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Uniformly Perfect Sets, Green's Function, and Fundamental Domains

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Introduction.

Suppose $\Omega \subset \bar{\mathbb{C}}$ is a domain conformally equivalent to $\mathbb{D}/\Gamma$ for some Fuchsian group Γ acting on $\mathbb{D} = \{z : |z| < 1\}$. Let $z_0 \in \Omega$ and let $G(z, z_0)$ be Green's function for Ω with pole at z_0. Define $S(z_0)$ to be the set of critical points of $G(z, z_0)$, that is

$$S(z_0) = \{z_j : \nabla G(z_j, z_0) = 0\}.$$

In Theorem 1 we characterize those domains Ω for which there exists a constant $c = c(\varepsilon)$ independent of z_0, such that

$$\sum_{z_j \in S(z_0)} G(z_j, z_0)^{1+\varepsilon} < c(\varepsilon) .$$

We will prove that this holds if and only if $\partial\Omega$ is a uniformly perfect set.

A set $K \subset \mathbb{C}$ is said to be uniformly perfect if there exists a constant $c > 0$ such that

$$K \cap \{z : c\, r \leq |z - a| \leq r\} \neq \emptyset \quad \text{for} \quad a \in K,\ 0 < r < \operatorname{diam}(K).$$

An equivalent definition is the following: There exists a constant $\eta > 0$ such that

$$\operatorname{cap}(K \cap \{z : |z - a| \leq r\}) \geq \eta\, r\ , \ \text{for } a \in K,\ 0 < r < \operatorname{diam}(K),$$

where cap denotes logarithmic capacity, see [9].

We will also need another characterization. Let Γ be a Fuchsian group such that $\mathbb{D}/\Gamma$ is conformally equivalent to Ω, then $\partial\Omega$ is uniformly perfect if and only if there exists $\rho > 0$ such that each disk in $\mathbb{D}$ of hyperbolic radius ρ contains no two Γ-equivalent points, [9].

Examples of uniformly perfect sets are the standard Cantor sets. A Cantor set can be constructed as follows: Fix $\alpha, 0 < \alpha < 1/2$, and take from the unit square four "corner" squares of side length α. Then take from each square four "corner" squares of side length α^2. By continuing this process, at the n-th stage we get 4^n squares Q_j^n of side length α^n. The limit set $E = E(\alpha)$ is a Cantor set.

Consider now the domain whose boundary is the Cantor set E. It is shown in [1] that to each square Q_j^n can be associated a critical point z_j^n such that $G(z_j^n, \infty) \simeq \omega(\infty, Q_j^n \cap E, \bar{\mathbb{C}} \backslash E)$ and then that

$$\sum_{z_j \in S(\infty)} G(z_j, \infty) = \infty .$$

However, by Theorem 1, $\sum_{z_j \in S(\infty)} G(z_j, \infty)^{1+\varepsilon} < \infty$ for all $\varepsilon > 0$.

The main motivation for these results is Widom's Theorem, which states that

$$\sum_{z_j \in S(z_0)} G(z_j, z_0) < \infty \text{ if and only if } H_\lambda^\infty(\mathbb{D}) \neq \{0\} \text{ for all } \lambda \in \Gamma^*,$$

where H_λ^∞ denotes the space of bounded analytic functions in $\mathbb{D}$ satisfying $f \circ \gamma = \lambda(\gamma) f$ for all $\gamma \in \Gamma$, [12].

Also in the spirit of these results, there is a theorem by J.L. Fernández involving uniformly perfect sets. If $\Omega \cong \mathbb{D}/\Gamma$ has Geeen's function then the Poincaré series $\sum_{\gamma \in \Gamma} 1 - |\gamma(0)|$ converges. What he proves in his paper [2] is that if $\partial\Omega$ is uniformly perfect, then for some $\delta > 0$,

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(0)|)^{1-\delta} < \infty .$$

In Theorem 2 we give a characterization of domains whose boundary is a uniformly perfect set in terms of the geometry of the fundamental domain. We will prove that $\partial\Omega$ is uniformly perfect if and only if there exists a fundamental domain whose boundary is a quasicircle. In fact, there is a fundamental domain $\mathcal{F}$ such that $\partial\mathcal{F}$ is chord arc and $\partial\mathcal{F} \cap \partial\mathbb{D}$ is also uniformly perfect.

We recall here the definition of quasicircles and chord arc curves. C is a K-quasicircle if it satisfies the three point condition $|z_3 - z_1| < c|z_2 - z_1|$ for some $c = c(K) > 0$ and any three points on C with z_3 on the arc of smaller diameter between z_1 and z_2, [6]. Equivalent definitions can be given in terms of quasiconformal mappings.

Suppose now γ is a locally rectifiable Jordan curve. For any two points $z_1, z_2 \in \gamma$, let $\ell(\gamma(z_1, z_2))$ denote the length of the arc in γ from z_1 to z_2 of smaller diameter. The curve γ is chord arc if there exists a constant $M \geq 0$ such that

$$\ell(\gamma(z_1, z_2)) \leq M|z_1 - z_2| \ , \quad \text{for all } z_1, z_2 \in \gamma.$$

To prove Theorem 2 we construct a simply connected domain $\Omega_0 \subset \Omega$ satisfying certain metric properties. The preimage of Ω_0 under the covering map will be the required fundamental domain.

Pommerenke proves in [7] that if $\sum_{z_j \in S(z_0)} G(z_j, z_0) < \infty$ then

$$\text{mes}(\partial\mathcal{F}_{z_0} \cap \partial\mathbb{D}) > 0 \ ,$$

where $\mathcal{F}_{z_0}$ denotes the normal fundamental domain of Ω with base point at z_0. Another theorem by Pommerenke involving the "size" of $\partial\mathcal{F} \cap \partial\mathbb{D}$ for a particular domain is the following: If Ω is a domain for which Green's function exists and $\mathcal{F}$ denotes the Green's fundamental domain, then $\text{cap}(\partial\mathcal{F} \cap \partial\mathbb{D}) > 0$, *cf.* [8].

Finally, we would like to point out the connection between Theorem 1 and 2 with the conjecture stated by P. Jones in [5].

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1. Uniformly Perfect Sets and Green's Function.

Theorem 1. *Suppose Ω is a domain on the Riemann sphere $\bar{\mathbb{C}}$. Let $z_0 \in \Omega$ and let $G(z, z_0)$ be Green's function for Ω with pole at z_0. Denote by $\beta_{z_0}(t)$ the first Betti number of the domain $\Omega_t = \{z \in \Omega : G(z, z_0) > t\}$. Then $\partial\Omega$ is uniformly perfect if and only if there exists $\varepsilon > 0$ such that*

$$\int_0^\infty t^\varepsilon \beta_{z_0}(t)dt < c \ , \quad \textit{for all } z_0 \in \Omega \ , \tag{1.1}$$

and for some constant $c = c(\eta) > 0$ depending only on the uniformly perfect constant η. Moreover, whenever this holds, we have

$$(1.2) \qquad \sum_{\nabla G(z_j,z_0)=0} F(G(z_j,z_0)) < c(\eta, A) \quad \text{for all } z_0 \in \Omega ,$$

for all increasing function F such that $F(0) = 0$ and

$$\int_0 \frac{F(t)}{t^2} \leq A < +\infty .$$

REMARK. Note that if Ω is a domain whose boundary $\partial\Omega$ is a uniformly perfect set, then $\text{cap}(\partial\Omega) > 0$ and therefore there exists Green's function for Ω. Moreover, it can be shown by applying Wiener's criterion that Ω is regular for the Dirichlet problem. Therefore, as a consequence of Theorem 1 we get that if (1.1) holds for some $\varepsilon > 0$, then Ω is regular and for all $\varepsilon > 0$,

$$(1+\varepsilon)\int_0^\infty t^\varepsilon \beta_{z_0}(t)dt = \sum_{\nabla G(z_j,z_0)=0} G(z_j,z_0)^{1+\varepsilon} < c(\varepsilon) \ , \ z_0 \in \Omega.$$

PROOF OF THEOREM 1: We start by proving sufficiency. Suppose $\Omega \subset \bar{\mathbb{C}}$ is a domain whose boundary $\partial\Omega$ is a uniformly perfect set, that is

$$(1.3) \ \text{cap}(\partial\Omega \cap \{z : |z-\xi| < r\}) \geq \eta r \ , \ \text{for } \xi \in \partial\Omega, \ 0 < r < \text{diam}(\partial\Omega),$$

for some constant $\eta > 0$.

First, we want to make an observation which will give us the uniform bound in (1.2). If φ is a Moebius transformation and K is a set satisfying (1.3), then $\varphi(K)$ satisfies (1.3) for some constant η^* depending only on η (see Proposition 1 in [9]). Thus, it is enough to prove the theorem in the case $\partial\Omega$ has diameter 1 and the pole z_0 for Green's function is ∞. We will define $G(z) = G(z, \infty)$.

We now need an estimate on harmonic measure. Let w_0 be a point in Ω such that $\delta(w_0) = \text{dist}\{w_0, \partial\Omega\} < 1$ and let ξ_0 be the closest point on $\partial\Omega$ to w_0. Consider the ball $B(w_0, r_0)$ of radius $r_0 = 5\delta(w_0)$ centered at w_0. We will prove

$$(1.4) \qquad \omega(\infty, B(w_0,r_0) \cap \partial\Omega, \Omega) \geq c\, G(w_0)$$

for some $c = c(\eta)$.

