Revista Matemática Iberoamericana
Vol. 8, n.º 2, 1992

A Picard type theorem for quasiregular mappings of $\mathbb{R}^n$ into n–manifolds with many ends

Ilkka Holopainen and Seppo Rickman

1. Introduction.

Quasiregular mappings are defined as the quasiconformal mappings by replacing the homeomorphism requirement by continuity. More precisely, a continuous map $f : G \to \mathbb{R}^n$, where G is an open set in $\mathbb{R}^n$ and $n \geq 2$, is *quasiregular* if $f \in W^1_{n,\mathrm{loc}}(G)$ and there exists K, $1 \leq K < \infty$, such that

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e.}$$

Here $W^1_{n,\mathrm{loc}}(G)$ is the space of maps that are locally in the Sobolev space W^1_n of L^n–integrable maps with distributional first order derivatives in L^n. Furthermore, $|f'(x)|$ is the operator norm of the formal derivative of f at x defined in terms of the partial derivatives, and $J_f(x) = \det f'(x)$. With this definition a quasiregular map turns out to be differentiable a.e. The definition extends easily to the case $f : M \to N$, where M and N are oriented Riemannian n–manifolds. We say that a continuous map $f : M \to N$ is *locally quasiregular* if for each $x \in M$ there exist neighborhoods U of x and V of $f(x)$ and charts $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$ is quasiregular in the above sense. Then $T_x f$, the differential of f at x, exists a.e., and f is called *quasiregular* if there exists K, $1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x) \quad \text{a.e.}$$

In this paper we call f K*-quasiregular* if it satisfies the above conditions with K.

Quasiregular maps constitute a natural generalization into n real dimensions of the analytic functions of one complex variable. For example, a Picard type theorem was established for $n \geq 3$ in 1980 in the following form.

Theorem 1.1. [R1]. *For each $n \geq 3$ and each $K \geq 1$ there exists a positive integer $q_0(n, K)$ such that every K-quasiregular map $f : \mathbb{R}^n \to \mathbf{S}^n \setminus \{a_1, \dots, a_q\}$, where $q \geq q_0(n, K)$ and $a_1, \dots, a_q$ are distinct, is constant.*

Theorem 1.1 is known to be sharp for $n = 3$ in the following sense.

Theorem 1.2. [R5]. *For each positive integer p there exists a nonconstant $K(p)$-quasiregular map $f : \mathbb{R}^3 \to \mathbb{R}^3$ omitting at least p points.*

Both these results can be extended considerably to yield a defect relation [R2], [R6], [R7] together with a sharpness result [R6].

The purpose of this paper is to give an affirmative answer to a question posed by M. Gromov, namely, whether one can put any Riemannian metric on $\mathbf{S}^n \setminus \{a_1, \dots, a_q\}$ and still get the constantness of the K-quasiregular map. We formulate our result as follows:

Theorem 1.3. *For each $n \geq 3$ and each $K \geq 1$ there exists a positive integer $q_0(n, K)$ such that the following holds. Let N be an oriented compact differentiable n-manifold and let $a_1, \dots, a_q$, $q \geq q_0(n, K)$, be distinct points in N. Suppose $M = N \setminus \{a_1, \dots, a_q\}$ is given any Riemannian metric $\langle\, ,\rangle$. Then any K-quasiregular map $f : \mathbb{R}^n \to M$ is constant.*

Gromov's question was partly solved in [H1]. For the proof in [H1] a regularity condition was needed on the metric which guaranteed the existence of certain path families. Up to very recently, all existing proofs of Theorem 1.1, including the extension of it in [H1], contain estimates in terms of moduli of path families (see [R2], [R4], [R7]). The heart of the matter has been to get a rapid growth on a measure ν in $\mathbb{R}^n$ in terms of the number of omitted points. For a Borel set E, $\nu(E)$ is the average covering number of $f \mid E$ over a fixed $(n-1)$–sphere in the target.

The main ingredient to the problem is the paper [EL] by A. Eremenko and J. Lewis where they present a purely potential theoretic proof for Theorem 1.1. They obtain this as an application of a general statement on $\mathcal{A}$–harmonic functions (see Section 2 for definitions). In [EL] there is to each $\mathcal{A}$–harmonic function w attached a measure μ, and this measure can be estimated by the growth behavior of w. In some sense such measures μ replace measures ν described above. To get a proof for Theorem 1.3 a technique developed in [H1] is used to produce certain n–harmonic functions in M with prescribed behavior near the ends of M. In addition to this we need from [EL] their Lemma 1 which relates the growth behavior of the measure μ to that of the corresponding $\mathcal{A}$–harmonic function. The rest of the proof follows more or less ideas established in [R1] and [R4]. An alternate way to get a proof for Theorem 1.3 is to use the construction in Section 3 and the main theorem (Theorem 1) from [EL]. However, for our purpose we can avoid a great deal of the complications in the proof of [EL, Theorem 1]. We therefore feel that it is justified to present a direct proof were only a part of the paper [EL] is used.

