^{n+|\alpha|+|\gamma|+1}}$$

for  $|x| > 2|y|$ , ( $j = 0$  or  $1$ ).

We call  $T_i^1$  the operator given by convolving with  $K_i^1$  and  $T_i^2$  the operator given with convolution with  $K_i^2$ . Theorem 12 in [FS] says that the operators  $\{T_i^1\}$ ,  $\{T_i^2\}$  map  $H^p \rightarrow H^p$ . The main result in this article is the  $H^p$  boundedness of the bilinear product

$$B(f, g) = \sum_i (T_i^1 f)(T_i^2 g)$$

where  $f, g$  lie in suitable Hardy spaces. We have the following theorem.

**Theorem.** *Suppose  $\{T_i^1\}$ ,  $\{T_i^2\}$ ,  $i = 1, 2, \dots, N$  are Calderón-Zygmund operators on  $\mathbb{R}^n$  as above. Fix  $p, q \leq 1$  and let*

$$B(f, g) = \sum_{i=1}^N (T_i^1 f)(T_i^2 g).$$

*Assume that for some  $k \geq 0$  integer, for all multi-indices  $|\alpha| \leq k$  and for all  $f$   $H^p$ -atoms and  $g$   $H^q$ -atoms the moments*

$$\int x^\alpha B(f, g)(x) dx = 0, \quad |\alpha| \leq k.$$

*Then  $B$  can be extended to a bounded operator from  $H^p \times H^q \rightarrow H^r$  where*

$$\frac{n}{n+k+1} < r \leq \frac{n}{n+k} \quad \text{and} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

**Remarks.**

- (1) Note that if  $f$  is an  $H^p$ -atom and  $g$  is an  $H^q$ -atom the integral defining the moments of  $B(f, g)$  is well defined, for  $(T_i f)(x) \in L^p$  for all  $p > 1$  and decays like  $|x|^{-[n(1/p-1)]-n-1}$  as  $|x| \rightarrow \infty$ . Therefore, the functions

$x^\alpha(T_i^1 f)(x)(T_i^2 g)(x)$ ,  $|\alpha| \leq k$  are  $L^1$  around 0 and decay like

$$|x|^{-[n(1/p-1)] - [n(1/q-1)] - 2n - 2 + |\alpha|} \quad \text{as } |x| \rightarrow \infty.$$

The exponent above is less than  $-n$  as long as  $|\alpha| \leq k$  and  $r \leq n/(n+k)$ . The operator  $B(f, g)$  is certainly well defined when  $f$  and  $g$  are finite sums of atoms and the assumptions make sense for the range of  $p, q$  and  $r$  as above. The conclusion is that  $B(f, g)$  maps  $H^p \times H^q \rightarrow H^r$  as long as it has  $r$  moments vanishing.

- (2) Our theorem is not vacuous if we assume that  $k \geq n$ . Otherwise,  $k \leq n-1$  implies that  $n/(n+k+1) \geq 1/2$  and thus  $r > 1/2$  which is impossible since  $r = (1/p + 1/q)^{-1} \leq 1/2$ .

Before we begin the proof of our theorem we state and prove a lemma that will be the main tool in the treatment of all the terms that will appear in the decomposition of  $B(f, g)$  except the main term.

Suppose  $\phi$  is a smooth function and  $f$  an  $H^p$  distribution ( $p \leq 1$ ). We are interested in computing the value of the constant  $C_\phi$  in the following inequality

$$\left| \int f(x)\phi(x) dx \right| \leq C_\phi$$

where, by  $\int f(x)\phi(x) dx$  we denote the action of the distribution  $f$  on the test function  $\phi$ . We have the following

**Lemma 1.** *Let  $\phi$  be sufficiently smooth. For any  $f \in H^p$ ,  $p \leq 1$  there exists an  $L^p$  function  $f^+$  independent of  $\phi$  with  $\|f^+\|_{L^p} \leq C_p \|f\|_{H^p}$  such that*

$$\left| \int f(x)\phi(x) dx \right| \leq N_{x_0}(\phi) f^+(x_0), \quad \text{all } x_0 \in \mathbb{R}^n,$$

where

$$N_{x_0}(\phi) = \sum_{s=0}^N \int |x - x_0|^{2s} |\Delta^s \phi|(x) dx$$

for some  $N$  depending on  $n$  and  $p$  only.

**PROOF.** We call  $N_{x_0}(\phi)$  the «norm» of  $\phi$ .

To prove the lemma we use the atomic decomposition of  $f$ . Assume first that  $f = a_{Q_0}$  is an atom supported in the unit cube  $Q_0$  of  $\mathbb{R}^n$ . We will show that

$$(1.1) \quad \left| \int a_{Q_0}(x)\phi(x) dx \right| \leq N_{x_0}(\phi) h(x_0)$$

where

$$h(x_0) = c(3 + |x_0|)^{-2s} \quad \text{and } s \in \mathbb{Z}^+$$

is chosen such that  $2s > n/p$ . We are assuming that  $a_{Q_0}$  has at least  $2s - n$  moments vanishing.

To verify (1.1) we consider two cases

*Case 1.*  $x_0 \in 3Q_0$  (triple of  $Q_0$ ). Then

$$\left| \int a_{Q_0}(x)\phi(x) dx \right| \leq C \int |\phi| dx \leq h(x_0)N_{x_0}(\phi).$$

*Case 2.*  $x \notin 3Q_0$ .

We choose a  $\zeta(x)$  in  $C^\infty(\mathbb{R}^n)$  function, supported in  $|x| < 1$  and we call  $\zeta_{x_0}(x) = \zeta(2(x - x_0)/|x_0|)$ . Then  $\zeta_{x_0}(x)$  is supported in  $|x - x_0| \leq |x_0|/2$ . Note that for  $s > n/2p$ ,  $\Delta^{-s}a_{Q_0}$  is defined because  $a_{Q_0}$  has enough vanishing moments. We have

$$\begin{aligned} \int a_{Q_0} \phi dx &= \int (\Delta^{-s}a_{Q_0})(\Delta^s\phi) dx \\ &= \int \zeta_{x_0}(\delta^{-s}a_{Q_0})(\Delta^s\phi) dx + \int (1 - \zeta_{x_0})(\Delta^{-s}a_{Q_0})(\Delta^s\phi)(x) dx \\ &= I + II. \end{aligned}$$

Term *II* is the easiest to treat. First we claim that  $\|\Delta^{-s}a_{Q_0}\|_{L^\infty} \leq C$ . To check this, write

$$\Delta^{-s}a_{Q_0}(x) = C \int_{Q_0} |x - y|^{-n+2s} a_{Q_0}(y) dy.$$

Use the fact that  $a_{Q_0}(y)$  has at least  $-n + 2s > 0$  moments vanishing to subtract a suitable polynomial  $\sum_{|\alpha| < -n+2s} C_\alpha |x|^{-n+2s-|\alpha|} y^\alpha$  from  $|x - y|^{-n+2s}$  so that

$$\left| |x - y|^{-n+2s} - \sum_{|\alpha| < -n+2s} C_\alpha |x|^{-n+2s-|\alpha|} y^\alpha \right| \leq C|y|^{-n+2s}$$

and thus get

$$|\Delta^{-s}a_{Q_0}(x)| \leq C \int_{Q_0} |y|^{-n+2s} dy \leq C.$$

Then we estimate term *II* by

$$\begin{aligned} II &\leq \int_{|x-x_0| \geq |x_0|/2} \|\Delta^{-s}a_{Q_0}\|_{L^\infty} |\Delta^s\phi| dx \\ &\leq C|x_0|^{-2s} \int_{|x-x_0| \geq |x_0|/2} |x-x_0|^{2s} |\Delta^s\phi(x)| dx \\ &\leq h(x_0)N_{x_0}(\phi). \end{aligned}$$

