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Boundary value problems in Lipschitz cylinders for three-dimensional parabolic systems

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Abstract. We consider initial-boundary value problems for a parabolic system in a Lipschitz cylinder. When the space dimension is three, we obtain estimates for the solutions when the lateral data taken from the best possible range of L^p-spaces.

Introduction.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and for $0 < T < \infty$, let $\Omega_T = \Omega \times (0, T)$ be a Lipschitz cylinder. Consider the parabolic system

$$\frac{\partial \vec{u}}{\partial t} = \mu \Delta \vec{u} + (\lambda + \mu) \nabla (\text{div } \vec{u}) \quad \text{in } \Omega_T \tag{0.1}$$

where $\vec{u} = (u^1, u^2, u^3)$ and $\mu > 0, \lambda > -2\mu/3$ are constants. In this paper, we study the solvability of the initial-Dirichlet problem for (0.1)

$$\text{(IDP)} \qquad \begin{cases} \vec{u}\big|_{\Sigma_T} = \vec{g} \in L^p(\Sigma_T),\ L^p_1(\Sigma_T) \text{ or } \Lambda^\alpha_0(\Sigma_T)\,, \\ \vec{u}\big|_{t=0} = \vec{0}\,. \end{cases}$$

where $\Sigma_T = \partial\Omega \times (0,T)$ denotes the lateral boundary of Ω_T, $L_1^p(\Sigma_T)$ denotes the space of functions on Σ_T having one spatial and half of a time derivative in L^p and $\Lambda_0^\alpha(\Sigma_T)$ denotes the space of Hölder continuous functions of order α (in parabolic sense) which vanish on $\partial\Omega \times \{t = 0\}$. We also consider the initial traction problem for (0.1)

$$\text{(ITP)} \qquad \begin{cases} \lambda(\text{div } \vec{u})N + \mu\big((\nabla\vec{u}) + (\nabla\vec{u})^{tr}\big)N\big|_{\Sigma_T} = \vec{g} \in L^p(\Sigma_T)\,, \\ \vec{u}\big|_{t=0} = \vec{0}\,, \end{cases}$$

where $N = (N_1, N_2, N_3)$ denotes the outward unit normal to $\partial\Omega$ and $(\cdot)^{tr}$ denotes the transpose of a matrix.

Our estimates for solutions of (0.1) will be given in terms of the parabolic maximal function denoted by $(\cdot)^*$. We will use $\partial_t^{1/2}$ to denote half of a time derivative. Throughout this paper C will denote constants depending at most on the Lipschitz character of $\partial\Omega, T, \lambda, \mu$ and p.

We now state our main results (see Section 1 below for the definition of function spaces appearing in these results).

Theorem A. (Dirichlet Problem with L^p data). *There exists $\varepsilon > 0$, such that, given any $\vec{g} \in L^p(\Sigma_T)$, $2 - \varepsilon < p \le \infty$, there exists a unique $\vec{u}$ in Ω_T satisfying* (0.1), (IDP) *and* $(\vec{u})^* \in L^p(\Sigma_T)$. *Moreover,* $\|(\vec{u})^*\|_{L^p(\Sigma_T)} \le C\, \|\vec{g}\|_{L^p(\Sigma_T)}$.

Theorem B. (Dirichlet Problem with Λ_0^α data). *There exists $\alpha_0 > 0$, such that, given any $\vec{g} \in \Lambda_0^\alpha(\Sigma_T)$, $0 < \alpha < \alpha_0$, there exists a unique $\vec{u}$ in Ω_T satisfying* (0.1), (IDP) *and $\vec{u} \in \Lambda_0^\alpha(\overline{\Omega} \times [0,T])$. In fact, we have* $\|\vec{u}\|_{\Lambda_0^\alpha(\overline{\Omega}\times[0,T])} \le C\, \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}$.

Theorem C. (Dirichlet Problem with L_1^p data). *There exists $\varepsilon > 0$, such that, given any $\vec{g} \in L_1^p(\Sigma_T)$, $1 < p < 2 + \varepsilon$, there exists a unique $\vec{u}$ in Ω_T satisfying* (0.1), (IDP) *and* $(\nabla\vec{u})^* \in L^p(\Sigma_T)$. *Moreover,* $\|(\nabla\vec{u})^*\|_{L^p(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^p(\Sigma_T)} \le C\, \|\vec{g}\|_{L_1^p(\Sigma_T)}$.

Theorem D. (Traction Problem with L^p data) *There exists $\varepsilon > 0$, such that, given any $\vec{g} \in L^p(\Sigma_T), 1 < p < 2 + \varepsilon$, there exists a unique $\vec{u}$ satisfying* (0.1), (ITP) *and $(\nabla\vec{u})^* \in L^p(\Sigma_T)$. In fact, we have*

$$\|(\nabla\vec{u})^*\|_{L^p(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^p(\Sigma_T)} \le C\, \|\vec{g}\|_{L^p(\Sigma_T)}\,.$$

The results above were announced in [BS2]. The boundary value problems for (0.1) in Lipschitz cylinders were studied in [S], where the solvability of (IDP) and (ITP) was obtained for $p = 2$ on $\Omega \times (0,T)$ where Ω is any bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$.

For the system of elastostatics,

$$\mu \Delta \vec{u} + (\lambda + \mu)\nabla(\text{div } \vec{u}) = \vec{0}$$

in Lipschitz domains, the corresponding results were obtained for $p = 2$, $n \geq 3$ in [DKV2] and for p optimal, $n = 3$ in [DK2].

Our argument follows closely that for elliptic systems in three dimensions in [DK2]. In the parabolic case, the dimension of the time variable t should be counted twice. Hence, we are working in a space of (homogeneous) dimension five. This is where the difficulty lies when we try to apply the argument from [DK2]. However, since our domains are cylindrical, we are able to overcome this difficulty. This is done by using Fourier analysis in the time variable to show that we may solve the initial-Dirichlet problem with data in mixed norm L^p spaces

$$L^{p,q}(\Sigma_T) = L^q\big((0,T); L^p(\partial\Omega)\big)$$

where p is close to 2 and $q \in (1,2]$. Since the time index may be arbitarily close to one, the extra dimensions do not introduce any additional difficulties.

We remark that the techniques we use in this paper also yield some interesting new results for the heat equation in Lipschitz cylinders in all dimensions. Indeed, using the $L^p(\partial\Omega)$-valued multiplier argument in Section 2, the L^p estimates in [B] and interpolation, it is not very hard to establish the solvability of the initial-Dirichlet problem for the heat equation with boundary data in $L^{p,q}(\Sigma_T)$ for $2 \leq p < \infty$ and $p/2 < q < \infty$ and the initial-Neumann problem with boundary data in $L^{p,q}(\Sigma_T)$ for $1 < p \leq 2$ and $1 < q < p/(2-p)$. We leave the details to interested readers.

The outline of this paper is as follows. Section 1 contains notations and definitions that will be used throughout the paper. In Section 2, we study the boundary potential operators on mixed norm spaces. In Section 3, we study the initial traction problem with atomic data. Our main results are proved in Section 4.

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1. Notations and definitions.

We retain the notations used in the Introduction. In particular, Ω will be a bounded Lipschitz domain in $\mathbb{R}^3$, $\Omega_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. We will use P and Q for points on $\partial\Omega$, X and Y for points in Ω and t and s will be the time variables.

For v defined on Ω_T, the nontangential maximal function of v is defined by

$$(v)^*(P,t) = \sup_{(Y,s)\in\gamma(P,t)} |v(Y,s)| \tag{1.1}$$

where $\gamma(P,t)$ is the parabolic nontangential approach region defined by

$$\gamma(P,t) = \{(Y,s):\ |Y-P| + |t-s|^{1/2} < 2\,\mathrm{dist}\{Y,\partial\Omega\}\} \cap \Omega_T. \tag{1.2}$$

We will use $\vec{u}$ and $\vec{f}$ to denote functions taking their values in $\mathbb{R}^3$. In the initial Dirichlet problem, the statement $\vec{u}\big|_{\Sigma_T} = \vec{g}$ is interpreted in the sense of nontangential limit

$$\lim_{\substack{(Y,s)\to(P,t)\\(Y,s)\in\gamma(P,t)}} \vec{u}(Y,s) = \vec{g}(P,t) \qquad \text{a.e. on} \quad \Sigma_T$$

with respect to the surface measure on Σ_T.

Similarly, for the initial traction problem, by

$$\lambda(\mathrm{div}\,\vec{u})N + \mu((\nabla\vec{u}) + (\nabla\vec{u})^{tr})N\,\big|_{\Sigma_T} = \vec{g},$$

we mean that

$$\lambda(\mathrm{div}\,\vec{u})\big|_{\Sigma_T}N + \mu((\nabla\vec{u}) + (\nabla\vec{u})^{tr})\big|_{\Sigma_T}\cdot N = \vec{g}.$$

We say $\vec{u}$ has initial value $\vec{0}$, write $\vec{u}\big|_{t=0} = \vec{0}$, if $\vec{u}(X,t) \to \vec{0}$ uniformly on every compact subset of Ω as $t \to 0^+$.

For a smooth function f which decays rapidly at $-\infty$, we define the fractional integral of order σ by

$$I_\sigma(f)(t) = \frac{1}{\Gamma(\sigma)}\int_{-\infty}^{t} \frac{f(s)}{(t-s)^{1-\sigma}}ds, \qquad 0 < \sigma \le 1, \tag{1.3}$$

and fractional derivative of order σ by

$$\partial_t^\sigma(f)(t) = \frac{\partial}{\partial t}I_{1-\sigma}(f)(t), \qquad 0 < \sigma < 1, \tag{1.4}$$

where Γ is the gamma function.

For u defined in a neighborhood of $\partial\Omega$, we define the tangential gradient of u on $\partial\Omega$ by

$$\nabla_{\tan} u = \nabla u - \langle \nabla u, N \rangle N. \tag{1.5}$$

We now give the definition of the spaces $L^{p,q}(\Sigma_T), \Lambda_0^\alpha(\Sigma_T)$, $\Lambda_0^\alpha(\overline{\Omega} \times [0,T])$, $L_1^p(\Sigma_T)$ and $L_1^{p,q}(\Sigma_T)$.

Definition 1.6. *For $1 \leq p, q < \infty$, we define the mixed norm space*

$$L^{p,q}(\Sigma_T) = L^q((0,T), L^p(\partial\Omega))$$
$$= \left\{ \vec{f}\colon \quad \|\vec{f}\|_{L^{p,q}(\Sigma_T)} = \Big(\int_0^T \Big(\int_{\partial\Omega} |\vec{f}|^p \Big)^{q/p} \Big)^{1/q} < +\infty \right\}.$$

Definition 1.7. *For $0 < \alpha < 1$, we let*

$$\|\vec{f}\|_{\Lambda^\alpha(\Sigma_T)} = \sup_{\Sigma_T} |\vec{f}| + \sup_{(P,t)\neq(Q,s)} \frac{|\vec{f}(P,t) - \vec{f}(Q,s)|}{(|P-Q| + |t-s|^{1/2})^\alpha}$$

and we define the space $\Lambda_0^\alpha(\Sigma_T)$ by

$$\Lambda_0^\alpha(\Sigma_T) = \{ \vec{f} : \vec{f}(P,0) = \vec{0} \text{ for } P \in \partial\Omega \text{ and } \|\vec{f}\|_{\Lambda^\alpha(\Sigma_T)} < +\infty \}.$$

The space $\Lambda_0^\alpha(\overline{\Omega} \times [0,T])$ is defined in a similar fashion.

