REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 9, N.° 1, 1993

Semigroup Commutators under differences, II

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0. The scope of the paper.

This is the second instalment of my previous paper with the same title, [1]. This paper consist of two different parts. The first part is devoted to improvements of the results developed in [1]. These improvements are explained in Section 0.1 below and developed in sections 1 to 5, and 9 to 10; they are in fact technically distinct from [1] and rely on a systematic use of "microlocalisation" in the context of Hörmander-Weyl calculus. These paragraphs can therefore be read quite independently from [1].

The second part studies a different problem and is, in its aim, fairly disjoint from [1]. This problem is explained in Section 0.2 below and developed in sections 6 to 8. The techniques used however in sections 6 to 8 (and also in Section 10 which in its scope is attached to the first part) are very close to the techniques of [1]. I feel that the reader would find it very difficult to follow these sections without being familiar with [1].

0.1. Pseudodifferential operators and the geometric problem.

The main technical estimate in [1] was the estimate (0.2) that asserted that

$$(0.1) \qquad \|[\cdots[A^\sigma, S_1], S_2], \cdots], S_k] f\|_m \leq C\, \|A^{\sigma-k/2} f\|_{m+n_1+\cdots+n_k}\ ,$$

when $f \in C_0^\infty$.

Here $[x,y] = xy - yx$ are as usual the commutators of two operators, $\|\cdot\|_\alpha$ indicate the usual Sobolev norms in $H_\alpha = \{f : \Lambda^\alpha f \in L^2(\mathbb{R}^n)\}$ ($\Lambda = (1 - \sum \partial^2/\partial x_i^2)^{1/2}$), $\sigma \in \mathbb{C}$, and $A = a^\omega(x,D) + \lambda_0$ for some large $\lambda_0 > 0$ and $0 \leq a(x,\xi) \in S^2_{1,0}$ and finally $S_j = s_j^\omega(x,D)$ with $s_j \in S^{n_j}_{1,0}$. It will turn out that a systematic use of Weyl calculus [10] (rather than ordinary $S^m_{1,0}$ pseudodifferential calculus) will be convenient in several places and will therefore be used interchangeably with pseudodifferential calculus.

The estimate (0.1) was proved in [1] for sums of square (-Hörmander) operators: $A = \sum X_j^* X_j$ where X_j are C^∞ fields. This estimate was not even proved for a general second order self adjoined differential operator of positive characteristic (*cf.* [1], (0.1)). Indeed, as far as I can tell the problem is as hard in this case as for a general pseudodifferential. As a result the main geometric theorem in [1] (and all the rest for that matter) was established only for Hörmander operators.

In this paper I shall give a complete proof of (0.1) for $A = a^\omega(x,D) + \lambda_0$ in full generality but only for $k = 1$. This will be done in sections 1 to 4 in the context of Hörmander's-$S(m,g)$ calculus with $A \in S(1/h^2, g)$. I shall also show that (0.1) holds (and this is easy because of previous work of R. Beals, *cf.* also the appendix at the end of this paper) for arbitrary k but with an A that is polyhomogeneous and subelliptic with a loss of one derivative (*cf.* Section 9 for the appropriate definitions).

Using the above results we shall show in Section 10 that in the main geometric theorem of [1] we can relax the sum of squares condition for the "top operator" L_1 (the set-up was $\|L_2^\beta f\| \leq C\,(\|L_1^\alpha f\| + \|f\|)$), which can therefore be an arbitrary self adjoint differential operator

$$L_1 \equiv a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \cdots$$

The above estimate (0.1) for $k = 1$ has a number of other more "esoteric" consequences, *e.g.* the boundedness of the operators

$$A^{i\sigma},\ e^{i\sigma A^{1/2}} : H_\alpha \to H_\alpha; \qquad \alpha, \sigma \in \mathbb{R}$$

i.e. the imaginary powers of A and the corresponding wave operators. These facts will be proved in sections 5 and 6.

0.2. The Beals characterisation and the $S^m_{\rho,\delta}$.

In Section 8, I will give the following characterisation of pseudodifferential operators (which is but a variant of the characterisations given by R. Beals [3]).

Criterion. *Let T be an arbitrary linear operator $T : C_0^\infty(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$ and let $1/2 \le \rho \le 1$ and $m \in \mathbb{R}$ be such that*

$$\| [[\cdots [T, E_1], E_2] \cdots, E_k] \|_{\alpha \to \alpha + k\rho - m} \le C, \quad k \ge 0, \alpha \in \mathbb{R} \tag{0.2}$$

where $E_j \in S^0_{1,0}$ are arbitrary. Then $\varphi T \varphi \in OPS^m_{\nu,\nu}$ for all $\nu \in [1-\rho, \rho]$ and all $\varphi \in C_0^\infty$.

Here we use of course the standard Hörmander notations for $S^m_{\rho,\delta}$ (*cf.* [2]) and $\| \cdot \|_{\alpha \to \beta}$ indicates the operator norm between the corresponding Sobolev spaces $H_\alpha(\mathbb{R}^d)$. The C in (0.2) depends, of course, on α and k as well as on the E_j's.