Term  $I$  can be handled with another integration by parts.

$$\begin{aligned} &= \int \Delta^s(\zeta_{x_0} \Delta^{-s} a_{Q_0}) \phi \, dx \\ &= \int \left( \sum_{j=0}^{2s-1} c_j (D_x^{2s-j} \zeta_{x_0}) D_x^j (\Delta^{-s} a_{Q_0}) + \zeta_{x_0} a_{Q_0} \right) \phi \, dx, \end{aligned}$$

where  $D_x^j F$  is a sum of derivatives of  $F$  in  $x$  of total order  $j$ . Since  $a_{Q_0}$  and  $\zeta_{x_0}$  have disjoint supports their product is zero.

Finally, we claim that  $D_x^j (\Delta^{-s} a_{Q_0})(x)$  decays like  $|x_0|^{-j}$  as  $|x_0| \rightarrow \infty$  for  $|x - x_0| \leq |x_0|/2$ . We can see this by the following argument. We have

$$\Delta^{-s} a_{Q_0}(x) = \int |x - y|^{-n+2s} a_{Q_0}(y) \, dy$$

and therefore

$$D_x^j (\Delta^{-s} a_{Q_0})(x) = \int D_x^j (|x - y|^{-n+2s}) a_{Q_0}(y) \, dy.$$

Note that the function  $|x - y|^{-n+2s}$  is smooth near  $x$  since  $y \in Q_0$  and  $|x| \sim |x_0| \notin 3Q_0$ . Using the fact that  $a_{Q_0}$  has  $-n + 2s$  moments vanishing we can subtract the Taylor polynomial of degree  $-n + 2s$  of  $D_x^j (|x - y|^{-n+2s})$  at  $x$  to get

$$\begin{aligned} &D_x^j (\Delta^{-s} a_{Q_0})(x) \\ &= \int \left\{ D_x^j (|x - y|^{-n+2s}) - \sum_{|\alpha| \leq -n+2s} \frac{\partial^\alpha}{\partial x^\alpha} (D_x^j |x|^{-n+2s}) \frac{y^\alpha}{\alpha!} \right\} a_{Q_0}(y) \, dy. \end{aligned}$$

The expression inside the curly brackets above decays like

$$|x - y|^{-n+2s-j-(-n+2s)} \sim |x|^{-j} \quad \text{as } |x| \rightarrow \infty.$$

Since  $|x - x_0| \leq |x_0|/2$  we have  $|x| \sim |x_0|$  and thus we get the required estimate

$$|D_x^j (\Delta^{-s} a_{Q_0})(x)| \leq C |x_0|^{-j} \quad \text{as } |x_0| \rightarrow \infty.$$

Clearly, we also have  $|D_x^{2s-j} \zeta_{x_0}| \leq C |x_0|^{-2s+j}$ .

Summing on  $j$  we obtain

$$\left| \sum_{j=0}^{2s-1} c_j (\Delta^{2s-j} \zeta_{x_0}) D_x^j (\Delta^{-s} a_{Q_0}) \right| \leq C |x_0|^{-2s}$$

and thus

$$I \leq C |x_0|^{-2s} \int |\phi| \, dx \leq \zeta(x_0) N_{x_0}(\phi).$$

Putting estimates *I* and *II* together we get the desired conclusion for Case 2. Our lemma is now proved in the case where  $f = a_{Q_0}$  is an atom supported in the unit cube centered at the origin. To obtain an estimate for a general atom  $a_Q$  we use translations and dilations. Observe that the norm  $N_{x_0}$  satisfies the following properties:

$$N_{x_0}\left(\frac{1}{t^n}\phi\left(\frac{\cdot}{t}\right)\right) = N_{x_0/t}(\phi)$$

$$N_{x_0}(\phi(\cdot + y)) = N_{x_0+y}(\phi).$$

For a general cube  $Q$ , the properties above and (1.1) give

$$\left| \int a_Q(x)\phi(x) dx \right| \leq |Q|^{-1/p} N_{x_0}(\phi) h\left(\frac{x_0}{|Q|^{1/n}}\right).$$

Summing over all  $Q$  we obtain

$$\left| \int f(x)\phi(x) dx \right| \leq N_{x_0}(\phi) f^+(x_0)$$

where

$$f^+(x_0) = \sum \lambda_Q |Q|^{-1/p} h\left(\frac{x_0}{|Q|^{1/n}}\right).$$

We easily check that  $\|f^+\|_{L^p} \leq C \|f\|_{H^p}$ ,

$$\begin{aligned} \left( \int f^+(x_0)^p dx_0 \right)^{1/p} &= \left( \int \left( \sum \lambda_Q |Q|^{-1/p} h\left(\frac{x_0}{|Q|^{1/n}}\right) \right)^p dx_0 \right)^{1/p} \\ &\leq \left( \sum \lambda_Q^p \int |Q|^{-1} h^p\left(\frac{x_0}{|Q|^{1/n}}\right) dx_0 \right)^{1/p} \\ &\leq \|h\|_{L^p} \left( \sum \lambda_Q^p \right)^{1/p} \\ &\leq C_p \|f\|_{H^p}. \end{aligned}$$

Our lemma is now proved.

## 2. Beginning of the Proof

Fix  $p, q, r$  as in the statement of the theorem. Fix a smooth compactly supported function  $\phi \geq 0$  in  $\mathbb{R}^n$  and define

$$\phi_{t, x_0}(x) = \frac{1}{t^n} \phi\left(\frac{x_0 - x}{t}\right)$$

where  $x_0 \in \mathbb{R}^n$  fixed. We will show that

$$\left\| \sup_{t>0} \left| \int \phi_{t,x_0}(x) B(f,g)(x) dx \right| \right\|_{L^r} \leq C \|f\|_{H^p} \|g\|_{H^q}$$

for  $f, g$  finite sums of  $H^p$  and  $H^q$  atoms respectively.

Without loss of generality we may assume that the support of  $\phi$  lies in  $\{x: |x| < 1\}$ . Fix a smooth cut-off  $\eta(x)$  supported in  $|x| < 4$  such that  $\eta \equiv 1$  on  $|x| < 2$ . Call for simplicity  $\eta_0(x) = \eta(x_0 - x)/t$ ,  $\eta_1(x) = 1 - \eta_0(x)$ . The reader should remember the dependence of  $\eta_0$  and  $\eta_1$  on  $t$ . We write  $B(f, g) = B_1 + B_2 + B_3 + B_4$  where

$$\begin{aligned} B_1 &= B(\eta_0 f, \eta_0 g), \\ B_2 &= B(\eta_1 f, g), \\ B_3 &= B(f, \eta_1 g), \\ B_4 &= -B(\eta_1 f, \eta_1 g). \end{aligned}$$

We treat term  $B_4$  first,

$$\begin{aligned} (2.1) \quad & - \int \phi_{t,x_0} B_4 dx \\ &= \sum \int \phi_{t,x_0}(x) (T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0)) (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) dx \\ &+ \sum_i T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x) (T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0)) dx \\ &+ \sum_i T_i^1(\eta_1 f)(x_0) \int \phi_{t,x_0}(x) (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) dx \\ &+ \sum_i T_i^1(\eta_1 f)(x_0) T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x) dx. \end{aligned}$$

We claim that for any fixed  $x$  such that  $|x - x_0| \leq t$  we have that,

$$\sup_{t>0} |T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0)| \leq C f^+(x_0)$$

where  $f^+(x_0)$  is an  $L^p$  function of  $x_0$  with  $\|f^+\|_{L^p} \leq C \|f\|_{H^p}$ .

