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A wavelet characterization for weighted Hardy Spaces

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Abstract. In this paper, we give a wavelet area integral characterization for weighted Hardy spaces $H^p(\omega)$, $0 < p < \infty$, with $\omega \in A_\infty$. Our wavelet characterization establishes the identification between $H^p(\omega)$ and $T_2^p(\omega)$, the weighted discrete tent space, for $0 < p < \infty$ and $\omega \in A_\infty$. This allows us to use all the results of tent spaces for weighted Hardy spaces. In particular, we obtain the isomorphism between $H^p(\omega)$ and the dual space of $H^{p'}(\omega)$, where $1 < p < \infty$ and $1/p + 1/p' = 1$, and the wavelet and the Carleson measure characterizations of BMO_ω. Moreover, we obtain interpolation between A_∞-weighted Hardy spaces $H^{p_1}(\omega)$ and $H^{p_2}(\omega)$, $1 \leq p_1 < p_2 < \infty$.

1. Introduction.

In this paper, we give a wavelet area integral characterization for weighted Hardy spaces $H^p(\omega)$, $0 < p < \infty$, with $\omega \in A_\infty$. Coifman and Meyer had earlier given a wavelet characterization for H^1, [9]. Our proof differs from [9], in that it follows from two good-λ inequalities between the non-tangential maximal function and the area integral function with respect to some wavelets. At the same time, our wavelet characterization establishes the identification between $H^p(\omega)$ and $H_0^p(\omega)$, the weighted discrete tent space, for $0 < p < \infty$ and $\omega \in A_\infty$. This allows us to use all the results of tent spaces for weighted Hardy spaces. In particular, we obtain the isomorphism between $H^p(\omega)$ and the dual space of $H^{p'}(\omega)$, where $1 < p < \infty$ and $1/p + 1/p' = 1$, and the wavelet and the Carleson measure characterizations of BMO_ω. Moreover, we obtain interpolation between A_∞-weighted Hardy spaces $H^{p_1}(\omega)$ and $H^{p_2}(\omega)$, $1 \le p_1 < p_2 < \infty$.

In Section 2, we will give the two good-λ inequalities and their proofs. In Section 3, we will give the wavelet characterization of weighted Hardy spaces and its corollaries.

2. Good-λ Inequalities.

A dyadic multiscale analysis of $L^2(\mathbb{R}^d)$ with respect to lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ is defined as an increasing sequence V_j of closed subspaces of $L^2(\mathbb{R}^d)$ with the following four properties [7]:

(1) $\bigcap V_j = \{0\}$, $\bigcup V_j$ is dense in $L^2(\mathbb{R}^d)$,
(2) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,
(3) for every $f \in V_0$ and every $\gamma \in \mathbb{Z}^d$, we have $f(x-\gamma) \in V_0$,
(4) there exist two constants $C_2 > C_1 > 0$ and a function $g \in V_0$ such that V_0 is the closed linear span of $g(x-\gamma)$, $\gamma \in \mathbb{Z}^d$ and

$$C_1\Big(\sum_{\gamma\in\mathbb{Z}^d} |\alpha_\gamma|^2\Big)^{1/2} \le \Big\| \sum_{\gamma\in\mathbb{Z}^d} \alpha_\gamma g(x-\gamma)\Big\|_2 \le C_2\Big(\sum_{\gamma\in\mathbb{Z}^d} |\alpha_\gamma|^2\Big)^{1/2}.$$

Denoting by W_j the orthogonal complement of V_j in V_{j+1}. There are $2^d - 1$ functions ψ_m, $1 \le m < 2^d$, such that $\psi_m(x-\gamma)$, $\gamma \in \mathbb{Z}^d$, $1 \le m < 2^d$ form an orthonormal basis of W_0, [7]. These functions ψ_m, $1 \le m < 2^d$ are called analyzing wavelets if they satisfy certain decay and moment vanishing conditions.

I. Daubechies discussed the existence of compactly supported wavelets in [4]. In fact, she showed that for any $n \in \mathbb{Z}^+$, there is a collection of functions $\{\psi^\varepsilon, \phi : \ \varepsilon = 1, 2, \ldots, 2^d - 1\}$ on $\mathbb{R}^d$ such that for some dyadic multiscale analysis $\{V_j\}$, $\phi \in V_0$ satisfies the property (4) and ψ^ε, $\varepsilon = 1, 2, \ldots, 2^d - 1$, $\in W_0$ are the wavelets corresponding to $\{V_j\}$. Moreover, they have the following properties

a) $\psi^\varepsilon \in C^1$,
b) ψ^ε is compactly supported, say, for some integer $m \ge 1$, supp $\psi^\varepsilon \subset [-m, m]^d$,

c) The collection $\{2^{jd/2}\psi^\varepsilon(2^j x-\gamma):\ j\in\mathbb{Z},\ \gamma\in\mathbb{Z}^d$, and $\varepsilon=1,2,\ldots,2^d-1\}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$,

d) $\displaystyle\int \psi^\varepsilon(x)x^k\,dx=0$, for $k=0,1,\ldots,n$,

e) ϕ is continuous and compactly supported, say, supp $\phi\subset[0,l]^d$ for some integer l,

f) For every $1\le\varepsilon<2^d$, $\psi^\varepsilon(x)$ is a finite linear combination of $\{\phi(x-\gamma),\ \gamma\in\mathbb{Z}^d\}$, *i.e.* there exist $m_\varepsilon\in\mathbb{Z}$ and $b^\varepsilon_\gamma\in\mathbb{R}$, $-m_\varepsilon\le\gamma\le m_\varepsilon$, such that

$$\psi^\varepsilon(x)=\sum_{\gamma=-m_\varepsilon}^{m_\varepsilon} b^\varepsilon_\gamma\phi(x-\gamma),$$

g) $\displaystyle\int\phi\,dx\neq 0$.

In this paper, we work with this collection of functions.