Since (1.4) is invariant under conformal mappings we can assume $\delta(w_0) = 1/3$. By (1.3), there exists a probability measure μ supported on

$$E = \partial\Omega \cap \{\xi : |\xi - \xi_0| \leq \frac{1}{3}\}$$

such that the logarithmic potential

$$\mathcal{U}^\mu(z) = \int_E \log \frac{1}{|z-\xi|} d\mu(\xi)$$

satisfies

$$\mathcal{U}^\mu(z) \leq c_0(\eta) \;\; \text{for all } z \in \bar{\mathbb{C}}\,,$$
$$\mathcal{U}^\mu(z) \leq 0 \;\; \text{if } z \notin B(w_0, r_0)\,.$$

(Note that if $z \notin B(w_0, r_0)$, then $|z - \xi| \geq 1$, for all $\xi \in E$). Thus, by the maximum principle,

$$\omega(w_0, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq \frac{1}{c_0(\eta)} \mathcal{U}^\mu(w_0) \geq \frac{\log 3/2}{c_0(\eta)},$$

and by Harnack's inequality,

$$\omega(z, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq c_1(\eta) \;\; \text{if } |z - w_0| = \frac{1}{6}\,. \tag{1.5}$$

We consider now the function

$$F(z) = \log \frac{1}{|z - w_0|} - \mathcal{U}^\mu(z)\,.$$

$F(z)$ is harmonic in $\Omega \backslash \{w_0\}$, bounded at ∞ and $|F(z)| \leq c_2(\eta)$ on $\partial\Omega$. Writing

$$G(z, w_0) = F(z) + (G(z, w_0) - F(z))$$

and applying the maximum principle to the harmonic function in Ω, $G(z, w_0) - F(z)$, we get

$$G(z, w_0) \leq c_2(\eta) \;\; \text{if } |z - w_0| = \frac{1}{6}\,.$$

If we combine this result with (1.5) and we apply again the maximum principle to the functions $\omega(z, B(w_0, r_0) \cap \partial\Omega, \Omega)$ and $G(z, w_0)$ in the domain $\Omega \backslash \{z : |z - w_0| \leq 1/6\}$ we have

$$\omega(\infty, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq c(\eta)\, G(\infty, w_0).$$

Since $G(\infty, w_0) = G(w_0, \infty)$, we obtain (1.4) as required.

Define now S to be the set of critical points of $G(z)$. Since S lies in the convex hull of $\partial\Omega$, *cf.* [11], which in this case is contained in $|z| \leq 1$, it is easy to see that if $z_j \in S$ then $G(z_j) \leq t_0$ for some constant $t_0 = t_0(\eta)$.

Next, consider the set of level curves $\{G(z) = t\}$, for all $t < t_0$. For each $t < t_0$ fixed, let N_t be the number of connected components of $\{G(z) = t\}$ and let C_j^t be the j-th component. Denote by K_j^t the subset of $\partial\Omega$ contained inside the curve C_j^t. To simplify the notation we set

$$\omega(E) = \omega(\infty, E, \Omega) \quad \text{for all } E \subset \partial\Omega\,.$$

We will prove that there exists a constant $c = c(\eta) > 0$ such that

$$\omega(K_j^t) > c\,t\,, \quad j = 1, \ldots, N_t, \tag{1.6}$$

but we will first show why sufficiency follows from this result.

Since $\partial\Omega$ is uniformly perfect and in particular regular we have

$$\partial\Omega = \bigcup_{j=1}^{N_t} K_j^t\,, \quad \text{for every } t < t_0\,.$$

Thus, by (1.6)

$$1 = \omega(\partial\Omega) = \sum_{j=1}^{N_t} \omega(K_j^t) > N_t\, c\, t$$

and therefore $N_t < 1/c\,t$ for all $t < t_0$.

Now let $G_t(z)$ be Green's function for the domain $\Omega_t = \{z : G(z) > t\}$ with pole at ∞. The number of critical points of $G_t(z)$ is $N_t - 1$, *cf.* [3], which is exactly the first Betti number of Ω_t, $\beta(t)$. On the other hand, note that

$$G_t(z) = G(z) - t,$$

therefore the number of critical points of $G(z)$ contained in Ω_t is equal to the number of critical points of $G_t(z)$, that is $N_t - 1$. So, for all $t < t_0$,

$$\#\{z_j : z_j \in S,\ G(z_j) > t\} < \frac{1}{ct} .$$

Suppose F is an increasing function such that $F(0) = 0$ and

$$\int_0^{t_0} \frac{F(x)}{x^2} dx \le A < +\infty.$$

Then, if $t_0 = 2^{k_0}$,

$$\begin{aligned}\sum_{z_j \in S} F(G(z_j)) &= \sum_{k=-k_0}^{\infty} \sum_{\substack{z_j \in S \\ 2^{-k-1} \le G(z_j) \le 2^{-k}}} F(G(z_j)) \\ &\le \frac{1}{c} \sum_{k=-k_0}^{\infty} 2^{k+1} F(2^{-k}) \le C ,\end{aligned}$$

where C is a constant depending on η and A. In particular, taking $F(x) = x^{1+\varepsilon}$ we get

$$\sum_{z_j \in S} G(z_j)^{1+\varepsilon} \le c(\eta, \varepsilon).$$

Also, as we pointed out before

$$\#\{z_j : z_j \in S, G(z_j) > t\} = \beta(t).$$

Integration by parts gives

$$c(\eta, \varepsilon) \ge \sum_{z_j \in S} G(z_j)^{1+\varepsilon} = (1+\varepsilon) \int_0^{\infty} t^{\varepsilon} \beta(t) dt, \quad \varepsilon > 0 .$$

We now turn to the proof of (1.6). Let $t < t_0$ and let $w_0 \in \Omega$ be a point inside the curve C_j^t such that if $d = \operatorname{dist}\{C_j^t, K_j^t\}$,

$$\operatorname{dist}\{w_0, C_j^t\} = 5 \operatorname{dist}\{w_0, K_j^t\} = \frac{5d}{6}.$$

Consider the ball $B(w_0, r_0)$ of radius $r_0 = 5d/6$ centered at w_0. Then $B(w_0, r_0)$ is inside the curve C_j^t and by (1.4)

$$\omega(B(w_0, r_0) \cap \partial\Omega) \geq c\, G(w_0).$$

Thus

$$\omega(K_j^t) \geq \omega(B(w_0, r_0) \cap \partial\Omega) \geq c\, G(w_0) \geq c' t. \tag{1.7}$$

To prove the last inequality we consider the point $w_1 \in \Omega$ which is at distance $d/3$ from C_j^t and $2d/3$ from K_j^t. Let $r_1 = 2d/3$ and let D be the subset of the ball $B(w_1, r_1)$ contained inside the curve C_j^t. Denote by $\tilde{D}$ the connected component of D which contains w_1. Then

$$\omega(w_1, C_j^t \cap \tilde{D}, \tilde{D}) \geq c_0$$

for some universal constant $c_0 > 0$.

By the maximum principle applied to the function $\omega(z, C_j^t \cap \tilde{D}, \tilde{D})$ and $G(z)$ on the domain $\tilde{D}$, we get $G(w_1) \geq c_0\, t$. Hence, by Harnack's principle $G(w_0) > ct$, for some universal constant c. Thus (1.7) holds and this ends the proof of the sufficiency.

To prove necessity suppose $\int_0^\infty t^\varepsilon \beta_{z_0}(t) < c$ for some $c > 0$ and for all $z_0 \in \Omega$. Then, there exists $M > 0$ such that $\beta_{z_0}(t) = 0$ if $t > M$, *i.e.* the domain

$$\Omega_{M,z_0} = \{z \in \Omega : G(z, z_0) > M\}$$

is simply connected. This is equivalent to saying that there is a ball of radius $r = r(M)$ which does not contain equivalent points, (to see this, write $G(\pi(\xi), z_0) = \log(1/|B(\xi)|)$ where $B(\xi)$ is the Blaschke product with zeros at $\{\gamma(0)\}_{\gamma\in\Gamma}$ and $\pi : \mathbb{D} \mapsto \Omega$ is the universal covering map with with $\pi(0) = z_0$, [7]). Therefore, Ω is a domain whose boundary is a uniformly perfect set.

2. Uniformly Perfect Sets and Fundamental Domains.

Theorem 2. *Suppose $\Omega \cong \mathbb{D}/\Gamma$ is a domain on the Riemann sphere $\bar{\mathbb{C}}$. Then $\partial\Omega$ is a uniformly perfect set if and only if there exists a fundamental domain whose boundary is a quasicircle.*

Furthermore, in this case, we can construct a fundamental domain $\mathcal{F}$, such that

(i) *$\partial\mathcal{F}$ is chord arc,*
(ii) *$\partial\mathcal{F} \cap \mathbb{T}$ is a uniformly perfect set.*

PROOF OF THEOREM 2: To prove sufficiency we construct a simply connected domain $\Omega_0 \subset \Omega$ satisfying certain metric properties. To be more precise:

Lemma 1. *Suppose $\Omega \subset \bar{\mathbb{C}}$ is an unbounded domain whose boundary K is a uniformly perfect set. Then, there are crosscuts $\{L_j\}$ on Ω such that $\Omega_0 = \Omega \backslash \{L_j\}$ is a simply connected domain and*

(2.1) *L_j is a chord arc curve, for every j,*
(2.2) $\text{dist}\{L_j, L_k\} \geq c_1 \min\{\text{diam}(L_j), \text{diam}(L_k)\}$, *for $j \neq k$,*
(2.3) *For any $L \in \{L_j\}$, if z_1, z_2 denote the endpoints of L and $z \in L$, we have*

$$\text{dist}\{z, K\} \geq c_2 \min\{|z - z_1|, |z - z_2|\}.$$

The chord arc constant M as well as c_1 and c_2 only depend on the uniformly perfect constant of K, η.