We verify this last assertion. Fix an  $i$  and write

$$\begin{aligned} & T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0) \\ &= \int \left( 1 - \eta\left(\frac{x_0 - y}{t}\right) \right) (K_i^1(x - y) - K_i^1(x_0 - y)) f(y) dy. \end{aligned}$$



Call

$$\Phi(y) = \left(1 - \eta\left(\frac{x_0 - y}{t}\right)\right) (K_i^1(x - y) - K_i^1(x_0 - y)).$$

$\Phi$  is a smooth function of  $y$  and by Lemma 1 we get

$$|T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0)| \leq N_{x_0}(\Phi) f^+(x_0).$$

It suffices to prove that  $\sup_{t>0} |N_{x_0}(\Phi)| \leq C$  to verify our claim.

By the basic estimates for  $\{K_i^1\}$  we get that

$$|\Delta_y^\alpha K_i^1(x - y) - \Delta_y^\alpha K_i^1(x_0 - y)| \leq \frac{C|x - x_0|}{|x - y|^{n+2|\alpha|}}.$$

Since  $|x - y| \sim |x_0 - y|$  and  $|x - x_0| \leq t$ , we get

$$\sup_{t>0} |N_{x_0}(\Phi)| \leq C \sup_{t>0} \sum_{s=0}^N \int_{|y-x_0|>2t} |y - x_0|^{2s} \frac{|x - x_0|}{|x - y|^{n+2s+1}} dy \leq C.$$

We denote by

$$(T_i^1)_* f(x_0) = \sup_{t>0} \left| \int \left(1 - \eta\left(\frac{x_0 - y}{t}\right)\right) K_i^1(x_0 - y) f(y) dy \right|$$

the smoothly truncated maximal singular integral of  $f$ . By [FS] we get that  $\|(T_i^1)_* f\|_{H^p} \leq C \|f\|_{H^p}$ . We now use (2.1) to estimate  $B_4$  as follows

$$\begin{aligned} \left| \int \phi_{t,x_0} B_4 dx \right| &\leq \sum_i f^+(x_0) g^+(x_0) \int \phi_{t,x_0} dx \\ &\quad + \sum_i f^+(x_0) (T_i^2)_* g(x_0) \int \phi_{t,x_0} dx \\ &\quad + \sum_i (T_i^1)_* f(x_0) g^+(x_0) \int \phi_{t,x_0} dx \\ &\quad + \sum_i (T_i^1)_* f(x_0) (T_i^2)_* g(x_0) \int \phi_{t,x_0} dx. \end{aligned}$$

Since the right hand side above is independent of  $t$

$$(2.2) \sup_{t>0} \left| \int \phi_{t,x_0} B_4 dx \right| \leq \sum_i \{f^+(x_0) g^+(x_0) + (T_i^1)_* f(x_0) g^+(x_0) + f^+(x_0) (T_i^2)_* g(x_0) + (T_i^1)_* f(x_0) (T_i^2)_* g(x_0)\}.$$

We raise the right hand side in (2.2) to the power  $r$  and we integrate with respect to  $x_0$ . We then apply Hölder's inequality to the right hand side with exponents  $p/r$  and  $q/r$ . We finally get that

$$\begin{aligned} \left\| \sup_{t>0} \left| \int \phi_{t,x_0} B_4 dx \right| \right\|_{L^r} &\leq \sum_i \{ \|f^+\|_{L^p} \|g^+\|_{L^q} + \|(T_i^1)*f\|_{L^p} \|g^+\|_{L^q} \\ &\quad + \|f^+\|_{L^p} \|(T_i^2)*g\|_{L^q} + \|(T_i^1)*f\|_{L^p} \|(T_i^2)*g\|_{L^q} \} \\ &\leq C \|f\|_{H^p} \|g\|_{H^q} \end{aligned}$$

and this is the required estimate for the term  $B_4$ . We now estimate the term  $B_3$ . We have

$$\int \phi_{t,x_0} B_3 dx = \int \phi_{t,x_0} B(f, \eta_1 g) dx = I_1 + I_2$$

where

$$I_1 = \sum_i \int \phi_{t,x_0}(x) T_i^1(f)(x) (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) dx,$$

$$I_2 = \sum_i T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x) T_i^1(f)(x) dx.$$

Let us start with the term  $I_1$ . Fix an  $i$  and call

$$\Phi(x) = \phi_{t,x_0}(x) (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)).$$

$\Phi$  is a smooth function of  $x$ . We claim that  $N_{x_0}(\Phi) \leq Cg^+(x_0)$ . To prove the claim, we first estimate

$$\begin{aligned} \left| \frac{\partial^\beta}{\partial x^\beta} (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) \right| &= \left| \int \eta_1(y) \left( \frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0-y)) \right) g(y) dy \right| \\ &= \left| \int \Psi(y) g(y) dy \right| \\ &\leq N_{x_0}(\Psi) g^+(x_0). \end{aligned}$$

where we set

$$\Psi(y) = \eta_1(y) \left( \frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0-y)) \right).$$

An easy calculation, using the basic estimates for  $\{K_i^2\}$ , shows that for  $|x - x_0| \leq t$ ,

$$\begin{aligned} N_{x_0}(\Psi) &\leq \sum_{s=0}^N \int_{|y-x_0|>2t} |y-x_0|^{2s} \left| \Delta^s \left( \eta_1(y) \frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0-y)) \right) \right| dy \\ &\leq Ct^{-|\beta|}. \end{aligned}$$

Therefore

$$\left| \frac{\partial^\beta}{\partial x^\beta} (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) \right| \leq Ct^{-|\beta|} g^+(x_0).$$

Now,

$$\begin{aligned} N_{x_0}(\Phi) &\leq \sum_{s=0}^N \int |x-x_0|^{2s} |\Delta^s(\phi_{t,x_0}(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)))| dx \\ &\leq C \sum_{s=0}^N \int_{|x-x_0|<t} |x-x_0|^{2s} \sum_{j=0}^{2s} |D_x^{2s-j} \phi_{t,x_0}| |D_x^j(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))| dx \end{aligned}$$

where by  $D_x^j F$  we denote any derivative of  $F$  in  $x$  of total order  $j$ .

The above can be dominated by

$$C \sum_{s=0}^N \int_{|x-x_0|<t} |x-x_0|^{2s} \sum_{j=0}^{2s} t^{-n-(2s-j)} t^{-j} dx g^+(x_0) \leq Cg^+(x_0).$$

This estimate finishes the proof of the claim.

