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Weak Type (1,1) Estimates of the Maximal Function for the Laguerre Semigroup in Finite Dimensions

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Abstract

We prove that, in arbitrary finite dimensions, the maximal operator for the Laguerre semigroup is of weak type (1,1). This extends Muckenhoupt's one-dimensional result.

Introduction

Let (F, ν) be some positive measure space. A semigroup $\{T^t\}_{0 \leq t < \infty}$ is understood to be a family of bounded linear operators, defined simultaneously on $L^p(\nu)$, $1 \leqslant p \leqslant \infty$, satisfying measurability in t and the semigroup axioms $T^{t+s} = T^t T^s$ and $T^0 = I$ is the identity. The semigroup $\{T^t\}$ is called a «symmetric diffusion semigroup» if it has the following properties

(I)	$\|T^t f\|_p \leqslant \|f\|_p, \quad 1 \leqslant p \leqslant \infty$	(Contraction)
(II)	T^t is self-adjoint on $L^2(\nu)$	(Symmetry)
(III)	$T^t f \geqslant 0$ if $f \geqslant 0$	(Positivity)
(IV)	$T^t 1 = 1$	(Conservation)
(V)	$\lim_{t \to 0^+} T^t f = f$ in $L^2(\nu)$ for all $f \in L^2(\nu)$	(Strong continuity on L^2).

The maximal operator M associated with the semigroup $\{T^t\}$ is defined by

$$Mf(x) = \sup_{t>0} |T^t f(x)|.$$

Under the assumptions (I), (II), and (V), the maximal theorem due to E. M. Stein ([St]) states that

(*a*) M is bounded on $L^p(\nu)$, *i.e.* $\|Mf\|_p \leqslant A_p \|f\|_p$, $1 < p \leqslant \infty$.

(*b*) If $f \in L^p(\nu)$, $1 < p < \infty$, then $\lim_{t \to 0+} T^t f(x) = f(x)$ a.e.

An operator is said to be of weak type (1,1) if it maps $L^1(\nu)$ boundedly into $L^{1,\infty}(\nu)$, where $L^{1,\infty}(\nu)$ is defined by means of its quasinorm

$$\|f\|_{1,\infty} = \sup_{\beta>0} \beta\nu\{x: \ |f(x)| > \beta\}.$$

It seems to be unknown whether weak type (1,1) holds for M, as a substitute result in the maximal theorem when $p = 1$. However, the Hopf-Dunford-Schwartz ergodic theorem ([DS] p. 690) gives that the «maximal average operator» $\tilde{M}$, defined by

$$\tilde{M}f(x) = \sup_{s>0} \left| \frac{1}{s} \int_0^s T^t f(x)\, dt \right|,$$

is of weak type (1,1) if the semigroup $\{T^t\}$ satisfies (I).

In several concrete settings, the maximal operator M of a symmetric diffusion semigroup has been investigated and found to be of weak type (1,1). This is done by well-known methods when the semigroup is defined as convolution with an «approximate identity» and the underlying measure is translation invariant. For instance the Poisson and Gaussian kernels in $(\mathbb{R}^n, dx)$ define semigroups of that kind. On the other hand, semigroups defined in terms of «classical orthogonal polynomials» have to be handled differently. For the ultraspherical harmonics, the Hermite polynomials, and the Laguerre polynomials —each with the appropriate measure space in one dimension —it has been proved that the corresponding maximal operators are of weak type (1,1). For the ultraspherical harmonics this is a joint result by B. Muckenhoupt and E. M. Stein ([MS]), whereas it is due to Muckenhoupt ([M]) for the Hermite and Laguerre semigroups. In [Sj1], P. Sjögren proves the same result for the Hermite semigroup in higher (finite) dimensions, and the Laguerre semigroup in higher dimensions is treated in the present paper.

The setting is as follows (*cf.* [M]). Let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0 \text{ for every } i\}$ and denote by γ_α the measure on $\mathbb{R}^d_+$ given by the density

$$\gamma_\alpha(x) = \prod_{i=1}^{d} x_i^\alpha e^{-x_i}.$$

For $\alpha > -1$, the d-dimensional Laguerre polynomials

$$L_m^\alpha(x) = \prod_1^d L_{m_i}^\alpha(x_i), \qquad m = (m_1, \dots, m_d) \in \mathbb{N}^d,$$

form a complete orthogonal system in $L^2(\gamma_\alpha)$. The definition of the m_i-th Laguerre polynomial, $L_{m_i}^\alpha$, used here will be the one used in [Sz]. For a function f in $L^1(\gamma_\alpha)$, its «Laguerre-Poisson integral» $M_r^\alpha f$ is defined for $0 < r < 1$ by

$$M_r^\alpha f(x) = \int K_r^\alpha(x, y) f(y)\, d\gamma_\alpha(y),$$

where

$$K_r^\alpha(x, y) = \prod_{i=1}^d K_r^\alpha(x_i, y_i), \qquad x, y \in \mathbb{R}_+^d,$$

$$K_r^\alpha(x_i, y_i) = \sum_{n=0}^\infty r^n L_n^\alpha(x_i) L_n^\alpha(y_i)/\Gamma(\alpha+1)\binom{n+\alpha}{n}.$$

The kernel may also be expressed in terms of the standard Bessel function J_α,

$$K_r^\alpha(s, t) = \frac{(-rst)^{-\alpha/2}}{1-r} \exp\left(\frac{-r(s+t)}{1-r}\right) J_\alpha\left(\frac{2\sqrt{-rst}}{1-r}\right),$$

which can be estimated to give

$$(0.1)\, K_r^\alpha(s, t) \sim \begin{cases} (1-r)^{-\alpha-1} \exp\left(\dfrac{-r(s+t)}{1-r}\right), & 0 < 4rst < (1-r)^2, \\ \dfrac{(4rst)^{-\alpha/2-1/4}}{(1-r)^{1/2}} \exp\left(\dfrac{-r(s+t)+2\sqrt{rst}}{1-r}\right), & 4rst \geqslant (1-r)^2. \end{cases}$$

Here, and in the sequel, $f \sim g$ means $c \leqslant f/g \leqslant C$ for some positive constants c and C. In (0.1) the constants depend only on α. Thus, for r and x fixed, the function $K_r^\alpha(x, \cdot)$ is bounded. Therefore, the Laguerre-Poisson integral $M_r^\alpha f$ is well defined for each $f \in L^1(\gamma_\alpha)$, $0 < r < 1$. In particular, if a function in $L^2(\gamma_\alpha)$ has the Laguerre expansion $\Sigma a_m L_m^\alpha$, then its Laguerre-Poisson integral is the function of x and r which for fixed $r (0 < r < 1)$ has the Laguerre expansion $\Sigma r^{|m|} a_m L_m^\alpha$, $|m| = \Sigma m_i$ (see [M]).

The Laguerre semigroup, with parameter $\alpha > -1$, is defined by

$$T^t = M_{e^{-t}}^\alpha, \quad 0 < t < \infty, \quad \text{and} \quad T^0 = I.$$

This defines a symmetric diffusion semigroup, whose infinitesimal generator is

$$Lu \equiv \Sigma(x_i \partial^2 u/\partial x_i^2 + (1 + \alpha - x_i)\, \partial u/\partial x_i),$$

i.e. $u(x, t) = T^t f(x)$ satisfies $\partial u/\partial t = Lu$ and $u(x, 0) = f(x)$.

By virtue of Stein's maximal theorem, the operator

$$M^\alpha f(x) = \sup_{0<r<1} |M_r^\alpha f(x)|$$

is bounded on $L^p(\gamma_\alpha)$, $1 < p \leqslant \infty$. The following theorem is the main result in this paper.

Theorem 1. *For each finite dimension d and $\alpha > -1$ there is a constant $c = c(d, \alpha)$ such that*

$$\|M^\alpha f\|_{1,\infty} \leqslant c\|f\|_1 \quad \textit{for each} \quad f \in L^1(\gamma_\alpha).$$

Corollary. $\lim_{r\to 1-} M_r^\alpha f(x) = f(x)$ a.e. *if $f \in L^1(\gamma_\alpha)$.*

The almost everywhere convergence is as usual a consequence of the estimate for the maximal operator. Everywhere pointwise convergence is easily verified for polynomials. Since they form a dense subset of $L^1(\gamma_\alpha)$, the proof of the Corollary is then quite standard and we omit it. The use of polynomials was suggested to the author by Fulvio Ricci.

There is a natural way to define γ_α and K_r^α, and hence M^α, for multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i > -1$. For such α, Theorem 1 still holds and the proof is only slightly modified.

It is noticeable that in the proof of Theorem 1 the constant c depends heavily on the dimension. Both for the Hermite and the Laguerre semigroup, it is an open problem to obtain constants independent of the dimension.

Some conventions: c and C denote various positive constants, which usually depend on the dimension and α. In case c depends on a parameter k we indicate this by writing c_k. We abbreviate $f \leqslant cg$ by $f \lessdot g$. Usually we will not indicate the underlying space for measures, *e.g.* $d\gamma_\alpha(x)$ means the measure $(\Pi x_i^\alpha e^{-x_i})\,dx$ on $\{x_i > 0, \text{ all } i\}$ where the dimension will be clear from x.