The Beals theorem that we refer to appeared for the first time in [3] (*cf.* also [2]). Essentially the same proof was given later in [5]. In [5] the authors work in the context of classical pseudodifferential operators and their assumption is

$$\| [T, X_1, \cdots, X_k] \|_{(1-\rho)k + m \to 0} \le C \tag{0.3}$$

with $\rho = 1$, or $1/2$ and where X_j are C^∞ fields on $\mathbb{R}^d$. The proofs in [3], [4] and [5] easily generalise and give (under the hypothesis (0.3)) the same conclusion

$$T \in \bigcap_{1-\rho \le \nu \le \rho} OPS^m_{\nu,\nu} = \mathcal{B}^m_\rho .$$

Incidentally, standard pseudodifferential calculus can be used and it follows that conversely every $T \in \mathcal{B}^m_\rho$ satisfies the commutator estimates (0.2) and (0.3). This implies in particular that $\mathcal{B}^m_\rho$ can be defined in a coordinate free way (*i.e.* on a manifold). The reference [5] is perhaps the easiest for the reader who is not familiar with (φ, Φ) calculus.

Using the above criterion we shall prove in Section 8 the following **Theorem.** *Let* $A = a^\omega(x,D) \in OPS^2_{1,0}$ *with symbol* $a(x,\xi) \geq 0$*, let* $\sigma \in \mathbb{C}$, $\mathrm{Re}\,\sigma \leq 0$*, and let us assume that* A *is subelliptic:*

$$\|f\|^2_{1-\delta} \leq C\,(Af,f) + C_1\,\|f\|^2$$

with $0 \leq \delta \leq 1/2$*. Then for all* $\varphi \in C_0^\infty$ *and* $\lambda > 0$ *large enough the operator* $\tilde{A}^\sigma = \varphi(A+\lambda)^\sigma\varphi$*, (this is just a banal modification to reduce the problem to compact supports) satisfies*

$$\tilde{A}^\sigma \in \mathcal{B}^{2\mathrm{Re}\,(\sigma(1-\delta))}_{1-\delta}\ .$$

The proof of this theorem will be given in sections 7 and 8. It is interesting to compare the above result with the final theorem in Beals [3]. Beals theorem (if the proof is pushed to its limit) will give a better conclusion since it will show that the correspondig parametrix belongs to $S^{2\mathrm{Re}\,(\sigma(1-\delta))}_{1-\delta,\delta}$. Beals theorem is also better in so far that it can deal with operators of higher order $S^m_{1,0}$, $m \geq 2$ and does not require that the symbol is positive (but only that the principal symbol takes values in an appropriate sector).

Our theorem above has however some advantages, the most significant of which is that it can deal with general symbols (and not only polyhomogeneus ones as seems to be the case in Beals). The other advantage is an advantage of the method of the proof (which is different from Beals' method) rather than of the result. Indeed, in our considerations, we can replace the Sobolev norms $H_\alpha(\mathbb{R}^d)$ by the corresponding L^p-Sobolev norms

$$H^p_\alpha = \{f : \quad \Lambda^\alpha f \in L^p\}\,, \qquad 1 < p < \infty\,,$$

and the estimates are relatively insensitive to that change, provided, that the original operator is a *differential* operator with positive characteristic. In view of the fact that the Hörmander classes $S^0_{\rho,\delta}$ do not in general stabilise L^p, results of this kind are perharps of some interest.

Finally other functions than the complex powers (with non positive real part) of A can be treated with our methods. It easily follows, for instance, that under the same conditions, and with the same notations as in our Theorem, we have

$$(0.4) \qquad \varphi e^{-zA}\varphi \in \mathcal{B}^0_{1-\delta}\,, \qquad |z| \leq L,\ |\mathrm{Arg} z| \leq \frac{\pi}{2} - \varepsilon_0$$

uniformly in z and any fixed $\varepsilon_0 > 0$. And also that

$$\tilde{A}^\sigma \in \mathcal{B}_{1-\delta}^{2\mathrm{Re}\,\sigma}\,, \qquad \mathrm{Re}\,\sigma \geq 0\,. \tag{0.5}$$

1. The Hörmander metrics.

This section is purely technical and contains nothing new. I simply collect a number of comments and elaborations on the $S(m,g)$ calculus as presented in L. Hörmander book [6]. All the notations will be (unless otherwise stated) identical to those of [6]. The facts that I shall need will be enumerated below. The proofs are just cross references in [6] and will be briefly explained after each fact.

(A) In [6], Lemma 18.4.4 can be improved to (with the same notations): given ν then the number of balls B_μ that intersects B_ν is bounded by N_ε.

This slightly stronger local finiteness property will simplify several of our arguments. When we examine the proof of the above lemma in [6], which is to be found in [6], Lemma 1.4.9, we see that this stronger property is in fact implicit in that proof.

(B) The choice of the balls U_ν, U'_ν defined in [6] just after relation (18.4.13) can be refined in the following way: we can choose (for $k = 1,2,\dots$ given in advance)

$$U_\nu = U_\nu^{(0)} \subset U_\nu^{(1)} \subset \dots U_\nu^{(k)}\,, \quad U_\nu^{(j)} = \{x : g_{x_\nu}(x - x_\nu) < c_j\}$$

for $j = 0,1,\dots,k$, in such a way that the balls $U_\nu^{(k)}$ have the local finiteness property of (A). Furthermore the ε in Lemma 18.4.4 and $c_0 > 0$ the radius of U_ν can be chosen small enough, so as to guarantee that

$$U_\mu \cap U_\nu^{(j)} \neq \emptyset \text{ implies } U_\mu \subset U_\nu^{(j+1)}\,, \quad \forall \nu,\mu\,, \quad j = 1,2,\dots,k-1\,.$$

Observe that in the elaborations and proofs of [6], sections 18.4 and 18.5 the two balls $U_\nu^{(j)} \subset U_\nu^{(j+1)}$, for any $j = 0,1,\dots,k-1$, could be used in the place of the paire $U_\nu \subset U'_\nu$ of [6]. The point to watch, and which is vital for us, is what lies between relations (18.4.19) and (18.4.21) in [6].