PROOF OF LEMMA 1: Without loss of generality we can assume $K \subset [0,1] \times [0,1]$. Let $k_0 \in \mathbb{Z}^+$ be a large number which will be fixed later. For each $n \in \mathbb{Z}^+$, let $\langle Q \rangle_n$ be the set of closed dyadic cubes of side length $\ell_n = 2^{-k_0 n}$. (A dyadic cube $Q \subset [0,1] \times [0,1]$ is a cube of the form $Q = [2^{-j}k, 2^{-j}(k+1)] \times [2^{-i}\ell, 2^{-i}(\ell+1)]$ for some $j, k, i, \ell \in \mathbb{Z}^+$).

Define

$$K_n = \{Q \in \langle Q \rangle_n : Q \cap K \neq \emptyset\}.$$

Note that $K_n \supset K_{n+1}$ and $K = \cap_1^\infty K_n$.

To prove the lemma we will use an induction argument. At each stage n, we will construct a family of curves $\langle L \rangle_n$ satisfying $\langle L \rangle_n \subset \langle L \rangle_{n+1}$ and such that

(2.4) $K_n \cup \langle L \rangle_n$ is a connected set. The endpoints of the curves in $\langle L \rangle_n$ are either vertices or middle points of edges of cubes in K_n,

(2.5) Any $L \in \langle L \rangle_n$ is a chord arc curve with chord arc constant depending on k_0, (2.6) If $L_j, L_k \in \langle L \rangle_n$,

$$\mathrm{dist}\{L_j, L_k\} \geq c_1 \ \min\{\mathrm{diam}(L_j), \mathrm{diam}(L_k)\},$$

(2.7) If z_1, z_2 are the endpoints of $L \in \langle L \rangle_n$, for all $z \in L$, we have

$$\mathrm{dist}\{z, K_n\} \geq c_2 \ \min\{|z - z_1|, |z - z_2|\}.$$

The constants c_1, c_2 only depend on k_0.

At the limit we will get the required crosscuts $\{L_j\}$.

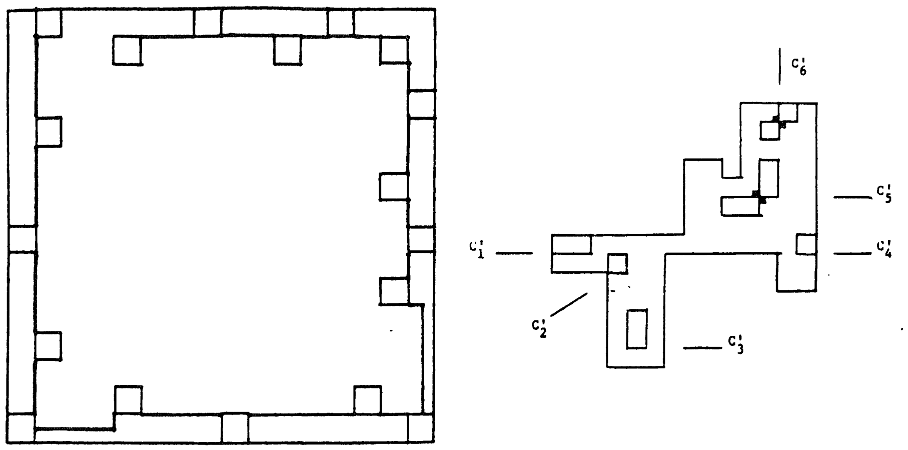

Fig. 1.

We start the proof by giving a lemma. We will consider simply connected domains P which are unions of dyadic cubes of the same side length and such that ∂P is a Jordan curve.

Lemma 2. *Fix $n_0 \in \mathbb{Z}^+$. Let P be a domain as above composed by no more than n_0 cubes of side length ℓ. Choose any collection of such cubes and call its connected components $\{C_j\}$. Then, we can construct chord arc curves $\{L_j\}$, contained in the interior of P, such that*

(i) $\{C_j\} \cup \{L_j\}$ *is a connected set and the endpoints of each curve in* $\{L_j\}$ *are either vertices or middle points of edges of cubes in* $\{C_j\}$,

(ii) *There are constants* $c_i > 0,\ i = 1, 2, 3$ *such that*

$$c_1\ell \leq \operatorname{diam}(L_j) \leq c_2\ell\ ,\ \textit{for all}\ j$$
$$\operatorname{dist}(L_j, L_k) \geq c_3\ell\ ,\ \textit{for}\ j \neq k.$$

The chord arc constant, c_1, c_2 *and* c_3 *only depend on* n_0.

PROOF OF LEMMA 2: Suppose P is the unit cube. Choose $k \in \mathbb{Z}^+$ so that $2^{-2k} \leq n_0$. In particular, P is the union of 2^{-2k} cubes of side length 2^{-k}. If all the components $\{C_j\}$ are cubes in the grid of size 2^{-k} either touching ∂P or at distance 2^{-k} from ∂P, the statement in the lemma is obvious (Figure 1). The idea is to reduce the general case to this one by using the fact that ∂P is a chord arc curve with chord arc constant only depending on n_0.

Normalize so that $\operatorname{diam}(P) = 1$, and suppose first that P minus the collection of cubes we choose inside P is still a connected set. Let $C_1, \dots, C_N$ be the connected components of this set of cubes. In fact, we can assume they are simply connected. In general, ∂C_j may have self intersections, so, what we do is to replace C_j by C_j', where C_j' is obtained by adding two small cubes of side length $\ell/8$ at the vertices where ∂C_j intersects itself, in case there are any. If not, let $C_j' = C_j$. Then $\partial C_j'$ is also a chord arc curve with chord arc constant only depending on n_0.

Let f be a bilipschitz map in $\mathbb{C}$ which sends P onto the unit cube Q_0, see [10]. Since we have normalized so that diam $P = 1$, the bilipschitz constant of f only depends on the chord arc constant of ∂P. The images $f(\partial C_j')\ ,\ j = 1, \dots, N$ are chord arc and for $j \neq k$,

$$\operatorname{dist}\{f(C_j'), f(C_k')\} \geq c\ \operatorname{dist}\{C_j', C_k'\} \geq c\,\ell$$

for some c depending on the bilipschitz constant. Now draw disjoint neighborhoods U_j around each $f(C_j')$ with $\operatorname{dist}\{U_j, U_k\} \geq c\,\ell/2,\ j \neq k$ (Figure 2). Take a grid of small size in Q_0. Again, there exist bilipschitz maps $f_j : Q_0 \mapsto Q_0,\ j = 1, \dots, N$, such that

(1) $f_j(f(C_j)) \subseteq S_j$ where S_j is a cube on the grid in Q_0 ,
(2) $f_j(z) = z$, for every $z \in Q_0 \backslash U_j$.

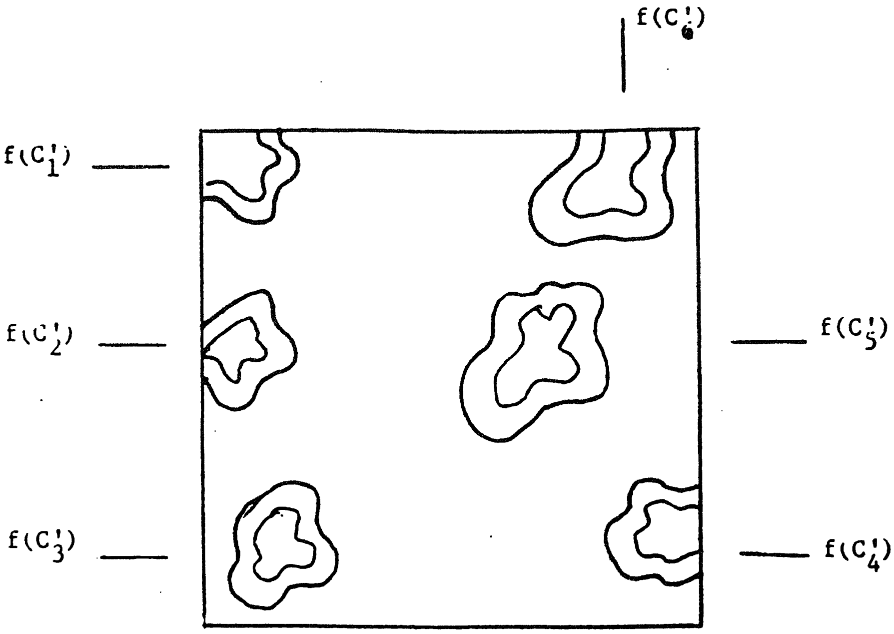

Fig. 2.

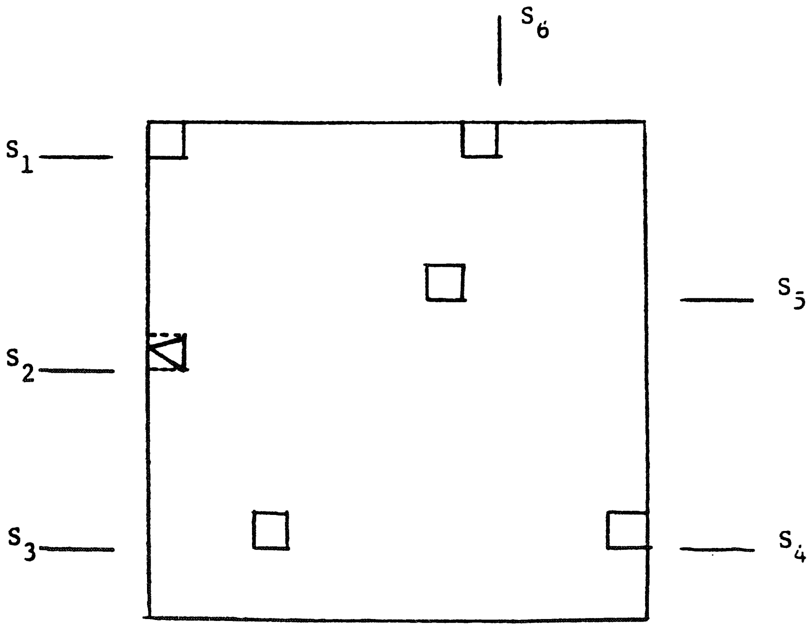

Fig. 3.