1. Sketch of the proof

The flavour of the proof of Theorem 1 is not «semigroup theoretical» —the semigroup property is not used at all. On the other hand, the fact that M_r^α is given by a symmetric positive kernel and leave constant functions invariant (implying (I)-(IV)) is essential.

The proof is split into several lemmas. First it is proved that M^α is of weak type $(1, 1)$ when α is an integer or half-integer. If $2\alpha = n - 2$, for some $n = 1, 2, \dots$, this is done by a transformation to the Hermite case in $\mathbb{R}^{dn}$ and an application of Sjögren's result. Next, for general α the supremum is restricted to the intervals $0 < r < 1/2$ and $1/2 \leqslant r < 1$, respectively. The main part is the proof that the operator

$$M_1^\alpha f(x) = \sup_{1/2 \leq r < 1} |M_r^\alpha f(x)|$$

is of weak type (1,1). For this, the course of action is as follows.

For any pair of disjoint subsets I' and I'' of $I \equiv \{1, \ldots, d\}$ let

$$A(I', I'') = \{(x, y):\ y_i \leqslant x_i/D \text{ for } i \in I', x_i/D < y_i < Dx_i \text{ for } i \in I'',\\ y_i \geqslant Dx_i \quad \text{for } i \notin I' \cup I''\},$$

where D is a large constant. The operator M_r^α can be written as a finite sum of operators, each given by the kernel $K_r^\alpha(x, y)$ restricted to one of the (pairwise disjont) regions $A(I', I'')$. Of course, it then suffices to prove weak type (1,1) for each of the corresponding maximal functions.

When $I' = I'' = \emptyset$ the corresponding maximal operator turns out to be bounded on $L^1(\gamma_\alpha)$. This implies (see Proposition 1), because of the product structure and the fact that $K_r^\alpha(x, y) \geqslant 0$, that it remains to prove weak type (1,1) for the cases $I' \cup I'' = I$. If $I' = \emptyset$, *i.e.* $x_i/D < y_i < Dx_i$ for all i, this is done by a comparison (locally) with the case $\alpha = -1/2$, which is already clear. The proof for the remaining case $I' \neq \emptyset$, $I' \cup I'' = I$ is rather long and technical. It is divided into five steps, of which we give a brief summary.

In Step 1 the kernel is estimated. Dependig on whether i belongs to I' or not, basic estimates of $K_r^\alpha(x_i, y_i)$ are obtained. These imply an appropriate division of the space into regions where additional (local) estimates are made. In this way a sum emerges, to which the summation theorem in $L^{1,\infty}$ due to E. M. Stein and N. Weiss ([SW]) is applied. The range of $1 - r$ is split into the intervals $[4^{-N-1}, 4^{-N}[$, and in Step 2 the task of estimating the terms in the sum above is reduced to prove that, for $0 < a < 1$,

$$\sum_{N=0}^{\infty} a^N \gamma_0 \{x \in Q:\ U_{N,k} g(x) > a^N \beta\} \leqslant \frac{A_k}{\beta} \|g\|_{L^1(dy)}, \qquad \beta > 0. \tag{1.1}$$

Here, the $U_{N,k}$'s are operators obtained in the series of estimates, Q is a «dyadic» rectangle, and g is any non-negative function supported in Q and its surrounding rectangles. For x in Q, the size of x_i is about 4^{N_i}. Assuming without loss of generality that N_1 is the largest of the N_i's, we split the sum in (1.1) into two parts, $N \leqslant N_1$ and $N > N_1$. In Step 3, the sum over $N > N_1$ is estimated by an application of Proposition 2. This proposition gives an inequality similar to that in (1.1), where the operators are averages over subsets belonging to hierarchical partitions. The sum over $N \leqslant N_1$ is estimated term by term in Step 4, where a covering argument is crucial. The covering argument is justified in Step 5.

2. Transformation to the Hermite Case

For certain α it is possible to prove the theorem by a transformation to the corresponding operator given by Hermite expansions and then apply the result in [Sj1].

Lemma 1. *$M^{\alpha}: L^1(\gamma_\alpha) \to L^{1,\infty}(\gamma_\alpha)$ is bounded for $2\alpha = -1, 0, 1, \ldots$.*

For a given finite dimension n we denote by μ the measure on $\mathbb{R}^n$ given by the density $e^{-|y|^2}$. Let H_k denote the k-th Hermite polynomial (see [Sz]) and define, for $0 < r < 1$, the kernels

$$P_r(x, y) = \prod_{i=1}^{n} P_r(x_i, y_i), \qquad x, y \in \mathbb{R}^n,$$

where

$$P_r(x_i, y_i) = \sum_{k=0}^{\infty} r^k H_k(x_i) H_k(y_i) / \sqrt{\pi}\, 2^k k!.$$

Define the operators N_r, $0 < r < 1$, and the corresponding maximal operator N by

$$N_r f(x) = \int P_r(x, y) f(y)\, d\mu(y),$$
$$N f(x) = \sup_{0<r<1} |N_r f(x)|.$$

The theorem proved in [Sj1] can then be stated as follows.

Theorem A ([Sj1]) *$N: L^1(\mu) \to L^{1,\infty}(\mu)$ is bounded.*

Now, if $2\alpha = n - 2$ $(n = 1, 2, \ldots)$ let μ live on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{dn}$ and define $G: \mathbb{R}^{dn} \to \mathbb{R}^d_+$ by

$$G(x^1, \ldots, x^d) = (|x^1|^2, \ldots, |x^d|^2), \qquad x^i \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^{dn}} f \circ G(x)\, d\mu(x) = c \int_{\mathbb{R}^d_+} f(y)\, d\gamma_\alpha(y) \tag{2.1}$$

for each $f \in L^1(\gamma_\alpha)$.

Moreover, expanding the function $L_k^{\alpha}(|x|^2)$, $x \in \mathbb{R}^n$, with respect to the orthogonal basis $\{H_{k_1}(x_1) \cdots H_{k_n}(x_n)\}_{k_i \in \mathbb{N}}$ in $L^2(\mu, \mathbb{R}^n)$, we get the formula

$$L_k^{\alpha}(|x|^2) = \sum_{k_1 + \cdots + k_n = k} a_{k_1, \ldots, k_n} H_{2k_1}(x_1) \cdots H_{2k_n}(x_n), \qquad x \in \mathbb{R}^n \tag{2.2}$$

for $2\alpha = n - 2$, $n = 1, 2, \ldots$.

PROOF OF LEMMA 1. Let $2\alpha = n - 2$ for some $n = 1, 2, \ldots$. Assume $f \in L^2(\gamma_\alpha)$ and let

$$f(x) = \sum_{k_i = 0, 1, \ldots} b_{k_1, \ldots, k_d} L^\alpha_{k_1}(x_1) \cdots L^\alpha_{k_d}(x_d) \equiv \sum_k b_k L^\alpha_k(x)$$

be the Laguerre expansion of f. From (2.2) we have, for $y \in \mathbb{R}^{dn}$,

$$\begin{aligned} f \circ G(y) &= \sum_k b_k L^\alpha_{k_1}(|y^1|^2) \cdots L^\alpha_{k_d}(|y^d|^2) \\ &= \sum_k b_k \prod_{i=1}^d \left(\sum_{\substack{l_j \\ l_1 + \cdots + l_n = k_i}} a_{l_1, \ldots, l_n} H_{2l_1}(y^i_1) \cdots H_{2l_n}(y^i_n) \right) \cdot \end{aligned}$$

The product is a sum of Hermite polynomials in $\mathbb{R}^{dn}$ of degree $2|k|$. Since (see [M])

$$N_r\left(\sum_k a_k H_k \right) = \sum_k r^{|k|} a_k H_k$$

this gives

$$\begin{aligned} N_r(f \circ G)(y) &= \sum_k r^{2|k|} b_k L^\alpha_k(|y^1|^2, \ldots, |y^d|^2) \\ &= M^\alpha_{r^2} f(|y^1|^2, \ldots, |y^d|^2) \\ &= (M^\alpha_{r^2} f) \circ G(y). \end{aligned}$$

Since both $K^\alpha_r(x, \cdot)$ and $P_r(x, \cdot)$ are bounded functions (see [M]), we get

$$|N_r(f \circ G)| = |(M^\alpha_{r^2} f) \circ G| \tag{2.3}$$

for all $f \in L^1(\gamma_\alpha)$, by taking the limit for $f_j \in L^2(\gamma_\alpha)$ such that $f_j \to f$ in $L^1(\gamma_\alpha)$.

Taking the supremum over $0 < r < 1$ on both sides in (2.3) gives

$$N(f \circ G) = (M^\alpha f) \circ G, \qquad f \in L^1(\gamma_\alpha). \tag{2.4}$$

Finally, from (2.1), (2.4), and Theorem A we get

$$\begin{aligned} c\gamma_\alpha \{ x \in \mathbb{R}^d_+ : M^\alpha f(x) > \beta \} &= \mu \{ y \in \mathbb{R}^{dn} : (M^\alpha f) \circ G(y) > \beta \} \\ &= \mu \{ y \in \mathbb{R}^{dn} : N(f \circ G)(y) > \beta \} \\ &\leqslant \frac{C}{\beta} \| f \circ G \|_{L^1(\mu)} \\ &= \frac{C}{\beta} \| f \|_{L^1(\gamma_\alpha)}, \end{aligned}$$

which proves Lemma 1.

3. The Case $1/2 \leqslant r < 1$

In this and the next section we intend to prove the following lemma —the main part of the theorem.