All this is fairly automatic from [6] and the proof will be left to the reader. Let me simply say that the reader whether he likes it or not will have to really understand [6] sections 18.4 and 18.5 if he wishes to follow what is happening. This applies especially here and in the next few pages.

(C) Let $a_i \in S(m_i, g)$, $(i = 1, 2)$, be such that $\operatorname{supp} a_1 \cap \operatorname{supp} a_2 = \emptyset$, let $b \in S(m_1 m_2, g)$ be such that $b^\omega = a_1^\omega a_2^\omega$, then we actually have $b \in S(m_1 m_2 h^N, g)$, $N \geq 0$.

This is contained in [6], Theorem 18.5.4.

(D) Let U_ν, U'_ν be as in [6] just after (18.4.13) and let $a \in S(m, g)$. We shall say that a is strongly concentrated "at ν in $S(m, g)$" if it satisfies the condition

$$|a|_s^g(\omega) \leq C_{k,s}\, m(\omega)\, (1 + d_\nu(\omega))^{-k}\,, \quad \forall k, s, \omega\,. \tag{1.1}$$

We shall say that a is concentrated (without the adjective strongly) "at ν in $S(m, g)$" if the same estimate (1.1) holds but *only* for $\omega \notin U'_\nu$ (Definition in [6] just after relation (18.4.13)).

The "subtlety" of the above notion lies in the fact that the balls U_ν are defined by the metric g while the distance d_ν is defined by the metric g^A (or g^σ in our case). In [4] Beals introduced an analogous notion which he then exploited in the special case when $g = g^\sigma$.

Observe that the above definition depends on the particular choice of U_ν, U'_ν. The conclusions that this property of concentration will allow us to draw will, on the other hand, be independent of that choice (*cf.* especially property (E) and Section 2 below). So, therefore, at the end, it will be irrelevant with respect which particular U_ν, U'_ν we are making the definition. The above notion will prove itself to be useful in the following properties.

(E) Let $a_\nu \in S(m, g)$ $(\nu \in \mathbb{N})$ be a family of operators so that a_ν is concentrated at ν in $S(m, g)$ for each ν, and that furthermore these conditions are verified *uniformly* in ν. Then the family $\sum a_\nu$ is "absolutely summable" in the sense that we have

$$\sum |a_\nu|_s^g(\omega) \leq C_s\, m(\omega)\,, \qquad \forall \omega\,. \tag{1.2}$$

If we demand that (a_ν) should be *strongly* concentrated and consider the case $m \equiv 1$, then the above statement is an immediate consequence of [6], Lemma 18.4.8. The modifications needed for the proof when m is arbitrary are obvious. If we only impose the weaker property of concentration (rather than strong concentration), then we have to split the sum in (1.2) as follows

$$\sum_{\omega \in U'_\nu} + \sum_{\omega \notin U'_\nu} .$$

(This is essentially the argument of [6] between the relations (18.4.19)-(18.4.21)). The first of the two sums is bounded by $C\, m(\omega) \sup_\nu \|a_\nu\|$ because of the local finiteness of our partition. To controle the second sum we apply the same argument (*cf.* [6], Lemma 18.4.8) as before.

(F) Let $a^{(i)} \in S(m_i, g)$, $i = 1, 2$, and let $a \in S(m_1 m_2, g)$ be such that $a^\omega = a^{(1)\omega} a^{(2)\omega}$. Let us suppose further that for some fixed ν and *either* $i = 1$, *or* $i = 2$ (or both) we have supp $a_\nu^{(i)} \subset U_\nu$. Then a is strongly concentrated at ν in $S(m_1 m_2, g)$.

(F′) We impose the same conditions on $a^{(1)}$, $a^{(2)}$, a as in (F) (with say supp $a^{(1)} \subset U_\nu$). In additions we demand that $U'_\nu \cap \operatorname{supp} a_\nu^{(2)} = \emptyset$. Then a is concentrated at ν in $S(m_1 m_2 h^N, g)$ for all $N \geq 0$. (Here $U_\nu \subset U'_\nu$ are as in [6] just after relation (18.4.13)).

In other words we are in an "arbitrary small class" (with respect to h) provided that we are prepared to sacrifice the property of *strong* concentration. This property should be thought as an elaboration of both (F) and (C).