The composition $F = f_1 \circ \cdots \circ f_N \circ f$ is a bilipschitz map in $\mathbb{C}$ which sends P onto the unit cube Q_0 and each component C_j into a cube S_j of the grid (Figure 3). If the size of the grid is small enough we can find bilipschitz maps $\{g_j\}$ which will push the cubes S_j's towards the boundary reducing the general case to the one considered at the beginning of the proof. Let $G = g_N \circ \cdots \circ g_1 \circ f_N \circ \cdots \circ f_1 \circ f$. It is important to note that the size of the grid depends on $N = N(n_0)$, the chord arc constant of each $\partial(f(C_j'))$ depends as well on n_0 and therefore the bilipschitz constant of G will only depend on n_0.

Next, let $\{\gamma_j\}$ be the set of lines contained in Q_0 which connect the disjoint cubes S_j, $j = 1, \ldots, N$. Choose $\{\gamma_j\}$ so that their endpoints are either vertices or middle points of edges of the S_j's. Call these endpoints $\{v_j\}$. Without loss of generality we can assume that $F^{-1}(v_j)$ is either a vertex or a middle point of an edge of a cube of some C_j. Let $L_j = F^{-1}(\gamma_j)$. Then $\{L_j\}$ are chord arc curves satisfying (i) and (ii) in Lemma 2.

Suppose now that P minus the collection of cubes we choose inside P is a disconnected set. Then we decompose P into polygons $\{P_k\}$ and apply the argument above to the connected components $\{C_j\}$ contained inside each P_k. By adding some extra lines connecting the different P_k's, we get the required $\{L_j\}$.

We now turn to the proof of Lemma 1. The situation is the following: we approach K by a sequence K_n, $n \geq 0$, $K_n \supset K_{n+1}$ and $K = \cap_0^\infty K_n$. Each K_n is in general a disconnected union of dyadic cubes of side length ℓ_n with $\ell_{n+1}/\ell_n = 2^{-k_0}$. At each stage n, we will construct a family of chord arc curves $\langle L \rangle_n$ which connect the disjoint components of K_n and satisfy (2.4)-(2.7). By perturbing the curves slightly in $\langle L \rangle_{n-1}$ we will get that each curve in $\langle L \rangle_{n-1}$ is contained in a curve in $\langle L \rangle_n$. The limit will be the required $\{L_j\}$.

Clearly, if $n = 0$, K_0 is the unit cube Q_0, and $\langle L \rangle_0 = \emptyset$.

For $n = 1$, take the grid of size 2^{-k_0} in Q_0. K_1 is the set of cubes in the grid which contain points of K. By applying Lemma 2 to $P = Q_0$ and the collection of cubes K_1, we get the set of curves $\langle L \rangle_1$ which connect the disjoint components of K_1 and satisfy (i) and (ii) with constants only depending on k_0. Thus, $\langle L \rangle_1$ obviously satisfy (2.4)-(2.6). It is also clear by the construction of $\langle L \rangle_1$ that (2.7) holds too. We can choose $\langle L \rangle_1$ to be minimal in the following sense: $K_1 \cup \langle L \rangle_1$ is connected, but for every $L \in \langle L \rangle_1, K_1 \cup \langle L_1 \rangle \backslash L)$ is disconnected.

Now, given $\langle L \rangle_{n-1}$ satisfying (2.4)-(2.7), we will construct $\langle L \rangle_n$.

The set K_n is obtained by partitioning K_{n-1} into smaller cubes of

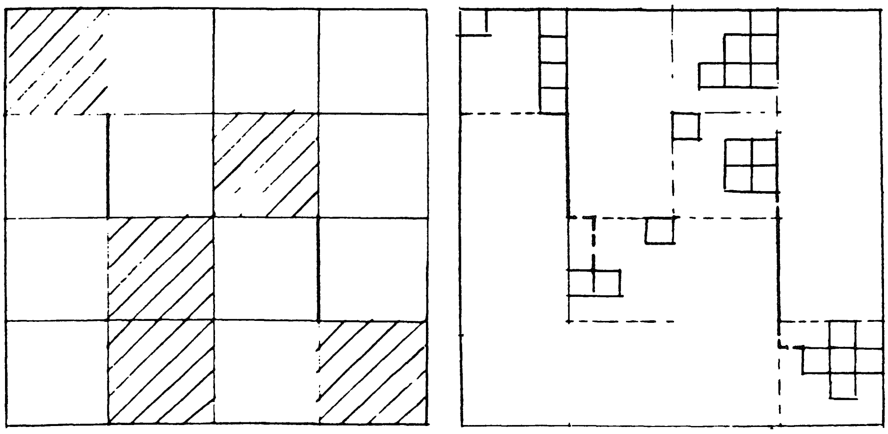

Fig. 4.

side length $2^{-k_0 n}$ and by choosing those which contain points of K. The curves in $\langle L\rangle_{n-1}$ connect the different components of K_{n-1}, thus we just have to extend these curves so they hit K_n and then apply Lemma 2 inside each component of K_{n-1} to connect the subset of K_n contained in it (Figure 4). The main problem is that as n grows, the number of cubes inside the components may get arbitrarily big and we would loose control of the constants. To avoid this, we will decompose each component of K_{n-1} into "disjoint" pieces and then apply the lemma to each of them. So, let C be a connected component of K_{n-1}. It is very easy to get a family of cubes $\{\tilde{Q}_j\}$ and connected sets $\{P_j\}$ in C such that $C = \cup P_j, P_j \subset 3\tilde{Q}_j$ and $(P_j)^\circ \cap (P_k)^\circ = \phi, j \neq k$ ($(P_j)^\circ$ denotes the interior of P_j).

For reasons which will be clear later we would like each P_j to be simply connected and satisfy:

(2.8) ∂P_j has no self-intersections,

(2.9) The concentric cube $(1+1/8)\tilde{Q}_j$ is contained in $(P_j)^\circ$.

To get such P_j's we only have to modify slightly the ones in the decomposition above.

If (2.8) fails for some P_j, we add two small cubes of side length $\ell_{n-1}/4$ at each vertix where ∂P_j intersects itself. If these cubes were contained in some $P_k \neq P_j$, we remove them from P_k.

If (2.9) fails, then we just add an ε-neighborhood around $\tilde{Q}_j \subset P_j$ with $\varepsilon = \ell_{n-1}/4$. Note that since $(2\tilde{Q}_j \cap C) \subset P$, the ε-neighborhood does not intersect any other $P_k \neq P_j$.

Thus, by adding some cubes we can assume that the P_j's in the decomposition above are simply connected and satisfy (2.8) and (2.9). So, we will replace any curve L in $\langle L\rangle_{n-1}$ which hits C by a curve $\tilde{L}$ which travels along L and stops when L hits $\cup P_j$ for the first time. For convenience of notation, we will set L equal to $\tilde{L}$ and we will continue referring to them as curves in $\langle L\rangle_{n-1}$.

Next, fix P_j and look at all k's such that $\partial P_k \cap \partial P_j \neq \emptyset$. For each such k, choose a point $z_{jk} \in \partial P_j \cap \partial P_k$ which is not an endpoint of any curve in $\langle L\rangle_{n-1}$. We define the set V_j as follows:

A point $z \in \partial P_j$ is in V_j if $z = z_{jk}$ for some z_{jk} as above or z is the endpoint of a curve $L \subset \langle L\rangle_{n-1}$ which hits ∂P_j. In this last case if the point z is also in ∂P_k for some $k \neq j$, we include it either in V_j or V_k. The whole point of this is to extend L, whether we do it inside P_j or P_k it does not matter. The points z_{jk}'s will be the link between the different polygons P_j's. Note that all the points in V_j can be chosen so that they are vertices or middle points of edges of cubes.

We will drop the "j's" in the notation until we need them again.

Let n_0 be the number of points in the set $V \subset \partial P$ and consider the cube $\tilde{Q}$ attached to P. We have constructed P so that P° contains the concentric cube $Q' = (1+1/8)\tilde{Q}$, *cf.* (2.9). The following remark is an immediate consequence of the definition of uniformly perfect sets.

REMARK. Suppose K is a (η) uniformly perfect set. Consider a cube Q with $Q \cap K \neq \emptyset$. Let ℓ be the side length of Q and let Q' denote the concentric cube of side length $(1+1/8)\ell$. Then, given any $N \in \mathbb{Z}^+$, there exists $k = k(\eta, N) \in \mathbb{Z}^+$ such that the grid of size $2^{-k}\ell$ in Q' contains N cubes $Q_1, \ldots, Q_N$ satisfying

(i) $Q_j \cap K \neq \emptyset$, $j = 1, \ldots, N$,

(ii) $\operatorname{dist}\{Q_j, Q_k\} \geq 10\, 2^{-k}\ell$, $j \neq k$.

Thus, there is $k_0 = k_0(\eta)$ such that the grid of size $\ell_n = 2^{-k_0}\ell_{n-1}$ inside $Q' = (1+1/8)\tilde{Q}$ contains at least n_0 cubes satisfying (i) and (ii). Call F to this family of cubes and note that they are cubes of the n-th generation K_n. Note also that n_0 is bounded by a universal constant, so let us fix k_0 so that (i) and (ii) holds for this constant. Next, we will build curves $\{L_k\}, k = 1, \ldots, n_0$ connecting each point $z \in V$ to one of the cubes in F. These curves will be polygonal lines inside P with

$$\text{(2.10)} \qquad \begin{aligned} &\text{(i)} \quad \operatorname{diam}(L_k) \leq c_0\, \ell_n \text{ for some universal constant } c_0 > 0, \\ &\text{(ii)} \quad \operatorname{dist}\{L_j, L_k\} \geq 10\, \ell_n\,. \end{aligned}$$

To do this, choose z_1, Q_1 such that

$$\operatorname{dist}\{z_1, Q_1\} \leq \operatorname{dist}\{z_j, Q_k\} \quad \text{for all } z_j \in V, Q_k \in F.$$

By distance here we mean the length of the shortest polygonal line which joins z_1 to Q_1 and it is contained in the grid of size ℓ_n in P. Let L_1 be such a line.