Definition 1.8. *For $1 < p < \infty$, $L_1^p(\partial\Omega \times \mathbb{R})$ denotes the closure of the space*

$$\{ \vec{v} : \vec{v} = \vec{u}\big|_{\partial\Omega\times\mathbb{R}},\ \vec{u} \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}) \}$$

with respect to the norm,

$$\|\vec{v}\|_{L_1^p(\partial\Omega\times\mathbb{R})} \equiv \|\nabla_{\tan}\vec{v}\|_{L^p(\partial\Omega\times\mathbb{R})} + \|\partial_t^{1/2}\vec{v}\|_{L^p(\partial\Omega\times\mathbb{R})} + \|\vec{v}\|_{L^p(\partial\Omega\times\mathbb{R})}.$$

We then define

$$L_1^p(\Sigma_T) = \left\{ \vec{f}\colon \quad \vec{f} = \vec{g}\big|_{\Sigma_T},\ \vec{g} \in L_1^p(\partial\Omega \times \mathbb{R}) \text{ and } \vec{g}(P,t) = \vec{0} \text{ for } t < 0 \right\}.$$

The norm is given by

$$\|\vec{f}\|_{L_1^p(\Sigma_T)} = \inf \left\{ \|\vec{g}\|_{L_1^p(\partial\Omega\times\mathbb{R})}\colon\ \vec{g}\big|_{\Sigma_T} = \vec{f} \text{ and } \vec{g}(P,t) = 0 \text{ for } t < 0 \right\}.$$

We may define $L_1^{p,q}(\partial\Omega \times \mathbb{R})$ *and* $L_1^{p,q}(\Sigma_T)$ *similarly, using the norm*

$$\|\vec{v}\|_{L_1^{p,q}(\partial\Omega\times\mathbb{R})} \equiv \|\nabla_{\tan}\vec{v}\|_{L^{p,q}(\partial\Omega\times\mathbb{R})} + \|\partial_t^{1/2}\vec{v}\|_{L^{p,q}(\partial\Omega\times\mathbb{R})} + \|\vec{v}\|_{L^{p,q}(\partial\Omega\times\mathbb{R})}.$$

We close this section with the following perturbation theorem which will be very useful to us.

Theorem 1.9. *Let M be a measure space with a positive measure ν. Let $0 < \sigma_0 < 1/2$ and S: $L^p(M,\nu) \to L^p(M,\nu)$ be a bounded linear operator, with norm bounded by B, for $|1/2 - 1/p| < \sigma_0$. Also assume that $S: L^2 \to L^2$ in invertible and $\|S^{-1}\|_{(2,2)} \leq A$. Then, there exist $\delta > 0$, $C > 0$ depending only on σ_0, A and B, such that, S: $L^p \to L^p$ is invertible and $\|S^{-1}\|_{(p,p)} \leq C$ for $|1/2 - 1/p| < \delta$.*

We remark that Theorem 1.9 is a consequence of more general results of G. David and S. Semmes (unpublished). Similar results also have been given by A. P. Calderón [C]. Theorem 1.9 may be proven by applying Calderón's result to the positive operators S^*S and $S\,S^*$. See also W. Cao and Y. Sagher [CS] for a more general result.

2. Layer Potentials.

In this section we study the boundary layer potentials on mixed norm spaces. In particular, we show that the double layer potential on the lateral boundary is bounded on $L^{p,q}(\Sigma_T)$ for $1 < p, q < \infty$ and is invertible on $L^{p,q}(\Sigma_T)$ for $p \in (2-\varepsilon_0, 2]$, $q \in (1,2]$ where $\varepsilon_0 > 0$. These results are used to solve the initial-Dirichlet problem with data in $L^{p,q}(\Sigma_T)$ for $p \in (2-\varepsilon_0, 2)$ and $q \in (1,2]$.

We begin by introducing a matrix of fundamental solutions for the system (0.1), $\Gamma(X,t) = \left(\Gamma_{jk}(X,t)\right)_{3\times 3}$ where

$$\Gamma_{jk}(X,t) = \delta_{jk}\,\frac{e^{-|X|^2/(4\pi\mu t)}}{(4\pi\mu t)^{3/2}} + \frac{\partial^2}{\partial x_j \partial x_k}\int_{\mu t}^{(\lambda+2\mu)t} \frac{e^{-|X|^2/(4\pi s)}}{(4\pi s)^{3/2}}\,ds$$

when $t > 0$ and $\Gamma_{jk}(X,t) = 0$ for $t \leq 0$. We have the following estimate

$$(2.1) \qquad \left|\frac{\partial^{|\alpha|+\alpha_0}}{\partial X^\alpha \partial t^{\alpha_0}}\,\Gamma(X,t)\right| \leq \frac{C}{(|X| + |t|^{1/2})^{3+|\alpha|+2\alpha_0}}\,.$$

Given $\vec{f} \in L^q(\mathbb{R}, L^p(\partial\Omega))$, $1 < p < \infty$, $1 < q < \infty$, define the single layer potential

$$\vec{u} = \mathcal{S}(\vec{f})(X,t) = \int_{-\infty}^{t} \int_{\partial\Omega} \Gamma(X - Q, t - s)\vec{f}(Q,s)dQds\,. \tag{2.2}$$

Let

$$\frac{\partial \vec{u}}{\partial \nu} = \lambda(\text{div } \vec{u})N \;+\; \mu\big((\nabla\vec{u}) + (\nabla\vec{u})^{tr}\big)N \tag{2.3}$$

be the traction operator on the lateral boundary. If $\vec{u} = \mathcal{S}(\vec{f})$, then

$$\frac{\partial \vec{u}_\pm}{\partial \nu} = \big(\pm\frac{1}{2}I + K_\nu\big)\vec{f} \qquad \text{on } \partial\Omega \times \mathbb{R}\,,$$

where the subscripts $+$ and $-$ indicate nontangential limits taken from $\Omega \times \mathbb{R}$ and from $\overline{\Omega} \times \mathbb{R}$ respectively, I denotes the identity operator and

$$K_\nu \vec{f}(P,t) = \text{p.v.} \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu(P)}(P - Q, t - s)\vec{f}(Q,s)dQds$$

is a singular integral (see [S]).

Lemma 2.4. *K_ν is bounded on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for any $1 < p, q < \infty$.*

PROOF. First, if $p = q$, $1 < p < \infty$, the lemma follows from the theorem of R. Coifman, A. McIntosh and Y. Meyer [CMM] on the boundedness of the Cauchy integral on Lipschitz curves and the argument of E. Fabes and N. Rivière [FR, Theorem 1.1].

Next, consider the case of $1 < q < p$. Let $B = L^p(\partial\Omega)$. We may view K_ν as a singular integral operator on B-valued functions on $\mathbb{R}^1$. Write

$$K_\nu \vec{f}(t) = \text{p.v.} \int_{-\infty}^{\infty} K_\nu(t - s)\vec{f}(s)ds\,, \quad \vec{f} \in L^q(\mathbb{R}, B)\,.$$

Using estimate (2.1), it is not difficult to see that, for $\vec{g} \in B$,

$$|K_\nu(t)\vec{g}|_B \le \frac{C}{t}\,|M_{\partial\Omega}(\vec{g})|_B$$

and

$$|\frac{d}{dt}K_\nu(t)\vec{g}|_B \le \frac{C}{t^2}\,|M_{\partial\Omega}(\vec{g})|_B$$

where $M_{\partial\Omega}$ denotes the Hardy-Littlewood maximal function on $\partial\Omega$. Thus,

$$\|K_\nu(t)\|_{B\to B} \le \frac{C}{t} \quad \text{and} \quad \|\frac{d}{dt}K_\nu(t)\|_{B\to B} \le \frac{C}{t^2}\,.$$

Therefore, $K_\nu(t)$ is a standard Calderón-Zygmund kernel. Since $K_\nu : L^p(\mathbb{R}, B) \to L^p(\mathbb{R}, B)$ is bounded, we have $K_\nu : L^q(\mathbb{R}, B) \to L^q(\mathbb{R}, B)$ is bounded for $1 < q < p$, by the standard Calderón-Zygmund argument.

Finally, note that the same argument as above also shows that K_ν^*, the adjoint of K_ν, is bounded on $L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$ for $1 < q \le p$. Thus, by duality, we obtain the boundedness of K_ν on $L^q(\mathbb{R}, L^p(\partial\Omega))$ when $q > p$.

For $\vec{f} \in L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$, define the double layer potential

$$\tag{2.5} \begin{aligned} \vec{v}(X,t) &= \mathcal{K}(\vec{f})(X,t) \\ &= \int_{-\infty}^{t}\int_{\partial\Omega}\left\{\frac{\partial}{\partial\nu_Q}\Gamma(X-Q,t-s)\right\}^{tr}\vec{f}(Q,s)\,dQds\,. \end{aligned}$$

Then $\vec{v}_\pm = (\mp\frac{1}{2}I + \widetilde{K}_\nu)\vec{f}$ on $\partial\Omega\times\mathbb{R}$ where $\widetilde{K}_\nu = RK_\nu^*R$ and R is the reflection defined by $R(\vec{f})(P,t) = \vec{f}(P,-t)$.

Lemma 2.6. *Let* $1 < p, q < \infty$ *and* $\vec{f} \in L^q(\mathbb{R}, L^p(\partial\Omega))$.
(i) *Let* $\vec{u} = \mathcal{S}(\vec{f})$, *then*

$$\|(\nabla\vec{u})^*\|_{p,q} + \|(\partial_t^{1/2}\vec{u})^*\|_{p,q} \le C\,\|\vec{f}\|_{p,q}\,,$$

where $\|\cdot\|_{p,q}$ *denotes the norm in* $L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$.
(ii) *Let* $\vec{v} = \mathcal{K}(\vec{f})$, *then*

$$\|(\vec{v})^*\|_{p,q} \le C\,\|\vec{f}\|_{p,q}\,.$$

PROOF. Using the argument of Fabes-Rivière [FR, Theorem 1.11], we can show that for any $q_0 > 1$,

$$(\nabla\vec{u})^*(P,t) \le C_{q_0}\Big[M_1\big(M_{\partial\Omega}(Kf)\big)(P,t) + M_1\big(M_{\partial\Omega}(|f|^{q_0})\big)^{1/q_0}(P,t)\Big]$$

where K is a singular integral operator of the same type as K_ν and M_1 is the Hardy-Littlewood maximal function on $\mathbb{R}^1$. It follows from the argument in the proof of Lemma 2.4 that K is bounded on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for $1 < p,q < \infty$. Also, note that an easy modification of the argument in [FSt] by C. Fefferman and E. M. Stein yields that M_1 is bounded on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for $1 < p,q < \infty$. The estimate for $(\nabla\vec{u})^*$ then follows easily. The estimates for $(\partial_t^{1/2}\vec{u})^*$ and $(\vec{v})^*$ follow in a similar manner. We omit the details.