For the proof of (F) the relevant passage in Hörmander is what lies a dozen lines after relation (18.6.6) and goes on until Theorem 18.6.6. In fact in that passage one essentially finds the proof of our statement for $m_1 \equiv m_2 \equiv 1$. Indeed let $m_1 \equiv m_2 \equiv 1$ and supp $a^{(1)} \subset U_\nu$ and let us proceed as in Hörmander and decompose $a^{(2)} = \sum a_\mu$ so that (in Hörmander's notations):

$$a = \sum_\mu a_{\nu\mu} , \quad a^\omega_{\nu\mu} = a^{(1)\omega} a^\omega_\mu .$$

(We ignore the complex conjugation of Hörmander here). The estimate

$$|a_{\nu\mu}(\omega)| \leq c_k \, |1 + M(\omega)|^{-k} = c_k \, |1 + d_\nu(\omega) + d_\mu(\omega)|^{-k} \tag{1.3}$$

(when $m_1 \equiv m_2 \equiv 1$ and where M is as in [6] bottom of p. 167, vol. III, 1985), then holds and if we use the uniform polynomial growth of M_μ (notations of proof of [6], Lemma 18.4.8) established in [6], Lemma 18.4.8, we obtain that

$$\sum_\mu |a_{\nu\mu}(\omega)| \leq C_k \, |1 + d_\nu(\omega)|^{-k} .$$

This is the required estimate (1.1) for $s = 0$. The modification for arbitrary m_1, m_2 is clear, we just have to insert the factor

$$|a_{\nu\mu}(\omega)| \leq c_k \; m_1(\omega) \, m_2(\omega) \, |1 + M(\omega)|^{-k}$$

in the estimate (1.3). This can clearly be done by the definition of $a_{\nu\mu}$ (*cf.* the same passage of Hörmander: bottom of p. 167, vol. III, 1985 edition, and also the estimate [6], (18.4.12)). To pass to the estimate for the more general seminorms $|a_{\nu\mu}|_s^g$, $(s \geq 1)$, we have to improve (1.3) exactly the way Hörmander does in (18.6.7) and (18.6.8). We obtain

$$|a_{\nu\mu}(\omega)| \leq C_k \, (1 + d_{\nu\mu})^{-k} \, |1 + M(\omega)|^{-N}$$

which is essentially [6], (18.6.9) except that we retain all the information given by [6], (18.6.7) and [6], (18.6.8). We then reason exactly as in [6] (the passage that follows relation [6], (18.6.8)) and we are done in the case $m_1 \equiv m_2 \equiv 1$. The general m_1, m_2 are treated similarly.

To obtain the refinement that is presented in (F′) we have to combine the above argument with the passage in [6] between relations (18.4.19) and (18.4.21) (*i.e.* p. 148-149, vol. III, 1985 edition) what is shown there is that we can improve by an arbitrary power of h^N provided that we are away from the "support". More specifically in the relation that defines $a_{\nu\mu}(x, \xi)$ (bottom of [6], p. 167) if we know that $(x, \xi) \notin U'_\nu$ we can obtain the following improvement to the estimate (1.3)

$$|a_{\nu\mu}(\omega)| \leq C_k \, h^N(\omega) \, (1 + M(\omega))^{-k} , \qquad \omega \notin U'_\nu . \tag{1.4}$$

This is explained in [6] in the passage between relations (18.4.19)-(18.4.21). In that passage we set $V = W \oplus W$, the metric is $G = g \oplus g$ (*i.e.* $g_1 = g_2 = g$) and $A = 2\,\sigma(\hat{x}, \hat{\xi}, \hat{y}, \hat{\eta})$ as in [6], p. 152. Observe that since $g_1 = g_2$ the metric G is now temperate everywhere on $W \oplus W$

and not only on the diagonal. This makes the reasonning easier for it implies that with $\omega = (x,\xi)$, $\omega' = (y,\eta)$, we have

$$|e^{i\sigma(D_x,D_\xi;D_y,D_\eta)/2}a^{(1)}(\omega)a_\mu(\omega')| \le C_k\, H^N(1+d_\nu(\omega)+d_\mu(\omega'))^{-k}\ ,$$

$\forall(\omega,\omega') \notin U'_\nu \times U'_\mu$. From this (1.4) follows immediately by setting $\omega = \omega'$ (since then $H(\omega,\omega') \sim h(\omega) \sim h(\omega')$). This outlines the proof when $m_1 \equiv m_2 \equiv 1$ and $s = 0$. The proof of the estimates in the general case follows by the same modifications as before. Once the estimate (1.4) has been proved (F′) follows since (C) guarantees that $a \in S(m_1 m_2 h^N, g)$, $\forall N \ge 0$.

2. The localisation of the commutator estimate.

Let $U_\nu \subset U_\nu^{(1)} \subset \dots \subset U_\nu^{(k)}$ be as in (B) ($k = 5$ will in fact suffice). Assume that $A = a^\omega(x,D)$, $B_\nu^{(j)} = b_\nu^{(j)\omega}(x,D)$ with $a, b_\nu^{(j)} \in S(m,g)$, $(j = 1,2,\dots,s)$. Let us also make the hypothesis that supp $b_\nu^{(j)} \subset U_\nu^{(p)}$ for some $1 \le p < k$ and let us denote

$$(2.1)\qquad \begin{aligned} A'_\nu &= \sum_\lambda \Big\{(a\varphi_\lambda)^\omega : \quad U_\lambda \cap U_\nu^{(p+1)} \ne \emptyset\Big\} \\ \tilde{A}_\nu &= \sum_\lambda \Big\{(a\varphi_\lambda)^\omega : \quad U_\lambda \cap U_\nu^{(p+1)} = \emptyset\Big\} \end{aligned}$$

where $\sum \varphi_\lambda \equiv 1$ is a Hörmander partition of unity subordinated to the covering $\{U_\nu\}$ as in [6], Lemma 18.4.4.