Draw now a tube T_1 running parallel to $L_1 \cup Q_1$ of radius $9\,\ell_n$. Note that T_1 does not contain any cube in F except Q_1. Moreover, since $F \subset P^\circ$ if the size of the grid is small enough (and we can assume it is), T_1 does not disconnect P, so we can apply the same construction to $P \backslash T_1$ which has $(n_0 - 1)$ points on its boundary and $(n_0 - 1)$ cubes of the family F in its interior. Choose z_2, Q_2 as before, but now the distance is measured with respect to the grid in $P \backslash T_1$. Get L_2 and draw the tube T_2. By continuing this process, we get polygonal lines $\{L_k\}, k = 1, \ldots, n_0$ such that each L_k joins V_k to Q_k inside P and (2.10) holds.

We point out again that the size of the grid which allows us to perform this last construction only depends on n_0. Now, replace L_k by Γ_k, where Γ_k is obtained by following L_k until it first hits a cube of K_n. Obviously Γ_k also satisfies (2.10). For convenience of notation call L_k to Γ_k. Next, we repeat this construction inside each component of K_{n-1}. The polygonal lines we get either extend the curves in $\langle L \rangle_{n-1}$ so that the sets $\{(K_n \cap C) : C \text{ is a component of } K_{n-1}\}$ remain connected or they connect the sets $\{P_j\}$ inside each component C. Now all that is left is to join the components of K_n inside each of the P_j's. To do this, fix P and consider the lines $\{L_k\}$ contained in P. For each k, we draw an auxiliary line $\tilde{L}_k$ running parallel to L_k at distance ℓ_n and with endpoints on the same cube as L_k. Call the thin tube in between T_k. Apply now Lemma 2 to $P' = P \backslash \cup T_k$ and to the collection of cubes $K_n \cap P'$. Get new lines $\{L'_j\}$ travelling inside P' satisfying (i)-(iii) in Lemma 2. Note that these lines are at distance at least ℓ_n from each of the L_k's. Finally, remove the extra lines $\{\tilde{L}_k\}$, and perform the same construction inside each polygon $P \subset C$ for all connected components C of K_{n-1}. The lines $\langle L\rangle_{n-1} \cup \{L_k\} \cup \{L'_k\}$ will be the lines of the n-th generation $\langle L\rangle_n$. Again, we can choose $\langle L\rangle_n$ to be minimal in the sense described before.

Before returning to the proof of Theorem 2 we need the following characterization of domains with uniformly perfect boundary. Let Ω be a domain on $\bar{\mathbb{C}}$ conformally equivalent to $\mathbb{D}/\Gamma$ and let π be the universal covering map. The Poincaré metric on Ω is defined by

$$\lambda(w)|dw| = \frac{|dz|}{1-|z|^2}\,, \qquad w = \pi(z).$$

Pommerenke shows in [9] that $\partial\Omega$ is uniformly perfect if and only if there exists a constant $c > 0$ such that

$$\frac{c}{d(w)} \le \lambda(w) \le \frac{1}{d(w)} \tag{2.11}$$

where $d(w) = \operatorname{dist}\{w, \partial\Omega\}$.

To prove Theorem 2 we can assume again $\operatorname{diam}(\partial\Omega) = 1$.

Consider the simply connected domain Ω_0 and the crosscuts $\{L_j\}$ constructed in Lemma 1. We start by showing that any such crosscut L is chord arc in the Poincaré metric on Ω.

Let z_1, z_2 be the endpoints of L and let $z(s)$ denote the arc length parametrization of L. Choose the point $\tilde{z} \in L$ such that

$$\ell(L(z_1, \tilde{z})) = \ell(L(\tilde{z}, z_2))\,.$$

Define L_1 to be the arc of L from z_1 to $\tilde{z}$. Let w_1, w_2 be any two points on L_1, then

$$w_i = z(s_i) \qquad \text{where} \qquad s_i = \ell(L(z_1, w_i)), \qquad i = 1, 2.$$

By (2.1) L is chord arc in the Euclidean metric, with chord arc constant $M = M(\eta)$. Thus, we have for all $z \in L_1$

$$|z - z_2| \ge \frac{1}{M}\, \ell(L(z, z_2)) \ge \frac{1}{M}\, \ell(L(z_1, z)) \ge \frac{1}{M}|z_1 - z|.$$

On the other hand, by (2.3)

$$\operatorname{dist}\{z, K\} \geq c_2 \min\{|z - z_1|, |z - z_2|\}, \quad \text{for } z \in L$$

for some $c_2 = c_2(\eta)$. Thus

$$\operatorname{dist}\{z, K\} \geq c\ \ell(L(z_1, z))$$

where $c = c(\eta)$.

By convenience of notation we will use the letter $c = c(\eta)$ to denote different constants, all of them depending only on η.

We now estimate the length of the arc of L from w_1 to w_2 in the Poincaré metric. By (2.11) and the inequalities above, we obtain

$$\begin{aligned}
\rho(L(w_1, w_2)) &\leq \int_{L(w_1,w_2)} \frac{|dz|}{\operatorname{dist}\{z, K\}} \\
&\leq c \int_{L(w_1,w_2)} \frac{|dz|}{\ell(L(z_1, z))} \\
&= c \int_{s_1}^{s_2} \frac{ds}{s} \\
&= c\ \log \frac{s_2}{s_1} \\
&\leq c\ \log\left(1 + \frac{\ell(L(w_1, w_2))}{|z_1 - w_1|}\right) \\
&\leq c\ \log\left(1 + \frac{|w_1 - w_2|}{|z_1 - w_1|}\right).
\end{aligned}$$

Next, let σ denote the geodesic from w_1 to w_2 in the Poincaré metric. Parametrizing σ by the arc length, $z = z(s)$, that is $s = \ell(\sigma(w_1, z))$, we get

$$\begin{aligned}
\rho(w_1, w_2) &\geq c \int_{\sigma(w_1,w_2)} \frac{|dz|}{\operatorname{dist}\{z, K\}} \geq c \int_{\sigma(w_1,w_2)} \frac{|dz|}{|z - z_1|} \\
&\geq c \int_0^{\ell(\sigma(w_1,w_2))} \frac{ds}{|z_1 - w_1| + s} \geq c\ \log\left(1 + \frac{|w_1 - w_2|}{|z_1 - w_1|}\right).
\end{aligned}$$

Therefore, there exists a constant $C = C(\eta)$ such that

$$\rho(L(w_1, w_2)) \leq C\rho(w_1, w_2).$$

Similar computations show that the same holds for any two points w_1, w_2 on L. Hence, L is chord arc in the Poincaré metric with chord arc constant $C = C(\eta)$.

Now, let $\pi : \mathbb{D} \mapsto \Omega$ be the universal covering map of Ω with $\pi(0) = \infty$. We can define a single valued analytic branch of π^{-1} in Ω_0. Set $\mathcal{F} = \pi^{-1}(\Omega_0)$. Then $\mathcal{F}$ is a fundamental domain. We will prove that $\partial\mathcal{F}$ is chord arc.

Since each crosscut L_j is a Jordan curve, the conformal map π^{-1} extends homeomorphically to L_j, for all j. So, if L_j^+ and L_j^- denote the two sides of L_j, we can define the set of curves $\{\gamma_j\}$ by

$$\{\gamma_j\} = \{\pi^{-1}(L_j^+)\} \cup \{\pi^{-1}(L_j^-)\}.$$

Every $\gamma \in \{\gamma_j\}$ is a curve on $\mathbb{D}$ with endpoints on $\partial\mathbb{D}$. Note that $\partial\mathcal{F}$ is the union of a set on $\partial\mathbb{D}$ (may be empty) and the collection of $\{\gamma_j\}$.

To prove $\partial\mathcal{F}$ is chord arc we will show that every γ_j is chord arc and then by using the metric properties of Ω_0 we will prove that the whole boundary $\partial\mathcal{F}$ is chord arc.

Before continuing we need to introduce some notation.

Let $\xi_1, \xi_2 \in \partial\mathbb{D}$. Consider the Möbius transformation $\Phi : \mathbb{R}^2_+ \mapsto \mathbb{D}$ such that $\Phi(0) = \xi_1$ and $\Phi(\infty) = \xi_2$. For any θ, $0 < \theta < \pi/2$, define

$$\Gamma_\theta = \Phi\{z : \theta \leq \arg z \leq \pi - \theta\}.$$

We will refer to Γ_θ as a cone in $\mathbb{D}$ with endpoints ξ_1, ξ_2.

It is a well known result that any curve $\gamma \subset \mathbb{D}$ with endpoints $\xi_1, \xi_2 \subset \partial\mathbb{D}$ which is chord arc in the hyperbolic metric with chord arc constant M satisfies

(i) $\gamma \subset \Gamma_\theta$ for some $\theta = \theta(M)$,
(ii) γ is chord arc in the Euclidean metric with chord arc constant $M' = M'(M)$ [4].

As we proved before, each L_j is chord arc in the hyperbolic metric with chord arc constant depending on η, thus the same holds for γ_j. By the result above we deduce that each γ_j is contained in a cone Γ_θ for some $\theta = \theta(\eta)$, and that γ_j is chord arc in the Euclidean metric with chord arc constant $M = M(\eta)$.

Next, we will prove

$$\rho(\gamma_j, \gamma_k) \geq c(\eta) \qquad \text{for every } j, k, \qquad j \neq k$$

for some constant $c(\eta) > 0$.