Let $B=\prod_{i=1}^d(\frac{\alpha_i}{2^\nu},\frac{\alpha_i+1}{2^\nu})$ be a dyadic cube in $\mathbb{R}^d$. We write

$$\psi_B(x)=2^{\nu d/2}\psi(2^\nu x-\alpha),\qquad \text{where } \alpha=(\alpha_1,\alpha_2,\ldots,\alpha_d)\in\mathbb{Z}^d$$

and

$$2kB=\prod_{i=1}^d\left(\frac{\alpha_i-k}{2^\nu},\frac{\alpha_i+k}{2^\nu}\right).$$

By property c), any testing function f can be written as

$$f(x)=\sum_\varepsilon\sum_{B\text{ dyadic}} a^\varepsilon_B\psi^\varepsilon_B(x),$$

where $a^\varepsilon_B=\langle f,\psi^\varepsilon_B\rangle$. Setting

$$N_{2k}f(x)=\sup_{\substack{Q\text{ dyadic}\\ 2kQ\ni x}}|\langle f,\phi_Q\rangle|\,|Q|^{-1/2}$$

$$Nf(x)=\sup_{\substack{Q\text{ dyadic}\\ Q\ni x}}|\langle f,\phi_Q\rangle|\,|Q|^{-1/2}$$

$$S_{2k}f(x)=\Big(\sum_\varepsilon\sum_{\substack{B\text{ dyadic}\\ 2kB\ni x}}|a^\varepsilon_B|^2|B|^{-1}\Big)^{1/2}$$

$$D_{2k}f(x) = \sup_{\varepsilon} \sup_{2kB \ni x} |\langle f, \psi_B^\varepsilon \rangle |B|^{-1/2}| = \sup_{\varepsilon} \sup_{2kB \ni x} |a_B^\varepsilon |B|^{-1/2}| .$$

We have

Theorem 2.1. *There exist constants $r_0 > 0$, and $k \in \mathbb{Z}^+$, such that for any test function f, which is a finite linear combination of $\{\psi_B^\varepsilon :$ B dyadic, $\varepsilon = 1, 2, \ldots, 2^d - 1\}$, for any $\lambda > 0$, and $0 < r < r_0$,*

$$|\{x : Nf(x) > 3\lambda,\ g_r^*(x) \le 1/2\}| \le C r^2 |\{x : Nf(x) > \lambda\}| ,$$

where $g_r = \chi_{\{S_{2k}f > r\lambda\}}$ and g_r^* is the Hardy-Littlewood maximal function of g_r, C is a constant.

Theorem 2.2. *There exist constants $C > 0$, $\delta_0 > 0$, and $k \in \mathbb{Z}^+$, such that for any $0 < \delta < \delta_0$, for any $\lambda > 0$, and for any test function f which is a finite linear combination of $\{\psi_B^\varepsilon :\ B$ dyadic, $\varepsilon = 1, 2, \ldots, 2^d - 1\}$,*

$$|\{x : S_k f(x) > 2\lambda,\ N_4 f(x) \le \delta\lambda,\ D_{2k}f(x) \le \delta\lambda\}| \le C\delta^2 |\{x : S_{2k}f(x) > \lambda\}| .$$

For simplicity, we only prove it for the one-dimensional case. The argument can be extended directly to higher dimensions.

In the following, we denote ψ^1 by ψ. All cubes Q, Q_1, B, etc. are dyadic cubes. All C's are constants, they need not to be equal in each appearance.

Lemma 2.3. *If $|B| \le |Q|$ or $2lQ \cap 2mB = \emptyset$, $\langle \psi_B, \phi_Q \rangle = 0$.*

The proof is trivial. $|B| = 2^{-i}$, $|Q| = 2^{-j}$, for some $i, j \in \mathbb{Z}$. We have $\psi_B \in W_i$, $\phi_Q \in V_j$. When $|B| \le |Q|$, *i.e.* $j \le i$, $V_j \subseteq V_i$. By definition, $W_i \subset V_{i+1}$ is the orthogonal complement of V_i in V_{i+1}. Therefore, $V_j \perp W_i$, which implies $\langle \psi_B, \phi_Q \rangle = 0$. On the other hand, because supp $\psi_B \subset 2mB$, supp $\phi_Q \subset 2lQ$, the condition $2lQ \cap 2mB = \emptyset$ implies supp $\psi_B \cap$ supp $\phi_Q = \emptyset$. This proves Lemma 2.3.

We first prove Theorem 2.1. Taking

$$f = \sum a_B \psi_B ,$$

where only finite number of a_B is nonvanishing. For any $x \in \{x : Nf(x) > \lambda\}$, there is a dyadic cube $Q \ni x$, such that

$$|\langle f, \phi_Q \rangle| \, |Q|^{-1/2} > \lambda .$$

This implies $Q \subset \{x : Nf(x) > \lambda\}$. Therefore $\{Nf > \lambda\}$ is a union of a collection $\mathcal{Q}_1$ of dyadic cubes. Meanwhile,

$$\begin{aligned} |\langle f, \phi_Q \rangle |Q|^{-1/2}| &\le \sum |a_B| \, |\langle \psi_B, \phi_Q \rangle| \, |Q|^{-1/2} \\ &\le C \sum |a_B| \, |Q|^{-1/2} \longrightarrow 0 , \quad \text{when } |Q| \longrightarrow +\infty . \end{aligned}$$

So we can pick up a collection $\mathcal{Q}$ of maximal dyadic cubes out of $\mathcal{Q}_1$, and

$$\{Nf > \lambda\} = \bigcup_{Q \in \mathcal{Q}} Q$$

is a disjoint union. Theorem 2.1 follows from the following Lemma. Taking $k = m + 2l$ in the definition of g_r ,