Then by (F′), $\tilde{A}_\nu B_\nu^{(1)}$ is concentrated at ν in $S(m^2h^N;g)$ with an $N \ge 1$ that can be given in advance; but then, by a succesive application of (F), $\tilde{A}_\nu B_\nu^{(1)} B_\nu^{(2)}$ (respectively, $\tilde{A}_\nu B_\nu^{(1)} \dots B_\nu^{(j)}$) is strongly concentrated at ν in $S(m^3h^N;g)$ (respectively, $S(m^{j+1}h^N;g)$) with the same N.

On the other hand the following two sums are absolutely summable in the sense of (E):

$$\begin{aligned} A' &= \sum A'_\nu B_\nu^{(1)} \dots B_\nu^{(s)}\ , \\ \tilde{A} &= \sum \tilde{A}_\nu B_\nu^{(1)} \dots B_\nu^{(s)}\ . \end{aligned}$$

This is because supp $b_\nu^{(s)} \subset U_\nu^{(p)}$ and thus each term of the above summation is concentrated (even strongly) at ν in $S(m^{s+1};g)$, and so (E) applies. But, by what we have said just above, $\tilde{A} = \sum \tilde{A}_\nu B_\nu^{(1)} \dots B_\nu^{(s)}$ is in fact absolutely summable in $S(m^{s+1}h^N; g)$ for any $N \geq 1$ arbitrary large. It follows in particular that

$$A \sum B_\nu^{(1)} \dots B_\nu^{(s)} \equiv A' \quad \text{mod-}S(m^{s+1}h^N; g)\,, \quad N \geq 0\,. \tag{2.2}$$

We shall apply these facts to localise a specific expression involving commutators. Let $A = a^\omega$ with $a \in S(m; g)$ and let us follow our notational convention of [1] and denote by $E = e^\omega$ where $e \in S(1, g)$ is arbitrary. The various e's and E's that appear below are not necessarily all the same. Let φ_ν be a partition of unity as in (2.1) and let $E \in OPS(1; g)$ and $A_\nu = (a\,\varphi_\nu)^\omega$. Let E'_ν be defined from E the way A'_ν was defined from A in (2.1) with $p = 0$. We obtain therefore from (2.2) that

$$EA = \sum_\nu EA_\nu \equiv \sum_\nu E'_\nu A_\nu \quad \text{mod-}S(h^N m; g)\,, \qquad N \geq 0\,.$$

But then it follows that

$$\begin{aligned} EAE &= \sum EA_\nu E \equiv (\sum E'_\nu A_\nu)E \\ &= \sum E'_\nu A_\nu E \quad \text{mod-}S(h^N m; g) \end{aligned} \tag{2.3}$$

because multiplication is distributive over absolute summation. A simple application of (C) allows to conclude on the other hand that

$$A_\nu \tilde{E}_\nu \in S(mh^N; g) \qquad (N \geq 1,\ \text{uniformly in } \nu)\,.$$

This together with (F) (we do not need to use (F′) anymore !) implies that $E_\nu A_\nu \tilde{E}_\nu$ is (strongly) concentrated at ν in $S(mh^N; g)$ and we can therefore sum these terms in $S(mh^N; g)$ by Section 1. The conclusion is that

$$\begin{aligned} \sum_\nu E'_\nu A_\nu E &= \sum_\nu E'_\nu A_\nu E'_\nu + \sum_\nu E'_\nu A_\nu \tilde{E}_\nu \\ &\equiv \sum E'_\nu A_\nu E'_\nu\,, \quad \text{mod-}S(mh^N; g)\,, \qquad (N \geq 1)\,. \end{aligned}$$

Combining this with (2.3) we conclude that

$$EAE = \sum E'_\nu A_\nu E'_\nu \quad \text{mod-}S(h^N m; g), \quad (N \geq 0)\,.$$

We shall use this idea again to the two products of

$$\sum_\nu [E'_\nu A_\nu E'_\nu, E] = \sum_\nu (E'_\nu A_\nu E'_\nu E - EE'_\nu A_\nu E'_\nu)$$

and we obtain

$$\begin{aligned}[EAE, E] &\equiv \sum [E'_\nu A_\nu E'_\nu, E] \\ &\equiv \sum [E'_\nu A_\nu E'_\nu, E''_\nu] \quad \text{mod-}S(mh^N; g) \qquad (N \geq 0)\end{aligned}$$

where E''_ν is constructed from E (the same way A'_ν was constructed from A) as in (2.1) with $p = 3$.

We shall carry this process one step further and finally deduce that

$$[EAE, E] = \sum I_\nu [E'_\nu A_\nu E'_\nu, E''_\nu] I_\nu \quad \text{mod-}S(mh^N; g)$$

where $I_\nu = i_\nu^\omega$: $i_\nu = \left\{\sum \varphi_\lambda : \ U_\lambda \cap U_\lambda^{(5)} \neq \emptyset\right\}$.