If $\gamma_j = \pi^{-1}(L_j^+)$ and $\gamma_k = \pi^{-1}(L_j^-)$, then this is just an easy consequence of the fact that Ω does not contain closed geodesics of arbitrary small length in the hyperbolic metric.

If $\gamma_j = \pi^{-1}(L_j)$ and $\gamma_k = \pi^{-1}(L_k)$, all we have to show is that $\rho(L_j, L_k) \geq c(\eta)$. For, let $w_1 \in L_j$ and $w_2 \in L_k$. Denote by σ the geodesic in the hyperbolic geometry from w_1 to w_2. Suppose that $\operatorname{diam}(L_j) \leq \operatorname{diam}(L_k)$ and that z_1 is the endpoint of L_j closest to w_1. Parametrizing σ by the arc length $z = z(s)$ we get

$$\begin{aligned} \rho(w_1, w_2) &\geq c(\eta) \int_\sigma \frac{|dz|}{\operatorname{dist}\{z, K\}} \\ &\geq c(\eta) \int_\sigma \frac{|dz|}{|z - z_1|} \\ &\geq c(\eta) \int_0^{\ell(\sigma)} \frac{ds}{s + |z_1 - w_1|} \\ &= c(\eta) \log\left(1 + \frac{\ell(\sigma)}{|z_1 - w_1|}\right). \end{aligned}$$

By (2.2),

$$\ell(\sigma) \geq \operatorname{dist}\{L_j, L_k\} \geq c_1(\eta) \operatorname{diam}(L_j)$$

and since L_j is chord arc

$$|z_1 - w_1| \leq \ell(L_j) \leq M(\eta) \operatorname{diam}(L_j).$$

Hence $\rho(L_j, L_k) \geq c'(\eta)$ as required.

Summarizing, for all j, γ_j satisfies

$$\text{(2.12)} \quad \begin{array}{lll} \text{(i)} & \gamma_j \subset \Gamma_\theta, & \text{for some} \quad 0 < \theta < \pi/2, \\ \text{(ii)} & \gamma_j \ \text{is chord arc with chord arc constant} \ M, & \\ \text{(iii)} & \rho(\gamma_j, \gamma_k) \geq c, & \text{for } j, k \quad j \neq k, \end{array}$$

with constants only depending on η.

It is now very easy to see that (i)-(iii) imply $\partial\mathcal{F}$ is chord arc.

To prove (ii) in Theorem 2 we need to introduce some more notation.

As we remarked before $\partial\mathcal{F} = \{\gamma_j\} \cup E$ where E is a set on $\partial\mathbb{D} = \mathbb{T}$. For every γ_j, let e_j be the arc of $\mathbb{T}$ joining the endpoints of γ_j (Figure 5).

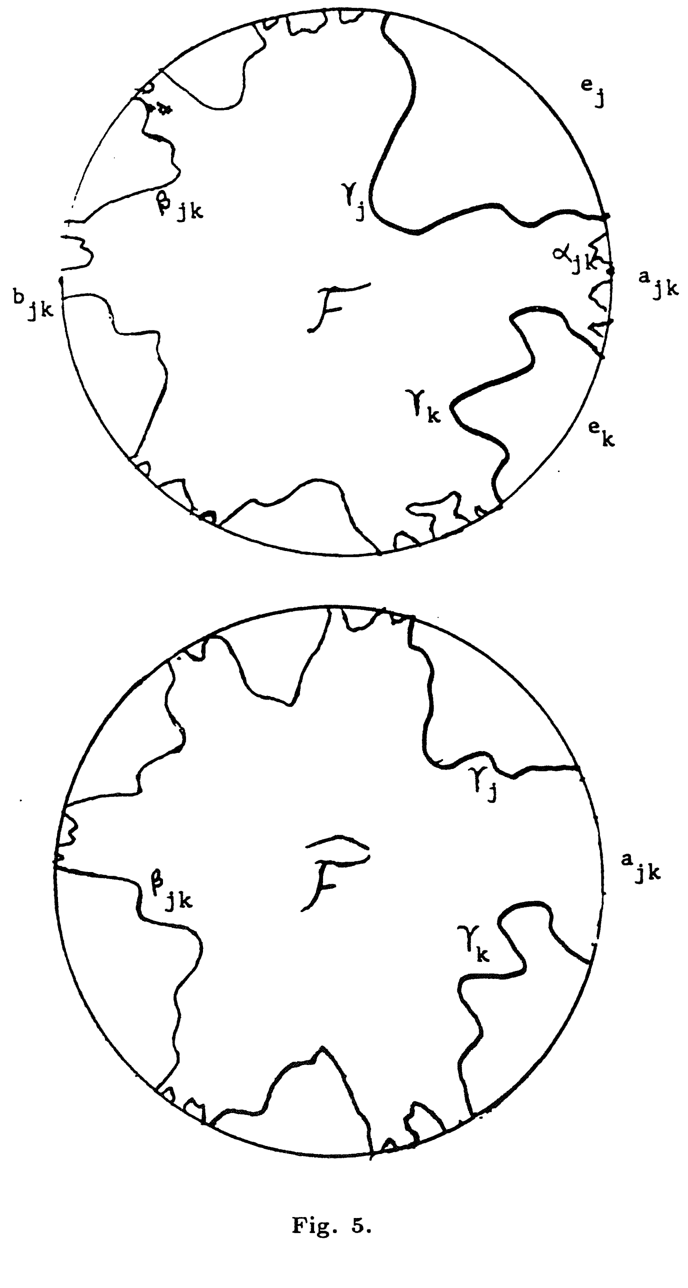

Fig. 5.

Also, for every γ_j, γ_k we denote by α_{jk} the arc of $\partial\mathcal{F}$ of smaller diameter between γ_j and γ_k and by β_{jk} the other arc. The arcs of $\mathbb{T}$ joining the endpoints of α_{jk} and β_{jk} will be denoted by a_{jk} and b_{jk} respectively. In particular,

$$\mathbb{T} = e_j \cup a_{jk} \cup e_k \cup b_{jk}.$$

The goal is to prove

$$(2.13) \qquad \ell(a_{jk}) \geq c(\eta)\min\{\ell(e_j), \ell(e_k)\}, \qquad \text{for all } j,k$$

and

$$(2.14) \qquad \frac{1}{2\pi}\ell(e_j) \leq 1 - \sigma(\eta), \qquad \text{for all } j,$$

for some constants $c(\eta) > 0$, $\sigma(\eta) > 0$.

If we think of $\partial\mathcal{F} \cap \mathbb{T}$ as the set obtained by removing the arcs $\{e_j\}$ successively (as we do with a Cantor set) it is clear that (2.13) and (2.14) imply $\partial\mathcal{F} \cap \mathbb{T}$ is a uniformly perfect set.

The idea is to prove

(i) There exists a point $\xi_{jk} \in \mathcal{F}$ such that

$$(2.15) \qquad \omega(\xi_{jk}, \alpha_{jk}, \mathcal{F}) \geq c_0, \qquad \text{for } j,k$$

$$(2.16) \qquad \omega(\xi_{jk}, \beta_{jk}, \mathcal{F}) \geq c_0, \qquad \text{for } j,k$$

for some $c_0 = c_0(\eta) > 0$.

(ii) There exists $\sigma_0 = \sigma_0(\eta) > 0$ such that

$$(2.17) \qquad \omega(0, \gamma_j, \mathcal{F}) \leq 1 - \sigma_0(\eta), \qquad \text{for all } j.$$

The fact that every γ_j is a chord arc curve contained in a "cone", (2.12), will allow us to show

$$(2.18) \qquad \omega(\xi_{jk}, a_{jk}, \mathbb{D}) \geq c'(\eta)$$

$$(2.19) \qquad \omega(\xi_{jk}, b_{jk}, \mathbb{D}) \geq c'(\eta)$$

$$(2.20) \qquad \omega(0, e_j, \mathbb{D}) \leq 1 - \sigma'(\eta)$$

for all j,k, and as an easy consequence we will get (2.13) and (2.14).

We start by proving (2.15). For, we will show that the equivalent estimate holds in the domain Ω_0. Suppose first that $\gamma_j = \pi^{-1}(L^+)$ and $\gamma_k = \pi^{-1}(L^-)$ for some crosscut L. Denote by A_1 and A_2 the two connected components of $\partial\Omega_0\backslash L$. Note that by the construction of Ω_0 given in Lemma 1, we have

$$\operatorname{diam}(A_i) \geq \delta \operatorname{diam}(L), \qquad i = 1,2$$

for some $\delta = \delta(\eta) > 0$. Also note that (2.15) and (2.16) are equivalent to

$$\begin{aligned} \omega(z, A_1, \Omega_0) &\geq c_0(\eta) \\ \omega(z, A_2, \Omega_0) &\geq c_0(\eta) \end{aligned} \tag{2.21}$$

for some $z \in \Omega_0$.

Without loss of generality we can assume $\operatorname{diam}(L) = 1$. To prove (2.21) we will use an extremal length argument.

Suppose L is the unit interval. Then (2.2) and (2.3) impy that the set $\Omega_0\backslash L$ is outside the "diamond" D_0,

$$D_0 = \{z : |\arg z| \leq \varepsilon\} \cap \{z : |\arg(z-1)| \leq \varepsilon\}$$

where $\varepsilon = \varepsilon(\eta)$.