Lemma 2.4. *There exists a constant $r_0 > 0$, such that for any $0 < r < r_0$, and any $Q \in \mathcal{Q}$,*

$$|\{x \in Q : \; Nf(x) > 3\lambda, \; g_r^*(x) \le 1/2\}| \le C \, r^2 |Q| .$$

Setting

$$E = \{Nf > 3\lambda, \; g_r^* \le 1/2\} \cap Q ,$$

without loss of generality, we suppose $|E| \neq 0$. Otherwise the proof of Lemma 2.4 will be done. Taking $Q \in \mathcal{Q}$, we have

$$|\langle f, \phi_Q \rangle |Q|^{-1/2}| > \lambda$$

and for any $Q_1 \supseteq Q^*$, where Q^* is the father dyadic cube of Q,

$$|\langle f, \phi_{Q_1} \rangle |Q_1|^{-1/2}| \le \lambda .$$

Then for any $x \in E$,

$$\begin{aligned} Nf(x) &= \sup_{Q_1 \ni x} |\langle f, \phi_{Q_1} \rangle |Q_1|^{-1/2}| \\ &= \sup_{\substack{Q_1 \subseteq Q \\ Q_1 \ni x}} |\langle f, \phi_{Q_1} \rangle |Q_1|^{-1/2}| . \end{aligned} \tag{2.5}$$

Now for

$$f_Q = \sum_{Q \subset 2kB} a_B \psi_B \,,$$

we estimate

$$|\langle f_Q, \phi_{Q_1}\rangle |Q_1|^{-1/2}|\,, \qquad \text{for } Q_1 \subseteq Q\,.$$

Lemma 2.6. *For $Q_1 \subseteq Q$,*

$$|\langle f_Q, \phi_{Q_1}\rangle |Q_1|^{-1/2}| \leq \lambda + C \inf_{x \in Q} S_{2k} f(x)\,.$$

Proof.

$$\begin{aligned}
&\langle f_Q, \phi_{Q_1} |Q_1|^{-1/2} - \phi_{Q^*} |Q^*|^{-1/2}\rangle \\
&\qquad\qquad = \sum_{Q \subset 2kB} a_B \langle \psi_B, \phi_{Q_1} |Q_1|^{-1/2} - \phi_{Q^*} |Q^*|^{-1/2}\rangle.
\end{aligned}$$

Suppose

$$Q_1 = \left(\frac{\alpha}{2^\nu}, \frac{\alpha+1}{2^\nu}\right), \; B = \left(\frac{\beta}{2^\mu}, \frac{\beta+1}{2^\mu}\right), \; \text{and } Q^* = \left(\frac{\gamma}{2^\iota}, \frac{\gamma+1}{2^\iota}\right)\,.$$

Then $Q_1 \subset Q^*$ implies $\iota < \nu$ and $\left|\frac{\gamma}{2^\iota} - \frac{\alpha}{2^\nu}\right| \leq \frac{1}{2^\iota}$, $Q \subset 2kB$ implies $|Q| \leq 2k|B|$, then $2^{-\iota} \leq 4\,k\,2^{-\mu}$, *i.e.* $\mu \leq \iota + \log_2 4k$. Because $\psi \in C^1$, and ψ is compactly supported,

$$\begin{aligned}
&|B|^{1/2} |\langle \psi_B, \phi_{Q_1} |Q_1|^{-1/2} - \phi_{Q^*} |Q^*|^{-1/2}\rangle| \\
&\qquad = \left| \int \psi(2^\mu x - \beta)(2^\nu \phi(2^\nu x - \alpha) - 2^\iota \phi(2^\iota x - \gamma))\, dx \right| \\
&\qquad = \left| \int \psi(2^\mu x - \beta)\phi(2^\nu x - \alpha)\, d(2^\nu x - \alpha) \right. \\
&\qquad\qquad \left. - \int \psi(2^\mu x - \beta)\phi(2^\iota x - \gamma)\, d(2^\iota x - \gamma) \right| \\
&\qquad = \left| \int (\psi(2^{\mu-\nu} x + 2^{\mu-\nu}\alpha - \beta) \right. \\
&\qquad\qquad \left. - \psi(2^{\mu-\iota} x + 2^{\mu-\iota}\gamma - \beta))\phi(x)\, dx \right| \\
&\qquad \leq \int_0^l \|\phi\|_\infty \|\psi'\|_\infty |2^{\mu-\nu} x + 2^{\mu-\nu}\alpha - 2^{\mu-\iota} x - 2^{\mu-\iota}\gamma|\, dx
\end{aligned}$$

$$\leq C\, 2^{\mu-\iota}.$$

Therefore,

$$|\langle f_Q, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle| \leq \sum_{Q\subset 2kB} |a_B||B|^{-1/2}\, C\, 2^{\mu-\iota}$$

$$\leq C \inf_{x\in Q} S_{2k}f(x)\Big(\sum_{\mu=-\infty}^{\iota+\log_2 4k} (2^{\mu-\iota})^2\Big)^{1/2}$$

$$\leq C \inf_{x\in Q} S_{2k}f(x)\,.$$

Because $|E| \neq 0$, there exists $x \in E \subset Q$, such that $g_r^*(x) \leq 1/2$, which implies $S_{2k}f(x) \leq r\lambda$. Taking $r_0 = 1/C$, where C is the constant appeared in the last inequality, by Lemma 2.3, we have

$$(2.7)\quad |\langle f_Q, \phi_{Q_1}\rangle |Q_1|^{-1/2}| \leq \lambda + Cr\lambda \leq 2\lambda\,, \qquad \text{for all } 0 < r \leq r_0\,.$$