Given $\vec{f} \in L^{p,q}(\Sigma_T)$, let $\vec{g}$ denote the extension of $\vec{f}$ by zero to $\partial\Omega \times \mathbb{R}$. We define $K_\nu\vec{f} \equiv K_\nu\vec{g}$. Clearly, $\pm\frac{1}{2}I + K_\nu$ is bounded on $L^{p,q}(\Sigma_T)$ for $1 < p,q < \infty$. From [S, Theorem 4.3.1], we know that $\pm\frac{1}{2}I + K_\nu$ is also invertible on $L^2(\Sigma_T)$. Using Theorem 1.9 and an $L^p(\partial\Omega)$-valued multiplier argument, we are able to extend this result.

Theorem 2.7. *There exists $\varepsilon_0 > 0$ such that $\pm\frac{1}{2}I + K_\nu$ is invertible on $L^{p,q}(\Sigma_T)$ for $p \in [2, 2+\varepsilon_0)$ and $q \in [2,\infty)$.*

PROOF. We give the proof for $\frac{1}{2}I + K_\nu$. The invertibility of $-\frac{1}{2}I + K_\nu$ follows in the same manner.

Let $a \geq 0$, consider the parabolic system

$$\frac{\partial\vec{u}}{\partial t} + a\vec{u} = \mu\,\Delta\vec{u} + (\lambda+\mu)\nabla(\text{div }\vec{u}). \tag{2.8}$$

Note that $\Gamma_a(X,t) = e^{-at}\Gamma(X,t)$ is a matrix of fundamental solutions for (2.8). For $\vec{f} \in L^q(\mathbb{R}, L^p(\partial\Omega))$, let

$$\mathcal{S}^a(\vec{f})(X,t) = \int_{-\infty}^t \int_{\partial\Omega} \Gamma_a(X-Q, t-s)\vec{f}(Q,s)dQds\,.$$

Clearly,

$$\frac{\partial}{\partial\nu}\mathcal{S}^a_\pm(\vec{f}) \;= \big(\pm\frac{1}{2}I + K^a_\nu\big)(\vec{f}) \quad \text{a.e. on } \partial\Omega\times\mathbb{R}\,,$$

where $K^a_\nu(\vec{f}) = e^{-at}K_\nu e^{at}(\vec{f})$. Thus, to show $\frac{1}{2}I + K_\nu$ is invertible on $L^{p,q}(\Sigma_T)$, it suffices to show $\frac{1}{2}I + K^a_\nu$ is invertible on $L^{p,q}(\Sigma_T)$.

First, we shall show that if a is large, then $\frac{1}{2}I + K^a_\nu$ is invertible on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for p near 2 and $q \in (1,\infty)$. Let $S_a = \frac{1}{2}I + K^a_\nu$ and

$$K_\nu(P,Q,t-s) = \frac{\partial\Gamma}{\partial\nu(P)}(P-Q,t-s)\,.$$

Then

$$S_a\vec{f}(P,t) = \frac{1}{2}\vec{f}(P,t) + \text{p.v.}\int_{-\infty}^{t}\int_{\partial\Omega} e^{-a(t-s)}K_\nu(P,Q,t-s)\vec{f}(Q,s)dQds\,.$$

Taking the partial Fourier transform in the t variable, we obtain

$$(S_a\vec{f})^\wedge(P,i\tau) = \frac{1}{2}(\vec{f})^\wedge(P,i\tau) + \text{p.v.}\int_{\partial\Omega}(K_\nu)^\wedge(P,Q,-a+i\tau)(\vec{f})^\wedge\,(Q,i\tau)\,dQ$$

where

$$(\vec{f})^\wedge(P,i\tau) = \int_{\mathbb{R}} e^{i\tau t}\vec{f}(P,t)dt, \qquad \tau\in\mathbb{R}$$

and

$$(K_\nu)^\wedge(P,Q,-a+i\tau) = \int_{\mathbb{R}} e^{(-a+i\tau)t}\,K_\nu(P,Q,t)dt, \quad \tau\in\mathbb{R}\,.$$

For $z\in\mathbb{C}, \operatorname{Re} z\le 0$, let $m(z): L^p(\partial\Omega)\to L^p(\partial\Omega)$ be defined by

$$m(z)(\vec{h})(P) = \frac{1}{2}\,\vec{h}(P) + \text{p.v.}\int_{\partial\Omega}(K_\nu)^\wedge(P,Q,z)\vec{h}(Q)dQ$$

for $\vec{h}\in L^p(\partial\Omega)$. Then

$$(S_a\vec{f})^\wedge(P,i\tau) = m\,(-a+i\tau)((\vec{f})^\wedge(\cdot,i\tau))(P)\,.$$

It follows from the theorem of R. Coifman, A. McIntosh and Y. Meyer [CMM] that $m(-a+i\tau)$ is bounded on $L^p(\partial\Omega)$ for $1<p<\infty$ and $\|m(-a+i\tau)\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)}\le C_a$ with constant C_a independent of τ. Moreover, using estimate (2.1), it is not difficult to see that

$$\left|\frac{d^\alpha}{d\tau^\alpha}m\,(-a+i\tau)\vec{h}\right| \le \frac{C_a}{(1+|\tau|)^\alpha}\,M_{\partial\Omega}(\vec{h})$$

for integer $\alpha\ge 1$. Hence,

$$\left\|\frac{d^\alpha}{d\tau^\alpha}m\,(-a+i\tau)\right\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \le \frac{C_a}{(1+|\tau|)^\alpha} \tag{2.9}$$

for integer $\alpha \geq 0$. On the other hand, it follows from the Rellich identity (see [S, Lemma 4.3.13], also [BS1, Proposition 2.2]) that, if $a \geq C_0 > 0$, then

$$\|\vec{h}\|_{L^2(\partial\Omega)} \leq C\, \|m(-a+i\tau)\vec{h}\|_{L^2(\partial\Omega)}\,. \tag{2.10}$$

Now, since $m(0)$ is a Fredholm operator on $L^2(\partial\Omega)$ with index zero [DKV2, Theorem 2.7] and $m(-a+i\tau) - m(0)$ is compact on $L^2(\partial\Omega)$ by (2.1), we see that $m(-a+i\tau)$ is also a Fredholm operator with index zero. Thus, by (2.10), $m(-a+i\tau)$ is invertible on $L^2(\partial\Omega)$ and $\|m^{-1}(-a+i\tau)\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq C$ for $a \geq C_0$. This implies that S_a is invertible on $L^2(\partial\Omega\times\mathbb{R})$, hence on $L^p(\partial\Omega\times\mathbb{R})$ for $|p-2| < \varepsilon_1$, by Theorem 1.9. It also follow from Theorem 1.9 that, $m(-a+i\tau)$ is invertible on $L^p(\partial\Omega)$ and $\|m^{-1}(-a+i\tau)\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \leq C_a$ for $|p-2| < \varepsilon_2$ where $C_a > 0$ and $\varepsilon_2 > 0$ are independent of τ. Moreover, by (2.9) and (2.10), we have

$$\|\frac{d^\alpha}{d\tau^\alpha} m^{-1}(-a+i\tau)\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \leq \frac{C_a}{(1+|\tau|)^\alpha} \tag{2.11}$$

for $|p-2| < \varepsilon_2$ and integer $\alpha \geq 0$.

To proceed, we let $B = L^p(\partial\Omega)$, $|p-2| < \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Since

$$(S_a^{-1}\vec{f})^\wedge(P, i\tau) = m^{-1}(-a+i\tau)\big((\vec{f})^\wedge(\cdot, i\tau)\big)(P)$$

for $\vec{f} \in L^2(\partial\Omega\times\mathbb{R})$, it follows from a standard argument and (2.11) that S_a^{-1} is associated with an $L(B)$-valued Calderón-Zygmund kernel where $L(B)$ denotes the bounded linear operators on B. Since $S_a^{-1} : L^p(\mathbb{R}, B) \to L^p(\mathbb{R}, B)$ is bounded, a standard Calderón-Zygmund argument yields that $S_a^{-1} : L^q(\mathbb{R}, B) \to L^q(\mathbb{R}, B)$ is bounded for $1 < q < p$, *i.e.* $\frac{1}{2}I + K_\nu^a : L^q\,(\mathbb{R}, L^p(\partial\Omega)) \to L^q(\mathbb{R}, L^p(\partial\Omega))$ is invertible for $|p-2| < \varepsilon_0$ and $1 < q < p$. The case where $p < q < \infty$ follows by duality.

Finally, to see that $\frac{1}{2}I + K_\nu^a$ is invertible on $L^{p,q}(\Sigma_T)$ for $p \in [2, 2+\varepsilon_0)$ and $q \in [2, \infty)$, let $\vec{g} \in L^{p,q}(\Sigma_T)$, and $\vec{G}$ be the extension of $\vec{g}$ by zero to $(\partial\Omega\times\mathbb{R})$. Since $\vec{G} \in L^q(\mathbb{R}, L^p(\partial\Omega))$, there exists $\vec{f} \in L^q(\mathbb{R}, L^p(\partial\Omega))$ such that $(\frac{1}{2}I + K_\nu^a)\vec{f} = \vec{G}$ on $(\partial\Omega\times\mathbb{R})$. We wish to show that $\vec{f}$ vanishes when $t < 0$.

To proceed, we let $\vec{u} = \mathcal{S}^a(\vec{f})$. It follows from (2.8) and integration by parts that

$$\begin{aligned} &\int_{T_0}^t \int_{\Omega_j} a|\vec{u}|^2 + \frac{1}{2}\int_{\Omega_j} |\vec{u}(X,t)|^2 - |\vec{u}(X,T_0)|^2 dX \\ &\qquad + \int_{T_0}^t \int_{\Omega_j} \lambda\,|\operatorname{div}\vec{u}|^2 + \frac{1}{2}|\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2 \\ &= \int_{T_0}^t \int_{\partial\Omega_j} \frac{\partial\vec{u}}{\partial\nu}\cdot\vec{u} \end{aligned} \tag{2.12}$$

where $T_0 < t < 0$ and $\bar{\Omega}_1 \subset \bar{\Omega}_2 \subset \cdots \subset \Omega$ is a sequence of smooth domains approximating Ω, [V]. Let $j \to \infty$ in (2.12), since $\|(\vec{u})^*\|_2 + \|(\nabla\vec{u})^*\|_2 \le C\,\|\vec{f}\|_2 \le C_{T_0}\,\|\vec{f}\|_{p,q}$, we obtain

$$\int_\Omega |\vec{u}(X,t)|^2 dX \le \int_\Omega |\vec{u}(X,T_0)|^2 dX\,.$$

Integrating in T_0, we have

$$\begin{aligned} \int_\Omega |\vec{u}(X,t)|^2 dX &\le \int_{T_0-1}^{T_0}\int_\Omega |\vec{u}(X,s)|^2 dX\,ds \\ &\le C\,\|(\vec{u})^*\|_{L^{p,q}\left(\partial\Omega\times(T_0-1,T_0)\right)} \\ &\le C\,\|\vec{f}\|_{L^{p,q}\left(\partial\Omega\times(-\infty,T_0)\right)} \end{aligned}$$

which goes to zero as $T_0 \to -\infty$. Hence, $\vec{u} \equiv \vec{0}$ in $\Omega \times (-\infty, 0)$. Thus, $\vec{u} = \vec{0}$ a.e. on $\partial\Omega \times (-\infty, 0)$. This, together with a similar argument in the complement, implies that $\vec{u} \equiv \vec{0}$ in $\overline{\Omega}^c \times (-\infty, 0)$. It then follows that

$$(-\frac{1}{2}I + K_\nu)\vec{f} = \frac{\partial\vec{u}_-}{\partial\nu} = \vec{0} \quad \text{on } \partial\Omega\times(-\infty,0)\,.$$

Therefore,

$$\vec{f} = (\frac{1}{2}I + K_\nu) - (-\frac{1}{2}\,I + K_\nu)\vec{f} = \vec{0} \quad \text{on } \partial\Omega\times(-\infty,0)\,.$$

From Theorem 2.7 and its proof, we draw three corollaries.