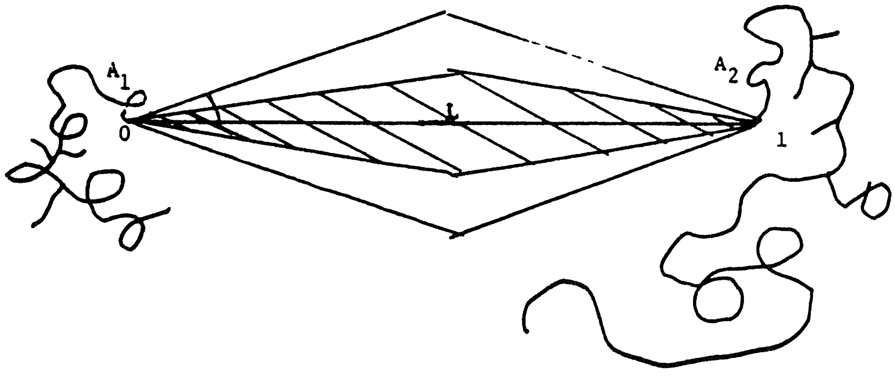

Fig. 6.

Denote by D the concentric "diamond" correspondent to the angle $\varepsilon/2$ (Figure 6). We will show

(i) For all $z \in \partial D$,

$$\omega(z, A_1 \cup A_2, \Omega) \geq c_1(\eta)$$

for some $c_1 = c_1(\eta) > 0$.

(ii) There exists a point $z_1 \in \partial D$ and a constant $c_2(\eta) > 0$ such that

$$\omega(z_1, A_1, \Omega_0) \geq c_2(\eta)$$

and similarly, there exists $z_2 \in \partial D$ such that

$$\omega(z_2, A_2, \Omega_0) \geq c_2(\eta).$$

Thus, by continuity there exists a point $z \in \partial D$, for which (2.21) holds.

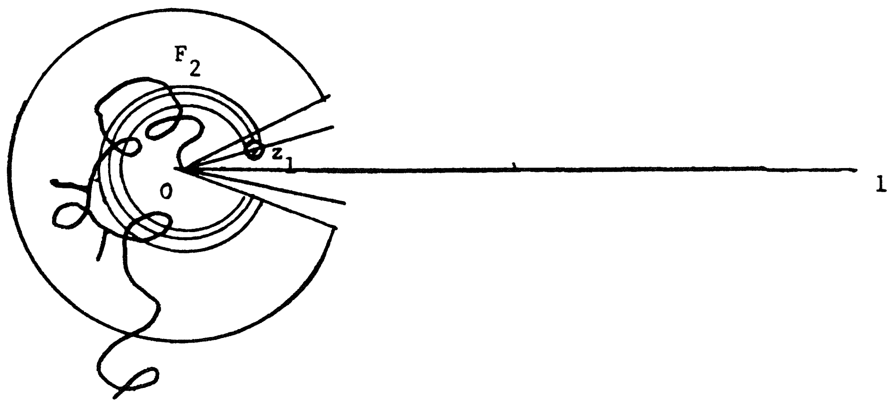

Fig. 7.

We start by proving (ii). Suppose $d = \text{dist}\{A_1, A_2\}$. In particular, A_2 is outside the ball $|z| < d$ (Figure 7). Let $r = \min\{d, \delta\}$, and let $z_1 = r/2\ e^{i\varepsilon/2}$. Denote by K the ball of radius $y_1/2$ ($z_1 = x_1 + iy_1$) centered at z_1. Consider the family of curves F_1 which join K to A_1 in Ω_0, and denote by $\lambda(F_1)$ the extremal length of F_1. Then by Beurling's Theorem

$$\omega(z_1, A_1, \Omega_0) \geq c\, e^{-\pi\lambda(F_1)} \tag{2.22}$$

for some universal constant c.

Let ξ_1, ξ_2 be the two points where K intersects ∂D, and define the family of curves $F_2 = \{\gamma_r : |\xi_1| < r < |\xi_2|\}$ where $\gamma_r(\theta) = re^{i\theta}$, $\varepsilon/2 \leq \theta \leq 2\pi - \varepsilon/2$. By the choice of r, each curve F_2 contains an arc which is an element of F_1. Therefore, if $\rho(z) \geq 0$ is a measurable function admissible for $F_1, \rho \in A(F_1)$, *i.e.* $\int_\gamma \rho ds \geq 1$ for all locally rectifiable curves $\gamma \in F_1$ (ds denotes the arc length on γ), then

$$\int_{\gamma_r} \rho \, ds \geq 1 \qquad \text{for all } \gamma_r \in F_2 \, .$$

By Hölder's inequality

$$\int_{\gamma_r} \rho^2 ds \geq \frac{1}{\ell(\gamma_r)}$$

where $\ell(\gamma_r)$ denotes the length of γ_r. Thus

$$\begin{aligned} \iint \rho^2 dx \, dy &\geq \int_{|\xi_1|}^{|\xi_2|} \int_{\gamma_r} \rho^2 ds \, dr \\ &\geq \int_{|\xi_1|}^{|\xi_2|} \frac{1}{\ell(\gamma_r)} dr \geq \int_{|\xi_1|}^{|\xi_2|} \frac{1}{2\pi r} dr = \frac{1}{2\pi} \log \frac{|\xi_2|}{|\xi_1|} . \end{aligned}$$

Since $|\xi_2|/|\xi_1| = c$ where $c = c(\varepsilon)$, we get $\lambda(F_1) \leq c'(\varepsilon)$. Hence, by (2.22) we obtain

$$\omega(z_1, A_1, \Omega_0) \geq c \, e^{-\pi c'} = c_2(\varepsilon) \, .$$

The same argument gives the correspondent result for z_1. So (ii) is proved.

Note now that it is enough to show (i) for any point $z_0 = x_0 + iy_0 \in \partial D$ with $0 < x_0 < 1/2, y_0 > 0$. For, let $\tilde{A}_1 = A_1 \cap \{z : |z| \leq \delta\}$. Then by the maximum principle we get

$$\omega(z_0, A_1 \cup A_2, \Omega_0) \geq \omega(z_0, \tilde{A}_1, \bar{\mathbb{C}} \backslash \{L \cup \tilde{A}_1\}).$$

Thus if $\Omega_1 = \bar{\mathbb{C}} \backslash \{L \cup \tilde{A}_1\}$, all we have to prove is

$$\omega(z_0, \tilde{A}_1, \Omega_1) \geq c_1(\eta). \tag{2.23}$$

For, let $\xi_0 = \delta/2 \, e^{i\varepsilon_0/2}$ and let B be the ball of radius $v_0/2$, ($\xi_0 = u_0 + iv_0$) centered at ξ_0 (Figure 8).

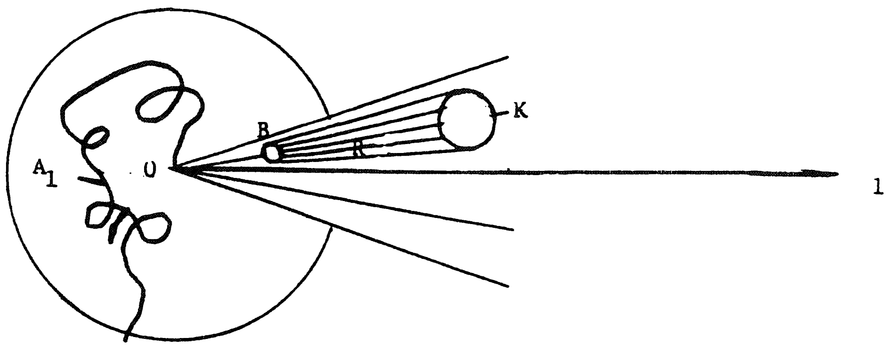

Fig. 8.

By Harnack's inequality,

$$\omega(\xi, \tilde{A}_1, \Omega_1) \geq k\,\omega(\xi_0, \tilde{A}_1, \Omega_1)$$

for all $\xi \in B$ and some universal constant k. Thus, by the maximum principle, for all $z \in \Omega_1 \backslash B$, in particular for z_0, we get

$$\omega(z, \tilde{A}_1, \Omega_1) \geq k\,\omega(z, \partial B, \Omega_1 \backslash B)\,\omega(\xi_0, \tilde{A}_1, \Omega_1). \tag{2.24}$$

Using a similar argument to the one in the proof of (ii) we get

$$\omega(\xi_0, \tilde{A}_1, \Omega_1) \geq c(\varepsilon). \tag{2.25}$$

Next, we show

$$\omega(z_0, \partial B, \Omega_1 \backslash B) \geq c(\eta). \tag{2.26}$$

For, let K be the ball of radius $y_0/2$ centered at z_0 and let F_1 be the family of curves joining K to ∂B in the domain $\Omega_1 \backslash B$.

Consider now the region R shown in Figure 8 and the rays $\{\gamma_\theta\}$ from ∂B to K with $\theta_1 \leq \theta \leq \theta_2$

$$\gamma_\theta(r) = \{re^{i\theta} : r_1 < r < r_2\}$$

where $r_2 \simeq |z_0|$ and $r_1 \simeq |\xi_0| = \delta/2$. Since $\gamma_\theta \in F$, for all $\rho \in A(F)$ we have

$$\int_{\gamma_\theta} \rho \, ds \geq 1 .$$

By Hölder's inequality, we get

$$\int_{r_1}^{r_2} \rho^2(re^{i\theta}) r \, dr \geq \frac{1}{\log r_2/r_1}.$$

Thus

$$\iint \rho^2 dx \, dy \geq \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \rho^2 r \, dr \, d\theta$$
$$\geq (\theta_2 - \theta_1) \frac{1}{\log r_2/r_1} \geq c(\varepsilon, \delta) .$$

Since $\varepsilon = \varepsilon(\eta)$ and $\delta = \delta(\eta)$ we get (2.26) which with (2.24) and (2.25) give us the required estimate (2.23). This concludes the proof of (i).

So, we have proved (2.21) in the case the line L is the unit interval.