Now setting

$$E_1 = \{x:\ g_r^*(x) \leq 1/2\} \cap 2\,kQ\,,$$
$$E_2 = \{x:\ S_{2k}f(x) \leq r\lambda\} \cap 10\,kQ\,,$$

and

$$U_i = V \cap \cup_{x\in E_i}\{B:\ 2kB \ni x\},$$

where $i = 1, 2$,

$$V = \{B:\ B \subset 2kQ,\ |B| < |Q| \text{ and } Q \not\subset 2kB\}\,.$$

Setting

$$f_1 = \sum_{B\in U_1} a_B \psi_B\,,$$

we prove

Lemma 2.8. *For any $x \in E$, $Nf_1(x) \geq \lambda$.*

PROOF. Setting

$$V_{Q_1} = \{B:\ |Q_1| < |B|, \text{ and } 2\,l\,Q_1 \cap 2\,mB \neq \emptyset\}\,,$$

we have $V_{Q_1} \subset \{B:\ Q_1 \subset 2kB\}$. Taking $x \in Q_1 \subset Q$, by Lemma 2.3,

$$\begin{aligned}\langle f, \phi_{Q_1}\rangle |Q_1|^{-1/2} &= \sum_{\substack{x\in 2kB\\ B\in V_{Q_1}}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &= \sum_{\substack{x\in 2kB \\ B\in V_{Q_1}\setminus\{B:\ Q\subset 2kB\}}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &\quad + \sum_{\substack{x\in 2kB\\ Q\subset 2kB}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}.\end{aligned}$$

It is easy to check that

$$V_{Q_1} \setminus \{B:\ Q \subset 2kB\} \subset V .$$

Therefore,

$$\begin{aligned}\langle f, \phi_{Q_1}\rangle |Q_1|^{-1/2} &= \sum_{\substack{x\in 2kB\\ B\in V}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &\quad + \sum_{\substack{x\in 2kB\\ Q\subset 2kB}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}.\end{aligned}$$

By (2.5) and (2.7),

$$\sup_{x\in Q_1\subset Q} \Big| \sum_{\substack{x\in 2kB\\ B\in V}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \Big| \geq \lambda ,$$

for any $x \in E$. Because

$$\langle f_1, \phi_{Q_1}\rangle |Q_1|^{-1/2} = \sum_{\substack{x\in 2kB\\ B\in U_1}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} ,$$

and for any $x \in E \subset E_1$,

$$\{B:\ B \in U_1,\ x \in 2kB\} = \{B:\ B \in V,\ x \in 2kB\} ,$$

we have

$$N f_1(x) = \sup_{Q_1 \ni x} |\langle f_1, \phi_{Q_1}\rangle |Q_1|^{-1/2}| \geq \lambda , \qquad \text{for all } x \in E .$$

Now we can start to prove Lemma 2.4. Because

$$f_1 = \sum_{B \in U_1} a_B \psi_B \,,$$

$$S_{2k} f_1(x) \le S_{2k} f(x)\,, \qquad \text{for all } x \in \mathbb{R}\,.$$

So

$$\int_{E_2} (S_{2k} f_1(x))^2 \, dx \le \int_{E_2} (S_{2k} f(x))^2 \, dx \le C\, r^2\, \lambda^2 |Q|\,,$$

and

$$\begin{aligned}\int_{E_2} (S_{2k} f_1(x))^2 \, dx &= \int_{E_2} \sum_{\substack{x \in 2kB \\ B \in U_1}} |a_B|^2 |B|^{-1} \, dx \\ &= \sum_{B \in U_1} |a_B|^2 |B|^{-1} |2kB \cap E_2|\,.\end{aligned}$$

For any $B \in U_1$, there exists $x \in E_1$, such that $B \in V$ and $2kB \ni x$, therefore $2kB \subset 10\, kQ$, so,

$$2kB \cap \{x : \ S_{2k} f(x) \le r\lambda\} = 2kB \cap E_2 \,.$$

The fact $x \in E_1$ implies $g_r^*(x) \le 1/2$, then

$$|2kB, S_{2k} f > r\lambda| \le \frac{1}{2}\, |2kB|\,.$$

Therefore we have

$$|2kB, S_{2k} f(x) \le r\lambda| = |2kB \cap E_2| \ge \frac{1}{2}\, |2kB|\,.$$

Consequently,

$$\begin{aligned}\int (S_{2k} f_1(x))^2 \, dx &= \sum_{B \in U_1} |a_B|^2 |B|^{-1} |2kB| \\ &\le 2 \int_{E_2} (S_{2k} f_1(x))^2 \, dx \\ &\le C\, r^2\, \lambda^2 |Q|\,.\end{aligned}$$

On the other hand, for any $g = \sum c_B \psi_B$, we have

$$\begin{aligned}\int (S_{2k}g)^2\,dx &= \int \sum_{x\in 2kB} |c_B|^2 |B|^{-1}\,dx \\ &= C\sum |c_B|^2 = C\int |g|^2\,dx\,.\end{aligned}$$

Because $Nf \le f^*$, where f^* is the Hardy-Littlewood maximal function of f, we have

$$\begin{aligned}\lambda^2|E| &\le \int_E (Nf_1(x))^2\,dx \le C\int f_1^2(x)\,dx \\ &= C\int (S_{2k}f_1(x))^2\,dx \le C\,r^2\,\lambda^2|Q|\,.\end{aligned}$$

So

$$|E| \le C\,r^2|Q| \qquad \text{for } 0 < r < r_0\,.$$

This completes the proof of Lemma 2.4 and then Theorem 2.1.