2. Quasiregular mappings and $\mathcal{A}$-harmonic functions.

Throughout this section let $G \subset \mathbb{R}^n$ be an open set and let

$$\mathcal{A} : G \times \mathbb{R}^n \to \mathbb{R}^n$$

be a mapping defined by

$$\mathcal{A}(x,h) = \big(\theta(x)h \cdot h\big)^{n/2-1}\theta(x)h \tag{2.1}$$

where $\theta : G \to \mathrm{GL}(\mathbb{R}^n, \mathbb{R}^n)$ is a Borel map with the following properties. For all $x \in G$ the linear map $\theta(x)$ is self-adjoint and there are constants $0 < \alpha \leq \beta < \infty$, called *structure constants* of $\mathcal{A}$, such that

$$\alpha^{2/n}\,|h|^2 \leq \theta(x)h \cdot h \leq \beta^{2/n}\,|h|^2 \tag{2.2}$$

for almost every $x \in G$ and all $h \in \mathbb{R}^n$. It follows from (2.1) and (2.2) that $\mathcal{A}$ has the following additional properties for almost every $x \in G$

and all $h, k \in \mathbb{R}^n$:

$$\mathcal{A}(x,h)\cdot h \geq \alpha\,|h|^n \tag{2.3}$$

$$|\mathcal{A}(x,h)\cdot k| \leq \beta\,|h|^{n-1}\,|k| \tag{2.4}$$

$$\begin{aligned}\big(\mathcal{A}(x,h)-\mathcal{A}(x,k)\big)\cdot(h-k) &\geq \frac{\alpha}{2}\,|h-k|^2\left(|h|^{n-2}+|k|^{n-2}\right)\\ &\geq \alpha\,2^{1-n}\,|h-k|^n\,.\end{aligned}\tag{2.5}$$

For the proof of these properties we refer to [BI]. A continuous function $u \in W^1_{n,\mathrm{loc}}(G)$ is said to be *$\mathcal{A}$-harmonic* in G if it is a weak solution of the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = 0 \tag{2.6}$$

in G, that is,

$$\int_G \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(G)$. The equation (2.6) is the Euler–Lagrange equation of the variational integral

$$\int_G F(x,\nabla u)\,dm\,, \quad F(x,h) = n^{-1}\big(\theta(x)h\cdot h\big)^{n/2}\,.$$

Therefore $\mathcal{A}$-harmonic functions are also called F-extremals in the literature.

An upper semicontinuous function $v : G \to \mathbb{R}\cup\{-\infty\}$ is called *$\mathcal{A}$-subharmonic* in G if for all domains $D \subset\subset G$ and all functions $h \in C(\bar{D})$ which are $\mathcal{A}$-harmonic in D the condition $h \geq v$ on ∂D implies $h \geq v$ in D. If u and v are $\mathcal{A}$-subharmonic in G, then obviously $\max\{u,v\}$ and $\lambda u + \mu$, $\lambda \geq 0$, $\mu \in \mathbb{R}$, are also $\mathcal{A}$-subharmonic in G. A lower semicontinuous function $u : G \to \mathbb{R}\cup\{\infty\}$ is said to be *$\mathcal{A}$-superharmonic* if $-u$ is $\mathcal{A}$-subharmonic.

There is a close connection between $\mathcal{A}$-harmonic functions and quasiregular maps which is proved by Yu. G. Reshetnyak, see *e.g.* [Re, Theorem 11.2]. Namely, if f is a quasiregular mapping of G into another open set $G' \subset \mathbb{R}^n$ and u is $\mathcal{A}$-harmonic in G', then $u\circ f$ is $f^\#\mathcal{A}$-harmonic in G where $f^\#\mathcal{A}$, the *pullback of* $\mathcal{A}$, is of type (2.1) and depends on $\mathcal{A}$ and f. We shall make use of this important result in the following case to where it easily extends. Let G' be an open subset of an oriented Riemannian n-manifold M. A function $u \in C(G') \cap W^1_{n,\mathrm{loc}}(G')$ is said to be *n-harmonic* in G' if

$$\int_{G'} \langle |\nabla u|^{n-2} \nabla u, \nabla \varphi \rangle \, dm = 0$$

for all $\varphi \in C_0^\infty(G')$. Here $\langle\, ,\, \rangle$ is the Riemannian metric of M. Suppose that $f : G \to G'$ is a K-quasiregular map. Then $u \circ f$ is $\mathcal{A}$-harmonic in G where $\mathcal{A}$ is defined by (2.1) with

$$\theta(x) = \begin{cases} J_f^{2/n}(x) T_x f^{-1} T_x f^{-1*}, & \text{if } J_f(x) > 0, \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

Here $T_x f^{-1*} : \mathbb{R}^n \to T_{f(x)}M$ is the transpose of the linear map $T_x f^{-1}$. We can choose

$$\alpha = 1/K \qquad \text{and} \qquad \beta = K^{n-1} \tag{2.7}$$

as structure constants of $\mathcal{A}$. We can apply the invariance property to chart mappings as follows. For each $x \in M$, choose a neighborhood U of x and a 2-bilipschitz chart $\varphi : U \to B^n(0,r)$. Then φ^{-1} is quasiconformal and $u \circ \varphi^{-1}$ is $\mathcal{A}$-harmonic in $B^n(0,r)$ if u is n-harmonic in U. In this way we easily obtain some basic properties for n-harmonic functions in U such as Hölder continuity, Harnack's inequality, and Harnack's principle.

During the last few years $\mathcal{A}$-harmonic functions have been extensively studied in a more general setting than the one introduced here, see [GLM], [HK]. The study has also been extended to Riemannian n-manifolds to create a classification theory of manifolds based on the existence of $\mathcal{A}$-harmonic functions with various properties [H1], [H2], [HR]. To simplify the presentation in this paper, we consider $\mathcal{A}$-harmonic functions only in the Euclidean n-space and n-harmonic functions on Riemannian n-manifolds.