Lemma 2. *The operator*

$$M_1^\alpha f(x) = \sup_{1/2 \leq r < 1} |M_r^\alpha f(x)|$$

is bounded from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$ for all $\alpha > -1$.

We split the proof into three more lemmas, where we restrict the kernel to different areas of integration. A simple result about non-negative kernels will be used several times. Therefore, it is convenient to state it in a general form.

Let (E, μ) and (F, ν) be measure spaces, with μ and ν non-negative and σ-finite. Further, let I be a countable subset of $\mathbb{R}$. Suppose that, for $r \in I$, $A_r(x, y)$ and $B_r(x', y')$ are non-negative measurable functions on $E \times E$ and $F \times F$, respectively, such that

$$A_r(x, \cdot) \in L^\infty(\mu) \quad \text{and} \quad B_r(x', \cdot) \in L^\infty(\nu) \quad \text{for all} \quad r \in I, \quad x \in E, \quad x' \in F.$$

Then the following operators are well defined, giving rise to measurable functions, on $L^1(\mu)$, $L^1(\nu)$, and $L^1(\mu \times \nu)$, respectively

$$Af(x) = \sup_{r \in I} \left| \int A_r(x, y) f(y)\, d\mu(y) \right|, \qquad x \in E,$$

$$Bf(x') = \sup_{r \in I} \left| \int B_r(x', y') f(y')\, d\nu(y') \right|, \quad x' \in F,$$

$$A \otimes Bf(x, x') = \sup_{r \in I} \left| \int A_r(x, y) B_r(x', y') f(y, y') d\mu(y) d\nu(y') \right|, \ (x, x') \in E \times F.$$

Proposition 1. *If $A: L^1(\mu) \to L^{1,\infty}(\mu)$ boundedly with norm C_A and $B: L^1(\nu) \to L^1(\nu)$ boundedly with norm C_B then $A \otimes B: L^1(\mu \times \nu) \to L^{1,\infty}(\mu \times \nu)$ is bounded with norm $C_A C_B$.*

Proof. Since the kernels are non-negative, it is enough to consider $f \geqslant 0$. Then

$$\begin{aligned} A \otimes Bf(x, x') &\leqslant \sup_{r \in I} \int A_r(x, y) \left(\sup_{r \in I} \int B_r(x', y') f(y, y')\, d\nu(y') \right) d\mu(y) \\ &= AF_{x'}(x) \end{aligned}$$

where

$$F_{x'}(y) = Bf_y(x'), \qquad f_y(y') = f(y, y').$$

Thus

$$\begin{aligned}(\mu \times \nu)\{(x, x'): A \otimes Bf(x, x') > \beta\} &\leqslant \int \mu\{x: AF_{x'}(x) > \beta\}\, d\nu(x') \\ &\leqslant C_A \beta^{-1} \int \|F_{x'}\|_{L^1(d\mu(y))}\, d\nu(x') \\ &= C_A \beta^{-1} \int \left(\int Bf_y(x')\, d\nu(x') \right) d\mu(y) \\ &\leqslant C_A C_B \beta^{-1} \int \|f_y\|_{L^1(d\nu(y'))}\, d\mu(y) \\ &= C_A C_B \beta^{-1} \|f\|_{L^1(\mu \times \nu)},\end{aligned}$$

which proves the proposition.

Notice that the condition «I countable» (to ensure measurability) will not cause us any problem, since the kernels we are dealing with are nice enough to enable us to restrict the supremum to *e.g.* r rational.

Now, by Proposition 1, the following three lemmas will imply Lemma 2.

Lemma 3. *For large D, the operator*

$$A^\alpha f(x) = \int \sup_{1/2 \le r < 1} K_r^\alpha(x, y) \chi_{\{y_i \ge Dx_i,\, \text{all}\, i\}} |f(y)|\, d\gamma_\alpha(y)$$

is bounded on $L^1(\gamma_\alpha)$.

Lemma 4. *For any $D > 1$, the operator*

$$B^\alpha f(x) = \sup_{1/2 \le r < 1} \left| \int K_r^\alpha(x, y) \chi_{\{x_i/D < y_i < Dx_i,\, \text{all}\, i\}} f(y)\, d\gamma_\alpha(y) \right|$$

maps $L^1(\gamma_\alpha)$ boundedly into $L^{1,\infty}(\gamma_\alpha)$.

Lemma 5. *For large D, the operator*

$$C^\alpha f(x) = \sup_{1/2 \le r < 1} \left| \int K_r^\alpha(x, y) \chi_{\{y_i < Dx_i,\, \text{all}\, i,\, \text{some}\, y_i \le x_i/D\}} f(y)\, d\gamma_\alpha(y) \right|$$

maps $L^1(\gamma_\alpha)$ boundedly into $L^{1,\infty}(\gamma_\alpha)$.

Throughout the remaining proofs, (0.1) will be used frequently. Therefore we set ($s, t \in \mathbb{R}_+, 0 < r < 1$)

$$D_r^\alpha(s, t) = (1 - r)^{-\alpha - 1} \exp\left(\frac{-r(s + t)}{1 - r}\right)\chi_{\{4rst < (1 - r)^2\}}$$

$$E_r^\alpha(s, t) = \frac{(4rst)^{-\alpha/2 - 1/4}}{(1 - r)^{1/2}} \exp\left(\frac{-r(s + t) + 2\sqrt{rst}}{1 - r}\right)\chi_{\{4rst \geq (1 - r)^2\}}.$$

With these notations, (0.1) says that

$$K_r^\alpha(s, t) \sim D_r^\alpha(s, t) + E_r^\alpha(s, t) \qquad (s, t \in \mathbb{R}_+) \tag{3.1}$$

and, consequently, we have

$$K_r^\alpha(x, y) \sim \sum_{I \subseteq \{1, \ldots, d\}} \left(\prod_{i \in I} D_r^\alpha(x_i, y_i)\right)\left(\prod_{i \notin I} E_r^\alpha(x_i, y_i)\right), \qquad x, y \in \mathbb{R}_+^d. \tag{3.2}$$

PROOF OF LEMMA 3. The lemma follows if we estimate $K_r^\alpha(x, y)$, when $y_i \geqslant Dx_i$, $1/2 \leqslant r < 1$, by a function $A(x, y)$ such that

$$\int A(x, y)\chi_{\{x_i \leq y_i/D \text{ all } i\}}\, d\gamma_\alpha(x) \leqslant c.$$

Of course, it is enough to do this in one dimension.

For fixed x and y in $\mathbb{R}_+$ let

$$D(r) = (1 - r)^{-\alpha - 1} \exp\left(\frac{-r(x + y)}{1 - r}\right), \qquad 1/2 \leqslant r < 1.$$

Then $D'(r)$ equals $(\alpha + 1 - (x + y)/(1 - r))$ times a positive factor. Hence, $D(r)$ takes its maximum at $1 - r = (x + y)/(1 + \alpha)$ if $(x + y)/(1 + \alpha) \leqslant 1/2$, which gives $D(r) \lessdot (x + y)^{-\alpha - 1}$. On the other hand, $D(r)$ is decreasing if $(x + y)/(1 + \alpha) > 1/2$, giving $D(r) \leqslant D(1/2) \lessdot 1$.

Thus, for all $x, y \in \mathbb{R}_+$ and $1/2 \leqslant r < 1$, we have

$$D_r^\alpha(x, y) \lessdot 1 + (x + y)^{-\alpha - 1}. \tag{3.3}$$

In order to estimate $E_r^\alpha(x, y)$, we set, for fixed x and y with $y \geqslant Dx$,

$$G(r) = (1 - r)^{-1/2} \exp\left(\frac{-r(x + y) + 2\sqrt{rxy}}{1 - r}\right) \equiv (1 - r)^{-1/2} \exp f(r),$$

$1/2 \leqslant r < 1$.

Then

$$f'(r) = \frac{1}{(1 - r)^2}\left(\sqrt{xy/r} - (x + y) + \sqrt{rxy}\right) = \operatorname{pos}\left(\sqrt{y/x} - \sqrt{r}\right)\left(\sqrt{x/y} - \sqrt{r}\right)$$

and

$$G'(r) = \operatorname{pos}(1/2 + (1-r)f'(r)).$$

With D large enough, $G'(r) = 0$ implies

$$(1-r) = 2(x+y) - 2\sqrt{xy}\,(1+r)/\sqrt{r} \gtrsim y\left(2 - 4\sqrt{2/D}\right) \gtrsim y.$$

Moreover, since f is decreasing for $y \geqslant 2x$,

$$f(r) \leqslant f(1/2) = -(x+y) + 2\sqrt{2xy} \leqslant x.$$

Thus, if $G'(r) = 0$ then

$$G(r) \lesssim y^{-1/2} \exp f(r) \lesssim (xy)^{-1/4} e^{x}.$$

Since $G(1/2) \lesssim \exp\left(-(x+y) + 2\sqrt{2xy}\right)$ and $\lim_{r \to 1-} G(r) = 0$, this gives

$$E_r^{\alpha}(x,y) \lesssim (xy)^{-\alpha/2-1/4} \exp\left\{-(x+y) + 2\sqrt{2xy}\right\} + (xy)^{-\alpha/2-1/2} e^{x}.$$

This and (3.3) give the desired estimate of $K_r^{\alpha}(x,y)$.