An application of Lemma 1 gives that

$$|I_1| \leq \sum_i N_{x_0}(\Phi)(T_i^1 f)^+(x_0) \leq C \sum_i g^+(x_0)(T_i^1 f)^+(x_0).$$

Since the right hand side does not depend on  $t$ , the  $\sup_{t>0} |I_1|$  satisfies the same estimate. Hölder's inequality will give that

$$\begin{aligned} \left\| \sup_{t>0} |I_1| \right\|_{L^r} &\leq C \sum_i \|g^+\|_{L^q} \|(T_i^1 f)^+\|_{L^p} \\ &\leq C \sum_i \|g\|_{H^q} \|T_i^1 f\|_{H^p} \\ &\leq C \|f\|_{H^p} \|g\|_{H^q}. \end{aligned}$$

We now continue with the term  $I_2$ . Term  $I_2$  satisfies the estimate

$$|I_2| \leq C \sum_i (T_i^2)_* g(x_0) (T_i^1 f)^*(x_0),$$

where by  $(T_i^1 f)^*$  we denote some smooth maximal function of  $T_i f$  and the same argument as before will give that

$$\left\| \sup_{t>0} |I_2| \right\|_{L^r} \leq C \|g\|_{H^q} \|f\|_{H^p}.$$

This estimate concludes the treatment of term  $B_3$ . We deal similarly with term  $B_2$ .

### 3. The Main Term

We are now left with the term  $B_1$  which is the main term of the bilinear operator  $B$ . We start by introducing some notation. For  $0 \leq \delta \leq 1$ , we denote by  $\Lambda_\delta$  the Lipschitz space of all bounded functions  $f$  on  $\mathbb{R}^n$  with

$$\sup_{x \in \mathbb{R}^n} \sup_{h \in \mathbb{R}^n} |h|^{-\delta} |f(x+h) - f(x)| = \|f\|_{\Lambda_\delta} < +\infty.$$

For  $m \in \mathbb{Z}$ ,  $m \geq 1$  and  $0 \leq \delta \leq 1$ , we denote by  $\Lambda_\delta^m$  the space of all bounded functions  $f$  on  $\mathbb{R}^n$  whose partials of order  $m$  exist and are in  $\Lambda_\delta$  and whose partials of order up to  $m-1$  are bounded. The  $\|\cdot\|_{\Lambda_\delta^m}$  norm of a function is defined as the sum of the  $\|\cdot\|_{\Lambda_\delta}$  norms of its partials of order  $m$ . It is easy to see that functions in  $\Lambda_\delta^m$  satisfy

$$\left| f(x+h) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha f}{\partial x^\alpha}(x) \frac{h^\alpha}{\alpha!} \right| \leq \frac{\|f\|_{\Lambda_\delta^m}}{m!} |h|^{m+\delta} \quad \text{for all } x \in \mathbb{R}^n, h \in \mathbb{R}^n.$$

For  $m, l \in \mathbb{Z}^+ \cup \{0\}$  and  $0 \leq \gamma, \delta \leq 1$  we define the space  $\Lambda_{\gamma, \delta}^{m, l}$  of all bounded functions  $b(y, z)$  on  $\mathbb{R}^n \times \mathbb{R}^n$  which are in  $\Lambda_\gamma^m$  as functions of  $y$  and in  $\Lambda_\delta^l$  as functions of  $z$  and satisfy the condition:

$$(3.1) \quad \left| b(y+h, z+k) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha}{\partial y^\alpha} b(y, z+k) \frac{h^\alpha}{\alpha!} - \sum_{|\beta| \leq l} \frac{\partial^\beta}{\partial z^\beta} b(y+h, z) \frac{k^\beta}{\beta!} + \sum_{|\alpha| \leq m} \sum_{|\beta| \leq l} \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial^\beta}{\partial z^\beta} b(y, z) \frac{h^\alpha}{\alpha!} \frac{k^\beta}{\beta!} \right| \leq C |h|^{m+\gamma} |k|^{l+\delta}$$

for all  $y, z, k, h \in \mathbb{R}^n$ .

This double Lipschitz condition above is equivalent to either one of the statements below

$$(3.2) \quad |h|^{-m-\gamma} \left( b(y+h, z) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha}{\partial y^\alpha} b(y, z) \frac{h^\alpha}{\alpha!} \right)$$

is in  $\Lambda_\delta^l$  as a function of  $z$

$$(3.3) \quad |k|^{-l-\delta} \left( b(y, z+k) - \sum_{|\beta| \leq l} \frac{\partial^\beta}{\partial z^\beta} b(y, z) \frac{k^\beta}{\beta!} \right)$$

is in  $\Lambda_\gamma^m$  as a function of  $y$  for all  $y, z, k, h \in \mathbb{R}^n$ ,  $|h|, |k| \leq C$ .

We will now state and prove the main lemma needed to estimate term  $B_1$ .

**Lemma 2.** *Suppose that  $p, q \leq 1$  and let  $m = [n(1/p - 1)]$ ,  $l = [n(1/q - 1)]$ . Suppose that the compactly supported function  $b(y, z)$  is in  $\Lambda_{\gamma, \delta}^{m, l}$ . For  $x_0 \in \mathbb{R}^n$  let*

$$S(f, g)(x_0) = \sup_{t > 0} \left| \iint f(y)g(z) \frac{1}{t^{2n}} b\left(\frac{x_0 - y}{t}, \frac{x_0 - z}{t}\right) dy dz \right|.$$

Then  $S(f, g)$  maps  $H^p \times H^q \rightarrow L^r$  where  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$  as long as

$$1 \geq \gamma > n\left(\frac{1}{p} - 1\right) - \left[ n\left(\frac{1}{p} - 1\right) \right]$$

and

$$1 \geq \delta > n\left(\frac{1}{q} - 1\right) - \left[ n\left(\frac{1}{q} - 1\right) \right].$$

**PROOF.** Let  $a_Q(y)$  be an  $H^p$ -atom and  $b_R(z)$  be an  $H^q$ -atom. We will first estimate  $S(a_Q, b_R)(x_0)$ . Let  $y_Q$  be the center of the cube  $Q$  and  $z_R$  the center of the cube  $R$ . We have the following four basic estimates:

*Case 1.*  $x_0 \in 3Q$  and  $x_0 \in 3R$ , then

$$S(a_Q, b_R)(x_0) \leq C|Q|^{-1/p}\chi_{3Q}(x_0)|R|^{-1/q}\chi_{3R}(x_0).$$

*Case 2.*  $x_0 \in 3Q$  and  $x_0 \notin 3R$ , then

$$S(a_Q, b_R)(x_0) \leq C|Q|^{-1/p}\chi_{3Q}(x_0) \frac{|R|^{-1/q+1+(l+\delta)/n}}{\text{dist}\{x_0, R\}^{n+l+\delta}}.$$

*Case 3.*  $x_0 \notin 3Q$  and  $x_0 \in 3R$ , then

$$S(a_Q, b_R)(x_0) \leq C \frac{|Q|^{-1/p+1+(m+\gamma)/n}}{\text{dist}\{x_0, Q\}^{n+m+\gamma}} |R|^{-1/q}\chi_{3R}(x_0).$$

Case 4.  $x_0 \notin 3Q$  and  $x_0 \notin 3R$ , then

$$S(a_Q, b_R)(x_0) \leq C \frac{|Q|^{-1/p+1+(m+\gamma)/n}}{\text{dist}\{x_0, Q\}^{n+m+\gamma}} \frac{|R|^{-1/q+1+(l+\delta)/n}}{\text{dist}\{x_0, R\}^{n+l+\delta}}.$$

We indicate how to prove the basic estimates.