3. Construction of n-harmonic functions in M.

Let N be an oriented compact differentiable n-manifold and let $M = N \setminus \{a_1, a_2, \dots, a_q\}$ be equipped with an arbitrary Riemannian metric $\langle\, ,\, \rangle$. We suppose that the points $a_i \in N$ are distinct and write $P = \{a_1, \dots, a_q\}$. In this paper a *condenser* in N will be a pair (G, C)

where $G \neq N$ is an open set in N and C is either a compact subset of G or a closed set in $G \cap M$ such that $C \cup (G \cap P)$ is compact. The *n-capacity* of (G, C) is defined by

$$\operatorname{cap}_n(G, C) = \inf_u \int_{G \cap M} |\nabla u|^n \, dm$$

where the infimum is taken over all $u \in C(\bar{G} \cap M) \cap W^1_{n,\mathrm{loc}}(G \cap M)$ with $u = 0$ in $\partial G \cap M$ and $u = 1$ in $C \cap M$. If the infimum is attained by some function u within the class, we call u an *n-capacity function* of (G, C). In that case u will be n-harmonic in $(G \cap M) \setminus C$. From now on we assume that $\langle \, , \rangle$ is given such that the deleted set $P = \{a_1, \dots, a_q\}$ is of zero n-capacity, *i.e.* $\operatorname{cap}_n(G, P) = 0$ for all open sets $G \subset N$ containing P. Let C be a smooth $(n-1)$-submanifold of N which divides N into two domains U_1 and U_2, one of them, say U_1, containing a_1, and U_2 containing the points $a_2, \dots a_q$. Write $U = U_1 \setminus \{a_1\} \subset M$ and $V = U_2 \setminus \{a_2, \dots, a_q\} \subset M$. This notation will be used throughout the paper.

In this section we construct n-harmonic functions in M that in some sense correspond to functions $-\log|x - a_j|$ in the case $M = \mathbf{S}^n \setminus \{a_1, \dots, a_q\}$ equipped with the Euclidean metric.

Lemma 3.1. *There are functions $h_j \in C(V \cup C)$, $j = 2, \dots, q$, with the following properties:*

(3.2) $\quad h_j$ *is n-harmonic in V,*

(3.3) $\quad h_j = 0$ *on C,*

(3.4) $\quad \sup h_j = \infty$ *(near a_j), and*

(3.5) $\quad h_j$ *is bounded in some neighborhood of a_k, $k \neq j$.*

PROOF: Fix j and choose a decreasing sequence $C_i \subset U_2$ of compact connected sets with nonempty interiors such that $\cap_i C_i = \{a_j\}$. For each i, there exists a function $w_i \in C(V \cup C)$ which is n-harmonic in $V \setminus C_i$ with $w_i = 0$ in C and $w_i = \left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(1-n)}$ in C_i, and which minimizes the Dirichlet n-integral

$$\int_{V \setminus C_i} |\nabla u|^n \, dm$$

among all functions $u \in C(V \cup C)$ that coincide with w_i in $C \cup C_i$. For each a, $0 < a \le \left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(1-n)}$, $\min\{1, w_i/a\}$ is the n-capacity function of $(U_2, \{x : w_i(x) \ge a\})$. By [H1, 3.8],

$$\operatorname{cap}_n(U_2, \{x : w_i(x) \ge a\}) = a^{1-n}. \tag{3.6}$$

If $B \subset V$ is a compact topological n-ball, then it follows from a local Harnack inequality that for sufficiently large i

$$\max_B w_i \le c_1 \min_B w_i$$

where c_1 is independent of i. On the other hand,

$$\min_B w_i = \operatorname{cap}_n\left(U_2, \{x : w_i(x) \ge \min_B w_i\}\right)^{1/(1-n)} \le \operatorname{cap}_n(U_2, B)^{1/(1-n)},$$

and so

$$\max_B w_i \le c_1 \operatorname{cap}_n(U_2, B)^{1/(1-n)}$$

for large i. Therefore (w_i) is a locally uniformly bounded sequence in V and it follows from the Hölder continuity estimate [GLM, 4.7] that (w_i) is equicontinuous in V. Ascoli's theorem and a standard diagonal process then give a subsequence, still denoted by w_i, which converges locally uniformly in V to a function $h \in C(V)$. Since the class of n-harmonic functions is closed under uniform convergence [HK, 3.2], the limit function h is n-harmonic in V. By a boundary estimate due to V. G. Maz'ya [M, p. 236], h is continuous in $V \cup C$ and $h = 0$ in C. Therefore conditions (3.2) and (3.3) hold.

To show that h is nonconstant, take a small topological $(n-1)$-sphere $S \subset V$ about a_j such that it separates a_j and the rest of the points a_k. Then

$$\max_S w_i \le c_S \min_S w_i$$

for large i. Here the constant c_S depends on the choice of S. But now

$$\begin{aligned}\min_S w_i &\ge c_S^{-1} \max_S w_i \\ &= c_S^{-1} \operatorname{cap}_n\left(U_2, \{x : w_i(x) \ge \max_S w_i\}\right)^{1/(1-n)} \\ &\ge c_S^{-1} \operatorname{cap}_n(U_2, S)^{1/(1-n)} > 0.\end{aligned}$$

Thus h is not a constant.

Next we show that h is bounded near the points a_k, $k \neq j$. Let $S \subset V$ be as above. Suppose that h takes arbitrary large values near some point a_k, $k \neq j$. In particular, some w_i takes larger values near a_k than $m = \sup_i \max_S w_i$. Truncating w_i by m near the point a_k we obtain a function that coincides with w_i in $C \cup C_i$ but its Dirichlet n-integral over $V \setminus C_i$ is strictly smaller than that of w_i. This is a contradiction, and (3.5) holds.

Suppose that $\sup_V h = \lambda < \infty$. Then $\left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(n-1)} w_i \geq h/\lambda$ in $V \setminus C_i$ for all i. This implies that

$$\operatorname{cap}_n(U_2, C_i) \geq c > 0$$

since $\left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(n-1)} w_i$ is the n-capacity function of (U_2, C_i). In particular,

$$\operatorname{cap}_n(U_2, \{a_j\}) > 0$$

which is a contradiction since $\langle\, ,\, \rangle$ is assumed to be such that P is of zero n-capacity. Hence (3.4) is true.