To prove Theorem 2.2, we rewrite $S_{2k}f$ as

$$S_{2k}f(x) = \sup_{\substack{Q\ni x\\ Q \text{ dyadic}}} \Big(\sum_{Q\subset 2kB} |a_B|^2|B|^{-1}\Big)^{1/2}.$$

This equality holds for a.e. $x \in \mathbb{R}$. Therefore

$$\{x:\ S_{2k}f(x) > \lambda\} = \bigcup_{Q\in\Re_1} Q,$$

where

$$\Re_1 = \{Q:\ Q \text{ dyadic}, \Big(\sum_{Q\subset 2kB} |a_B|^2|B|^{-1}\Big)^{1/2} > \lambda\}\,.$$

Because $\sum_{Q\subset 2kB} |a_B|^2|B|^{-1} \longrightarrow 0$, as $|Q| \longrightarrow +\infty$, we can pick up a set $\Re$ of maximal dyadic cubes out of $\Re_1$, and

$$\{x:\ S_{2k}f(x) > \lambda\} = \bigcup_{Q\in\Re} Q\,.$$

Theorem 2.2 follows whenever we prove the following lemma. Taking $k = 8m + 8l$,

Lemma 2.9. *For any* $Q \in \Re$,

$$|\{x \in Q : S_k f(x) > 2\lambda, N_4 f(x) \leq \delta\lambda, D_{2k} f(x) \leq \delta\lambda\}| \leq C\delta^2 |Q| .$$

For any $Q \in \Re$, we have

$$(2.10) \qquad \begin{aligned} &\Big(\sum_{Q\subset 2kB} |a_B|^2 |B|^{-1}\Big)^{1/2} > \lambda, \quad \text{and} \\ &\Big(\sum_{Q^*\subset 2kB} |a_B|^2 |B|^{-1}\Big)^{1/2} \leq \lambda, \end{aligned}$$

where Q^* is the father dyadic cube of Q. Setting

$$V = \{B : B \text{ dyadic}, \; B \subset 2kQ^*, \; |B| < |Q^*| \text{ and } Q^* \not\subset 2kB\}$$

and

$$f_V = \sum_{B\in V} a_B \psi_B ,$$

and setting

$$E = \{x \in Q : S_k f(x) > 2\lambda, N_4 f(x) \leq \delta\lambda, D_{2k} f(x) \leq \delta\lambda\},$$

we have

Lemma 2.11. *For any* $x \in E$, $S_k f_V(x) > \lambda$.

PROOF. Taking $x \in E$, it is easy to check that for any $Q_1 \ni x$,

$$\{B : kB \supset Q_1, B \in V\} \cup \{B : 2kB \supset Q^*, kB \supset Q_1\} \supset \{B : kB \supset Q_1\}.$$

Therefore we have

$$\sum_{\substack{kB\supset Q_1\\ B\in V}} |a_B|^2 |B|^{-1} + \sum_{\substack{2kB\supset Q^*\\ kB\supset Q_1}} |a_B|^2 |B|^{-1} \geq \sum_{kB\supset Q_1} |a_B|^2 |B|^{-1}.$$

Because $S_k f(x) > 2\lambda$, *i.e.*

$$\sup_{Q_1 \ni x} \sum_{kB \supset Q_1} |a_B|^2 |B|^{-1} > 4\lambda^2,$$

there exists $Q_1 \ni x$, such that

$$\sum_{kB \supset Q_1} |a_B|^2 |B|^{-1} > 4\lambda^2.$$

By (2.10),

$$\begin{aligned} 4\lambda^2 &< \sum_{kB \supset Q_1} |a_B|^2 |B|^{-1} \\ &\le \sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} + \sum_{\substack{2kB \supset Q^* \\ kB \supset Q_1}} |a_B|^2 |B|^{-1} \\ &\le \sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} + \lambda^2. \end{aligned}$$

Consequently,

$$\sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} \ge 3\lambda^2,$$

and then

$$\sup_{Q_1 \ni x} \sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} > \lambda^2,$$

i.e.

$$S_k f_V(x) > \lambda.$$

Lemma 2.12. *There exists a constant $C > 0$, such that for any $x \in E$, $N_4 f_V(x) < C\delta\lambda$.*

Proof. Taking $x \in E$. Because $N_4 f(x) \le \delta\lambda$,

$$|\langle f, \phi_{Q^*}\rangle |Q^*|^{-1/2}| = |\sum_{2kB \supset Q^*} a_B \langle \psi_B, \phi_{Q^*}\rangle |Q^*|^{-1/2}| \le \delta\lambda.$$

Now for any Q_1, such that $4Q_1 \ni x$, and $|Q_1| \le |Q^*|$, checking as for the case in Lemma 2.8, we have

$$\langle f, \phi_{Q_1}\rangle |Q_1|^{-1/2} = \sum_{B\in V} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} + \sum_{2kB\supset Q^*} a_B\langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}, \tag{2.13}$$

and by a same argument as that for Lemma 2.6,

$$\begin{aligned}
&\Big|\sum_{2kB\supset Q^*} a_B\langle\psi_B,\phi_{Q_1}\rangle|Q_1|^{-1/2} - \sum_{2kB\supset Q^*} a_B\langle\psi_B,\phi_{Q^*}\rangle|Q^*|^{-1/2}\Big| \\
&\le \sum_{2kB\supset Q^*} |a_B|\,|B|^{-1/2}|B|^{1/2}|\langle\psi_B, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle| \\
&\le C \sum_{2kB\supset Q^*} |a_B|\,|B|^{-1/2} \\
&\le C\, D_{2k}f(x) \le C\delta\lambda\,.
\end{aligned}$$

Therefore,

$$\Big|\sum_{2kB\supset Q^*} a_B\langle\psi_B,\phi_{Q_1}\rangle|Q_1|^{-1/2}\Big| \le \delta\lambda + C\delta\lambda\,.$$

From (2.13)

$$\Big|\sum_{B\in V} a_B\langle\psi_B,\phi_{Q_1}\rangle|Q_1|^{-1/2}\Big| \le 2\delta\lambda + C\delta\lambda\,.$$

For Q_1 such that $4Q_1 \ni x$ and $|Q_1| > |Q^*|$, because $x \in 4Q_1 \cap Q^*$, $Q^* \subset 4Q_1$. Therefore $Q_1 \subset 2kB/4$ implies that $Q^* \subset 2kB$. Meanwhile,

$$\begin{aligned}
\langle f_V, \phi_{Q_1}\rangle|Q_1|^{-1/2} &= \sum_{\substack{B\in V\\ Q_1\subset 2kB/4}} a_B\langle\psi_B,\phi_{Q_1}\rangle|Q_1|^{-1/2} \\
&= \sum_{\substack{B\in V\\ Q^*\subset 2kB}} a_B\langle\psi_B,\phi_{Q_1}\rangle|Q_1|^{-1/2}.
\end{aligned}$$

Then the definition of V gives

$$\langle f_V, \phi_{Q_1}\rangle|Q_1|^{-1/2} = 0\,.$$

This proves that $N_4 f_V(x) < (2 + C)\,\delta\,\lambda$, for $x \in E$.