PROOF OF LEMMA 4. Because of (3.2), it is enough to prove the lemma with $K_r^{\alpha}(x,y)$ replaced by the kernel

$$\left(\prod_{i \in I} D_r^{\alpha}(x_i, y_i)\right)\left(\prod_{i \notin I} E_r^{\alpha}(x_i, y_i)\right)$$

for a fixed, but arbitrary, I. The operator

$$f \to \int \left[\prod (1 + (x_i + y_i)^{-\alpha-1})\right] \chi_{\{x_i/D < y_i < Dx_i \text{ all } i\}} |f(y)|\, d\gamma_\alpha(y)$$

is easily seen to be bounded on $L^1(\gamma_\alpha)$ for any dimension. Thus (3.3) implies, by means of Proposition 1, that we only have to prove that the operator

$$E^{\alpha}f(x) = \sup_{1/2 \leq r < 1} \left| \int \left(\prod_{i=1}^{d} E_r^{\alpha}(x_i, y_i)\right) \chi_{\{x_i/D < y_i < Dx_i \text{ all } i\}} f(y)\, d\gamma_\alpha(y) \right| \tag{3.4}$$

is bounded from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$ for any dimension d, for $D > 1$ and $\alpha > -1$. We write (3.4) as

$$E^{\alpha}f(x) = \sup_{1/2 \leq r < 1} |E_r^{\alpha} f(x)|.$$

Again, it is enough to consider $f \geqslant 0$. For $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ let

$$Q^m = \{x \colon D^{m_i} < x_i \leqslant D^{m_i+1}, i = 1, \dots, d\}$$

and

$$\tilde{Q}^m = \{x\colon\ D^{m_i - 1} < x_i \leqslant D^{m_i + 2}, i = 1, \ldots, d\}.$$

Since $E_r^\alpha f(x) \sim E_r^{-1/2} f(x)$ for all x, we get, by applying Lemma 1,

$$\begin{aligned}\gamma_\alpha\{x \in Q^m\colon E^\alpha f(x) > \beta\} &\leqslant D^{(\alpha+1/2)\Sigma m_i}\gamma_{-1/2}\{x \in Q^m\colon\ E^{-1/2}f(x) > c\beta\}\\ &\leqslant D^{(\alpha+1/2)\Sigma m_i}\gamma_{-1/2}\{x\colon\ M^{-1/2}(f|_{\tilde{Q}^m})(x) > c\beta\}\\ &\leqslant \frac{D^{(\alpha+1/2)\Sigma m_i}}{\beta}\|f|_{\tilde{Q}^m}\|_{L^1(\gamma_{-1/2})}\\ &\sim \frac{1}{\beta}\int f|_{\tilde{Q}^m}\,d\gamma_\alpha.\end{aligned}$$

Summation over m yields that E^α is bounded from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$, which proves Lemma 4.

So far we have used the Hermite result, but to prove Lemma 5 we have to investigate the kernel in more detail. Before the proof we recall a result (slightly restated) due to E. M. Stein and N. Weiss. We also state a proposition generalizing a special case of Lemma 1 in [Sj2].

Let (E, ν) be a measure space, with ν non-negative.

Theorem B ([SW], Lemma 2.3) *Suppose g_j, $j = 1, 2, \ldots$, are non-negative measurable functions on E, satisfying*

$$\nu\{x\colon\ g_j(x) > \alpha\} < Ac_j/\alpha, \qquad \alpha > 0,$$

where c_j are positive numbers such that $\sum c_j < \infty$ and $\sum c_j \ln(1/c_j) < \infty$. Then

$$\nu\Big\{x\colon\ \sum_j g_j(x) > \alpha\Big\} < 2AC/\alpha, \qquad \alpha > 0, \ \textit{for some positive number } C.$$

E is said to have a piece hierarchy if for each $N = 1, 2, \ldots$ there exists a partition of E into at most countably many disjoint subsets called N-pieces with the following properties

(i) Each N-piece is measurable with positive finite measure,
(ii) Any $(N+1)$-piece is contained in some N-piece.

Now, suppose E has a piece hierarchy. Denote the N-pieces by P_N^j, $j = 1, 2, \ldots$, and let $P_N(x)$, for $x \in E$, be the N-piece that contains x. Assume $^\wedge$ is a one-to-one map from $\{P_N^j\colon\ j = 1, 2, \ldots\}$ into itself for each N, such that

$$\nu(P_N^j) \sim \nu({}^\wedge P_N^j) \qquad \text{(uniformly in } N, j).$$

Define the operators T_N on $L^1(\nu)$ by

$$T_N g(x) = \frac{1}{\nu({}^\wedge P_N(x))} \int_{{}^\wedge P_N(x)} |g(y)| \, d\nu(y).$$

Proposition 2. *For any $0 < a < 1$ there is a constant $c = c(a)$ such that*

$$\sum_{N=1}^{\infty} a^N \nu\{x\colon\ T_N g(x) > a^N \beta\} \leqslant \frac{c}{\beta} \|g\|_{L^1(\nu)}, \qquad \beta > 0.$$

PROOF. Since $T_N g$ is constant on each P_N^j, the set $I = \{(N,j)\colon\ T_N g > a^N \beta$ on $P_N^j\}$ is well defined and the sum equals

$$\sum_{(N,j)\in I} a^N \nu(P_N^j) \sim \sum_{(N,j)\in I} (a^N - a^{N+1}) \nu({}^\wedge P_N^j). \tag{3.5}$$

Regard $[0,1] \times E$ as a measure space. Then, the last sum above is just the measure of the union of the pairwise disjoint sets

$$A_{N,j} \equiv]a^{N+1}, a^N] \times {}^\wedge P_N^j \subseteq [0,1] \times E, \qquad (N,j) \in I.$$

The family $\{A_{N,j}\colon\ (N,j) \in I\}$ is partially ordered by

$$A_{M,i} \leqslant A_{N,j} \quad \text{if and only if} \quad {}^\wedge P_M^i \subseteq {}^\wedge P_N^j \quad \text{and} \quad M \geqslant N.$$

Notice that every $A_{N,j}$ is included in a maximal set. In measuring the union of the sets $A_{N,j}$, we first measure the sets smaller (not strictly) than a given maximal set. We then sum over all maximal sets. This means that the sum to the right in (3.5) equals

$$\sum_{\substack{(N,j)\in I \\ A_{N,j} \text{maximal}}} \left(\sum_{\substack{(M,i)\in I \\ A_{M,i} \le A_{N,j}}} (a^M - a^{M+1}) \nu({}^\wedge P_M^i) \right). \tag{3.6}$$

The inner sum in (3.6) may be estimated by

$$\sum_{M \ge N} (a^M - a^{M+1}) \left(\sum_{\substack{i \\ {}^\wedge P_M^i \subseteq {}^\wedge P_N^j}} \nu({}^\wedge P_M^i) \right) \leqslant a^N \nu({}^\wedge P_N^j).$$

Since

$$\nu({}^\wedge P_N^j) < \frac{a^{-N}}{\beta} \int_{{}^\wedge P_N^j} |g| \, d\nu$$

for $(N,j) \in I$, (3.6) is less than

$$\frac{1}{\beta} \sum_{\substack{(N,j)\in I \\ A_{N,j} \text{maximal}}} \int_{{}^\wedge P_N^j} |g| \, d\nu.$$

Since these ${}^\wedge P_N^j$ are pairwise disjoint, this proves Proposition 2.

4. The Proof of Lemma 5

For $n = 0, \dots, d-1$ we let χ_n denote the characteristic function of the set

$$(4.1)\qquad \begin{aligned} x_i/D \le y_i < Dx_i, &\qquad i = 1, \dots, n \\ y_i \leqslant x_i/D, &\qquad i = n+1, \dots, d. \end{aligned}$$

By symmetry, it suffices to prove the lemma if we let, for $n = 0, \dots, d-1$,

$$C^\alpha f(x) = \sup_{1/2 \le r < 1} |C_r^\alpha f(x)|$$

with

$$C_r^\alpha f(x) = \int K_r^\alpha(x, y)\chi_n f(y)\, d\gamma_\alpha(y).$$

Moreover, for the same reason as in the proof of Lemma 4, it is enough to prove boundedness (from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$) for the corresponding maximal operator, still called C^α, where $K_r^\alpha(x_i, y_i)$ is replaced by $E_r^\alpha(x_i, y_i)$ for $i = 1, \dots, n$. We write

$$(4.2)\qquad C_r^\alpha(x, y) \equiv \prod_{i=1}^{n} E_r^\alpha(x_i, y_i) \prod_{i=n+1}^{d} K_r^\alpha(x_i, y_i) \equiv E_r^\alpha(x', y')K_r^\alpha(x'', y'')$$

with $x = (x', x'')$ in the obvious way. The proof will be split into 5 steps.