Similarly, we can find a decreasing sequence $K_i \subset U_1$ of compact connected sets with nonempty interiors such that $\cap_i K_i = \{a_1\}$ and a sequence of functions $u_i \in C(U \cup C)$ with the following properties

(3.7) u_i is n-harmonic in $U \setminus K_i$,

(3.8) $u_i = 0$ in C and $u_i = \left(\operatorname{cap}_n(U_1, K_i)\right)^{1/(1-n)}$ in K_i,

(3.9) $\operatorname{cap}_n\bigl(U_1, \{x : u_i(x) \geq a\}\bigr) = a^{1-n}$
for all a, $0 < a \leq \left(\operatorname{cap}_n(U_1, K_i)\right)^{1/(1-n)}$, and

(3.10) $u_i \to h_1$ locally uniformly in U.

The limit function h_1 is continuous in $U \cup C$, n-harmonic in U, $h_1 = 0$ in C, and $\sup_U h_1 = \infty$.

The main result of this section is the following lemma. The proof is similar to the proof of [H1, 4.13].

Lemma 3.11. *There exist n-harmonic functions v_j, $j = 2, \ldots, q$, in*

M and a positive constant κ such that

(3.12) $|v_j| \le \kappa$ *in* C,

(3.13) $|v_j - v_i| \le 2\kappa$ *in* U,

(3.14) $\sup_U v_j = \infty$,

(3.15) $\inf_V v_j = -\infty$,

(3.16) v_j *is bounded from below near* a_k, $k \ne 1, j$, *and*

(3.17) *if* $v_j(x) > \kappa$, *then* $x \in U$; *if* $v_j(x) < -\kappa$, *then* $x \in V$.

PROOF: Fix $j \in \{2, \dots, q\}$, and let C_i and w_i, $i = 1, 2 \dots$, be as in the construction of h_j. For each i, let $\nu = \nu_i$ be the largest integer such that

$$\nu \le \min\left\{ \left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(1-n)}, \left(\operatorname{cap}_n(U_1, K_i)\right)^{1/(1-n)} \right\}.$$

Then ν_i is increasing and tends to ∞ as $i \to \infty$. Write

$$G_\nu = N \setminus \overline{\{x : w_i(x) \ge \nu\}},$$
$$F_\nu = \{x : u_i(x) \ge \nu\},$$

and

$$\gamma_\nu = \left(\operatorname{cap}_n(G_\nu, F_\nu)\right)^{1/(1-n)}.$$

Let $e_\nu \in C(M)$ be a function which is n-harmonic in $G_\nu \setminus F_\nu$,

$$e_\nu | F_\nu = \gamma_\nu / 2,$$

and

$$e_\nu | \{x : w_i(x) \ge \nu\} = -\gamma_\nu / 2$$

such that it minimizes the Dirichlet n-integral over $G_\nu \setminus F_\nu$ among all functions taking these values in F_ν and $\{x : w_i(x) \ge \nu\}$. First we note that $-\gamma_\nu/2 \le e_\nu \le \gamma_\nu/2$ in $G_\nu \setminus F_\nu$. Applying Harnack's inequality to $e_\nu + \gamma_\nu/2$ and to $\gamma_\nu/2 - e_\nu$ we obtain that, in fact, $-\gamma_\nu/2 < e_\nu < \gamma_\nu/2$ in $G_\nu \setminus F_\nu$. Write $M_\nu = \max_C e_\nu$ and $m_\nu = \min_C e_\nu$. The function $\gamma_\nu^{-1}(e_\nu + \gamma_\nu/2)$ is the n-capacity function of (G_ν, F_ν). Applying [H1, 3.8] to this function yields

$$M_\nu - m_\nu = \operatorname{cap}_n\left(\{x : e_\nu(x) > m_\nu\}, \{x : e_\nu(x) \ge M_\nu\}\right)^{1/(1-n)}.$$

The sets $\{x : e_\nu(x) > m_\nu\}$ and $\{x : e_\nu(x) \geq M_\nu\}$ contain continua E_1 and E_2, respectively, that join C and some point of $F_{\nu_1} \cup P \cup \{x : w_1(x) \geq \nu_1\}$. Therefore

$$\begin{aligned} &\operatorname{cap}_n\bigl(\{x : e_\nu(x) > m_\nu\}, \{x : e_\nu(x) \geq M_\nu\}\bigr) \\ &\qquad \geq \inf_{E_1, E_2} \mathrm{M}_n\bigl(\Delta(E_1, E_2; G_{\nu_1} \setminus F_{\nu_1})\bigr) > 0 \end{aligned}$$

where E_1 and E_2 are as above. Here $\mathrm{M}_n\bigl(\Delta(E_1, E_2; G_{\nu_1} \setminus F_{\nu_1})\bigr)$ is the n-modulus of all curves in $G_{\nu_1} \setminus F_{\nu_1}$ connecting E_1 and E_2. Hence

$$M_\nu - m_\nu \leq \kappa \tag{3.18}$$

where $\kappa < \infty$ is independent of ν.