For the general case, consider the bilipschitz map f on $\mathbb{C}$ which sends the chord arc curve L of diameter 1 to the interval $[0, 1]$. Then, for all $z, w \in \mathbb{C}$,

$$\frac{1}{c(\eta)} \leq \frac{f(z) - f(w)}{|z - w|} \leq c(\eta)$$

where c only depends on the chord arc constant of L, $M = M(\eta)$.

Thus, the images of A_1 and A_2 under f are connected sets with $\operatorname{diam}(f(A_i)) \geq \delta(\eta)$, $i = 1, 2$. Furthermore, $f(A_2)$, $f(A_2)$ will stay outside a "diamond" of angle $\varepsilon = \varepsilon(\eta)$. Hence the results above apply to $f(A_1)$ and $f(A_2)$ in the domain $f(\Omega_0)$. This, together with the fact that if F is a family of curves and f is an M-bilipschitz map in $\mathbb{C}$ then

$$\frac{1}{k(M)} \lambda(F) \leq \lambda(f(F)) \leq k(M) \lambda(F) ,$$

give us (i) and (ii) in the general case.

As we remarked before, (i) and (ii) imply (2.21). So we have proved (2.15) and (2,16) whenever $\gamma_j = \pi^{-1}(L^+)$ and $\gamma_k = \pi^{-1}(L^-)$.

Suppose now $\gamma_j = \pi^{-1}(L_j^\pm)$ and $\gamma_k = \pi^{-1}(L_k^\pm)$ (whether it is $+$ or $-$ does not matter, the proof is practically the same). Assume

$\operatorname{diam}(\gamma_j) \geq \operatorname{diam}(\gamma_k)$ and $\operatorname{diam}(\gamma_k) = 1$. Denote by A_1, A_2, A_3 the three components of $\partial\Omega_0 \backslash (\gamma_j \cup \gamma_k)$, and suppose A_2 is the one which connects γ_j to γ_k. Again, we have

$$\operatorname{diam}(A_i) \geq \delta(\eta)\,, \qquad i = 1, 2, 3\,.$$

A similar argumet using extremal length and bilipschitz maps can be given to prove (2.15) and (2.16) in this case.

Next, we will prove (2.17). For, let $\gamma_j = \pi^{-1}(L^+)$, then (2.17) is equivalent to

$$\omega(\infty, L^+, \Omega_0) \leq 1 - \sigma_0(\eta).$$

By the maximum principle $\omega(\infty, L^+, \Omega_0) \leq \omega(\infty, L^+, \bar{\mathbb{C}} \backslash L)$. Suppose first $L = [0, 1]$. Standard estimates show that for all z with $|z| = 2$,

$$\omega(z, L^-, \bar{\mathbb{C}} \backslash [0, 1]) \geq c_0,$$

for some universal constant $c_0 > 0$. Thus, if $|z| = 2$, $B(z)$ is the ball of radius $1/2$ centered at z, and F_z is the family of curves joining $B(z)$ to L^-, we get $\lambda(F_z) \leq c_1$ for some universal constant $c_1 > 0$.

We now turn to the general case. Assume $\operatorname{diam}(L) = 1$, and let f be a bilipschitz map in $\mathbb{C}$ which sends L to the unit interval. Consider the curve $\Gamma = f^{-1}(|z| = 2)$. For each $\xi \in \Gamma$, set $K(\xi) = f^{-1}(B(z))$ and consider the family of curves joining $K(\xi)$ to L^-, that is $F_\xi = f^{-1}(F_z)$. Then

$$\omega(\xi, L^-, \bar{\mathbb{C}} \backslash L) \geq c\, e^{-\pi\lambda(F_\xi)} \geq c\, e^{-\pi c_2 \lambda(F_z)} \geq \sigma_0$$

where the constants c_2 and σ_0 depend on the chord arc constant of L, $M = M(\eta)$.

The next goal is to prove (2.18)-(2.20). It just says that (2.15)-(2.17) hold if we remove the arcs $\{\gamma_j\}$. This will be an easy consequence of the following result:

Let γ be a curve from 0 to 1 in $\mathbb{R}^2_+$, which is contained in a "cone" Γ_θ. The boundary of Γ_θ is the union of two arcs, C_1 and C_2. Denote by C the biggest arc of the boundary of the "cone" $\Gamma_\theta/2$ and by D the domain bounded by C and the interval $[0, 1]$. Then, for all $z \in \gamma$,

$$\omega(z, I, D) \geq c_0 \tag{2.27}$$

for some constant c_0 only depending on θ.

It is easy to prove that for all $z \in C_2$

$$\omega(z, I, D) \geq c_0\,.$$

Thus, by the maximum principle, (2.27) holds.

To prove (2.18), consider the domain $\hat{\mathcal{F}} \subset \mathbb{D}$ bounded by $\gamma_j, \gamma_k, \beta_{jk}$ and a_{jk} . In particular $\hat{\mathcal{F}} \supset \mathcal{F}$, and therefore any harmonic function in $\hat{\mathcal{F}}$ is also harmonic in $\mathcal{F}$. Let $u(z) = \omega(z, a_{jk}, \hat{\mathcal{F}})$, then

$$\omega(\xi_{jk}, a_{jk}, \hat{\mathcal{F}}) = \int_{\partial\mathcal{F}} u(\xi)\; d\omega_{\xi_{jk}}(\xi).$$

By (2.15) and (2.27), we get

$$\int_{\partial\mathcal{F}} u(\xi) d\omega_{\xi_{jk}}(\xi) \geq c(\eta)\,.$$

Thus, by the maximum principle

$$\omega(\xi_{jk}, a_{jk}, \mathbb{D}) \geq c(\eta)\,.$$

Similar arguments can be given to prove (2.19) and (2.20).

Note now that (2.14) is an immediate consequence of (2.20).

To prove (2.13), let us assume that it does not hold. Then it is easy to show that (2.18) implies

$$\omega(\xi_{jk}, b_{jk}, \mathbb{D}) \underset{j,k}{\mapsto} 0$$

which contradicts (2.19). This concludes the proof of (2.13) and (2.14) and therefore the proof of sufficiency.

To prove necessity, first we need to prove a lemma on quasidisks, that is, domains bounded by quasicircles.

Lemma 3. *Suppose $\mathcal{R} \subset \mathbb{R}^2_+ = \{y \geq 0\}$ is a K-quasidisk. Let z_0 be any point in $\bar{\mathcal{R}}\backslash\mathbb{R}$ and let B be the hyperbolic ball of radius* 1 *centered at z_0. Then $B \cap \mathcal{R}$ contains a ball of hyperbolic radius $\varepsilon > 0$, for some ε only depending on K. ($\bar{\mathcal{R}}$ denotes the closure of $\mathcal{R}$).*

PROOF OF LEMMA 3: Suppose first $z_0 \in \partial\mathcal{R}$. Consider the K-quasiconformal map $f : \mathbb{R}^2_+ \mapsto \bar{\mathcal{R}}$ with $f(0) = z_0$. Assume $z_0 = iy_0, y_0 > 0$. Let $\tilde{B}$ be the biggest Euclidean ball centered at z_0 contained in B, and let ρ_0 be its Euclidean radius. Then, it is easy to see that $\rho_0 = (1 - 1/e)\, y_0$.

Let $B_1 \subset f^{-1}(\tilde{B})$ be the ball centered at 0 and radius $r_1 = \min_{z\in\partial\tilde{B}} |f^{-1}(z)|$. Apply the distortion theorem in [6] to $f(B_1)$ to get a ball contained in $f(B_1)$ of radius

$$\rho_1 = \min_{z\in\partial B_1} |f(z) - z_0| \geq c(K)\,\rho_0 \,.$$

Now, let $D \subset B_1$ be the ball of radius $r_1/2$ centered at $\xi = i\, r_1/2$. Then $f(D)$ is contained in $\bar{\mathcal{R}}$ and it is tangential to z_0 and to $f(\partial B_1)$. Thus, by the distortion theorem, there is a ball $B_2 \subset f(D)$ of radius ρ_2 with

$$\rho_2 \geq c(K)\ \mathrm{diam}(f(D)) \geq c_1(K)\,\rho_0 \,.$$

Let B_ε be the ball concentric to B_2 and radius $\rho_2/2$. Then $\bar{B}_\varepsilon \subset \mathcal{R} \cap B$ and since

$$\frac{\rho_2}{2} \geq \frac{c_1(K)}{2}\rho_0 = c_2(K)\,y_0 \,,$$

we get that the hyperbolic radius of $B_\varepsilon, \varepsilon$, satisfies $\varepsilon \geq c_3(K)$.

A similar argument can be given in the case z_0 lies in the interior of $\mathcal{R}$.

We now turn to the proof of the theorem.

Let $\mathcal{F}$ be a fundamental domain for the domain $\Omega \subset \bar{\mathbb{C}}$ such that $\partial\mathcal{F}$ is a K-quasicircle.

Suppose the theorem fails, that is for each $\delta > 0$, there is $\xi_0 \in \bar{\mathcal{F}}$ and an element γ in the Fuchsian group Γ such that $\rho(\xi_0, \gamma(\xi_0)) \leq \delta$. Then

$$\rho(\gamma^n(\xi_0), \xi_0) \leq n\delta \leq 1 \quad \text{if} \quad n < \frac{1}{\delta} \,. \tag{2.28}$$

Consider the ball B of hyperbolic radius 1 centered at ξ_0. By Lemma 3 there is a ball $B_\varepsilon \subset B \cap \mathcal{F}$ of hyperbolic radius ε. Since $B_\varepsilon \subset \mathcal{F}$, the balls $B_\varepsilon, \gamma(B_\varepsilon), \ldots, \gamma^n(B_\varepsilon)$ are disjoint. Furthermore, they all have hyperbolic radius ε and by (2.28) they are contained in the ball of hyperbolic radius 2 centered at ξ_0. Since this cannot happen for arbitrarily big n's, we get a contradiction. Thus, Theorem 2 is proved.

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