Let us now examine more closely the operators I_ν and A_ν. The first observation is that

$$\sum \|I_\nu f\|^2 \leq C \|f\|^2, \qquad \forall f \in C_0^\infty . \tag{2.4}$$

This is best seen by considering vector valued symbols (*cf.* [6] relation 18.6.24). Alternatively (and equivalently) the estimate (2.4) can be obtained by taking the expectation on $\| \sum \pm I_\nu f\|$ (as is done in Section 8). Consider next the operator

$$\sum_\nu I_\nu A_\nu I_\nu$$

where again, by that we have said, the summation is absolute. To examine this operator we shall impose for the first time on $a(x, \xi)$, the symbol of A, our basic conditions. We shall assume that

$$a(x, \xi) \geq 0, \qquad a(x, \xi) \in S(1/h^2; g). \tag{2.5}$$

Let $\theta = \sum i_\nu^2$. Under the conditions (2.5), we then have

$$a - (a\theta)^\omega = a_1^\omega + a_2^\omega \tag{2.6}$$

where $a_1(x,\xi)$ takes pure imaginary values and $a_2(x,\xi) \in S(1,g)$. Similarly if we denote by $\Theta_{\pm 1} = (\theta^{\pm 1/2})^\omega$, then we have

$$\Theta_1 A \Theta_1 - (a\theta)^\omega = a_1^\omega + a_2^\omega \tag{2.7}$$

with similar conditions on a_1, a_2. Furthermore we have

$$\Theta_1 \cdot \Theta_{-1} \; ; \; \Theta_{-1} \cdot \Theta_1 \equiv I + ib^\omega(x,D) \quad \text{mod-}S(h^2;g)\,, \tag{2.8}$$

where $b \in S(h,g)$ is real. The last two relations (2.7) and (2.8) are obtained by $S(m,g)$ calculus (in (2.8) we in fact have $b \equiv 0$! One can compare this with the argument in [6], p. 171 just before Theorem 16.6.8). The relation (2.6) is also obtained by symbolic calculus but the presence of the infinite sum that is involved in the definition of a makes life slightly more complicated. There are many ways to deal with that infinite sum, the most elegant is, in my opinion, the one presented in [6] just after relation (18.6.24) where the author uses operator valued symbols.

I shall now show how the above considerations can be used to localise commutators. Our problem [*cf.* Section 0.1] is to show that for operators $A = a^\omega(x,D)$ with symbols that satisfy (2.5) we have

$$\|[A,E]f\|^2 \le C\,(Af,f) + C_1\,\|f\|^2\,, \qquad f \in C_0^\infty$$

or better still that

$$\|[EAE,E]f\|^2 \le C\,(Af,f) + C_1\,\|f\|^2\,, \qquad f \in C_0^\infty\,. \tag{2.9}$$

$(\cdot,\cdot)$ indicates of course the scalar product in L^2 and $\|\cdot\|$ the corresponding norm. The reason why we did all the work in this section was because we wanted to show that the estimate (2.9) is "localisable". Let me be more specific and let us assume that we can find some Hörmander covering (U_ν) of the (x,ξ) space as above for which the estimate (2.9) holds "for each ν separately", *i.e.* such that

$$\|[E_\nu\, A_\nu\, E_\nu\,,\, E_\nu]f\|^2 \le C\,(Af,f) + C_1\,\|f\|^2 \tag{2.10}$$

provided that $A_\nu = (ae_\nu)^\omega$, $E_\nu = e_\nu^\omega$ with $e_\nu \in S(1,g)$ uniformly in ν and supp $e_\nu \subset U'_\nu$. Then we shall deduce that the estimate (2.9) itself also holds.

The first thing to observe is that we have

$$\|[EAE,E]f\| \leq \|\sum_{\nu} I_\nu [E'_\nu A_\nu E'_\nu\,,\, E''_\nu] I_\nu f\| + C\,\|f\|\,. \tag{2.11}$$

We shall need the following

Lemma. *Let $J_\nu = j_\nu^w(x,D)$ with real valued $j_\nu \in S(1;g)$ (uniformly in ν) and* supp $j_\nu \subset U'_\nu$ *then there exists a constant C such that*

$$\|\sum J_\nu f_\nu\|^2 \leq C \sum \|f_\nu\|^2\,, \quad f_\nu \in C_0^\infty\,.$$

The proof of the lemma will be given presently. Let us draw the conclusions: From (2.11), the lemma, and the local hypothesis (2.10), we deduce inmmediately that $\|[EAE,E]f\|^2$ can be estimated by

$$\begin{aligned}&\sum_{\nu} \|[E'_\nu A_\nu E'_\nu\,,\, E''_\nu] I_\nu f\|^2 + \|f\|^2 \\ &\qquad \leq \sum_{\nu} (I_\nu A I_\nu f, f) + C\,\|f\|^2 + C\sum \|I_\nu f\|^2\,.\end{aligned} \tag{2.11$'$}$$

(The local hypothesis (2.10) is applied to each $I_\nu f$ separately). And this by (2.4), (2.6) and (2.7) can be estimated by

$$(A\Theta_1 f, \Theta_1 f) + C\,\|f\|^2\,.$$

The upshot is that we have

$$\|[EAE,E]f\|^2 \leq C_1\,(A\Theta_1 f, \Theta_1 f) + C_2\,\|f\|^2\,, \quad f \in C_0^\infty\,, \tag{2.12}$$

for some $C_1, C_2 > 0$. If we apply (2.12) to $f = \Theta_{-1}\varphi$ we obtain

$$\|[EAE,E]\Theta_{-1}\varphi\|^2 \leq\ C_1\,(A\varphi,\varphi) +\ C_2\,\|\varphi\|^2 \tag{2.13}$$

since $\Theta_{-1} \in OPS(1;g)$. But now we almost have our required estimate. Indeed set $\psi = [EAE,E]\varphi$, we have

$$\|\psi\| \leq \|\Theta_1\Theta_{-1}\psi\| + \|R[EAE,E]\varphi\| \tag{2.14}$$

with $R \in S(h,g)$ by (2.8). On the other hand

$$\|\Theta_{-1}\psi\| \leq C\,(\,\|[EAE,E]\Theta_{-1}\varphi\| + \|\varphi\|\,) \tag{2.15}$$

since $[\Theta_{-1}, [EAE, E]] \in OPS(1; g)$.