In Case 1 we just use the  $L^\infty$  bounds for atoms.

In Case 2 we need the following estimate which can be easily proved by integrating (3.3) over the  $y$ -support of  $b$ .

$$(3.4) \quad \int \left| b(y, z+k) - \sum_{|\beta| \leq l} \frac{\partial^\beta b}{\partial z^\beta}(y, z) \frac{k^\beta}{\beta!} \right| dy \leq C |k|^{l+\delta} \quad \text{for all } z, k \in \mathbb{R}^n.$$

Using the fact that  $b_R$  has moments up to order  $l$  vanishing we have

$$(3.5) \quad \begin{aligned} & \iint a_Q(y) b_R(z) \frac{1}{t^{2n}} b\left(\frac{x_0-y}{t}, \frac{x_0-z}{t}\right) dy dz \\ &= \int a_Q(y) \int b_R(z) \frac{1}{t^n} \left( b\left(\frac{x_0-y}{t}, \frac{x_0-z}{t}\right) \right. \\ & \quad \left. - \sum_{|\beta| \leq l} \frac{\partial^\beta b}{\partial z^\beta}\left(\frac{x_0-y}{t}, \frac{x_0-z_R}{t}\right) \frac{1}{\beta!} \left(\frac{z_R-z}{t}\right)^\beta \right) \\ & \quad \times \frac{dz}{t^n} dy. \end{aligned}$$

By (3.4) the above can be estimated by

$$\begin{aligned} & C |Q|^{-1/p} \chi_{3Q}(x_0) |R|^{-1/q} \frac{1}{t^n} \int_R \left| \frac{z-z_R}{t} \right|^{l+\delta} dz \\ & \leq C |Q|^{-1/p} \chi_{3Q}(x_0) \frac{|R|^{-1/q+1+(l+\delta)/n}}{t^{n+l+\delta}}. \end{aligned}$$

Note that (3.5) is nonzero if

$$t \geq \text{dist}\{(x_0, x_0), Q \times R\} \sim \text{dist}\{x_0, R\},$$

since  $x_0 \in 3Q$  and  $x_0 \notin 3R$ . The required estimate for  $S(a_Q, b_R)$  in Case 2 follows. Case 3 is similar.

To get the estimate in Case 4 we use the double Lipschitz estimate for  $b$ . We have

$$\begin{aligned}
 & \iint a_Q(y)b_R(z) \frac{1}{t^{2n}} b\left(\frac{x_0-y}{t}, \frac{x_0-z}{t}\right) dy dz \\
 &= \iint a_Q(y)b_R(z) \frac{1}{t^{2n}} \left\{ b\left(\frac{x_0-y}{t}, \frac{x_0-z}{t}\right) \right. \\
 & \quad - \sum_{|\alpha| \leq m} \frac{\partial^\alpha b}{\partial y^\alpha} \left(\frac{x_0-y_Q}{t}, \frac{x_0-z}{t}\right) \frac{1}{\alpha!} \left(\frac{y_Q-y}{t}\right)^\alpha \\
 & \quad - \sum_{|\beta| \leq l} \frac{\partial^\beta b}{\partial z^\beta} \left(\frac{x_0-y}{t}, \frac{x_0-z_R}{t}\right) \frac{1}{\beta!} \left(\frac{z_R-z}{t}\right)^\beta \\
 & \quad \left. + \sum_{|\alpha| \leq m} \sum_{|\beta| \leq l} \frac{\partial^\alpha \partial^\beta b}{\partial y^\alpha \partial z^\beta} \left(\frac{x_0-y_Q}{t}, \frac{x_0-z_R}{t}\right) \frac{1}{\alpha!} \right. \\
 & \quad \left. \times \left(\frac{y_Q-y}{t}\right)^\alpha \frac{1}{\beta!} \left(\frac{z_R-z}{t}\right)^\beta \right\} dy dz.
 \end{aligned} \tag{3.6}$$

Apply (3.1) to bound the above by

$$\begin{aligned}
 & C|Q|^{-1/p}|R|^{-1/q} \frac{1}{t^{2n}} \int_Q \int_R \left| \frac{y-y_Q}{t} \right|^{m+\gamma} \left| \frac{z-z_R}{t} \right|^{l+\delta} dy dz \\
 & \leq C t^{-2n-m-l-\gamma-\delta} |Q|^{-1/p+1+(m+\gamma)/n} |R|^{-1/q+1+(l+\delta)/n}.
 \end{aligned}$$

Note that (3.6) is nonzero as long as

$$t \geq \text{dist}\{(x_0, x_0), Q \times R\} \sim \text{dist}\{x_0, Q\} + \text{dist}\{x_0, R\},$$

because  $x_0 \notin 3Q$  and  $x_0 \notin 3R$ .

Since

$$\begin{aligned}
 & \text{dist}\{(x_0, x_0), Q \times R\}^{-2n-m-\gamma-l-\delta} \\
 & \leq C \text{dist}\{x_0, Q\}^{-n-m-\gamma} \text{dist}\{x_0, R\}^{-n-l-\delta},
 \end{aligned}$$

the required estimate for (3.6) follows immediately.

We have now proved our basic estimates and we continue with the proof of the lemma.

Let  $f \in H^p$ ,  $g \in H^q$  be finite sums of atoms

$$f = \sum \lambda_Q a_Q, \quad g = \sum \mu_R b_R$$

where  $\lambda_Q > 0$ ,  $\mu_R > 0$  and

$$\left(\sum \lambda_Q^p\right)^{1/p} \approx \|f\|_{H^p}, \quad \left(\sum \mu_R^q\right)^{1/q} \approx \|g\|_{H^q}.$$

Bound

$$S(f, g)(x_0) \leq \sum_Q \sum_R \lambda_Q \mu_R S(a_Q, b_R)(x_0)$$

by  $\Sigma_1(x_0) + \Sigma_2(x_0) + \Sigma_3(x_0) + \Sigma_4(x_0)$  where

$$\Sigma_j(x_0) = \sum_Q \sum_R \lambda_Q \mu_R S(a_Q, b_R)(x_0)$$

and the sum above is taken over all  $Q$  and  $R$  related to  $x_0$  as in case  $j$  above,  $1 \leq j \leq 4$ .