Corollary 2.13. *There exists* $\varepsilon_0 > 0$, *such that, given any* $\vec{g} \in L^{p,q}(\Sigma_T)$, $p \in [2, 2+\varepsilon_0)$, $q \in [2, \infty)$, *there exists a unique* $\vec{u}$ *satisfying* (0.1), (ITP) *and* $\|(\nabla\vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. *Moreover, we have*

$$\|(\nabla\vec{u})^*_{L^{p,q}(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \leq C\,\|\vec{g}\|_{L^{p,q}(\Sigma_T)}\,.$$

PROOF. The existence follows from the invertibility of $\frac{1}{2}I + K_\nu$ on $L^{p,q}(\Sigma_T)$ for $p \in [2, 2+\varepsilon_0)$ and $q \in [2, \infty)$.

To see the uniqueness, let $\vec{u}$ be a solution of (0.1) in Ω_T such that, $\partial\vec{u}/\partial\nu = \vec{0}$ on Σ_T and $\|(\nabla\vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. Let $\{\Omega_j\}$ be a sequence of smooth domain approximating Ω. Since $\vec{u}$ is smooth on $\overline{\Omega}_j \times [0, T]$, by the existence and uniqueness results for the traction problem with L^2 data [S], there exists $\vec{f}_j \in L^2(\partial\Omega_j \times (0, T))$, such that

$$\vec{u}(X,t) = \int_0^t \int_{\Omega_j} \Gamma(X-Q, t-s)\vec{f}_j(Q,s)\,dQ\,ds \tag{2.14}$$

for $(X, t) \in \Omega_j \times (0, T)$. Thus,

$$(\frac{1}{2}I + K_{\nu,j})\vec{f}_j = \frac{\partial\vec{u}}{\partial\nu} \quad \text{on } \partial\Omega_j \times (0,T)\,,$$

where $K_{\nu,j}$ is the corresponding operator on $\partial\Omega_j \times (0, T)$. Hence, by Theorem 2.7,

$$\|\vec{f}_j\|_{L^{p,q}(\partial\Omega_j\times(0,T))} \leq C\left\|\frac{\partial\vec{u}}{\partial\nu}\right\|_{L^{p,q}(\partial\Omega_j\times(0,T))} \to 0$$

as $j \to \infty$ where we used the assumption

$$\frac{\partial\vec{u}}{\partial\nu} = \vec{0} \quad \text{on } \Sigma_T \qquad \text{and} \qquad \|(\nabla\vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty\,.$$

It then follows from (2.14) that $\vec{u} \equiv \vec{0}$ on Ω_T and the uniqueness is proved.

Corollary 2.15. *There exists* $\varepsilon_1 > 0$, *such that, given any* $\vec{g} \in L^{p,q}(\Sigma_T)$, $p \in (2-\varepsilon_1, 2]$ *and* $q \in (1, 2]$, *there exists a unique* $\vec{u}$ *satisfying* (0.1), (IDP) *and* $\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. *Moreover,*

$$\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \leq C\,\|\vec{g}\|_{L^{p,q}(\Sigma_T)}\,.$$

PROOF. Note that, restricting $\widetilde{K}_\nu$ to functions supported on Σ_T, we have $(-\frac{1}{2}I + \widetilde{K}_\nu)\vec{f} = R_T(-\frac{1}{2}I + K_\nu^*)R_T(\vec{f})$ where $\vec{f} \in L^{p,q}(\Sigma_T)$ and $R_T(\vec{f})(P,t) = \vec{f}(P, T-t)$. Hence, by Theorem 2.7 and duality, $-\frac{1}{2}I + \widetilde{K}_\nu$ is invertible on $L^{p,q}(\Sigma_T)$, $p \in (2-\varepsilon_1, 2]$, $q \in (1,2]$ for some $\varepsilon_1 > 0$. The existence follows. The uniqueness follows from an approximation argument, similar to that in the proof of Corollary 2.13.

Corollary 2.16. *There exists $\varepsilon_2 > 0$, such that, given any $\vec{g} \in L_1^{p,q}(\Sigma_T)$, $p \in [2, 2+\varepsilon_2)$, $q \in [2,\infty)$, there exists a unique $\vec{u}$ satisfying* (0.1), (IDP) *and $\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} + \|(\nabla \vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. Moreover,*

$$\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} + \|(\nabla\vec{u})^*\|_{L^{p,q}(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \le C\,\|\vec{g}\|_{L_1^{p,q}(\Sigma_T)}.$$

PROOF. Clearly, the uniqueness follows from the uniqueness for the initial Dirichlet problem with $L^2(\Sigma_T)$ data. To see the existence, we shall show that $\mathcal{S}\big|_{\Sigma_T} : L^{p,q}(\Sigma_T) \to L_1^{p,q}(\Sigma_T)$ is invertible for $p \in [2, 2+\varepsilon)$, $q \in [2,\infty)$ where $\varepsilon_2 > 0$.

Let

$$V(\vec{f}) = \left(\nabla_{\tan}\mathcal{S}(\vec{f}),\ \partial_t^{1/2}\mathcal{S}(\vec{f}),\ \mathcal{S}(\vec{f})\right). \tag{2.17}$$

Then $V : L^q(\mathbb{R}, L^p(\partial\Omega)) \to \left[L^q(\mathbb{R}, L^p(\partial\Omega))\right]^5$ is bounded for $1 < p, q < \infty$. Taking the partial Fourier transform in the t variable in (2.17), as in the proof of Theorem 2.7, we obtain

$$(V\vec{f})^\wedge(P, i\tau) =$$
$$\left(\nabla_{\tan}\hat{\mathcal{S}}_{i\tau}((\vec{f})^\wedge(\cdot,i\tau))(P),\ (i\tau)^{1/2}\hat{\mathcal{S}}_{i\tau}((\vec{f})^\wedge(\cdot,i\tau)),\ \hat{\mathcal{S}}_{i\tau}((\vec{f})^\wedge(\cdot,i\tau))(P)\right)$$

where $\hat{\mathcal{S}}_{i\tau}$ is defined by

$$\hat{\mathcal{S}}_{i\tau}(\vec{h})(P) = \int_{\partial\Omega} (\Gamma)^\wedge(P,Q,i\tau)\vec{h}(Q)\,dQ, \quad \text{for } \vec{h} \in L^p(\partial\Omega).$$

Let

$$m_{i\tau}(\vec{h})(P) = \left(\nabla_{\tan}\hat{\mathcal{S}}_{i\tau}(\vec{h})(P),\ (i\tau)^{1/2}\hat{\mathcal{S}}_{i\tau}(\vec{h})(P),\ \hat{\mathcal{S}}_{i\tau}(\vec{h})(P)\right).$$

Then,

$$\left(V(\vec{f})\right)^\wedge(P, i\tau) = m_{i\tau}((\vec{f})^\wedge(\cdot, i\tau))(P)$$

From [S], we know that

$$\| \, m_{i\tau}(\vec{h}) \, \|_{\left[L^p(\partial\Omega)\right]^5} \;\; \leq C \, \|\vec{h}\|_{L^p(\partial\Omega)} \, , \qquad 1 < p < \infty \, ,$$

and

$$\|\vec{h}\|_{L^2(\partial\Omega)} \leq C \, \|m_{i\tau}(\vec{h})\|_{L^2(\partial\Omega)}$$

with constant C independent of τ. Moreover, using estimate (2.1), we have

$$\|\frac{d^\alpha}{d\tau^\alpha} m_{i\tau}\|_{L^p(\partial\Omega) \to L^p(\partial\Omega)} \leq \frac{C}{|\tau|^\alpha} \; \text{ for integer } \alpha \geq 0 \, .$$

Now, consider the operator V^*V. The same argument as in the proof of Theorem 2.7, implies that there exists $\varepsilon_2 > 0$, such that, V^*V : $L^q\big(\mathbb{R}, L^p(\partial\Omega)\big) \to L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$ is invertible for $p \in (2 - \varepsilon_2, 2 + \varepsilon_2)$ and $1 < q < \infty$. It follows that

$$\begin{aligned} \|\vec{f}\|_{L^q(\mathbb{R}, L^p(\partial\Omega))} &\leq C \, \|V^*V(\vec{f})\|_{L^q(\mathbb{R}, L^p(\partial\Omega))} \\ &\leq C \, \|V(\vec{f})\|_{\left[L^q(\mathbb{R}, L^p(\partial\Omega))\right]^5} \\ &\leq C \, \|\mathcal{S}(\vec{f})\|_{L_1^{p,q}(\partial\Omega \times \mathbb{R})} \, . \end{aligned}$$

By the definition of $L_1^{p,q}(\Sigma_T)$ norm, we have

$$\|\vec{f}\|_{L^{p,q}(\Sigma_T)} \;\leq C \, \|\mathcal{S}(\vec{f})\|_{L^{p,q}(\Sigma_T)} \, .$$

This implies that $\mathcal{S}\big|_{\Sigma_T}$: $L^{p,q}(\Sigma_T) \to L_1^{p,q}(\Sigma_T)$ is injective and has closed range. An approximation argument shows that the range is dense (see [S] for the case $p = q = 2$). Hence, we are done.

3. Main Lemma.

In this section, we will consider solutions to the initial-traction problem with atomic data on the lateral boundary. Our main result is Lemma 3.6 where we show that the nontangential maximal function of the gradient of such a solution is in L^1.