Step 1: *Estimates*

Since, for $y_i \sim x_i$,

$$E_r^\alpha(x_i, y_i) \leqslant \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}} \exp\left(x_i + \frac{-(\sqrt{x_i} - \sqrt{ry_i}\,)^2}{1-r}\right)$$

and

$$\frac{(\sqrt{x_i} - \sqrt{ry_i}\,)^2}{1-r} = \frac{(y_i - x_i/r)^2}{x_i(1-r)} \frac{r^2}{(1 + \sqrt{ry_i/x_i}\,)^2} \geqslant c \frac{(y_i - x_i/r)^2}{x_i(1-r)}$$

we get

$$(4.3)\quad E_r^\alpha(x', y') \leqslant \left[\prod_{i=1}^{n} \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}}\right] \exp\left(\sum_1^n x_i\right) \exp\left(-c\sum_1^n \frac{(y_i - x_i/r)^2}{x_i(1-r)}\right).$$

For $i = n+1, \dots, d$ we have

$$r(x_i + y_i) - 2\sqrt{rx_iy_i} \geqslant x_i\big(r - 2\sqrt{ry_i/x_i}\,\big) \geqslant x_i/4$$

if D is large enough. Hence, if $\alpha \geqslant -1/2$ we get

$$E_r^\alpha(x_i, y_i) \leqslant (1-r)^{-\alpha-1} \exp(-x_i/4(1-r)),$$

while for $-1 < \alpha < -1/2$ the following holds

$$E_r^\alpha(x_i, y_i) \leqslant \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}} \exp(-x_i/4(1-r)).$$

Since

$$D_r^\alpha(x_i, y_i) \leqslant (1-r)^{-\alpha-1} \exp(-x_i/4(1-r)),$$

these estimates imply

$$K_r^\alpha(x'', y'') \leqslant \exp\left(-\frac{1}{4(1-r)} \sum_{n+1}^d x_i\right) \prod_{n+1}^d \left(\frac{1}{(1-r)^{\alpha+1}} + \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}} \chi_{\{-1<\alpha<-1/2\}}\right). \tag{4.4}$$

For fixed x', we divide $\mathbb{R}_+^n$ into the rectangles

$$2^{k_i-1}\sqrt{x_i(1-r)} \leqslant |y_i - x_i/r| < 2^{k_i}\sqrt{x_i(1-r)}, \qquad k_i = 0, 1, \ldots,$$

where the left hand side is interpreted as 0 for $k_i = 0$.

Further, $\mathbb{R}_+^{d-n}$ is divided into strips (with respect to x'') where

$$4^{j-1}(1-r) \leqslant \sum_{n+1}^d x_i < 4^j(1-r), \qquad j = 0, 1, \ldots.$$

Again, $j = 0$ means 0 to the left.

Estimating the right hand side of (4.3) and (4.4) in these rectangles and strips, respectively, gives (we assume $f \geqslant 0$)

$$\int C_r^\alpha(x, y)\chi_n f(y)\, d\gamma_\alpha(y) \leqslant \sum_{j=0}^\infty a_j (1-r)^{-(d-n)(\alpha+1)} \chi_{\{x'': \sum x_i < 4^j(1-r)\}} \times \sum_{k_i=0,1,\ldots} b_k \left(\prod_1^n \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}}\right) \exp\left(\sum_1^n x_i\right) \int_{R_k(r,x')} F(y')\, d\gamma_\alpha(y'), \tag{4.5}$$

where

$$F(y') = \int f(y', y'')\, d\gamma_\alpha(y''),$$

$$R_k(r, x') = \{y' : |y_i - x_i/r| < 2^{k_i}\sqrt{x_i(1-r)}, x_i/D < y_i < Dx_i\},$$

$$a_j = 2^{(d-n)j} \exp(-4^{j-2}),$$

and

$$b_k = \exp\left(-c\sum_1^n 4^{k_i}\right).$$

The factor $2^{(d-n)j}$ in a_j is what remains from the case $-1 < \alpha < -1/2$.

Taking the supremum over $1 - r \in [4^{-N-1}, 4^{-N}[$ (written $1 - r \sim 4^{-N}$) in (4.5) yields

$$C^\alpha f(x) \leqslant \sum_j \sum_k \sup_{N=0,1,\ldots} T_{N,k,j} f(x) \equiv \sum_j \sum_k T_{k,j} f(x), \tag{4.6}$$

where

$$T_{N,k,j} f(x) = a_j b_k 4^{N(d-n)(\alpha+1)} \chi_{\{x'': \sum x_i < 4^{j-N}\}} \times \left(\prod_1^n 2^N x_i^{-\alpha-1/2}\right) \exp\left(\sum_1^n x_i\right) \sup_{1-r\sim 4^{-N}} \int_{R_k(r,x')} F(y')\, d\gamma_\alpha(y').$$

Hence, by iterating the result in Theorem B, we get Lemma 5 if we prove

$$\gamma_\alpha\{x\colon\ T_{k,j} f(x) > \beta\} \leqslant c_{k,j} \frac{1}{\beta} \|f\|_{L^1(\gamma_\alpha)}, \tag{4.7}$$

with $c_{k,j} = c^1_{k_1} \cdots c^n_{k_n} c^0_j$ such that

$$\sum_{l=0}^\infty c^i_l < \infty, \qquad \sum_{l=0}^\infty c^i_l \ln(1/c^i_l) < \infty, \qquad i = 0, \ldots, n. \tag{4.8}$$

Step 2: Localization

Let

$$S_{N,k} F(x') = \left(\prod_{i=1}^n 2^N x_i^{-\alpha-1/2}\right) \exp\left(\sum_1^n x_i\right) \sup_{1-r\sim 4^{-N}} \int_{R_k(r,x')} F(y')\, d\gamma_\alpha(y').$$

Then

$$\begin{aligned}
\gamma_\alpha\{x\colon\ T_{k,j} f(x) > \beta\} &\leqslant \sum_{N=0}^\infty \gamma_\alpha\{x\colon\ T_{N,k,j} f(x) > \beta\} \\
&\leqslant \sum_{N=0}^\infty \gamma_\alpha\left\{x''\colon\ \sum_{n+1}^d x_i < 4^{j-N}\right\} \\
&\qquad \times \gamma_\alpha\{x'\colon\ S_{N,k} F(x') > \beta a_j^{-1} b_k^{-1} 4^{-N(d-n)(\alpha+1)}\} \\
&\leqslant \sum_{N=0}^\infty 4^{(j-N)(d-n)(\alpha+1)} \\
&\qquad \times \gamma_\alpha\{x'\colon\ S_{N,k} F(x') > \beta a_j^{-1} b_k^{-1} 4^{-N(d-n)(\alpha+1)}\}.
\end{aligned}$$

Hence, if we prove

$$(4.9)\quad \sum_{N=0}^{\infty} a^N \gamma_\alpha\{x'\colon S_{N,k}F(x') > a^N\beta\} \leqslant \frac{A_k}{\beta}\|F\|_{L^1(d\gamma_\alpha(y'))}, \qquad 0 < a < 1,$$

then (4.7) follows (take $a = 4^{-(d-n)(\alpha+1)}$) with

$$(4.10)\qquad c_{k,j} \sim 4^{j(d-n)(\alpha+1)} 2^{(d-n)j} \exp(-4^{j-2}) \exp\left(-c\sum_1^n 4^{k_i}\right) A_k.$$

Thus, with A_k good enough, (4.9) would complete the proof of the lemma. Actually we have assumed here that $n = 1, \ldots, d-1$.

In the case $n = 0$, we have

$$C^\alpha f(x) \leqslant \sum_{j=0}^{\infty} \sup_{N=0,1,\ldots} 4^{Nd(\alpha+1)} 2^{dj} \exp(-4^{j-2}) \chi_{\{\sum x_i < 4^{j-N}\}} \|f\|_{L^1(\gamma_\alpha)}$$

$$\equiv \sum_{j=0}^{\infty} \sup_N \tilde{T}_{N,j} f(x).$$

With

$$N_0 = \min\{N\colon 4^{Nd(\alpha+1)} 2^{dj} \exp(-4^{j-2}) \|f\|_{L^1(\gamma_\alpha)} > \beta\},$$

we get

$$\gamma_\alpha\left\{x\colon \sup_N \tilde{T}_{N,j} f(x) > \beta\right\} \leqslant \gamma_\alpha\left\{x\colon \sum x_i < 4^{j-N_0}\right\}$$

$$\leqslant 4^{d(j-N_0)(\alpha+1)}$$

$$\leqslant 4^{dj(\alpha+1)} 2^{dj} \exp(-4^{j-2}) \frac{1}{\beta} \|f\|_{L^1(\gamma_\alpha)}.$$

These constants are good enough to imply, in view of Theorem B again, that C^α is bounded.

Next, to simplify the proof of (4.9), we reduce it to a local property, which in particular enables us to get rid of the α's. From now on we drop the primes, *i.e.* $x' = x \in \mathbb{R}^n_+$. Let $\{Q^m\colon m \in \mathbb{Z}^n\}$ be the partition of $\mathbb{R}^n_+$ into rectangles defined by

$$Q^m = \{x\colon D^{m_i} < x_i \leqslant D^{m_i+1},\ i = 1, \ldots, n\}$$

and let

$$\tilde{Q}^m = \{x\colon D^{m_i-1} < x_i \leqslant D^{m_i+2},\ i = 1, \ldots, n\}.$$

In proving (4.9), we may, without loss of generality, assume that $D = 4^M$ for some $M \in \mathbb{N}$. For a given $m \in \mathbb{Z}^n$, let $N_i \in \mathbb{Z}$ be such that

$$(4.11) \qquad D^{m_i+1} = 4^{N_i} \qquad i = 1, \ldots, n$$

and let, for $g \geqslant 0$,

$$(4.12) \qquad U^m_{N,k} g(x) = \left(\prod_{i=1}^n 2^{N-N_i}\right) \exp\left(\sum_1^n x_i\right) \sup_{1-r \sim 4^{-N}} \int_{|y_i - x_i/r| < 2^{k_i+N_i-N}} g(y)\,dy.$$

Then, if $x \in Q^m$

$$S_{N,k}F(x) \leqslant U^m_{N,k}(F|_{\tilde{Q}^m}\gamma_0)(x).$$

Hence, the sum in (4.9) is less than or equal to

$$(4.13) \qquad \sum_{m \in \mathbb{Z}^n} \left(\prod_1^n 4^{\alpha N_i}\right) \sum_{N=0}^\infty a^N \gamma_0 \{x \in Q^m \colon U^m_{N,k}(F|_{\tilde{Q}^m}\gamma_0)(x) > ca^N\beta\}.$$

If we prove that the sum over N in (4.13) is less than a constant B_k, not depending on m, times

$$\frac{1}{\beta} \int F|_{\tilde{Q}^m}\, d\gamma_0$$

we get (4.9) with $A_k \sim B_k$.