So putting (2.13), (2.14) and (2.15) together we obtain the required estimate

$$\|[EAE, E]\varphi\|^2 \le C\,(\,\|\Theta_{-1}\psi\| + \|\varphi\|\,)^2 \le C_1\,(A\varphi, \varphi) + C_2\,\|\varphi\|^2 .$$

It remains to give the proof of the lemma which is standard. Indeed

$$\|\sum J_\nu f_\nu\|^2 = \sum_{\nu,\mu}(K_{\mu\nu} f_\nu, f_\mu) \le \|K\| \sum \|f_\nu\|^2$$

where $K_{\nu,\mu} = J_\mu J_\nu$ and $\|K\|$ is the operator norm of the "Hilbert space matrix" $(K_{\nu,\mu})_{\nu,\mu}$ acting on $L^2 \otimes \ell^2$. The boundedness of that norm is a consequence of the estimate

$$\|K_{\nu,\mu}\| \le C\,(1 + d_{\nu\mu})^{-N} . \tag{2.16}$$

The proof of (2.16) can be found in [6], p. 168, just before relation (18.6.10) and in the few lines that follow. Observe that any of the standard proof that (2.16) implies the boundedness of the scalar valued matrix operator $(k_{\nu\mu})$ also works in the present vector valued case.

3. The Fefferman-Phong reduction of variables.

In this section, I shall give a proof of

$$\|[E(ea)^\omega E, E]f\|^2 \le C_1\,(a^\omega f, f) + C_2\,\|f\|^2 , \quad f \in C_0^\infty , \tag{3.1}$$

where $C_1, C_2 \ge 0$, E, e are as in Section 2 and $a(x,\xi) \in S(1/h^2; g)$ with $a \ge 0$. The main step of the proof is an inductive procedure (due to Fefferman-Phong) on the "essential" number of variables $x_1, \dots, x_n$, $\xi_1, \dots, \xi_n$ that appear in $a(\cdot,\cdot)$. To make this precise I shall say that $a(\cdot)$ depends only on k variables $0 \le k \le 2n$ if there exists $2n-k$ linear independent vectors $l_1, l_2, \dots, l_{2n-k}$ in the (x,ξ) space $T(\mathbb{R}^{2n})$ such that $da(l_j) = 0$, $j = 1, 2, \dots, 2n-k$ (*i.e.* a is constant along these vectors).

When a depends on 0 variables our estimate holds (the constants C_1, C_2 depend on a since (3.1) is not homogeneous in a !). Observe incidentally that the e inside $(ea)^\omega$ is imposed by technical reasons due to the above inductive procedure. In reality e can be absorbed in the E's outside since $E(ea)^\omega E \equiv Ea^\omega E$ mod-$S(1/h; g)$ and the $S(1/h; g)$ disappears after the commutator is taken.

Observe also that one situation in which our estimate (3.1) holds trivially is when $a = c^2$ with $c \in S(1/h; g)$ (real valued). Indeed, banal symbolic calculus shows in that case that

$$[E(ea)^\omega E, E] = E\,c^\omega + E\,.$$

It follows therefore that the left hand side of (3.1) can be estimated by $\|c^\omega f\|^2 + \|f\|^2$. On the other hand we have:

$$(c^\omega)^2 - (c^2)^\omega = a_1^\omega + a_2^\omega$$

with a_1 purely imaginary (in fact here we have $a_1 \equiv 0$!) and $a_2 \in S(1; g)$ (again by symbolic calculus). We can therefore estimate $\|c^\omega f\|^2 = ((c^\omega)^2 f, f)$ by $((c^2)^\omega f, f) + O(\|f\|^2)$ which gives our assertion.

The next observation is that in proving (3.1) we can reduce everything to the case when $g = g_0$ is a *constant* metric, *i.e.* a positive definite quadratic form on the $2n$ variables (x, ξ) for which $g/g^\sigma \leq \lambda^2$ where $0 < \lambda(= h) \leq 1$. This of course is the whole point of the localisation explained in [6, Lemma 18.4.4]. Indeed by what we did in the last section we see that our estimate (3.1) is "localisable" to each U_ν where the metric can be considered as constant.

More can in fact be demanded form the constant metric $g = g_0$. We can even ensure that $g = \lambda e$ where e is the euclidian metric $\sum(dx_i^2 + d\xi_i^2)$, and $0 < \lambda \leq 1$ as before. To see this we argue as in [6] in the first few lines of the proof of Lemma 18.6.10. Indeed we can, by a linear symplectic transformation T, reduce $g(x, \xi) = \sum \lambda_\nu (x_\nu^2 + \xi_\nu^2)$ with $\lambda = \sup \lambda_j$ and since our hypothesis on a, e is that $|a|_k^g \leq c_k \lambda^{-2}$, $|e|_k^g \leq c_k$, we can replace all the λ_j's by λ in the hypothesis. The estimate we wish to prove is

$$\|[e^\omega (ea)^\omega e^\omega, e^\omega] f\|^2 \leq C_1\,(a^\omega f, f) + C_2\,\|f\|^2\,, \quad f \in C_0^\infty\,.$$

The linear symplectic transformation has the following effect on the symbols and the corresponding operators

$$(e \circ T)^\omega = U^{-1} e^\omega U \quad , \quad (a \circ T)^\omega = U^{-1} a^\omega U$$

(*cf.* [6], Theorem 15.5.9) where $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary transformation. It is then clear that the above conjugation operation commutes with all the "elements" of our estimate and we are done.