We begin with some notations. For $P \in \partial\Omega$, $t \in \mathbb{R}$ and $R > 0$ small, let

$$\begin{aligned} J_R(P) &= \{Q \in \partial\Omega : \quad |Q - P| < R\}, \\ \Delta_R(P,t) &= J_R(P) \times (t - R^2, t + R^2), \\ D_R(P) &= \{X \in \Omega : |X - P| < R\}, \\ Z_R(P,t) &= D_R(P) \times (t - R^2, t + R^2). \end{aligned}$$

For a function $\vec{u}$ on Ω_T, we define

$$(\vec{u})^*_R(P,t) = \sup_{\substack{(Y,s)\in\gamma(P,t) \\ |Y-P|\le R}} |\vec{u}(Y,s)|$$

and

$$(\vec{u})^{*R}(P,t) = \sup_{\substack{(Y,s)\in\gamma(P,t) \\ |Y-P|\ge R}} |\vec{u}(Y,s)|.$$

Clearly, $(\vec{u})^*(P,t) \le (\vec{u})^*_R(P,t) + (\vec{u})^{*R}(P,t)$.

Lemma 3.1. (Cacciopoli Inequalities) *Suppose* $\vec{u}$ *is a solution of* (0.1) *in* $Z_{2R} = Z_{2R}(P_0, t_0)$. *Assume* $\partial\vec{u}/\partial\nu = \vec{0}$ *on* $\Delta_{2R} = \Delta_{2R}(P_0,t_0)$ *and* $\|(\vec{u})^*_R\|_{L^2(\Delta_{2R})} < \infty$. *Then*

$$\text{(i)} \qquad \iint_{Z_{3R/2}} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2 \le \frac{C}{R^2}\iint_{Z_{2R}} |\vec{u}|^2,$$

and

$$\text{(ii)} \qquad \iint_{Z_{3R/2}} |\partial_t^{1/2}(\vec{u}\eta)|^2 \le \frac{C}{R^2}\iint_{Z_{2R}} |\vec{u}|^2$$

where $\eta = \eta(t) \in C_0^\infty(\mathbb{R}), \eta \equiv 1$ *on* $(t_0 - R^2, t_0 + R^2)$, $\operatorname{supp}\eta \subset (t_0 - 2R^2, t_0 + 2R^2)$ *and* $|\eta'(t)| \le C/R^2$.

PROOF. By a parabolic rescaling, we may assume that $R = 1$. We write

$$\mu\,\Delta\vec{u} + (\lambda+\mu)\nabla(\operatorname{div}\vec{u}) = a_{ij}^{kl}\,\frac{\partial^2 u^l}{\partial x_i \partial x_j}$$

where $a_{ij}^{kl} = a_{ji}^{lk}$ are constants such that

$$a_{ij}^{kl}\,\frac{\partial u^l}{\partial x_j} N_i = \Big(\frac{\partial \vec{u}}{\partial \nu}\Big)^k$$

(see [DKV2, p. 800]). Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \equiv 1$ on $D_{3/2}$, $\operatorname{supp}\psi \cap \Omega \subset D_2$ and $|\nabla\psi| \le C$. It follows from equation (0.1), the vanishing of $\partial\vec{u}/\partial\nu$ on Δ_2 and integration by parts that

$$\int_{D_2} \frac{\partial u^k}{\partial t}\, u^k (\psi\eta)^2 + \int_{D_2} a_{ij}^{kl}\,\frac{\partial u^k}{\partial x_i}\,\frac{\partial u^l}{\partial x_j}(\psi\eta)^2 = -\int_{D_2} a_{ij}^{kl}\frac{\partial u^l}{\partial x_j}\, u^k\, 2\psi\,\frac{\partial \psi}{\partial x_i}\,\eta^2 .$$

Integrating in t, we obtain

$$\begin{aligned}\iint_{Z_2} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2(\psi\eta)^2 &\le C\iint_{Z_2} a_{ij}^{kl}\,\frac{\partial u^k}{\partial x_i}\,\frac{\partial u^l}{\partial x_j}(\psi\eta)^2 \\ &\le C\iint_{Z_2} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|\;|\vec{u}|\;|\psi\eta|\;|\nabla\psi|\;|\eta| \\ &\quad + C\iint_{Z_2} |\vec{u}|^2 .\end{aligned}$$

From this, (i) follows easily.

To see part (ii), we integrate by parts and obtain

$$\begin{aligned}&\iint_{Z_2} a_{ij}^{kl}\,\frac{\partial}{\partial x_i} I_{1/4}(u^k\eta)\,\frac{\partial}{\partial x_j}\,\partial_t^{3/4}(u^l\eta)\,\psi^2 \\ &\qquad = -\iint_{Z_2} I_{1/4}\,\Big(\frac{\partial u^l}{\partial t}\,\eta\Big)\partial_t^{3/4}(u^l\eta)\,\psi^2 \\ &\qquad\quad - \iint_{Z_2} a_{ij}^{kl}\,\frac{\partial}{\partial x_i}\,\big(I_{1/4}(u^k\eta)\big)\partial_t^{3/4}(u^l\eta)\,\frac{\partial}{\partial x_j}\,(\psi^2)\,.\end{aligned}$$

Note that

$$\begin{aligned}&\iint_{Z_2} a_{ij}^{kl}\,\frac{\partial}{\partial x_i} I_{1/4}(u^k\eta)\,\frac{\partial}{\partial x_j}\,\partial_t^{3/4}\,(u^l\eta)\,\psi^2 \\ &\qquad = \frac{1}{2}\iint_{Z_2} \frac{\partial}{\partial t}\Big(a_{ij}^{kl}\,\frac{\partial}{\partial x_i} I_{1/4}(u^k\eta)\,\frac{\partial}{\partial x_j}\,I_{1/4}\,(u^l\eta)\Big)\psi^2 \ge 0\end{aligned}$$

and

$$\iint_{Z_2} I_{1/4}\Big(\frac{\partial u^l}{\partial t}\eta\Big)\partial_t^{3/4}(u^l\eta)\ \psi^2 = \iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2 - \iint_{Z_2} I_{1/4}\Big(u^l\frac{\partial \eta}{\partial t}\Big)\partial_t^{3/4}(u^l\eta)\ \psi^2.$$

We have

$$\begin{aligned}\iint_{Z_2} &|\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2 \\ &\le C\Big(\iint_{Z_2} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2\Big)^{1/2}\Big(\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2\Big)^{1/2} \\ &\quad + C\Big(\iint_{Z_2} |\vec{u}|^2\Big)^{1/2}\Big(\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2\Big)^{1/2}.\end{aligned}$$

Thus,

$$\iint_{Z_2} |\partial_t^{1/2}(\vec{u}\eta)|^2\psi^2 \le C\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2 \quad \le C\iint_{Z_2} |\vec{u}|^2.$$

Part (ii) then follows.

Lemma 3.2. *Under the same assumptions as in Lemma* 3.1, *we have*

(i) $$\iint_{Z_R} |\nabla\vec{u}|^2 \le \frac{C}{R^2}\iint_{Z_{2R}} |\vec{u}|^2$$

(ii) $$\sup_{|t-t_0|<R^2}\Big(\int_{D_R} |\vec{u}(X,t)|^{5/2}dX\Big)^{2/5} \le \frac{C}{R^{13/10}}\Big(\iint_{Z_{2R}} |\vec{u}|^2\Big)^{1/2}$$

(iii) $$\iint_{\Delta_R} (\nabla\vec{u})_R^* + \iint_{\Delta_R} (\partial_t^{1/2}(\vec{u}\eta))_R^* \le C\,R^{1/2}\Big(\iint_{Z_{2R}} |\vec{u}|^2\Big)^{1/2}$$

where $\eta \in C_0^\infty(\mathbb{R}), \eta \equiv 1$ *on* $[t_0 - R^2, t_0 + R^2]$, supp $\eta \subset [t_0 - \frac{4}{3}R^2, t_0 + \frac{4}{3}R^2]$ *and* $|\eta'(t)| \le C/R^2$.

PROOF. By translation and rescaling, we may assume that $(P_0, t_0) = (0,0)$ and $R = 1$. Fix $\tau \in [1, 3/2]$, let $D_\tau = D_\tau(0)$ and then set

$$\vec{u}_\tau(X,t) = \int_{-4}^{t}\int_{D_\tau} \Gamma(X-Y, t-s)\vec{u}(Y,s)\eta'(s)dY\,ds$$

and $\vec{v} = \vec{u}_\tau - \vec{u}\eta$. Then

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} = \mu\, \Delta \vec{v} + (\lambda + \mu)\nabla(\text{div } \vec{v}) & \text{in } D_\tau \times (-4,4)\,, \\ \vec{v}\big|_{t=-4} = \vec{0}\,, \end{cases}$$

and $\|(\nabla \vec{u})^*\|_{L^2(\partial D_\tau \times(-4,4))} < \infty$. Hence, by the L^2-theory for the traction problem in [S], there exists $\vec{g}_\tau \in L^2(\partial D_\tau \times (-4,4))$, such that

$$\vec{v}(X,t) = \int_{-4}^{t} \int_{\partial D_\tau} \Gamma(X-Q, t-s)\vec{g}_\tau(Q,s)\, dQ ds$$

and

$$\|\vec{g}_\tau\|_{L^2(\partial D_\tau \times(-4,4))} \ \le C\ \|\frac{\partial \vec{v}}{\partial \nu}\|_{L^2(\partial D_\tau \times(-4,4))}$$

with C independent of τ.

Now, to see part (i), we use estimates (2.1) for $\Gamma(X,t)$ and obtain

$$\begin{aligned} \int_{-4}^{4}\int_{D_\tau} |\nabla \vec{v}|^2 &\le C \int_{-4}^{4}\int_{\partial D_\tau} |\vec{g}_\tau|^2 \\ &\le C \int_{-4}^{4}\int_{\partial D_\tau} \left|\frac{\partial \vec{v}}{\partial \nu}\right|^2 \\ &\le C \int_{-4}^{4}\int_{\partial D_\tau} |\nabla \vec{u}_\tau|^2 \ + \ C \int_{-4}^{4}\int_{D_2 \cap \partial D_\tau} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 \end{aligned}$$

where we used that $\partial \vec{u}/\partial \nu = \vec{0}$ on Δ_2. Therefore,

$$\begin{aligned} \int_{-1}^{1}\int_{D_1} |\nabla \vec{u}|^2 &\le C \int_{-4}^{4}\int_{D_\tau} |\nabla \vec{u}_\tau|^2 \ + \ C \int_{-4}^{4}\int_{D_\tau} |\nabla \vec{v}|^2 \\ &\le \int_{-4}^{4}\int_{D_\tau} |\vec{u}|^2 \ + \ \int_{-4}^{4} \int_{D_2 \cap \partial D_\tau} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2. \end{aligned}$$

Integrating in τ, we obtain

$$\begin{aligned} \int_{-1}^{1}\int_{D_1} |\nabla \vec{u}|^2 &\le C \int_{-4}^{4}\int_{D_2} |\vec{u}|^2 \ + \ C \int_{-4}^{4}\int_{D_{3/2}} |\nabla \vec{u} \ + \ (\nabla \vec{u})^{tr}|^2 \\ &\le \int_{-4}^{4}\int_{D_2} |\vec{u}|^2 \end{aligned}$$

where we used part (i) of Lemma 3.1. Part (i) is then proved.