For a fixed, but arbitrary, $m \in \mathbb{Z}^n$, let $Q = Q^m$ and $U_{N,k} = U^m_{N,k}$.

It remains to prove that, for $0 < a < 1$ and $g = g|_{\tilde{Q}} \geqslant 0$,

$$(4.14) \qquad \sum_{N=0}^\infty a^N \gamma_0 \{x \in Q \colon U_{N,k} g(x) > a^N \beta\} \leqslant \frac{A_k}{\beta} \|g\|_{L^1(dy)},$$

with A_k good enough (see (4.10) and (4.8)). Of course, A_k may depend on a.

Since the N_i's are related to m as in (4.11), $x \in Q$ clearly implies $x_i \sim 4^{N_i}$ for all i. For symmetry reasons, we may assume that

$$N_1 \geqslant \cdots \geqslant N_k \geqslant N_{k+1} \geqslant \cdots \geqslant N_n.$$

We split the sum in (4.14) into two parts, $\sum_{N > N_1}$ and $\sum_{N \leq N_1}$. In Step 3 we will apply Proposition 2 to $\sum_{N > N_1}$, while in Step 4 we estimate the sum $\sum_{N \leq N_1}$ term by term. In Step 4 we use a covering argument, which will be justified in Step 5.

We consider $k \in \mathbb{N}^n$ fixed.

Step 3: *Proof of* (4.14) *restricted to* $N > N_1$

For $x \in Q$, let

$$R_N(x) = \{y \in \mathbb{R}^n_+ \colon |y_i - x_i| \leqslant 3 \cdot 2^{k_i + N_i - N} \text{ for every } i\}.$$

Since $N \geqslant N_i$ for each i, this set contains all y with $|y_i - x_i/r| < 2^{k_i + N_i - N}$, $i = 1, \ldots, n$, if $1 - r \sim 4^{-N}$. Thus,

$$U_{N,k}g(x) \leqslant \left(\prod_1^n 2^{N-N_i}\right) \exp\left(\sum_1^n x_i\right) \int_{R_N(x)} g(y)\,dy.$$

Let

$$Q_\eta = \left\{x \in \mathbb{R}^n_+ : \ \eta \leqslant \sum_{i=1}^n x_i < \eta + 1\right\}, \qquad \eta = 0, 1, \ldots .$$

Suppose $x \in Q \cap Q_\eta$. Then $R_N(x)$ is included in the set

$$\tilde{Q}_\eta = \bigcup_{l=-a}^{a} Q_{\eta + l}, \qquad a = 3\Sigma 2^{k_i},$$

where $Q_\eta = Q_0$ for $\eta \leqslant 0$, so that

$$U_{N,k}g(x) \leqslant e^\eta \left(\prod_{i=1}^n 2^{k_i}\right) \frac{1}{|R_N(x)|} \int_{R_N(x)} g|_{\tilde{Q}_\eta}(y)\,dy, \qquad x \in Q \cap Q_\eta. \tag{4.15}$$

Here $|\bullet|$ denotes the volume in $\mathbb{R}^n$.

Let P_N, for $N = 0, 1, \ldots$, be the partition of $\mathbb{R}^n_+$ consisting of rectangles with sidelengths $6 \cdot 2^{k_i + N_i - N}$ $(i = 1, \ldots, n)$. Then $\{P_N\}$ defines a piece hierarchy in $\mathbb{R}^n_+$. Let $P_N(x)$ denote the rectangle in P_N which contains x, and $P^j_N(x)$, $j = (j_1, \ldots, j_n)$, $j_i = -1, 0, 1$, the rectangle in P_N obtained by moving one step to the left or right in the x_i-direction depending on whether $j_i = -1$ or $j_i = 1$. If $j_i = 0$ one does not move at all in the x_i-direction. With these notations, we have

$$R_N(x) \subseteq \bigcup_{\substack{j \\ j_i = -1, 0, 1}} P^j_N(x). \tag{4.16}$$

Let

$$U^j_N h(x) = \frac{1}{|P^j_N(x)|} \int_{P^j_N(x)} |h(y)|\,dy \qquad (j_i = -1, 0, 1).$$

For each choice of j we apply Proposition 2 to U^j_N to get, using (4.15) and (4.16),

$$\sum_{N > N_1} a^N \gamma_0 \{x \in Q : \ U_{N,k}g(x) > a^N \beta\}$$

$$\leqslant \sum_{\eta=0}^{\infty} \sum_{N > N_1} a^N e^{-\eta} \left| \left\{x \in Q \cap Q_\eta : \ \sum_j U^j_N(g|_{\tilde{Q}_\eta})(x) > c a^N \beta e^{-\eta} \prod 2^{-k_i}\right\} \right|$$

$$\leqslant \sum_{\eta=0}^{\infty} \frac{1}{\beta} \left(\prod 2^{k_i}\right) \|g|_{\tilde{Q}_\eta}\|_{L^1(dy)}$$

$$\leqslant \frac{\left(\sum 2^{k_i}\right)\left(\prod 2^{k_i}\right)}{\beta} \|g\|_{L^1(dy)}.$$

This proves that the inequality in (4.14), for $\sum\limits_{N \geq 0}$ replaced by $\sum\limits_{N > N_1}$, holds with

$$A_k \sim \prod_{i=1}^{n} 2^{2k_i}. \tag{4.17}$$

Step 4: *Proof of* (4.14) *restricted to* $N \leqslant N_1$

Let N be fixed, $0 \leqslant N \leqslant N_1$, and let

$$Q(z) = \{y \in \mathbb{R}^n_+ : \ |y_i - z_i| < 2^{k_i + N_i - N}, i = 1, \ldots, n\}$$
$$K(z) = \operatorname{con}(Q(z))$$

where $\operatorname{con}(A)$ means the conical hull of A, *i.e.* the intersection of all cones containing A. With these notations we have,

$$U_{N,k}g(x) = \left(\prod_{i=1}^{n} 2^{N-N_i}\right) \exp\left(\sum x_i\right) \sup_{1-r\sim 4^{-N}} \int_{Q(x/r)} g(y)\, dy$$
$$\leqslant \left(\prod_{i=1}^{n} 2^{N-N_i}\right) \exp\left(\sum x_i\right) \int_{K(x)} g(y)\, dy$$
$$\equiv \tilde{U}_{N,k}g(x).$$

Let $l_i = 2 \cdot 2^{k_i + N_i - N}$ be the sidelengths of $Q(x)$.

Our method of estimating

$$\gamma_0 \{x \in Q: \ \tilde{U}_{N,k}g(x) > \beta\} \tag{4.18}$$

depends on whether l_1 is larger or smaller (up to a constant) than the sidelength of Q in the x_1-direction. We get two cases.

Case 1: $2^{k_1} > 2^{N_1 + N}(D-1)/D^2$.

In this case we may estimate the integral in the definition of $\tilde{U}_{N,k}g(x)$ by $\|g\|_{L^1(dy)}$. Set

$$\beta' = \beta \Big/ \left(\prod_i 2^{N-N_i}\right) \|g\|_{L^1(dy)}.$$

Using the fact that the area of $Q \cap \left\{\sum\limits_1^n x_i = t\right\}$ is $\leqslant \prod\limits_{i=2}^{n} 4^{N_i}$ and that $N_i \leqslant N_1$ for all i, we get

$$\begin{aligned}\gamma_0\{x\in Q:\ \tilde{U}_{N,k}g(x)>\beta\} &\leqslant \gamma_0\Big\{x\in Q:\ \exp\Big(\sum x_i\Big)>\beta'\Big\}\\ &\leqslant \Big(\prod_{i=2}^{n}4^{N_i}\Big)\int_{\ln\beta'}^{\infty}e^{-t}\,dt\\ &\leqslant \frac{1}{\beta}\Big(\prod_{i=2}^{n}4^{N_i}\Big)\Big(\prod_{i=1}^{n}2^{N-N_i}\Big)\|g\|_{L^1(dy)}\\ &\leqslant \frac{1}{\beta}2^{(n-1)(N+N_1)}2^{N-N_1}\|g\|_{L^1(dy)}.\end{aligned}$$

Thus, in Case 1 we can estimate (4.18) by

$$\frac{c}{\beta}2^{(n-1)k_1}2^{N-N_1}\|g\|_{L^1(dy)}. \tag{4.19}$$

Case 2: $2^{k_1}\leqslant 2^{N_1+N}(D-1)/D^2$.