From Lemma 2.11 and Lemma 2.12,

$$E \subset \{x \in Q \,:\, N_4 f_V(x) \le C\delta\lambda,\, S_k f_V(x) > \lambda,\, D_{2k} f(x) \le \delta\lambda\}\,.$$

Obviously we have supp $N f_V \subset \alpha Q$, for some large constant α. Setting

$$E_1 = \alpha Q \cap \{N_4 f_V \le C\delta\lambda,\, D_{2k} f \le \delta\lambda\}$$

and

$$W = V \cap \cup_{x \in E_1} \{B \,:\, 2kB \ni x\},$$

defining

$$f_W = \sum_{B \in W} a_B \psi_B,$$

we have $E_1 \supset E$, and for any $x \in E$,

$$\begin{aligned} S_k f_W(x) &= \Big(\sum_{\substack{kB \ni x \\ B \in W}} |a_B|^2 |B|^{-1} \Big)^{1/2} \\ &= \Big(\sum_{\substack{kB \ni x \\ B \in V}} |a_B|^2 |B|^{-1} \Big)^{1/2} = S_k f_V(x) > \lambda\,. \end{aligned}$$

For any $x \in E_1$,

$$\begin{aligned} N_4 f_W(x) &= \sup_{4Q_1 \ni x} \Big| \sum_{\substack{B \in W \\ 2kB \ni x}} a_B \langle \psi_B, \phi_{Q_1} \rangle |Q_1|^{-1/2} \Big| \\ &= \sup_{4Q_1 \ni x} \Big| \sum_{\substack{B \in V \\ 2kB \ni x}} a_B \langle \psi_B, \phi_{Q_1} \rangle |Q_1|^{-1/2} \Big| \\ &= N_4 f_V(x) \le C\delta\lambda\,. \end{aligned}$$

And also for $x \notin \alpha Q$, we have

$$N f_W(x) = 0\,.$$

Lemma 2.14. *There exists a constant $C > 0$, such that for any $x \in \mathbb{R}$, $N f_W(x) \le C\delta\lambda$.*

PROOF. We need only prove for $x \in \alpha Q \cap E_1^c$. Setting

$$\begin{aligned}\Omega &= \alpha\, Q \cap E_1^c \\ &= \alpha\, Q \cap (\{x : N_4 f_V(x) > C\delta\lambda\} \cup \{x : D_{2k} f(x) > \delta\lambda\}).\end{aligned}$$

Then Ω is an open set. Therefore Ω is a union of a collection $\Im$ of disjoint open intervals. (In higher dimensional case, we use Whitney decomposition.) Taking $I \in \Im$, there exist at most two dyadic cubes C_1 and C_2, such that $|C_1| = |C_2| \sim |I|$ and $I \subset \overline{C_1} \cup \overline{C_2} \subset 4C_1$, $4C_1 \cap E_1 \neq \emptyset$. Now for any $x \in I$, setting

$$\mathcal{B} = \{B : B \in W, 2kB \ni x\},$$

if $\mathcal{B} = \emptyset$,

$$\begin{aligned}Nf_W(x) &= \sup_{Q_1 \ni x} \langle f_W, \phi_{Q_1} \rangle |Q_1|^{-1/2} \\ &= \sup_{Q_1 \ni x} \sum_{\substack{2kB \ni x \\ B \in W}} a_B \langle \psi_B, \phi_{Q_1} \rangle |Q_1|^{-1/2} = 0\,.\end{aligned}$$

In case $\mathcal{B} \neq \emptyset$, taking $B_1 \in \mathcal{B}$ such that

$$|B_1| \le 2 \inf_{\mathcal{B}} |B|\,,$$

then for any $B \in \mathcal{B}$, the fact that $x \in 2kB_1 \cap 2kB$ implies that

$$2kB_1 \subset 12\,kB\,.$$

By the definition of W, there is a $y \in E_1$, such that $2kB_1 \ni y$. Therefore we have $2kB_1 \cap \partial I \neq \emptyset$. And taking a dyadic cube $\mathbf{B}$ out of $2kB$, which has the same size as B_1, $\mathbf{B} \subset 2kB_1$, such that $\overline{\mathbf{B}} \cap \partial I \neq \emptyset$. Then $\mathbf{B} \subset 12k\, B$ for any $B \in \mathcal{B}$, *i.e.*

$$\mathcal{B} = \{B \in \mathcal{B} : \ \mathbf{B} \subset 12\,kB\}$$

and also $E_1 \cap 4\mathbf{B} \neq \emptyset$. Now for $Q_1 \ni x$, consider

$$\sum_{B \in W} a_B \langle \psi_B, \phi_{Q_1} \rangle |Q_1|^{-1/2}.$$

1) If $4Q_1 \cap E_1 \neq \emptyset$, from $N_4 f_W(x_0) < C\delta\lambda$ for any $x_0 \in E_1$,

$$|\sum_{B \in W} a_B \langle \psi_B, \phi_{Q_1} \rangle |Q_1|^{-1/2}| < C\delta\lambda\,.$$