To see part (ii), fix $t \in [-1,1]$. Since $\vec{u} = \vec{u}_\tau - \vec{v}$ on $D_1 \times [-1,1]$, we have

$$\|\vec{u}(\cdot,t)\|_{L^{5/2}(D_1)} \le \|\vec{u}_\tau(\cdot,t)\|_{L^{5/2}(D_\tau)} + \|\vec{v}(\cdot,t)\|_{L^{5/2}(D_\tau)} .$$

Using estimates (2.1) on $\Gamma(X,t)$, Minkowski's inequality and Young's inequality, it is not hard to see that

$$\|\vec{v}(\cdot,t)\|_{L^{5/2}(D_\tau)} \le C\,\|\vec{g}_\tau\|_{L^2(\partial D_\tau \times(-4,4))}$$

and

$$\|\vec{u}_\tau(\cdot,t)\|_{L^{5/2}(D_\tau)} \le C\,\|\vec{u}\|_{L^2(D_\tau\times(-4,4))} .$$

Thus,

$$\begin{aligned}\|\vec{u}(\cdot,t)\|_{L^{5/2}(D_1)} &\le C\,\|\vec{g}_\tau\|_{L^2(\partial D_\tau\times(-4,4))} + C\,\|\vec{u}\|_{L^2(D_\tau\times(-4,4))}\\ &\le C\,\|\vec{u}\|_{L^2(D_2\times[-4,4])} .\end{aligned}$$

Finally, part (iii) follows from the L^2-theory and part (i) in a similar fashion.

REMARK 3.3. The conclusions of lemmas 3.1 and 3.2 also hold if we replace the assumption $\partial\vec{u}/\partial\nu = \vec{0}$ on $\Delta_{2R}(P_0,t_0)$ by $\vec{u} = \vec{0}$ on $\Delta_{2R}(P_0, t_0)$. The proof is similar.

Lemma 3.4. *Suppose $\vec{u}$ is a solution of (0.1) in Ω_T, with $\|(\vec{u})^*\|_{L^2(\Sigma_T)} + \|(\nabla\vec{u})^*\|_{L^2(\Sigma_T)} < \infty$. Assume that , $\vec{u}\big|_{t=0} = \vec{0}$ and $\partial\vec{u}/\partial\nu = \vec{a}$ where $\vec{a}$ is a parabolic atom, i.e.,* $\operatorname{supp}\vec{a} \subset \Delta_r(P_0,t_0)$ *for some $(P_0,t_0) \in \Sigma_T$, $\|\vec{a}\|_{L^2(\Sigma_T)} \le 1/r^2$ and $\iint_{\Sigma_T} \vec{a} = \vec{0}$. Then, there exists $\varepsilon > 0$, such that*

$$\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \le C\, r^{2/p+2/q-3}$$

where $p \in (2-\varepsilon, 2]$ and $q \in (1,2]$.

PROOF. By the divergence theorem, we have, for $(X,t) \in \Omega_T$,

$$\vec{u}(X,t) = \mathcal{S}\Big(\frac{\partial\vec{u}}{\partial\nu}\Big)(X,t) - \mathcal{K}\big(\vec{u}\big|_{\Sigma_T}\big)(X,t) \tag{3.5}$$

where $\mathcal{S}$ and $\mathcal{K}$ are the single and double-layer potentials. Letting $X \to P \in \partial\Omega$ nontangentially in (3.5), we obtain

$$\left(\frac{1}{2}I + \widetilde{K}_\nu\right)\left(\vec{u}\big|_{\Sigma_T}\right) = \mathcal{S}\left(\frac{\partial \vec{u}}{\partial \nu}\right)\Big|_{\Sigma_T} = \mathcal{S}(\vec{a})\big|_{\Sigma_T}.$$

Recall that $\frac{1}{2}I + \widetilde{K}_\nu = R_T(\frac{1}{2}I + K_\nu)^* R_T$. Thus by Theorem 2.7 and duality, there exists $\varepsilon > 0$, such that $\frac{1}{2}I + \widetilde{K}_\nu$ is invertible on $L^{p,q}(\Sigma_T)$ for $p \in (2-\varepsilon, 2]$ and $q \in (1, 2]$. This, together with Corollary 2.15, implies that

$$\begin{aligned} \|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} &\le C\,\|\vec{u}\|_{L^{p,q}(\Sigma_T)} \\ &\le C\,\|(\tfrac{1}{2}I + \widetilde{K}_\nu)\vec{u}\|_{L^{p,q}(\Sigma_T)} \\ &\le C\,\|\mathcal{S}(\vec{a})\big|_{\Sigma_T}\|_{L^{p,q}(\Sigma_T)}. \end{aligned}$$

To estimate $\|\mathcal{S}(\vec{a})\big|_{\Sigma_T}\|_{L^{p,q}(\Sigma_T)}$, first note that

$$\begin{aligned} \|\mathcal{S}(\vec{a})\big|_{\Sigma_T}\|_{L^{p,q}(\Delta_{10r})} &\le C\,r^{2/p+2/q-2}\|\mathcal{S}(\vec{a})\big|_{\Sigma_T}\|_{L^2(\Delta_{10r})} \\ &\le C\,r^{2/p+2/q-1}\|\vec{a}\|_{L^2(\Sigma_T)} \\ &\le C\,r^{2/p+2/q-3}. \end{aligned}$$

Next, using estimate (2.1) and $\iint_{\Sigma_T} \vec{a} = \vec{0}$, we get

$$\left|\mathcal{S}(\vec{a})(P,t)\right| \le \begin{cases} \dfrac{C}{|P-P_0|^3}, & \text{if } |P-P_0| \ge 10\,r \text{ and } |t-t_0| < 100\,r^2, \\ \dfrac{C}{|t-t_0|^{3/2}}, & \text{if } |P-P_0| \le 10\,r \text{ and } |t-t_0| \ge 100\,r^2, \\ \dfrac{Cr}{\left(P-P_0| + |t-t_0|^{1/2}\right)^4}, & \text{if } |P-P_0| \ge 10\,r \text{ and } |t-t_0| \ge 100\,r^2. \end{cases}$$

An easy computation then gives the desired estimate.

We are now in a position to state and prove our main lemma which gives L^1-estimates for the nontangential maximal functions of solutions with atomic data.

Lemma 3.6. (Main Lemma) *Under the same assumptions as in Lemma* 3.4, *we have*

$$\iint_{\Sigma_T} (\nabla \vec{u})^* + \iint_{\Sigma_T} (\partial_t^{1/2} \vec{u})^* \leq C\,.$$

PROOF. By L^2-theory [S, Theorem 4.1.2]

$$\begin{aligned}\iint_{\Delta_{100r}(P_0,t_0)} (\nabla \vec{u})^* &\leq C\, r^2 \left(\iint_{\Delta_{100r}(P_0,t_0)} (\nabla \vec{u})^{*2} \right)^{1/2} \\ &\leq C\, r^2\, \|\vec{a}\|_{L^2(\Sigma_T)} \leq C\,.\end{aligned}$$

Thus, it is easy to see that the estimate for $(\nabla \vec{u})^*$ in the lemma will follow if we can prove that

$$\text{(3.9)} \qquad \iint_{\Lambda(R)} (\nabla \vec{u})^* \leq C\left(\frac{r}{R}\right)^{\alpha_0} \qquad \text{for some } \alpha_0 > 0$$

where $\Lambda(R) = \Delta_R(P_1, t_1)$ for some $(P_1, t_1) \in \Sigma_T$, $R \geq 5\,r$ and $\partial \vec{u}/\partial \nu = \vec{0}$ on $\Lambda(10\,R)$.

First, we estimate $(\nabla \vec{u})^{*R}$ on $\Lambda(R)$. Let $(Y, s) \in \gamma(P, t)$, $|Y - P| \geq R$, $(P, t) \in \Lambda(R)$. By interior estimates,

$$|\nabla \vec{u}(Y, s)| \leq \frac{C}{R^6} \iint_{\substack{|\zeta - s| < CR^2 \\ |Z - Y| \leq CR}} |\vec{u}(Z, \zeta)| dZ d\zeta \leq \frac{C}{R^5} \iint_{\Lambda(2R)} (\vec{u})^*\,.$$

Thus, if $(P, t) \in \Lambda(R)$,

$$(\nabla \vec{u})^{*R}(P, t) \leq \frac{C}{R^5} \iint_{\Lambda(2R)} (\vec{u})^*\,.$$

Consequently,

$$\iint_{\Lambda(R)} (\nabla \vec{u})^{*R} \leq \frac{C}{R} \iint_{\Lambda(2R)} (\vec{u})^* \leq C R^{3-2/q-2/p} \|(\vec{u})^*\|_{p,q}\,.$$

We have applied Hölder inequality for the last inequality. Choose $p \in (2 - \varepsilon, 2)$, ε as in Lemma 3.4 and q close to 1 such that

$$3 - \frac{2}{q} - \frac{2}{p} = -\alpha_1 < 0\,.$$

It then follows from Lemma 3.4 that

$$\iint_{\Lambda(R)} (\nabla\vec{u})^{*R} \le C\Big(\frac{r}{R}\Big)^{\alpha_1}, \quad \alpha_1 > 0\,.$$

Next, we estimate $(\nabla\vec{u})^*_R$ on $\Lambda(R)$. By Lemma 3.2,

$$\iint_{\Lambda(R)} (\nabla\vec{u})^*_R \le C\, R^{1/2}\left(\iint_{Z_{2R}(P_1,t_1)} |\vec{u}|^2\right)^{1/2}.$$

We use Hölder's inequality, then Lemma 3.4 and Lemma 3.2 (ii) and then Hölder's inequality again to obtain

$$\begin{aligned}\iint_{Z_{2R}(P_1,t_1)} |\vec{u}|^2 &= R^3\int_{|t-t_1|<4R^2}\left(\frac{1}{R^3}\int_{D_{2R}(P_1)} |\vec{u}|^2\right)\\ &\le C\,R^3\int_{|t-t_1|<4R^2}\left(\frac{1}{R^3}\int_{D_{2R}(P_1)} |u|^p\right)^{1/p}\\ &\qquad\left(\frac{1}{R^3}\int_{D_{2R}(P_1)} |u|^{5/2}\right)^{2/5}\\ &\le C\,R^{1/2-2/p}\int_{|t-t_1|<4R^2}\left(\int_{J_{3R}(P_1)} (\vec{u})^{*p}\right)^{1/p}\\ &\qquad\left(\iint_{Z_{3R}(P_1,t_1)} |\vec{u}|^2\right)^{1/2}\\ &\le C\,R^{5/2-2/p-2/q}\,\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)}\\ &\qquad\left(\iint_{Z_{3R}(P_1,t_1)} |\vec{u}|^2\right)^{1/2}.\end{aligned}$$

As before, choose $p\in(2-\varepsilon,2)$ and q close to 1 such that

$$\alpha_1 = \frac{2}{p}+\frac{2}{q}-3>0\,.$$

Let

$$a_R = R^{1/2}\Big(\iint_{Z_{2R}(P_1,t_1)} |\vec{u}|^2\Big)^{1/2}$$

and observe that

$$a_R^2 \le C\Big(\frac{r}{R}\Big)^{\alpha_1} a_{2R}\,. \tag{3.10}$$

By Lemma 3.4,

$$a_R \le CR\,\|(\vec{u})^*\|_{L^2(\Sigma_T)} \le C\Big(\frac{r}{R}\Big)^{-1}.$$

It follows from (3.10) that

$$a_R \le C\Big(\frac{r}{R}\Big)^{(\alpha_1-1)/2}$$

In general, if $a_R \le C\big(r/R\big)^{\beta_n}$, then, by (3.10) we obtain the improved estimate

$$a_R \le C\Big(\frac{r}{R}\Big)^{(\alpha_1+\beta_n)/2}$$

Consider the sequence $\{\beta_n\}, \beta_0 = (\alpha_1 - 1)/2$, $\beta_{n+1} = (\alpha_1 + \beta_n)/2$. Then

$$\beta_{n+1} - \beta_n = \frac{\alpha_1 - \beta_n}{2} \ge \frac{\alpha_1}{2} \quad \text{if} \quad \beta_n \le 0\,.$$

Therefore, there exists N such that $\beta_N > \alpha_1/4$. Hence,

$$a_R \le C\big(\frac{r}{R}\big)^{\alpha_1/4}$$

Let $\alpha_0 = \alpha_1/4$. The proof for (3.9) is complete.