Next we show that $\{x : e_\nu(x) > M_\nu\} \subset U$ and $U \subset \{x : e_\nu(x) > m_\nu\}$. Suppose there is a point $x_0 \in G_\nu \setminus (U \cup C)$ such that $e_\nu(x_0) = a > M_\nu$. Then the component A of $\{x \in M : e_\nu(x) > a\}$ whose boundary contains x_0 must be a punctured neighborhood of some a_k and A is entirely contained in $G_\nu \setminus (U \cup C)$. Replacing $e_\nu \mid A$ by a decreases the Dirichlet n-integral of e_ν which gives a contradiction. On the other hand, $e_\nu(x) \geq m_\nu$ for every $x \in U$. It follows from Harnack's inequality applied to $e_\nu - m_\nu$ in $U \setminus F_\nu$ that $e_\nu > m_\nu$ in U. Applying [H1, 3.8] to $\gamma_\nu^{-1}(e_\nu + \gamma_\nu/2)$ yields

$$\begin{aligned} \gamma_\nu/2 - M_\nu &= \operatorname{cap}_n\bigl(\{x : e_\nu(x) > M_\nu\}, F_\nu\bigr)^{1/(1-n)} \\ &\leq \operatorname{cap}_n(U_1, F_\nu)^{1/(1-n)} = \nu\,. \end{aligned}$$

Similarly,

$$\begin{aligned} \gamma_\nu/2 - m_\nu &= \operatorname{cap}_n\bigl(\{x : e_\nu(x) > m_\nu\}, F_\nu\bigr)^{1/(1-n)} \\ &\geq \operatorname{cap}_n(U_1, F_\nu)^{1/(1-n)} = \nu\,. \end{aligned}$$

We claim that $m_\nu \leq 0 \leq M_\nu$, that is, each e_ν takes the value 0 in C. Suppose that we can find ν such that, for instance, $\{x : e_\nu(x) \geq 0\} \subset U$. Then

$$\begin{aligned} \nu^{1-n} &= \operatorname{cap}_n(U, F_\nu) \\ &< \operatorname{cap}_n\bigl(\{x : e_\nu(x) > 0\}, F_\nu\bigr) \\ &= (2/\gamma_\nu)^{n-1} \\ &= \operatorname{cap}_n\bigl(G_\nu, \{x : e_\nu(x) \geq 0\}\bigr) \\ &< \operatorname{cap}_n(G_\nu, U \cup C) = \nu^{1-n}\,. \end{aligned}$$

This is a contradiction. The case $U \cup C \subset \{x : e_\nu(x) > 0\}$ can be treated similarly. Since $M_\nu - m_\nu \le \kappa$, we obtain $m_\nu \ge -\kappa$ and $M_\nu \le \kappa$.

We have proved

$$\nu - \kappa \le \nu + m_\nu \le \gamma_\nu/2 \le \nu + M_\nu \le \nu + \kappa .$$

Thus $u_i - \kappa \le e_\nu \le u_i + \kappa$ on $\partial(U \setminus F_\nu)$ and therefore

$$u_i - \kappa \le e_\nu \le u_i + \kappa \tag{3.19}$$

in $(U \cup C) \setminus F_\nu$, see [GLM, 4.18], [HK, 3.7]. We also have

$$-w_i - \kappa \le e_\nu \le -w_i + \kappa$$

in $\partial(G_\nu \setminus U) \cap M$. We want to show that

$$-w_i - \kappa \le e_\nu \le -w_i + \kappa \tag{3.20}$$

in $(G_\nu \setminus U) \cap M$. For large ν, the boundary of $G_\nu \setminus U$ contains some points of P. Therefore we can not use [GLM, 4.18] in this case. Suppose that there is a point $x_0 \in (G_\nu \setminus U) \cap M$ and $\varepsilon > 0$ such that $e_\nu(x_0) = -w_i(x_0) + \kappa + \varepsilon$. Let A be the open connected subset of $\{x \in M : e_\nu(x) > -w_i(x) + \kappa\}$ that contains x_0. Now e_ν and $-w_i$ are $n-$harmonic in A, $e_\nu = -w_i + \kappa$ in $\partial A \cap M$, and ∇e_ν and ∇w_i belong to $L^n(A)$. Since P is of zero $n-$capacity and $e_\nu + w_i - \kappa$ is bounded, we can find a sequence $\varphi_\ell \in C_0^\infty(A)$ such that $\|\nabla e_\nu + \nabla w_i - \nabla \varphi_\ell\|_{n,A} \to 0$ as $\ell \to \infty$. Therefore

$$\begin{aligned} &\int_A \langle\, |\nabla e_\nu|^{n-2}\, \nabla e_\nu, \nabla e_\nu + \nabla w_i \rangle \, dm \\ &\qquad = \int_A \langle |\nabla(-w_i)|^{n-2}\, \nabla(-w_i), \nabla e_\nu + \nabla w_i \rangle \, dm = 0 . \end{aligned}$$

We conclude that $e_\nu = -w_i + \kappa$ in A since

$$2^{1-n} \int_A |\nabla e_\nu + \nabla w_i|^n \, dm$$

$$\le \int_A \langle |\nabla e_\nu|^{n-2}\, \nabla e_\nu - |\nabla(-w_i)|^{n-2}\, \nabla(-w_i), \nabla e_\nu + \nabla w_i \rangle \, dm = 0$$

Hence no such x_0 and ε can exist. The left side inequality of (3.20) can be proved similarly.

As in the proof of Lemma 3.1 we find a subsequence of (e_ν) which converges locally uniformly in M to an n-harmonic function v_j. We can choose κ in (3.18) so large that it is independent of j. Then

$$h_1 - \kappa \le v_j \le h_1 + \kappa \tag{3.21}$$

in $U \cup C$, and

$$-h_j - \kappa \le v_j \le -h_j + \kappa \tag{3.22}$$

in $V \cup C$ for all $j = 2, \dots, q$. It follows from (3.21) and (3.22) that functions v_j, $j = 2, \dots, q$, and the constant κ satisfy the conditions of the Lemma.