We will show that each  $\Sigma_j$  is in  $L^r(dx_0)$  with  $L^r$  quasinorm bounded by  $C \|f\|_{H^p} \|g\|_{H^q}$ . Then we can sum on  $j$ ,  $1 \leq j \leq 4$ . Use Hölder's inequality with exponents  $p/r$ ,  $q/r$  to get

$$\begin{aligned} & \left( \int \Sigma_1(x_0)^r dx_0 \right)^{1/r} \\ & \leq C \left( \int \left( \sum \lambda_Q |Q|^{-1/p} \chi_{3Q}(x_0) \right)^p dx_0 \right)^{1/p} \left( \int \left( \sum \mu_R |R|^{-1/q} \chi_{3R}(x_0) \right)^q dx_0 \right)^{-1/q} \\ & \leq C \left( \int_{3Q} \sum \lambda_Q^p |Q|^{-1} dx_0 \right)^{1/p} \left( \int_{3R} \sum \mu_R^q |R|^{-1} dx_0 \right)^{1/q} \quad (\text{since } p, q \leq 1) \\ & \leq C \left( \sum \lambda_Q^p \right)^{1/p} \left( \sum \mu_R^q \right)^{1/q} \leq C \|f\|_{H^p} \|g\|_{H^q}. \end{aligned}$$

Similarly

$$\begin{aligned} & \left( \int \Sigma_4(x_0)^r dx_0 \right)^{1/r} \\ & \leq C \left( \int_{\text{dist}\{x_0, Q\} \leq C|Q|^{1/n}} \left( \sum \lambda_Q \frac{|Q|^{-1/p+1+(m+\gamma)/n}}{\text{dist}\{x_0, Q\}^{n+m+\gamma}} \right)^p dx_0 \right)^{1/p} \\ & \quad \left( \int_{\text{dist}\{x_0, R\} \geq C|R|^{1/n}} \left( \sum \mu_R \frac{|R|^{-1/q+1+(l+\delta)/n}}{\text{dist}\{x_0, R\}^{n+l+\delta}} \right)^q dx_0 \right)^{1/q} \\ & \leq C \left( \int_{\text{dist}\{x_0, Q\} \geq C|Q|^{1/n}} \sum \lambda_Q^p \frac{|Q|^{-1+p+(m+\gamma)p/n}}{\text{dist}\{x_0, Q\}^{(n+m+\gamma)p}} dx_0 \right)^{1/p} \\ & \quad \left( \int_{\text{dist}\{x_0, R\} \geq C|R|^{1/n}} \sum \mu_R^q \frac{|R|^{-1+q+(l+\delta)q/n}}{\text{dist}\{x_0, R\}^{(n+l+\delta)q}} dx_0 \right)^{1/q} \\ & \leq C \left( \sum \lambda_Q^p \right)^{1/p} \left( \sum \mu_R^q \right)^{1/q} \leq C \|f\|_{H^p} \|g\|_{H^q}, \end{aligned}$$



where we used the fact that  $(n + m + \gamma)p > n$  and

$$\int_{\text{dist}\{x_0, Q\} \geq C|Q|^{1/n}} \text{dist}\{x_0, Q\}^{-(n+m+\gamma)p} dx_0 \sim |Q|^{(n-(n+m+\gamma)p)/n} \sim |Q|^{1-p-(m+\gamma)p/n}.$$

The same way we can prove that

$$\left( \int \Sigma_2(x_0)^r dx_0 \right)^{1/r} \leq C \|f\|_{H^p} \|g\|_{H^p}$$

and

$$\left( \int \Sigma_3(x_0)^r dx_0 \right)^{1/r} \leq C \|f\|_{H^p} \|g\|_{H^q}$$

using a combination of the estimates above. Finally we approximate the general  $f \in H^p$  and  $g \in H^q$  by finite sums of atoms to finish the proof of the lemma.

We now continue the proof of our theorem by estimating the term  $B_1$ . We have

$$\int \phi_{t,x_0} B_1 dx = \iint f(y)g(z) \frac{1}{t^{2n}} b\left(\frac{x_0-y}{t}, \frac{x_0-z}{t}\right) dy dz$$

where we set

$$b(y, z) = \sum_i \eta(y)\eta(z) \int \phi(\sigma) K_i^1(y-\sigma) K_i^2(z-\sigma) d\sigma.$$

To apply Lemma 2 we need to prove that  $b(y, z) \in \Lambda_{\gamma, \delta}^{m, l}$  where  $m, \gamma, l, \delta$  as in Lemma 2.

Note that the assumption that  $B(f, g)$  has moments up to order  $k$  vanishing gives that the kernel of  $x^\alpha B(f, g)(x)$  is identically zero for all  $|\alpha| \leq k$ , i.e.

$$\sum_i \int x^\alpha K_i^1(y-x) K_i^2(z-x) dx = 0 \quad \text{for all } y, z \in \mathbb{R}^n.$$

We can therefore write

$$b(y, z) = \eta(x)\eta(y) \sum_i \int \left( \phi(\sigma) - \sum_{|\alpha| \geq k} \frac{\partial^\alpha \phi}{\partial y^\alpha}(\sigma) \frac{(\sigma-y)^\alpha}{\alpha!} \right) K_i^1(y-\sigma) K_i^2(z-\sigma) d\sigma.$$

The fact that  $b(y, z) \in \Lambda_{\gamma, \delta}^{m, l}$  where  $m, l, \gamma, \delta$  are as in Lemma 2, will be a consequence of the following two lemmas.

**Lemma 3.** *Let  $G$  be a function on  $\mathbb{R}^n$  of class  $\Lambda_1^s$  and let  $a(y, \sigma) = G(y - \sigma)$ .*

- (a) *If  $m, l$  are non-negative integers such that  $m + l = s$  and  $\gamma, \delta > 0$  such that  $\gamma + \delta = 1$ , then  $a(y, \sigma)$  is of class  $\Lambda_{\gamma, \delta}^{m, l}$ .*  
 (b) *If  $m, l$  are integers such that  $m + l = s - 1$  then  $a(y, \sigma)$  is of class  $\Lambda_{1, 1}^{m, l}$ .*

**Lemma 4.** *Let  $K$  be a convolution Calderón-Zygmund kernel on  $\mathbb{R}^n$ . If  $a(y, \sigma) \in \Lambda_{\gamma, \delta}^{m, l}$  then*

$$b(y, z) = \int a(y, \sigma)K(z - \sigma) d\sigma \in \Lambda_{\gamma, \delta}^{m, l}.$$

**PROOF.** We denote by  $\partial^\alpha G$  the partial derivative of  $G$  of order  $\alpha = (\alpha_1, \dots, \alpha_n)$ . To prove Lemma 3 (a), by (3.2), it suffices to show that the function

$$\begin{aligned} F(\sigma) &= \frac{1}{|h|^{m+\gamma}} \left( a(y+h, \sigma) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha a}{\partial y^\alpha}(y, \sigma) \frac{h^\alpha}{\alpha!} \right) \\ &= \frac{1}{|h|^{m+\gamma}} \left( g(y+h-\sigma) - \sum_{|\alpha| \leq m} \partial^\alpha G(y-\sigma) \frac{h^\alpha}{\alpha!} \right) \end{aligned}$$

is in  $\Lambda_\delta^l$  in the variable  $\sigma$  uniformly in  $y \in \mathbb{R}^n$ . This will be a consequence of the following two observations:

- (1)  $F(\sigma)$  is in  $\Lambda_0^l$  with norm  $\leq C|h|^{1-\gamma}$ .  
 (2)  $F(\sigma)$  is in  $\Lambda_1^l$  with norm  $\leq C|h|^{-\gamma}$ .

Interpolation will then give that  $F(\sigma)$  is in  $\Lambda_\delta^l$  with norm  $\leq C$ . (Recall  $\gamma + \delta = 1$ .)