We now give the proof that

$$\iint_{\Sigma_T} \Big(\partial_t^{1/2}\vec{u}\Big)^* \le C\,.$$

Again, we wish to show that

$$\iint_{\Lambda(R)} \Big(\partial_t^{1/2}\vec{u}\Big)^* \le C\Big(\frac{r}{R}\Big)^{\alpha_0}.$$

Note that we may only consider those $\Lambda(R) = \Delta_R(P_1, t_1)$ where $|t_1 - t_0| < CR^2$.

To proceed we let $\eta \in C_0^\infty(\mathbb{R}), \eta \equiv 1$ on $\big[t_1 - \frac{7}{6}R^2, t_1 + \frac{7}{6}R^2\big]$, $\operatorname{supp}\eta \subset \big[t_1 - \frac{4}{3}R^2, t_1 + \frac{4}{3}R^2\big]$ and $|\eta'(t)| \le C/R^2$. Then, by Lemma 3.2,

$$\iint_{\Lambda(R)} \Big(\partial_t^{1/2}(\vec{u}\eta)\Big)^* \le C\,R^{1/2}\Big(\iint_{Z_{2R}} |\vec{u}|^2\Big)^{1/2} + \frac{C}{R}\iint_{\Lambda(2R)} (\vec{u})^*$$

$$\le C\Big(\frac{r}{R}\Big)^{\alpha_0}.$$

The estimate for $\left(\partial_t^{1/2}(\vec{u}(1-\eta))\right)^*$ is easy. In fact, if $(P,t) \in \Lambda(R)$,

$$\left|\left[\partial_t^{1/2}(\vec{u}(1-\eta))\right]^*(P,t)\right| \le \frac{C}{R^3}\int_{t_0-R^2}^{t_1-R^2} |(\vec{u})^*(P,s)|ds$$

where we used $\vec{u}(Q,s) = \vec{0}$ if $s < t_0 - R^2$. Thus,

$$\begin{aligned}\iint_{\Lambda(R)} |\partial_t^{1/2}(\vec{u}(1-\eta))| &\le \frac{C}{R}\int_{t_0-R^2}^{t_1-R^2}\int_{I_R(P_1)} (\vec{u})^* \\ &\le C\,R^{3-2/p-2/q}\|\vec{u}\|_{L^{p,q}(\Sigma_T)} \\ &\le C\left(\frac{r}{R}\right)^{2/p+2/q-3}\end{aligned}$$

where $2/p + 2/q - 3 > 0$. The estimate for $(\partial_t^{1/2}\vec{u})^*$ then follows. This completes the proof of the main lemma.

We omit the proof of the following lemma which may be carried out using an argument similar to that of Lemma 3.6.

Lemma 3.11. *Suppose $\vec{u}$ is a solution of (0.1) in Ω_T with $\|(\vec{u})^*\|_{L^2(\Sigma_T)} + \|(\nabla\vec{u})^*\|_{L^2(\Sigma_T)} < \infty$ and $\vec{u}\big|_{t=0} = \vec{0}$. Assume that* supp $\vec{u}\big|_{\Sigma_T} \subset \Delta_r(P_0,t_0)$ *for some $(P_0,t_0) \in \Sigma_T$, r small and $\|\vec{u}\|_{L^2_1(\Sigma_T)} \le 1/r^2$. Then, there exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$,*

$$\iint_{\Sigma_T} \Big((\nabla\vec{u})^*(P,t) + (\partial_t^{1/2}\vec{u})^*(P,t)\Big) \cdot \Big(r + |P-P_0| + |t-t_0|^{1/2}\Big)^{\alpha}\, dP\,dt \le C\,r^{\alpha}.$$

4. Boundary Value Problems.

In this section we give the proofs for Theorem A, B, C, and D which were stated in the Introduction. We begin by introducing the matrix Green's function for (0.1) in Ω_T. Fix $X \in \Omega$, let $v^X(Y,s)$ be the matrix valued solution of (0.1) in Ω_T, which satisfies the boundary conditions

$$\begin{cases} v^X\big|_{\Sigma_T} = \Gamma(X-\cdot,\cdot)\big|_{\Sigma_T} \\ v^X\big|_{s=0} = 0\,. \end{cases}$$

By the L^2-theory for the Dirichlet problem with data in $L_1^2(\Sigma_T)$, such a solution exists and satisfies $\|(v^X)^*\|_{L^2(\Sigma_T)} + \|(\nabla v^X)^*\|_{L^2(\Sigma_T)} < \infty$. Let

$$G(X,Y,t,s) = G(X,Y,t-s) = \Gamma(X-Y,t-s) - v^X(Y,t-s)$$

be the matrix Green's function.

Lemma 4.1. *Let $X \in \Omega$, $P \in \partial\Omega$ and $|X-P| < 2\,\mathrm{dist}\{X,\partial\Omega\}$. Then, there exists $\alpha_0 > 0$, such that*

$$\iint_{\Sigma_T} |\nabla_Q G(X,Q,t)|\,(r + |Q-P| + t^{1/2})^\alpha dQds \le C\,r^\alpha$$

where $r = |X-P|$, $0 < \alpha < \alpha_0$.

PROOF. We may assume that r is small. Let $X^* \in \bar{\Omega}^C$ such that $|X^*-P| = r$ and $|X^*-P| < 2\,\mathrm{dist}\{X^*,\partial\Omega\}$. Let $\tilde{v}^X(Y,s)$ be the matrix valued solution of (0.1) in Ω_T such that

$$\begin{cases} \tilde{v}^X\big|_{\Sigma_T} = \Gamma(X-\cdot,\cdot) - \Gamma(X^*-\cdot,\cdot)\big|_{\Sigma_T} \\ \tilde{v}^X\big|_{s=0} = 0 \end{cases}$$

given by L^2-theory. We claim that there exist $\alpha_0 > 0$ such that

$$\iint_{\Sigma_T} |\nabla_Q \tilde{v}^X(Q,s)|(r + |Q-P| + s^{1/2})^\alpha dQds \le C\,r^\alpha\,, \tag{4.2}$$

for $0 < \alpha < \alpha_0$.

Assume (4.2) for a moment. We give the proof of the lemma. By the uniqueness for the initial-Dirichlet problem with L^2-data, it is easy to see that

$$G(X,Y,s) = \Gamma(X-Y,s) - \Gamma(X^*-Y,s) - \tilde{v}^X(Y,s)\,.$$

By estimate (2.1) for $\Gamma(X,t)$, we have, for $|Y-P| \ge 10$,

$$\begin{aligned} \big|\nabla_Y\big(\Gamma(X-Y,s) - \Gamma(X^*-Y,s)\big)\big| &\le \frac{Cr}{(|Y-P| + s^{1/2})^5} \\ \Big|\frac{\partial}{\partial s}\big(\Gamma(X-Y,s) - \Gamma(X^*-Y,s)\big)\Big| &\le \frac{Cr}{(|Y-P| + s^{1/2})^6} \end{aligned} \tag{4.3}$$

The lemma then follows easily from (4.2) and (4.3).

To prove (4.2), let

$$B(Q,t) = B^X(Q,t) = \Gamma(X-Q,t) - \Gamma(X^*-Q,t)\,, \quad (Q,t) \in \Sigma_T\,.$$

We may choose a partition of unity, $1 = \sum_{k\geq 0,\, j=1,2,\dots,N} \psi_{k,j}$ which satisfies

1) $\operatorname{supp}\psi_{k,j} \subset \Delta_{2^k r}(P_{k,j}, t_{k,j})$, $P_{0,j} = P$,
2) $2^k r \leq |P_{k,j} - P| + t_{k,j}^{1/2} \leq C\,2^k r$,
3) $|\psi_{k,j}| \leq 1$ and $|\nabla\psi_{k,j}| + 2^k r|\partial\psi_{k,j}/\partial t| \leq C\,(2^k r)^{-1}$.

Then

$$B(Q,t) = \sum \psi_{k,j} B(Q,t) = \sum 2^{-k} a_{k,j}(Q,t)$$

where $a_{k,j}(Q,t) = 2^k \psi_{k,j}(Q,t) B(Q,t)$. It follows from (4.3) and an argument in [B, Proposition 1.9] that

$$\|a_{k,j}\|_{L_1^2(\Sigma_T)} \leq \frac{C}{(2^k r)^2}\,.$$

Now, let $\tilde{v}_{k,j}$ be the solution to the initial-Dirichlet problem for (0.1) with $\tilde{v}_{k,j}\big|_{\Sigma_T} = a_{k,j}$. Clearly, $\tilde{v}^X = \sum 2^{-k}\tilde{v}_{k,j}$. By Lemma 3.11, we have, for $0 < \alpha < \alpha_0$,

$$\iint_{\Sigma_T} |\nabla_Q \tilde{v}_{k,j}(Q,s)|\big(2^k r + |Q - P_{k,j}| + |s - t_{k,j}|^{1/2}\big)^\alpha \leq C(2^k r)^\alpha\,.$$

Since

$$\begin{aligned}|Q-P| + |s|^{1/2} &\leq |Q - P_{k,j}| + |P_{k,j} - P\,| + |s - t_{k,j}|^{1/2} + |t_{k,j}|^{1/2}\\ &\leq C\,2^k r + |Q - P_{k,j}| + |s - t_{k,j}|^{1/2}\end{aligned}$$

it follows that

$$\iint_{\Sigma_T} |\nabla_Q \tilde{v}_{k,j}(Q,s)| \;\; (r + |Q-P| + s^{1/2})^\alpha dQds \leq C\,(2^k r)^\alpha.$$

Thus

$$\begin{aligned}\iint_{\Sigma_T} |\nabla_Q \tilde{v}(Q,s)| \;\; &\big(r + |Q-P| + s|^{1/2}\big)^\alpha dQds\\ &\leq C \sum 2^{-k}(2^k r)^\alpha \leq C\,r^\alpha\,.\end{aligned}$$

The claim (4.2) is proved and the proof is finished.