4. Measure attached to an $\mathcal{A}$-harmonic function.

Let $\mathcal{A}$ be of type (2.1) and let w be an $\mathcal{A}$-harmonic function in a domain $G \subset \mathbb{R}^n$. Then $w^+ = \max\{0, w\}$ is a continuous $\mathcal{A}$-subharmonic function in G which belongs to $W^1_{n,\mathrm{loc}}(G)$. By [HK, 3.14], w^+ is an *$\mathcal{A}$-subsolution* in G, *i.e.*

$$\int_G \mathcal{A}(x, \nabla w^+) \cdot \nabla \varphi \, dm \le 0$$

for all nonnegative $\varphi \in C_0^\infty(G)$, see also [GLM, 5.17]. Hence

$$\varphi \mapsto -\int_G \mathcal{A}(x, \nabla w^+) \cdot \nabla \varphi \, dm , \quad \varphi \in C_0^\infty(G),$$

is a positive linear functional. By the Riesz representation theorem, there exists a measure μ on G such that

$$-\int_G \mathcal{A}(x, \nabla w^+) \cdot \nabla \varphi \, dm = \int_G \varphi \, d\mu , \quad \varphi \in C_0^\infty(G). \tag{4.1}$$

In the following we denote by b_i, c_i, $i = 0, 1, \dots$, positive constants which depend only on n and the structure constants of $\mathcal{A}$.

Lemma 4.2. *Let w be $\mathcal{A}$-harmonic in G, let $\bar{B}^n(x, 2r) \subset G$, and suppose $w(z) = 0$ for some $z \in B^n(x, 7r/8)$. Then*

$$\frac{1}{c_0} M(w, x, 7r/8) \le M(-w, x, r) \le c_0 M(w, x, 8r/7)$$

where

$$M(w,x,t)=\max_{\bar{B}^n(x,t)} w\,.$$

PROOF: It is enough to prove the left hand inequality. Set $g = w + M(-w,x,r)$. Then g is nonnegative in $B^n(x,r)$ and we can apply Harnack's inequality to g in the ball $\bar{B}^n(x,7r/8)$ and get for some $c_2 > 1$ that

$$\begin{aligned} M(-w,x,r) = g(z) &\geq \frac{1}{c_2}\max_{\bar{B}^n(x,7r/8)} g \\ &= \frac{1}{c_2}\big(M(w,x,7r/8)+M(-w,x,r)\big)\,, \end{aligned}$$

and the lemma follows.

The next lemma is essentially Lemma 1 in [EL] for the special case $p = n$. For completeness we include the proof which is somewhat shorter in our case.

Lemma 4.3. *Let w be $\mathcal{A}$-harmonic in G, let $\bar{B}^n(x,2r) \subset G$, and suppose $w(z) = 0$ for some $z \in B^n(x,r/4)$. Then*

$$\frac{1}{c_1}\mu(x,r/2) \leq M(w,x,r)^{n-1} \leq c_1\mu(x,2r)\,, \tag{4.4}$$

where μ is the measure defined by (4.1) *and* $\mu(x,t) = \mu\big(\bar{B}^n(x,t)\big)$.

PROOF: To prove the left hand inequality of (4.4) let $\sigma \in C_0^\infty(G)$ be such that $0 \leq \sigma \leq 1$, spt $\sigma \subset B^n(x,3r/4)$, $\sigma \mid B^n(x,r/2) = 1$, and $|\nabla\sigma| \leq 8/r$. Then by (2.4) and Hölder's inequality,

$$\begin{aligned} \mu(x,r/2) &\leq \int_G \sigma\, d\mu = -\int_G \mathcal{A}(x,\nabla w^+)\cdot\nabla\sigma\, dm \\ &\leq \beta\int_{B^n(x,3r/4)} \left|\nabla w^+\right|^{n-1}|\nabla\sigma|\, dm \\ &\leq c_3\left(\int_{B^n(x,3r/4)} \left|\nabla w^+\right|^n dm\right)^{(n-1)/n}. \end{aligned} \tag{4.5}$$

By the so called standard estimate (see [GLM, 4.2]) and Lemma 4.2,

$$\begin{aligned}\int_{B^n(x,3r/4)} |\nabla w^+|^n \, dm &\le \int_{B^n(x,3r/4)} |\nabla w|^n \, dm \\ &\le c_4 \operatorname{osc}\big(w, B^n(x,7r/8)\big)^n \\ &\le c_5 \, M(w,x,r)^n . \end{aligned} \tag{4.6}$$

The left hand inequality of (4.4) follows from (4.5) and (4.6).

For the right hand inequality of (4.4) let h be the $\mathcal{A}$-harmonic function in $B^n(x,2r)$ such that $h - w^+ \in W^1_{n,0}\big(B^n(x,2r)\big)$. Then $0 \le w^+ \le h$. By Harnack's inequality applied to h we get

$$M(w,x,r) \le M(h,x,r) \le c_2 h(y), \quad y \in B^n(x,r). \tag{4.7}$$

Hölder continuity of w ([GLM, 4.7]) gives for $0 < \varrho \le r/4$ the estimate

$$\begin{aligned} M(w,z,\varrho) &\le \operatorname{osc}\big(w, B^n(z,\varrho)\big) \\ &\le c_5 \left(\frac{\varrho}{r}\right)^\gamma \operatorname{osc}\big(w, B^n(z,r/4)\big) \\ &\le c_5 \left(\frac{\varrho}{r}\right)^\gamma \operatorname{osc}\big(w, B^n(x,r/2)\big) \\ &\le 2c_5 \left(\frac{\varrho}{r}\right)^\gamma \max\{M(w,x,r/2), M(-w,x,r/2)\} \\ &\le c_6 \left(\frac{\varrho}{r}\right)^\gamma M(w,x,r), \end{aligned} \tag{4.8}$$

where we also used Lemma 4.2 and the fact that $w(z) = 0$. Here $\gamma \in]0,1[$ is a constant which depends only on n and the structure constants of $\mathcal{A}$. Choose $\varrho \le r/4$ maximal such that

$$c_6 \left(\frac{\varrho}{r}\right)^\gamma M(w,x,r) \le \frac{1}{2} \min_{\bar{B}^n(x,r)} h . \tag{4.9}$$