Both observations follow from Taylor's theorem. For some  $\xi_{y, h}$  between  $y$  and  $y+h$  we have that

$$F(\sigma) = |h|^{-m-\gamma} \left( \sum_{|\alpha|=m} (\partial^\alpha G(\xi_{y, h} - \sigma) - \partial^\alpha G(y - \sigma)) \frac{h^\alpha}{\alpha!} \right)$$

and therefore for a fixed  $\beta$  with  $|\beta| = l$  we have

$$\partial^\beta F(\sigma) = |h|^{-m-\gamma} \left( \sum_{|\alpha|=m} (\partial^{\alpha+\beta} G(\xi_{y, h} - \sigma) - \partial^{\alpha+\beta} G(y - \sigma)) \frac{h^\alpha}{\alpha!} \right).$$

Since  $|\alpha + \beta| = m + l = s$  and since  $\partial^\sigma G$  is in  $\Lambda_1$  if  $|\sigma| = s$  it follows that  $\partial^{\alpha+\beta} G$  is in  $\Lambda_1$  and thus  $(\partial^\beta F)(\sigma)$  is in  $\Lambda_0 = L^\infty$ , with norm

$$\leq C|h|^{-\gamma} \sum_{|\alpha|=s} \|\partial^\alpha G\|_{\Lambda_1} |\xi_{y, h} - y| \leq C|h|^{1-\gamma}.$$

Also, from the translation invariance of the Lipschitz norms it follows that  $\partial^{\alpha+\beta}G(\xi_{y,h} - \sigma)$  and  $\partial^{\alpha+\beta}G(y - \sigma)$  are in  $\Lambda_1$  in  $\sigma$ , and therefore the function  $\partial^\beta F(\sigma)$  is in  $\Lambda_1$  in  $\sigma$  with norm  $\leq C|h|^{-\gamma}$ . We proved that the arbitrary partial derivative  $\partial^\beta F$  of  $F$  of order  $l$  is in  $\Lambda_0$  with norm  $\leq C|h|^{1-\gamma}$  and in  $\Lambda_1$  with norm  $\leq C|h|^{-\gamma}$ . This concludes the proofs of the observations. Note that both norm estimates are independent of  $y$ .

Part (b) of Lemma 3 follows by a similar argument. We need to show that the function

$$\begin{aligned} F(\sigma) &= \frac{1}{|h|^{m+1}} \left( a(y+h, \sigma) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha a}{\partial y^\alpha}(y, \sigma) \frac{h^\alpha}{\alpha!} \right) \\ &= \frac{1}{|h|^{m+1}} \left( G(y+h-\sigma) - \sum_{|\alpha| \leq m} \partial^\alpha G(y-\sigma) \frac{h^\alpha}{\alpha!} \right) \end{aligned}$$

is of class  $\Lambda_1^l$  in the variable  $\sigma$  uniformly in  $y \in \mathbb{R}^n$ . For some  $\xi'_{y,h}$  between  $y$  and  $y+h$ , we have that

$$F(\sigma) = \frac{|h|^{-m-1}}{(m+1)!} \sum_{|\alpha|=m+1} \partial^\alpha G(\xi'_{y,h} - \sigma) \frac{h^\alpha}{\alpha!}.$$

Fix a multi-index  $\beta$  with  $|\beta| = l$ . Clearly

$$\partial^\beta F(\sigma) = \frac{|h|^{-m-1}}{(m+1)!} \sum_{|\alpha|=m+1} \partial^{\alpha+\beta} G(\xi'_{y,h} - \sigma) \frac{h^\alpha}{\alpha!}.$$

Since in this case  $|\alpha+\beta| = m+1+l = s$  and the partials of  $G$  of order  $s$  are in  $\Lambda_1$  it follows that  $\partial^\beta F$  is in  $\Lambda_1$  with norm independent of  $y$ . Therefore  $F$  is in  $\Lambda_1^l$  with norm independent of  $y$  and this concludes the proof of Lemma 3.

We now indicate how to prove Lemma 4. It is a well-known fact that convolution Calderón-Zygmund operators map the Lipschitz spaces  $\Lambda_\gamma^m$  into themselves. If  $a(y, \sigma) \in \Lambda_{\gamma,\delta}^{m,l}$  then by (3.2) the function

$$|h|^{-m-\gamma} \left\{ a(y+h, \sigma) - \sum_{|\alpha| < m} \frac{\partial^\alpha a}{\partial y^\alpha}(y, \sigma) \frac{h^\alpha}{\alpha!} \right\}$$

is in  $\Lambda_\delta^l$  in the variable  $\sigma$ . Convolution with  $K$  in  $\sigma$  will give that

$$|h|^{-m-\gamma} \left\{ b(y+h, z) - \sum_{|\alpha| \leq m} \frac{\partial^\alpha b}{\partial y^\alpha}(y, z) \frac{h^\alpha}{\alpha!} \right\} \in \Lambda_\delta^l \quad \text{in } z$$

and by (3.2) again we get that  $b(y, z) \in \Lambda_{\gamma,\delta}^{m,l}$ . This finishes the proof of Lemma 4.

To conclude the proof of our theorem, it suffices to check that the functions

$$a_i(y, \sigma) = \left\{ \phi(\sigma) - \sum_{|\alpha| \leq k} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y) \frac{(\sigma - y)^\alpha}{\alpha!} \right\} K_i^1(y - \sigma)$$

are in  $\Lambda_{\gamma, \delta}^{m, l}$ . Then Lemma 4 will give that  $b(y, z)$  is in  $\Lambda_{\gamma, \delta}^{m, l}$  also.

We write

$$a_i(y, \sigma) = \sum_{|\alpha| = k+1} \phi^{(\alpha)}(\xi_{\sigma, y}) \frac{(\sigma - y)^\alpha}{\alpha!} K_i^1(y - \sigma).$$

Fix  $i$  and  $\alpha$  with  $|\alpha| = k + 1$ . It suffices to show that

$$\phi^{(\alpha)}(\xi_{\sigma, y}) (\sigma - y)^\alpha K_i^1(y - \sigma) \in \Lambda_{\gamma, \delta}^{m, l}.$$

Since  $\phi^{(\alpha)}(\xi_{\sigma, y})$  is a smooth function of  $y$  and  $\sigma$  it is enough to show that

$$(\sigma - y)^\alpha K_i^1(y - \sigma) \in \Lambda_{\gamma, \delta}^{m, l}$$

for suitable  $m, l, \gamma, \delta$ .

As a function of the variable  $x$ , the function  $x^\alpha K_i^1(x)$  is smooth everywhere except possibly at zero and it behaves like  $|x|^{k+1}|x|^{-n} = |x|_1^{k-n}$  as  $|x| \rightarrow 0$ . Therefore,  $x^\alpha K_i^1(x) \in \Lambda^{k-n+1}$ . We are going to apply Lemma 3 with  $G(x) = x^\alpha K_i^1(x)$ .

Let  $m = \left\lceil n \left( \frac{1}{p} - 1 \right) \right\rceil$ ,  $l = \left\lceil n \left( \frac{1}{q} - 1 \right) \right\rceil$ . Since  $\frac{n+k}{n} \leq \frac{1}{r} \leq \frac{n+k+1}{n}$  it follows that  $m+l$  is either  $k-n$  or  $k-n-1$ . If  $m+l = k-n$  just pick  $\gamma > n \left( \frac{1}{p} - 1 \right) - m$  and  $\delta > n \left( \frac{1}{q} - 1 \right) - l$  such that  $\gamma + \delta = 1$  (this is possible since  $1 > n \left( \frac{1}{r} - 2 \right) - m - l$ ) and apply part (a) of Lemma 3 with  $s = k-n$ . If  $m+l = k-n-1 = s-1$ , apply part (b) of Lemma 3.