In Step 5 we will construct a covering of Q, consisting of pairwise disjoint convex cones K_j, $j\in\mathbb{N}$, such that

(i) If $x\in Q\cap K_j$ then $\tilde{Q}\cap K(x)\subseteq\tilde{K}_j$, where $\tilde{K}_j$ is the union of a bounded number of cones «near» K_j.

(ii) $\displaystyle\Big|K_j\cap\Big\{\sum_1^n x_i=t\Big\}\Big|_{n-1}\leqslant\Big(\sum_1^n 2^{k_i}\Big)^{n-1}\prod_2^n 2^{N_i-N}$ if $0<t\leqslant T\equiv\sum_1^n 4^{N_i}$.

Here $|\cdot|_{n-1}$ denotes $(n-1)$-dimensional area. Set

$$\beta_j=\beta\Big/\Big(\prod_{i=1}^{n}2^{N-N_i}\Big)\int_{\tilde{K}_j}g(y)\,dy.$$

Assuming this covering, we get

$$\begin{aligned}\gamma_0\{x\in Q:\ \tilde{U}_{N,k}g(x)>\beta\} &= \sum_j\gamma_0\{x\in Q\cap K_j:\ \tilde{U}_{N,k}g(x)>\beta\}\\ &\leqslant \sum_j\gamma_0\Big\{x\in Q\cap K_j:\ \exp\Big(\sum x_i\Big)>\beta_j\Big\}\\ &\leqslant \sum_j\int_{(T\wedge\ln\beta_j)\vee(T/D)}^{T}e^{-t}\Big|K_j\cap\Big\{\sum_1^n x_i=t\Big\}\Big|_{n-1}dt\\ &\leqslant \Big(\sum_1^n 2^{k_i}\Big)^{n-1}\Big(\prod_2^n 2^{N_i-N}\Big)\Big(\prod_1^n 2^{N-N_i}\Big)\frac{1}{\beta}\sum_j\int_{\tilde{K}_j}g(y)dy\\ &\leqslant \Big(\sum_1^n 2^{k_i}\Big)^{n-1}2^{N-N_1}\frac{1}{\beta}\|g\|_{L^1(dy)}.\end{aligned}$$

Combining this with the estimate in Case 1, (4.19), we get that (4.18) is less than or equal to

$$\frac{c}{\beta}\left(\sum_{i=1}^{n} 2^{k_i}\right)^{n-1} 2^{N-N_1}\|g\|_{L^1(dy)}.$$

Since this quantity is summable over $N \leqslant N_1$, the inequality in (4.14), for $\sum_{N \geq 0}$ replaced by $\sum_{N=0}^{N_1}$, follows with

$$A_k \sim \prod_{i=1}^{n} 2^{(n-1)k_i}.$$

Together with the result in Step 3 (see (4.17)), this yields (4.14), with

$$A_k \sim \prod_{i=1}^{n} 2^{k_i \max\{2, n-1\}}.$$

Obviously, A_k is good enough.

Apart from the covering, this proves the lemma.

Step 5: *The covering*

Let

$$K = \operatorname{con}(\tilde{Q}), \qquad L = K \cap \{x_1 = 4^{N_1}/D^2\}.$$

Notice that $x_i \sim 4^{N_i}$, $i = 1, \ldots, n$, for all $x \in L$.

Cover L with pairwise disjoint $(n-1)$-dimensional rectangles $\tilde{R}_j$, $j \in \mathbb{N}$, with sides parallel to the axes and of length

$$L_i = l_i\left(\sum_{j=1}^{n} 2^{k_j}\right)\Big/ 2^{k_i}, \qquad i = 2, \ldots, n.$$

Next, let

$$R_j = \tilde{R}_j \cap L, \qquad K_j = \operatorname{con}(R_j).$$

We have to verify (i) and (ii).

(i) Assume $x \in Q \cap K_j$. For arbitrary y and z in $L \cap K(x)$, the conditions $2^{k_1} \leqslant 2^{N_1+N}(D-1)/D^2$ and $N_1 \geqslant N_i$ imply that $y_i - z_i \leqslant L_i$. Hence,

$$L \cap K(x) \subseteq \bigcup_{n \in I(j)} R_n,$$

where $I(j)$ represents a bounded number of the R_n's near R_j. This clearly implies

$$\tilde{Q} \cap K(x) \subseteq \bigcup_{n \in I(j)} K_n \equiv \tilde{K}_j.$$

(ii) Take $0 < t \leqslant T$, $j \in \mathbb{N}$, and let

$$A = K_j \cap \left\{ \sum_1^n x_i = t \right\}.$$

Let A' be the orthogonal projection of A into the hyperplane $\{x_1 = 0\}$ and let R' be the smallest rectangle, with sides parallel to the axes, in $\{x_1 = 0\}$ containing A'. The sidelengths, L_i', of R' are given by

$$L_i' = \max_{x \in A} x_i - \min_{x \in A} x_i = t\left(\max_{x \in R_j} x_i \Big/ \sum_1^n x_l - \min_{x \in R_j} x_i \Big/ \sum_1^n x_l \right), \qquad (i = 2, \ldots, n).$$

Suppose the function $x_i / \sum x_l$ takes its maximum and minimum (over R_j) at a and b, respectively. Set $b_l = a_l + r_l$, $l = 1, \ldots, n$. Then

$$L_i' = t\left(a_i \sum r_l - r_i \sum a_l\right) \Big/ \sum a_l \sum b_l \lesssim a_i \sum r_l \Big/ \sum a_l - r_i$$

where we used the fact that $t \leqslant T \sim \sum b_l$.

By assumption, $N_l \leqslant N_i$ if $l > i$. Therefore,

$$a_i L_l \Big/ \sum a_l \leqslant L_l \leqslant L_i, \qquad l > i.$$

On the other hand, if $l \leqslant i$ then $N_l \geqslant N_i$ implies

$$a_i L_l \Big/ \sum a_l \leqslant a_i L_l / a_l \sim 4^{N_i - N_l} \left(\sum_{j=1}^n 2^{k_j} \right) 2^{N_l - N} \lesssim L_i, \qquad l \leqslant i.$$

Thus $L_i' \lesssim L_i$ $(i = 2, \ldots, n)$ and, finally,

$$\begin{aligned} |A|_{n-1} = \sqrt{n}\, |A'|_{n-1} &\lesssim \prod_2^n L_i' \\ &\lesssim \prod_2^n L_i \\ &= \left(\sum_1^n 2^{k_i} \right)^{n-1} \prod_2^n 2^{N_i - N}. \end{aligned}$$

The proof of Lemma 5 is complete.

5. The Case $0 < r < 1/2$

The following lemma completes the proof of Theorem 1.

Lemma 6. *The operator*

$$M_0^\alpha f(x) = \sup_{0<r<1/2} |M_r^\alpha f(x)|$$

maps $L'(\gamma_\alpha)$ *boundedly into* $L^{1,\infty}(\gamma_\alpha)$ *for all* $\alpha > -1$.

PROOF. Since $D_r^\alpha(x_i, y_i) \leqslant c$ for $0 < r < 1/2$, we may replace $K_r^\alpha(x, y)$ (see (3.2)) by

$$E_r^\alpha(x, y) = \prod_i E_r^\alpha(x_i, y_i).$$

Let

$$E^\alpha f(x) = \sup_{0<r<1/2} \left| \int E_r^\alpha(x, y) f(y)\, d\gamma_\alpha(y) \right| \cdot$$

We claim that $(0 < r < 1/2)$

(5.1) $E_r^\alpha(s, t) \leqslant c$ if $\alpha \leqslant -1/2$, $s < 1$ or $\alpha > -1/2$, $s < \alpha + 1/2$.

Assuming $s < 1$, we get

$$\frac{-r(s+t) + 2\sqrt{rst}}{1-r} \leqslant -\frac{\sqrt{rst}}{1-r} + \frac{-rt + 3\sqrt{rt}}{1-r} \leqslant -\sqrt{rst} + C,$$

and, if also $\alpha \leqslant -1/2$,

$$E_r^\alpha(s, t) \leqslant c(rst)^{-\alpha/2 - 1/4} \exp\left(-\sqrt{rst}\right) \leqslant C.$$

Next, assume $\alpha > -1/2$, $s < \alpha + 1/2$. The derivative $(\partial/\partial t)E_r^\alpha(s, t)$ $(t \geqslant (1-r)^2/4rs)$ equals a positive factor times

$$-rt + \sqrt{rst} - \left(\frac{\alpha}{2} + \frac{1}{4}\right)(1 - r),$$

which is negative for all r, s, t under consideration. Hence,

$$E_r^\alpha(s, t) \leqslant E_r^\alpha(s, (1-r)^2/4rs) \leqslant C$$

and (5.1) holds.

Because of (5.1) and Proposition 1 we may, when measuring the level sets $\{x \in \mathbb{R}_+^d : E^\alpha f(x) > \beta\}$, neglect the region where any $x_i < 1$ if $\alpha \leqslant -1/2$ or any $x_i < \alpha + 1/2$ if $\alpha > -1/2$. Therefore, we assume $x_i \geqslant c = c(\alpha)$, $i = 1, \dots, d$.