Let $y \in B^n(z,\varrho)$. Then inequalities (4.7)–(4.9) imply

$$\begin{aligned} \frac{1}{2c_2} h(z) &\le \frac{1}{2} \min_{\bar{B}^n(x,r)} h \le h(y) - \frac{1}{2} \min_{\bar{B}^n(x,r)} h \\ &\le h(y) - M(w,z,\varrho) \\ &\le h(y) - w^+(y) \le c_2 h(z) . \end{aligned} \tag{4.10}$$

Let $\varphi \in W^1_{n,0}(B^n(x,2r))$ be defined by $\varphi = \min\{h - w^+, c_2 h(z)\}$ and set $F = \{y \in B^n(x,2r) : \nabla\varphi(y) \neq 0\}$. By (4.10), Poincaré's inequality, and by (2.5) we get

$$
\begin{aligned}
h(z)^n \varrho^n &\le c_7 \int_{B^n(z,\varrho)} \varphi^n \, dm \le c_7 \int_{B^n(x,2r)} \varphi^n \, dm \\
&\le c_8 r^n \int_F |\nabla\varphi|^n \, dm \\
&\le c_9 r^n \int_F \big(\mathcal{A}(x,\nabla h) - \mathcal{A}(x,\nabla w^+)\big) \cdot \nabla\varphi \, dm \qquad (4.11) \\
&\le -c_9 r^n \int_{B^n(x,2r)} \mathcal{A}(x,\nabla w^+) \cdot \nabla\varphi \, dm \\
&= c_9 r^n \int_{B^n(x,2r)} \varphi \, d\mu \le c_9 c_2 r^n h(z) \mu(x,2r) .
\end{aligned}
$$

Since ϱ/r has a positive lower bound depending only on n and the structure constants of $\mathcal{A}$, we obtain the right hand inequality of (4.4) from (4.11) and (4.7).

5. Proof of Theorem 1.3.

Let $f : \mathbb{R}^n \to M$ be a nonconstant K-quasiregular mapping. If the Riemannian metric of M is given such that $\{a_1, \dots, a_q\}$ is of positive n-capacity, it is possible to construct a positive nonconstant n-harmonic function v in $f(\mathbb{R}^n)$ using the ideas from Section 3. Then $v \circ f$ is a nonconstant positive $\mathcal{A}$-harmonic function in $\mathbb{R}^n$ which is impossible by the Harnack inequality [GLM, 4.15]. Therefore we may assume that $\{a_1, \dots, a_q\}$ is of zero n-capacity. Let $v_2, \dots, v_q$ be the n-harmonic functions in M constructed in Section 3 and satisfying properties (3.12)–(3.17). Let $u_j = v_j \circ f$, $j = 2, \dots, q$. Then each u_j is an $\mathcal{A}$-harmonic function in $\mathbb{R}^n$, and the structure constants of $\mathcal{A}$ depend only on n and K. By Harnack's inequality, u_2 can not be bounded either below or above. Hence u_2 and $-u_2$ take arbitrary large values in $\mathbb{R}^n$, and it follows from (3.17) that there exists $x_0 \in \mathbb{R}^n$ where $f(x_0) \in C$. Set $w_j = u_j - u_j(x_0)$. We write $w = w_2$ and define the measure μ by (4.1).

Let λ_0 be a sufficiently large number so that obvious estimates in inequalities (5.3) and (5.5) are true. We shall choose λ_0 more precisely after (5.7). Since $M(w,x,s) \to \infty$ as $s \to \infty$, Lemma 4.3 shows that $\mu(x,s) \to \infty$ as $s \to \infty$. Take r so large that

$$\mu(x_0, r) \ge \lambda_0 . \qquad (5.1)$$

By Lemma 4.3,

$$\mu(x_0, r) \le c_1\, M(w, x_0, 2r)^{n-1}, \tag{5.2}$$

and by the properties (3.13), (3.17), and Lemma 4.2,

$$\begin{aligned} M(w, x_0, 2r) &\le M(w_j, x_0, 2r) + 4\kappa \\ &\le 2M(w_j, x_0, 2r) \\ &\le 2c_0\, M(-w_j, x_0, 3r), \qquad j = 3, \dots, q. \end{aligned} \tag{5.3}$$

From (5.2) and (5.3) we get

$$\mu(x_0, r) \le b_1 M(-w_j, x_0, 3r)^{n-1}, \qquad j = 2, \dots, q. \tag{5.4}$$

For $j = 2, \dots, q$, let $z_j \in \bar{B}^n(x_0, 3r)$ be a point such that

$$M(-w_j, x_0, 3r) = -w_j(z_j).$$

For $0 < \varrho < r$, Hölder continuity, Lemma 4.2, (3.13), and (3.17) imply

$$\begin{aligned} \operatorname{osc}\big(w_j, B^n(z_j, \varrho)\big) &\le b_2 \left(\frac{\varrho}{r}\right)^{\gamma} \operatorname{osc}\big(w_j, B^n(x_0, 4r)\big) \\ &\le 2b_2 \left(\frac{\varrho}{r}\right)^{\gamma} \max\{M(w_j, x_0, 4r), M(-w_j, x_0, 4r)\} \\ &\le 2b_2\, c_0 \left(\frac{\varrho}{r}\right)^{\gamma} M(w_j, x_0, 5r) \\ &\le 2b_2\, c_0 \left(\frac{\varrho}{r}\right)^{\gamma} \big(M(w, x_0, 5r) + 4\kappa\big) \\ &\le 4b_2\, c_0 \left(\frac{\varrho}{r}\right)^{\gamma} M(w, x_0, 5r), \qquad j = 2, \dots, q. \end{aligned} \tag{5.5}$$

From (5.4), (5.5), and Lemma 4.3 we obtain

$$\max_{\bar{B}(z_j, \varrho)} w_j \le -\left(\frac{\mu(x_0, r)}{b_1}\right)^{1/(n-1)} + b_3 \left(\frac{\varrho}{r}\right)^{\gamma} \mu(x_0, 10r)^{1/(n-1)} \tag{5.6}$$

for $0 < \varrho < r$ and $j = 2, \dots, q$.