2) If $4Q_1 \cap E_1 = \emptyset$, $\operatorname{dist}\{x, E_1\} \geq |Q_1|$. Because $4k\mathbf{B} \ni x$ and $4k\mathbf{B} \ni y$, $y \in E_1$, we have $4k|B_1| \geq |Q_1|$. And $4k\mathbf{B} \cap Q_1 \neq \emptyset$ implies $Q_1 \subset 12k\mathbf{B}$. Therefore, by 1) and by a same argument as that for Lemma 2.6,

$$\begin{aligned} |\sum_{\substack{B \in W \\ 2kB \ni x}} a_B \langle \psi_B, \phi_{Q_1} \rangle |Q_1|^{-1/2}| &\leq |\sum_{\substack{B \in W \\ 12kB \supset \mathbf{B}}} a_B < \psi_B, \phi_{\mathbf{B}} > |\mathbf{B}|^{-1/2}| \\ &\quad + |\sum_{\substack{B \in W \\ 12kB \supset \mathbf{B}}} a_B \langle \psi_B, \phi_{Q_1} |Q_1|^{-1/2} - \phi_{\mathbf{B}} |\mathbf{B}|^{-1/2} \rangle| \\ &\leq C\delta\lambda + C \sum_{B \in W} |a_B|\, |B|^{-1/2} \\ &\leq C\delta\lambda\,. \end{aligned}$$

This proves Lemma 2.14.

Now we can prove Lemma 2.9. Because $S_k f_W(x) > \lambda$ for $x \in E$, $N f_W(x) < C\delta\lambda$ for $x \in \alpha Q$ and supp $N f_W \subset \alpha Q$, we have

$$\lambda^2 m(E) \leq \int_E (S_k f_W(x))^2\, dx \leq C \int (N f_W)^2\, dx \leq C\delta^2 \lambda^2 m(Q)\,.$$

So,

$$m(E) \leq C\delta^2 m(Q)\,.$$

This proves Lemma 2.9 and then Theorem 2.2.

By the property f) it is easy to check that

$$D_{2k} f(x) \leq C N_{4k} f(x) \qquad \text{a.e. } x\,,$$

where $k = 8\,m + 8\,l$. And by a similar argument as that in [3], we can prove that

$$\|S_{2k} f\|_{L^p(\omega)} \approx \|S_{2l} f\|_{L^p(\omega)}\,,$$

for $0 < p < \infty$, $\omega \in A_\infty$ and $k, l \in \mathbb{Z}^+$. Also

$$|\{N_{2k} f > \lambda\}| \leq C\,|\{N_2 f > \lambda\}|\,.$$

Then, as a direct consequence of Theorem 2.1 and Theorem 2.2, we have

Corollary 2.15. *For any $\omega \in A_\infty$ and $0 < p < \infty$, there exist constants C_1 and C_2, such that for any test function f, which is a linear combination of $\{\psi_B^\varepsilon$, B dyadic, $\varepsilon = 1, 2, \dots, 2^d - 1\}$,*

$$C_1 \, \|N_2 f\|_{L^p(\omega)} \le \|S_2 f\|_{L^p(\omega)} \le C_2 \, \|N_2 f\|_{L^p(\omega)} \, .$$

REMARK. Because of property c), Corollary 2.15 is true for any f with $\|S_2 f\|_{L^p(\omega)} < \infty$.

The Main Results.

In this section, we will give the wavelet area integral characterization of the weighted Hardy spaces $H^p(\omega)$, $0 < p < \infty$, with $\omega \in A_\infty$, which establishes the identification between $H^p(\omega)$ and $H_0^p(\omega)$, the weighted discrete tent space. Therefore, a series of corollaries parallel to those of tent spaces follows [3]. Because most of the proofs are almost the same as those in [3], we omit them. For simplicity, we only discuss the one-dimensional case.

In Section 2, we proved that for $0 < p < \infty$, $\omega \in A_\infty$,

$$\|N_2 f\|_{L^p(\omega)} \approx \|S_2 f\|_{L^p(\omega)} \, .$$

Define

$$H_0^p(\omega) = \{f \; : \; N_2 f \in L^p(\omega)\} = \{f \; : \; S_2 f \in L^p(\omega)\} \, ,$$

with

$$\|f\|_{H_0^p(\omega)} = \|N_2 f\|_{L^p(\omega)} \, .$$

And for $H_0^p(\omega)$, $0 < p \le 1$, we define an atom of $H_0^p(\omega)$ to be a function a which satisfies that for some cube R,

$$a = \sum_{\substack{I \subset R \\ I \text{ dyadic}}} a_I \psi_I \tag{A1}$$

$$\sum_{I \subset R} |a_I|^2 \frac{\omega(I)}{|I|} \le \omega(R)^{1-2/p}. \tag{A2}$$

Because for a an atom of $H_0^p(\omega)$,

$$\begin{aligned}\int |a(x)|^2\omega(x)\,dx &\le \int |N_2a(x)|^2\omega(x)\,dx\\ &\le C\int |S_2a(x)|^2\omega(x)\,dx\\ &\le C\sum_{I\subset R}|a_I|^2\frac{\omega(I)}{|I|}\\ &\le C\,\omega(R)^{1-2/p},\end{aligned}$$

an atom of $H_0^p(\omega)$ is also an atom of $H^p(\omega)$. The space $H_0^p(\omega)$ can be viewed as a weighted discrete tent space. Therefore, using the same argument as in [3], we can get the following lemma.