We are now in a position to prove Theorem A and B.

PROOF OF THEOREM A. The case $2-\varepsilon < p \le 2$ is contained in Corollary 2.15. Consider the case $2 < p \le \infty$. The uniqueness follows the uniqueness for L^2-solutions [S]. To see the existence, note that, by Lemma 4.1,

$$\iint_{\Sigma_T} |\nabla_Q G(X,Q,t)| dQdt \le C .$$

Hence

$$\begin{aligned} |\vec{u}(X,t)| &= \Big| \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,t-s)\vec{u}(Q,s)dQds \Big| \\ &\le C\, \|\vec{u}\|_{L^\infty(\Sigma_T)} . \end{aligned}$$

Thus, $\|\vec{u}\|_{L^\infty(\Omega_T)} \le C\|\vec{u}\|_{L^\infty(\Sigma_T)}$. The existence then follows by interpolation.

PROOF OF THEOREM B. Given $\vec{g} \in \Lambda_0^\alpha(\Sigma_T)$, $0 < \alpha < \alpha_0$, α_0 as in Lemma 4.1. Let $\vec{u}$ be the solution of (0.1), $\vec{u}\big|_{\Sigma_T} = \vec{g}$ and $\vec{u}\big|_{t=0} = \vec{0}$. To show $\|\vec{u}\|_{\Lambda_0^\alpha(\Omega_T)} \le C\,\|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}$, it is enough to prove

$$|\vec{u}(X_1,t) - \vec{u}(X_2,t)| \le C\,|X_1 - X_2|^\alpha \;\; \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)} \tag{4.4}$$

and

$$|\vec{u}(X,t_1) - \vec{u}(X,t_2)| \le C\,|t_1 - t_2|^{\alpha/2} \;\; \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)} . \tag{4.5}$$

The estimate (4.5) is easy. In fact, write

$$\vec{u}(X,t) = \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,t-s)\vec{g}(Q,S)dQds$$

$$= \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,s)\vec{g}(Q,t-s)dQds ,$$

where we have put $\vec{g}(Q,t) = \vec{0}$ for $t < 0$. From this, (4.5) follows easily.

To see (4.4), it suffices to show

$$|\nabla_X \vec{u}(X,t)| \leq \frac{C\,\|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}}{\operatorname{dist}\{X,\partial\Omega\}^{1-\alpha}}\,. \tag{4.6}$$

We write

$$\begin{aligned}
\vec{u}(X,t) &= \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,s)\vec{u}(Q,t-s)dQds \\
&= \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,s)\ \big(\vec{u}(Q,t-s) - \vec{u}(P,t)\big)dQds \\
&\quad + \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,s)\vec{u}(P,t)dQds \\
&= \mathrm{I}(X,t) + \mathrm{II}(X,t)\,,
\end{aligned}$$

where $P \in \partial\Omega$ and $|X-P| < 2\operatorname{dist}\{X,\partial\Omega\}$. By interior estimates and Lemma 4.1, we have

$$|\nabla_X I(X,t)| \leq \frac{C\,\|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}}{\operatorname{dist}\{X,\partial\Omega\}^{1-\alpha}}\,.$$

To estimate ∇_X II (X,t), let

$$v(X,t) = \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,t-s)dQds\,.$$

We wish to show that

$$|\nabla v(X,T)| \leq \frac{C}{r^{1-\alpha}} \tag{4.7}$$

where $r = \operatorname{dist}\{X,\partial\Omega\}$. Clearly, (4.7), together with the estimate for $\nabla_X I(X,t)$ yields (4.6).

To show (4.7), note that, v is a solution of (0.1) in Ω_T, $v\big|_{\Sigma_T}$ is the identity matrix and $v\big|_{t=0} = 0$. Hence, by Theorem A, $|v(X,t)| \leq C$.

Let

$$w(X,t) = \int_0^t \int_\Omega \Gamma(X-Y,t-s)v(Y,s)\eta'(s)dYds$$

where $\eta(0) = 0$, $\eta(t) = 1$ for $t > T/2$ and $|\eta'(t)| \leq C$. Then, $|\nabla w| + |\partial_t^{1/2} w| \leq C$. Let $v_1 = w - v\eta$. Then, v_1 is a solution of (0.1) in

Ω_T, $v_1\big|_{t=0} = 0$. Note that $\|v_1\|_{L_1^{p,q}(\Sigma_T)} \leq C$ for $p > 2, q > 2$. Choose $p \in (2, 2+\varepsilon_2)$, ε_2 as in Corollary 2.16 and $q > 2$ large such that $2/p + 2/q < 1$. It then follows from interior estimates, Hölder's inequality and Corollary 2.16 that

$$\begin{aligned} |\nabla_X v_1(X,t)| &\leq \frac{C}{r^{2/p+2/q}} \; \|(\nabla v_1)^*\|_{L^{p,q}(\Sigma_T)} \\ &\leq \frac{C}{r^{2/p+2/q}} \; \|v_1\|_{L_1^{p,q}(\Sigma_T)} \\ &\leq \frac{C}{r^{2/p+2/q}} \, . \end{aligned}$$

Thus,

$$\begin{aligned} |\nabla_X v(X,T)| &\leq |\nabla_X v_1(X,T)| + |\nabla_X w(X,T)| \\ &\leq \frac{C}{r^{2/p+2/q}} + C \\ &\leq \frac{C}{r^{1-\alpha}} \end{aligned}$$

where we assumed $\alpha_0 \leq 1 - 2/p - 2/q$. Thus, (4.7) is then proved.

We now turn to the proof of Theorem D.

PROOF OF THEOREM D. The case $2 \leq p < 2 + \varepsilon$ is contained in Corollary 2.15.

For $1 < p < 2$. the existence follows from Lemma 3.6 and interpolation. To show the uniqueness, let $\vec{u}$ be a solution of (0.1) in Ω_T, $\partial\vec{u}/\partial\nu = \vec{0}$, $\vec{u}\big|_{t=0} = \vec{0}$ and $\|(\nabla\vec{u})^*\|_p < \infty$. First, we assume that Ω is a smooth domain. Given $\vec{F} \in C_0^\infty(\Omega_T)$. It is well known that, there exists $\vec{v} \in C^\infty(\overline{\Omega}_T)$, such that

$$\begin{cases} \dfrac{\partial\vec{u}}{\partial t} + \mu\,\Delta\vec{v} + (\lambda+\mu)\nabla(\operatorname{div}\vec{v}) = \vec{F}\,, \\ \qquad\qquad\qquad\qquad \dfrac{\partial\vec{v}}{\partial\nu}\Big|_{\Sigma_T} = \vec{0}\,, \\ \qquad\qquad\qquad\qquad \vec{v}\big|_{t=T} = \vec{0}\,. \end{cases}$$

It then follows from integration by parts that

$$\int_0^T \int_\Omega \vec{u}\cdot\vec{F} = \iint_{\Sigma_T} \vec{u}\cdot\frac{\partial\vec{v}}{\partial\nu} - \iint_{\Sigma_T} \frac{\partial\vec{u}}{\partial\nu}\cdot\vec{v} = \vec{0}\,.$$

Since $\vec{F}$ is arbitary, we have $\vec{u} \equiv \vec{0}$ on Ω_T.

Next, if Ω is a general Lipschitz domain, let $\{\Omega_j\}$ be a sequence of smooth domains approximating Ω such that $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ ([V], Theorem 1.12). Let $\Sigma_T^j = \partial\Omega_j \times (0,T)$. By the existence and uniqueness results on $\Omega_i \times (0,T)$, we have

$$\|(\nabla \vec{u})^*\|_{L^p(\Sigma_T^j)} \le C\,\|\frac{\partial \vec{u}}{\partial \nu}\|_{L^p(\Sigma_T^j)} \to 0 \quad \text{as } j \to \infty\,.$$

Hence $\vec{u} \equiv 0$ on Ω_T. The proof is complete.

To establish Theorem C, we need the following lemma.

Lemma 4.8. *Let $\vec{u}$ be a solution of* (0.1) *in Ω_T. Assume that*

$$\|(\nabla \vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty \quad \textit{for some} \quad 1 < p,q < \infty\,.$$

Then

$$\|(\vec{u})^*\|_{L^{\bar{p},\bar{q}}(\Sigma_T)} < \infty$$

where $1 < \bar{p},\bar{q} < \infty$, $1+2/\bar{p}+2/\bar{q} = 2/p+2/q$ *and* $0 < 1/p - 1/\bar{p} < 1/2$, $0 < 1/q - 1/\bar{q} < 1/2$.

PROOF. By interior estimates, we have

$$(\vec{u})^*(P,t) \le C(\vec{u}) + C\iint_{\Sigma_T} \frac{(\nabla \vec{u})^*\ (Q,s)}{(|Q-P| + |t-s|^{1/2})^3}\,dQds$$

where $C(\vec{u}) < \infty$ is a constant depending on $\vec{u}$. From this, the lemma follows from Minkowski's inequality and the theorem on fractional integrals in one dimension.

We are now ready to prove Theorem C.

PROOF OF THEOREM C. The case $2 \le p < 2+\varepsilon$ is contained in Corollary 2.16. For $1 < p < 2$, as before, the existence follows from Lemma 3.11 and interpolation. To see uniqueness, let $\vec{u}$ be a solution of (0.1) in Ω_T, $\vec{u}\big|_{\Sigma_T} = \vec{0}$, $\vec{u}\big|_{t=0} = \vec{0}$ and $\|(\nabla\vec{u})^*\|_{L^p(\Sigma_T)} < \infty$. Let $\vec{g} \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$. By Corollary 2.16, there exists $\vec{v}$ satisfying (0.1) in Ω_T. $\vec{v}\big|_{\Sigma_T} = \vec{g}\big|_{\Sigma_T}$, $\vec{v}\big|_{t=0} = \vec{0}$ and $\|(\nabla\vec{v})^*\|_{L^{p^*,q^*}(\Sigma_T)} < \infty$ for $p^* \in [2, 2+\varepsilon_2), q^* \in [2,\infty)$. Choose p^*, q^*, such that $2/p^* + 2/q^* = 5 - 4/p$ (we may assume that p close to 1). Then, by Lemma 4.8, $\|(\vec{u})^*\|_{L^{(p^*)',(q^*)'}(\Sigma_T)} <$

∞ where $1/p^* + 1/(p^*)' = 1$ and $1/q^* + 1/(q^*)' = 1$. It then follows from integration by parts and a limiting argument that

$$\int_0^T \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu}(Q,t)\vec{g}(Q,T-t)\,dQ\,dt = \int_0^T \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu}(Q,t)\vec{v}(Q,T-t)dQdt$$
$$= \int_0^T \int_{\partial\Omega} \vec{u}(Q,t)\frac{\partial \vec{v}}{\partial \nu}(Q,T-t)dQdt = \vec{0}\,.$$

So $\partial\vec{u}/\partial\nu = \vec{0}$ on Σ_T. Therefore, by the uniqueness for traction problem (Theorem D), $\vec{u} \equiv \vec{0}$ in Σ_T. The proof is finished.

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