By covering the ball $\bar{B}^n(x_0, 10r)$ by balls of radius $r/4$ we find a constant $d_n > 1$, depending only on n, such that if $\mu(x_0, 10r) > d_n \mu(x_0, r)$, then there exists $x_1 \in B^n(x_0, 10r)$ such that $\mu(x_1, r/4) \ge \mu(x_0, r)$. We may assume that q is so large that $\varrho < r$ can be chosen by the formula $(q-1)\varrho^n = 2(4r)^n$. Suppose that $\mu(x_0, 10r) \le d_n \mu(x_0, r)$.

From (5.6) we see that there exists an integer $q_0(n,K)$ such that if $q \geq q_0(n,K)$, then

$$\max_{\bar{B}^n(z_j,\varrho)} w_j \leq -\frac{1}{2}\left(\frac{\mu(x_0,r)}{b_1}\right)^{1/(n-1)} \leq -\frac{1}{2}(\lambda_0/b_1)^{1/(n-1)}. \tag{5.7}$$

From (5.7), (3.16), and (3.17) it follows that with a sufficient large λ_0 each set $fB^n(z_j,\varrho)$ is contained in a neighborhood of a_j such that these neighborhoods are disjoint. Hence the sets $B^n(z_j,\varrho)$ are disjoint. But the balls $B^n(z_j,\varrho)$ are all in $B^n(x_0,4r)$ which is impossible by the choice of ϱ. Therefore, if $q \geq q_0(n,K)$, then $\mu(x_0,10r) > d_n\mu(x_0,r)$, and so $\mu(x_1,r/4) \geq \mu(x_0,r) \geq \lambda_0$ for some $x_1 \in \bar{B}^n(x_0,10r)$. Since the support of μ is contained in $\{x \in \mathbb{R}^n : w(x) = 0\}$, we also have $w(z) = 0$ for some $z \in \bar{B}^n(x_1,r/4)$. We can then repeat the above by starting from the ball $B^n(x_1,r/2)$ instead of $B^n(x_0,r)$. We get a sequence $B^n(x_k,r/2^k)$ of balls converging to a point y with $\mu(x_k,r/2^k) \geq \lambda_0$. By Lemma 4.3 this means that w would be discontinuous at y, which is a contradiction. The theorem is proved.

References.

[BI] Bojarski, B., Iwaniec, T. Analytic foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$. *Ann. Acad. Sci. Fenn. Ser.* A-I *Math.* **8** (1983), 257-324.

[EL] Eremenko, A., Lewis, J. L. Uniform limits of certain $\mathcal{A}$-harmonic functions with applications to quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser.* A-I *Math.* **16** (1991), 361-375.

[GLM] Granlund, S., Lindqvist, P., Martio, O. Conformally invariant variational integrals. *Trans. Amer. Math. Soc.* **277** (1983), 43-73.

[HK] Heinonen, J., Kilpeläinen, T. $\mathcal{A}$-superharmonic functions and supersolutions of degenerate elliptic equations. *Ark. Mat.* **26** (1988), 87-105.

[H1] Holopainen, I. Nonlinear potential theory and quasiregular mappings on Riemannian manifolds. *Ann. Acad. Sci. Fenn. Ser.* A-I *Math. Diss.* **74** (1990), 1-45.

[H2] Holopainen, I. Positive solutions of quasilinear elliptic equations on Riemannian manifolds. *Proc. London Math. Soc.* (To appear).

[HR] Holopainen, I., Rickman S. Classification of Riemannian manifolds in nonlinear potential theory. (To appear).

[M] Maz'ya, V. G. On the continuity at a boundary point of solutions of quasi-linear elliptic equations. *Vestnik Leningrad Univ.* **3** (1976), 225-242. (English translation).

[Re] Reshetnyak, Yu. G. *Space mappings with bounded distortion.* - Translations of Mathematical Monographs, Amer. Math. Soc., **73**, 1989.

[R1] Rickman, S. On the number of omitted values of entire quasiregular mappings. *J. Analyse Math.* **37** (1980), 100-117.

[R2] Rickman, S. A defect relation for quasimeromorphic mappings. *Ann. of Math.* **114** (1981), 165-191.

[R3] Rickman, S. Value distribution of quasiregular mappings. - Proc. Value Distribution Theory, Joensuu 1981, *Lecture Notes in Math.*, **981**, 220-245 (1983).

[R4] Rickman, S. Quasiregular mappings and metrics on the n-sphere with punctures. *Comment. Math. Helv.* **59** (1984), 134-148.

[R5] Rickman, S. The analogue of Picard's theorem for quasiregular mappings in dimension three. *Acta Math.* **154** (1985), 195–242.

[R6] Rickman, S. Defect relation and its realization for quasiregular mappings. (In preparation).

[R7] Rickman, S. Quasiregular mappings. (In preparation).

Recibido: 13 de junio de 1991.

Ilkka Holopainen* and Seppo Rickman
Department of Mathematics
University of Helsinki
Hallituskatu 15, SF–00100 Helsinki
FINLAND

* This research was supported in part by grants from the Academy of Finland and the U. S. National Science Foundation (NSF DMS 9003438).