It follows that  $(\sigma - y)^\alpha K_i^1(y - \sigma)$  is in  $\Lambda_{\gamma, \delta}^{m, l}$  where  $\gamma$  and  $\delta$  satisfy the hypotheses of Lemma 1. The same is true for  $a_i(y, \sigma)$  and therefore for  $b(y, z)$  by Lemma 4.

#### 4. Applications and Examples

The vanishing moments properties of the bilinear operators

$$B(f, g) = \sum_j (T_j^1 f)(T_j^2 g)$$

can be written in terms of relations involving multipliers. Let  $m_j^1$  and  $m_j^2$  be the multipliers corresponding to the Calderón-Zygmund operators  $T_j^1$  and  $T_j^2$ .

$B(f, g)$  has mean value zero if and only if  $\widehat{B(f, g)}(0) = 0$ , *i.e.*

$$\sum (\widehat{T_j^1 f} * \widehat{T_j^2 g})(0) = 0.$$

This is equal to

$$\sum_j \int m_j^1(-\xi) \hat{f}(-\xi) m_j^2(\xi) \hat{g}(\xi) d\xi = 0$$

and since  $f$  and  $g$  are arbitrary the above is equivalent to the statement

$$\sum_j m_j^1(-\xi) m_j^2(\xi) = 0 \quad \text{for all } \xi \neq 0.$$

Similar reasoning shows that  $B$  has two moments zero if and only if the following identities hold

$$\begin{aligned} \sum_j m_j^1(-\xi) m_j^2(\xi) &= 0 \quad \text{for all } \xi \neq 0, \\ \sum_j m_j^1(-\xi) \left( \frac{\partial}{\partial y_i} m_j^2 \right)(\xi) &= 0 \quad \text{for all } \xi \neq 0, \quad i = 1, 2, \dots, n. \end{aligned}$$

The second identity can be replaced by

$$\sum_j \left( \frac{\partial}{\partial \xi_i} m_j^1 \right)(-\xi) m_j^2(\xi) = 0 \quad \text{for all } i$$

in view of the first identity and the product rule.

Generalizing the above, we get that  $B$  has all moments of order up to and including  $k$  vanishing if and only if

$$(4.1) \quad \sum_j m_j^1(-\xi) \left( \frac{\partial^m}{\partial \xi_i^m} m_j^2 \right)(\xi) = 0$$

holds for all  $\xi \neq 0$ ,  $i = 1, 2, \dots, n$ ;  $m = 0, 1, \dots, k$ . The identities above give us an easy way to decide whether a bilinear operator has vanishing moments. For example, using (4.1), it is trivial to check that the bilinear operator  $\tilde{J}(f, g) = R_1 f R_2 g - R_2 f R_1 g$  has integral zero. To include an example, we check that the bilinear operator  $\tilde{H}(f, g) = (R_1^2 f)(R_2^2 g) - 2(R_1 R_2 f)(R_2 R_1 g) + (R_2^2 f)(R_1^2 g)$  has vanishing first moments. We calculate (4.1) when  $m = 1$  and  $i = 1$ . Let  $T_1^1 = R_1^2$ ,  $T_1^2 = R_2^2$ ,  $T_2^1 = -2R_1 R_2$ ,  $T_2^2 = R_2 R_1$ ,  $T_3^1 = R_2^2$ ,  $T_3^2 = R_1^2$  and let  $m_1^1(\xi) = -\xi_1^2/|\xi|^2$ ,  $m_1^2(\xi) = -\xi_2^2/|\xi|^2$ ,  $m_2^1(\xi) = 2\xi_1 \xi_2/|\xi|^2$ ,  $m_2^2(\xi) = -\xi_1 \xi_2/|\xi|^2$ ,  $m_3^1(\xi) = -\xi_2^2/|\xi|^2$ ,  $m_3^2(\xi) = -\xi_1^2/|\xi|^2$  be the corresponding multipliers. Then

$$\sum_{j=1}^3 m_j^1(-\xi) \frac{\partial}{\partial y_1} m_j^2(\xi) = \left( -\frac{\xi_1^2}{|\xi|^2} \right) \left( \frac{2\xi_1 \xi_2^2}{|\xi|^4} \right) + \left( \frac{2\xi_1 \xi_2}{|\xi|^2} \right) \left( \frac{-\xi_2^3 + \xi_1^2 \xi_2}{|\xi|^4} \right) + \left( -\frac{\xi_2^2}{|\xi|^2} \right) \left( -\frac{2\xi_1 \xi_2^2}{|\xi|^4} \right) = 0.$$

An example of an operator with two vanishing moments is given on  $\mathbb{R}^1$  by the bilinear map  $B(f, g) = fg - (Cf)(Cg) + (Sf)(Sg)$ , where the operators  $Sf$  and  $Cf$  are defined on the Fourier side by

$$\widehat{Sf}(\xi) = \sin(\log |\xi|) \hat{f}(\xi)$$

and

$$\widehat{Cf}(\xi) = \cos(\log |\xi|) \hat{f}(\xi).$$

One can easily check using (4.1) that  $B$  has integral and first moments zero and hence by our theorem it maps  $H^p \times H^q \rightarrow H^r$  for  $p, q > 1$  and  $1/2 \geq r = (p^{-1} + q^{-1})^{-1} > 1/3$ .

Examples of bilinear operators with moments of all orders vanishing are given on  $\mathbb{R}^1$  by the maps

$$D_1(f, g) = fg - (Hf)(Hg)$$

$$D_2(f, g) = f(Hg) + (Hf)g$$

where  $H$  is the usual Hilbert transform. It follows from our theorem that  $D_1$  and  $D_2$  map  $H^p \times H^q \rightarrow H^r$  for all  $p, q \leq 1$  and  $r$  their harmonic mean.  $D_1$  and  $D_2$  are the real and imaginary parts of the holomorphic function  $(f + iHf)(g + iHg)$  and they can also be studied through complex analysis.

## References

- [C] Coifman, R. R. A Real Variable Characterization of  $H^p$ , *Studia Math.* **51**(1974), 269-274.
- [CLMS] Coifman, R. R., Lions, P. L., Meyer, Y. and Semmes, S. Compacit e par compensation et espaces de Hardy, to appear in *C. R. Acad. Sci. Paris*.
- [CM 1] Coifman, R. R. and Meyer, Y. On Commutators of Singular Integrals and Bilinear Singular Integrals, *Trans. Amer. Math. Soc.* **212**(1975).
- [CM 2] Coifman, R. R. and Meyer, Y. Non-Linear Harmonic Analysis Operator Theory and P.D.E., *Beijing Lectures in Harmonic Analysis, Annals of Math. Studies*, Princeton Univ. Press, 1986.
- [FS] Fefferman, C. and Stein, E. M.  $H^p$  Spaces of Several Variables, *Acta Math.* **228**(1972), 137-193.
- [M1] Miyachi, *Tohoku Math. J.* **35**(1983), 483-498.
- [M2] Miyachi, *Pacific Math. J.* **115**(1984), 165-175.

- [S] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.  
[SW] Stein, E. M. and Weiss, G. *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

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