Lemma 3.1. *Suppose $f\in H_0^p(\omega)$, with $0<p\le 1$ and $\omega\in A_\infty$. Then $f=\sum_{j=1}^\infty \lambda_j a_j$, with a_j $H_0^p(\omega)$-atoms, $\lambda_j\in\mathbb{C}$ and*

$$\sum |\lambda_j|^p \le C\,\|f\|^p_{H_0^p(\omega)}\,.$$

Now for any $f\in H_0^p(\omega)$, $0<p\le 1$, with $f=\sum\lambda_j a_j$ being its atomic decomposition,

$$\begin{aligned}\|f\|^p_{H^p(\omega)} = \|\sum \lambda_j a_j\|^p_{H^p(\omega)} &\le \sum |\lambda_j|^p\|a_j\|^p_{H^p(\omega)}\\ &\le C\sum|\lambda_j|^p \le C\,\|f\|^p_{H_0^p(\omega)}\,.\end{aligned}$$

Therefore, $H_0^p(\omega)\subset H^p(\omega)$ and $\|f\|_{H^p(\omega)}\le C\,\|f\|_{H_0^p(\omega)}$. We want to prove that for $f\in H_0^p(\omega)$, with $0<p<\infty$, $\omega\in A_\infty$, there exists a constant $C>0$, such that $\|f\|_{H^p(\omega)}\le C\,\|f\|_{H_0^p(\omega)}$. Setting

$$Mf(x)=\sup_{\Gamma(x)}|f*\phi_t(y)|\,,$$

where $\Gamma(x)=\{(y,t)\,:\,|y-x|<t\}$. And suppose $\omega\in A_{p_0}$ for some $p_0>1$, then we have

$$\|Mf\|_{L^{p_0}(\omega)}\le C\,\|f\|_{L^{p_0}(\omega)}\le C\,\|f\|_{H_0^{p_0}(\omega)}$$

and also

$$\|Mf\|_{L^1(\omega)}=\|f\|_{H^1(\omega)}\le C\,\|f\|_{H_0^1(\omega)}\,.$$

By interpolation, we obtain

$$\|Mf\|_{L^p(\omega)} = \|f\|_{H^p(\omega)} \le C\,\|f\|_{H_0^p(\omega)}\,, \qquad \text{for } 1 \le p \le p_0\,.$$

Because $A_{p_0} \subset A_q$ for $p_0 < q$, $\omega \in A_q$ for any $q > p_0$. Thus

$$\|f\|_{H^p(\omega)} \le C\,\|f\|_{H_0^p(\omega)}\,, \qquad \text{for } 1 \le p < \infty\,,$$

and then $H_0^p(\omega) \subset H^p(\omega)$ for $0 < p < \infty$. On the other hand,

$$N_2 f(x) = \sup_{\substack{2Q \ni x \\ Q \text{ dyadic}}} |\langle f, \phi_Q\rangle|\,|Q|^{-1/2} \le \sup_{\Gamma(x)} |f * \phi_t(y)| = Mf(x)\,.$$

Therefore,

$$\|f\|_{H_0^p(\omega)} = \|N_2 f\|_{L^p(\omega)} \le \|Mf\|_{L^p(\omega)} = \|f\|_{H^p(\omega)}\,,$$

for $0 < p < \infty$, $\omega \in A_\infty$. Then we have proved

Theorem 3.2. *For* $0 < p < \infty$, $\omega \in A_\infty$, $H^p(\omega) = H_0^p(\omega) = \{f \,:\, S_2 f \in L^p(\omega)\}$, *with*

$$\|f\|_{H^p(\omega)} \approx \|S_2 f\|_{L^p(\omega)}\,.$$

Theorem 3.2 establishes the identification between $H^p(\omega)$ and $H_0^p(\omega)$, a discrete tent space. Therefore, all the properties of tent spaces can be applied to the weighted Hardy spaces $H^p(\omega)$. Especially, we have the following consequences.

Corollary 3.3. $[H^{p_0}(\omega), H^{p_1}(\omega)]_\theta = H^p(\omega)$, *where* $1 \le p_0 < p < p_1 < \infty$ *with* $1/p = (1-\theta)1/p_0 + \theta/p_1$ *and* $[\cdot,\cdot]_\theta$ *is the complex method of interpolation described in* [2].

For $f = \sum f_I \psi_I$, where $f_I = \int f \psi_I\, dx$, define

$$c(f)(x) = \sup_{B \ni x} \left(\frac{1}{\omega(B)} \sum_{I \subset B} |a_I|^2 \frac{\omega(I)}{|I|} \right)^{1/2}$$

and

$$H_0^\infty(\omega) = \{f \,:\, c(f) \in L^\infty\}\,.$$

We have the following duality result.

Corollary 3.4.

1. *The following inequality holds, whenever* $f \in H^1(\omega)$ *and* $g \in H_0^\infty(\omega)$

$$\sum_{I \text{ dyadic}} |f_I g_I| \frac{\omega(I)}{|I|} \le C \int S_2 f(x)\, c(g)(x)\, \omega(x)\, dx\,,$$

where $f = \sum f_I \psi_I$, $g = \sum g_I \psi_I$.

2. *The pairing*

$$\langle f, g\rangle_\omega \longmapsto \sum f_I g_I \frac{\omega(I)}{|I|}$$

realizes $H_0^\infty(\omega)$ *as equivalent to the Banach space dual of* $H^1(\omega)$.

3. *Suppose* $1 < p < \infty$, *then the dual space of* $H^p(\omega)$ *is* $H^{p'}(\omega)$, *with* $1/p + 1/p' = 1$. *More precisely, the pairing*

$$\langle f, g\rangle_\omega = \sum f_I g_I \frac{\omega(I)}{|I|}$$

realizes $H^{p'}(\omega)$ *as equivalent with the dual space of* $H^p(\omega)$.

We have known that $\mathrm{BMO}_\omega = (H^1(\omega))^*$ realized by the pairing

$$(f, g) = \int f g\, dx = \sum f_I g_I\,.$$

Therefore, we can get as a consequence of the last corollary the following wavelet and also Carleson measure characterization of BMO_ω.

Theorem 3.5.

$$\mathrm{BMO}_\omega = \{f \,:\, f = \sum f_I \psi_I\,,\ \sup_{B \text{ ball}} \frac{1}{\omega(B)} \sum_{\substack{I \subset B \\ I \text{ dyadic}}} |a_I|^2 \frac{|I|}{\omega(I)} < \infty\}\,.$$

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