In the interval $2^{l_i - 1}\sqrt{x_i}/r \leqslant |y_i - x_i/r| \leqslant 2^{l_i}\sqrt{x_i}/r$ $(l_i = 0, 1, \dots)$, a simple calculation where we use $x_i \geqslant c(\alpha)$ gives the following estimate

$$E_r^\alpha(x_i, y_i) \leqslant a_{l_i} x_i^{-\alpha - 1/2} e^{x_i},$$

with

$$a_{l_i} = 4^{l_i|\alpha + 1/2|} \exp(-c2^{l_i}).$$

Thus, for $f \geqslant 0$, we get

$$(5.2)\quad E^\alpha f(x) \leqslant \sum_{l_i=0}^{\infty} \left(\prod_{i=1}^{d} a_{l_i}\right)\left(\prod_{i=1}^{d} x_i^{-\alpha-1/2} e^{x_i}\right) \sup_{0<r<1/2} \int_{|y_i - x_i/r| \leq 2^{l_i}\sqrt{x_i}/r} f(y)\, d\gamma_\alpha(y).$$

Let

$$Q(x) = \left\{ y \in \mathbb{R}_+^d : |y_i - x_i| \leqslant 2^{l_i}\sqrt{x_i} \text{ for all } i \right\}.$$

Then the conical hull of $Q(x)$, $\operatorname{con}(Q(x)) = K(x)$, contains the domain of integration in (5.2) for all $r > 0$. Set, for $f \geqslant 0$,

$$A_l f(x) = \left(\prod_i x_i^{-\alpha-1/2} e^{x_i}\right) \int_{K(x)} f(y)\, d\gamma_\alpha(y).$$

By Theorem B, Lemma 6 follows if we prove that, for some M,

$$(5.3)\qquad \gamma_\alpha\{x \in \mathbb{R}_+^d : A_l f(x) > \beta\} \leqslant \frac{C}{\beta}\left(\prod_{i=1}^{d} 2^{l_i}\right)^M \|f\|_{L^1(\gamma_\alpha)}.$$

For $k \in \mathbb{Z}^d$ let $Q^k = \{x \in \mathbb{R}_+^d : 2^{k_i} \leqslant x_i < 2^{k_i+1}, \text{ for all } i\}$. Of course, (5.3) would follow from

$$(5.4)\qquad \gamma_\alpha\{x \in Q^k : A_l f(x) > \beta\} \leqslant \frac{c_{k,l}}{\beta} \|f\|_{L^1(\gamma_\alpha)},$$

with

$$\sum_k c_{k,l} \leqslant \left(\prod_i 2^{l_i}\right)^M.$$

Depending on whether a sidelength of Q^k is greater or smaller than the corresponding sidelength of $Q(x)$, $x \in Q^k$, we will treat the coordinates in different ways. Therefore, for fixed k and l, we let

$$I' = \{i : k_i \geqslant 2(l_i + 1)\}, \qquad I'' = \{1, \dots, d\} \setminus I',$$

and write $x = (x', x'') \in \mathbb{R}_+^d$, $Q^k = Q^{k'} \times Q^{k''}$ in the obvious way. Further, with $d' = \# I'$, we define $K(x')$ and $A_l F$ quite analogously as above, for $x' \in \mathbb{R}_+^{d'}$ and functions $F \geqslant 0$ on $\mathbb{R}_+^{d'}$. Let

$$F(y') = \int f(y', y'')\, d\gamma_\alpha(y'').$$

Then

$$A_l f(x', x'') \leqslant \left(\prod_{i \in I''} x_i^{-\alpha-1/2} e^{x_i}\right) A_l F(x').$$

Notice that

$$\int_{Q^{k''}} \Big(\prod_{i\in I''} x_i^{-\alpha-1/2} e^{x_i}\Big)\, d\gamma_\alpha(x'') \sim \prod_{i\in I''} 2^{k_i/2}. \tag{5.5}$$

To prove (5.4) we will prove that (for $d' = 1, \ldots, d$)

$$\gamma_\alpha\{x' \in Q^{k'}:\ A_l F(x') > \beta\} \leqslant \frac{c_{k',l}}{\beta} \|F\|_{L^1(d\gamma_\alpha(y'))}. \tag{5.6}$$

According to Proposition 1, (5.4) follows from (5.5) and (5.6), with

$$c_{k,l} = c_{k',l} \prod_{i\in I''} 2^{k_i/2}. \tag{5.7}$$

(We interpret products and constants with indices over empty sets as 1.)

We will construct a covering of $Q^{k'}$ consisting of pairwise disjoint convex cones $\{K_j\}$ such that, for $c_l = \prod_{I'} 2^{l_i}$,

(i) If $x' \in Q^{k'} \cap K_j$ then $K(x') \subseteq \tilde{K}_j$, where $\tilde{K}_j$ is the union of $\sim c_l^{d'-1}$ cones «near» K_j.

(ii) $\left|K_j \cap \left\{\sum_{I'} x_i = t\right\}\right|_{d'-1} \leqslant c_l^{d'} \prod_{i\in I'\setminus\{i_0\}} 2^{k_i/2}$ if $0 < t \leqslant \sum_{I'} 2^{k_i+1} \equiv T$ and i_0 is given by $k_{i_0} = \max_{i\in I'} k_i$.

Suppose we have this covering. Then, with

$$\beta_j = c\beta 2^{(\alpha+1/2)\sum_{I'} k_i} \Big/ \int_{\tilde{K}_j} F\, d\gamma_\alpha,$$

we get

$$\begin{aligned}
\gamma_\alpha\{x' \in Q^{k'}:\ A_l F(x') > \beta\} &\leqslant \sum_j \gamma_\alpha\Big\{x' \in Q^{k'} \cap K_j:\ \exp\Big(\sum_{I'} x_i\Big) > \beta_j\Big\} \\
&\leqslant 2^{\alpha \Sigma k_i} \sum_j \int_{(T\wedge \ln \beta_j)\vee(T/2)}^{T} e^{-t} \Big|K_j \cap \Big\{\sum x_i = t\Big\}\Big|_{d'-1} dt \\
&\leqslant 2^{\alpha \Sigma k_i} c_l^{d'} \Big(\prod_{i\neq i_0} 2^{k_i/2}\Big) \frac{1}{\beta} 2^{-(\alpha+1/2)\Sigma k_i} \sum_j \int_{\tilde{K}_j} F\, d\gamma_\alpha \\
&\leqslant \frac{1}{\beta} c_l^{2d'-1} 2^{-k_{i_0}/2} \|F\|_{L^1(\gamma_\alpha)} \\
&\leqslant \frac{1}{\beta} c_l^{2d'-1} \Big(\prod_{I'} 2^{-k_i/2d'}\Big) \|F\|_{L^1(\gamma_\alpha)}.
\end{aligned}$$

Hence, the covering would imply (5.4), with (see (5.6) and (5.7))

$$c_{k,l} \sim \Big(\prod_{I'} 2^{l_i}\Big)^{2d'-1}\Big(\prod_{I'} 2^{-k_i/2d'}\Big)\Big(\prod_{I''} 2^{k_i/2}\Big).$$

Since

$$\sum_{k\in\mathbb{Z}^d}\Big(\prod_{k_i \geq 2(l_i+1)} 2^{-k_i/2d'}\Big)\Big(\prod_{k_i<2(l_i+1)} 2^{k_i/2}\Big) \leqslant \prod_{i=1}^{d} 2^{l_i},$$

this would complete the proof.

It remains to construct the covering. This will be very similar to the construction in the proof of Lemma 5. Therefore, after describing the cones, we just sketch the verifications of (i) and (ii).

From now on everything takes place in $\mathbb{R}_+^{d'}$, so we drop the primes on the elements. Let

$$\tilde{Q} = \bigcup_{x\in Q^{k'}} Q(x), \qquad K = \operatorname{con}(\tilde{Q}),$$

and

$$L = K \cap \{x\colon\ x_{i_0} = 2^{k_{i_0}} - 2^{l_{i_0}+k_{i_0}/2}\}.$$

Next, cover L with pairwise disjoint $(d'-1)$-dimensional rectangles $\tilde{R}_j$, $j\in\mathbb{N}$, with sides parallel to the axes and of length

$$L_i = 2^{l_i+k_i/2}, \qquad i\in I', \qquad i\neq i_0.$$

Let

$$R_j = \tilde{R}_j \cap L \quad \text{and} \quad K_j = \operatorname{con}(R_j).$$

(i) Suppose $x\in Q^{k'}\cap K_j$. The condition $k_i \geqslant 2(l_i+1)$ ensures that $y_i \sim 2^{k_i}$, $i\in I'$, for all $y\in\tilde{Q}$. This and the choice of i_0 imply that $y_i - z_i \lesssim c_l L_i$, whenever y and z are points in $K(x)\cap L$. This gives (i).

(ii) Let $A = K_j \cap \left\{\sum_i x_i = t\right\}$ and let A' be the orthogonal projection of A into $\{x_{i_0} = 0\}$. Further, let R' be the smallest rectangle, with sides parallel to the axes, in $\{x_{i_0} = 0\}$ containing A'. Imitating the verification of (ii) in Lemma 5, we get

$$L_i' \leqslant c_l L_i, \qquad i\in I', \qquad i\neq i_0,$$

where L_i' are the sidelengths of R'. This gives (ii).

The proof of Lemma 6 is complete.

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