From everything that we have done up to now we see that the proof of (3.1) is reduced to the proof of the following inductive step for $k = 1, 2, \ldots, 2n$.

(I_k): *All the metrics are of the form λe for some $0 < \lambda \le 1$. We shall assume that the estimate* (3.1) *holds if a depends on $k-1$ variables and we shall conclude that it holds for any a that depends on k variables.*

The proof is but a variant of the Fefferman-Phong argument. I shall follow closely the presentation given in [6] in the proof of Lemma 18.6.10.

Our original metric is $g = \lambda e$ and our conditions on a and e are (with apologies for the confusing notation!)

$$|e|_k^e \le C_k\, \lambda^{k/2}\,, \quad |a|_k^e \le C_k\, \lambda^{(k-4)/2}$$

(observe that all our estimates below have to be uniform in λ).

The metric g will be replaced by

$$G_{x,\xi} = mg = H(x,\xi)e$$

where

$$\frac{1}{H(\omega)} = \max\{1, a(\omega)^{1/2}, |a|_2^e(\omega)\}\,.$$

Since clearly by our hypothesis $H \ge \lambda$ we have $m \ge 1$ and clearly also $G/G^\sigma \le H^2 \le 1$. It follows therefore that to show that G is σ-temperate it suffices to show that it is slowly varying and invoque [6], Proposition 18.5.6 (or one can even give a direct proof, *cf.* [6]). The fact that G is slowly varying is proved in Hörmander (although I feel that the corresponding passage of the proof of Lemma 18.6.10 in [6] is unclear. Indeed I had to work somewhat to convince myself that it works ! Maybe the reader can do better). Be it as it may, we now have a σ-temperate metric G, and since $G \ge g$ we have $S(1;g) \subset S(1;G)$, but we also have $a \in S(1/H^2;G)$. To see this, since $m \ge 1$, it is enough to check that

$$|a|_k^e \le C\, H^{(k-4)/2}\,, \quad k = 0,1,2,3$$

(compare with [6, relations (18.6.13)-(18.6.13)$''$]). For $k = 0,2$ this follows by the definition of H and for $k = 1,3$ by the standard log-convexity of the $\|\cdot\|_\infty$ norm of the derivatives ($\|F'\|_\infty^2 \le C\,\|F\|_\infty \|F''\|_\infty$). What really has been done up to now is simply to transform all the data to a new metric $G = He$.

$$e \in S(1,G)\,, \quad a \in S(1/H^2, G) \tag{3.2}$$

and the λ has disappeared.

Our next step consists in a new localisation that will allow us to suppose that H is constant and so be able to use the inductive hypothesis. We consider a convering of the phase space by balls $U_\nu \subseteq U'_\nu$ as in (A) for the metric G. These balls are in fact *euclidean* balls centered around the points ω_ν of radius $c\,H_\nu^{-1/2}$ where $H_\nu = H(\omega_\nu)$ (for appropriate constants c).

Our strategy now is simple. We shall prove the estimate (3.1) when the e's ($e \in S(1;G)$) are such that supp $e \in U'_\nu$ for *some fixed* ν and when a only depends on k variables. This will be done under the inductive hypothesis that I_{k-1} holds. It is of course impossible (since incompatible with its constancy along certain directions) to assume that a also has supp $a \in U_\nu$ (hence the factor e in the $(ea)^\omega$ of our estimate).

There is one case that can be delt with immediately, this case is when $H_\nu = 1$. Indeed we then have $(ea) \in S(1; H_\nu e)$ and our estimate follows. It suffices therefore to analyse the case

$$1 \le \max\{H_\nu^2 a(\omega_\nu), H_\nu |a|_2^e(\omega_\nu)\} \le C\,. \tag{3.3}$$

The upper bound follows from (3.2). We consider then the function

$$f(z) = H_\nu^2 a(\omega_\nu + z/H_\nu^{1/2})$$

that satisfies $\max\{|f(0)|, |f|_2^e(0)\} \sim 1$ (in the sense of (3.3)). We shall apply to that function a slight variant (in the sense that the constants in (18.6.14) and (18.6.15) are different) of [6], Lemma 18.6.9. This allows us to decompose

$$f(x) = f_1(x) + g^2(x)\,, \quad |x| \le C \tag{3.4}$$

with the appropriate bounds on the derivatives of f_1 and g, and f_1 depending only on $k-1$ variables. To see that we actually gain one extra direction of constancy, we have to apply, in fact, the proof of the lemma and not just the lemma itself.

Finally we shall go back to the original symbol a and cut it off by a function $\varphi_k \in S(1; H_\nu e)$ constant along the same directions as a. This allows us to define $\tilde{a} = a\varphi_k \in S(1/H_\nu^2; H_\nu e)$ globaly. The localised estimate that we wish to show refers to $(ea)^\omega$ and not to a^ω itself, so by properly choosing φ_k, we can replace a by $\tilde{a}$ on the left hand side of (3.1) without changing anything. Therefore, since

$$e \in S(1; H_\nu e)\,, \quad \tilde{a} \in S(H_\nu^{-2}; H_\nu e)\,, \tag{3.5}$$

we have succeeded in reducing everything a constant conformal metric again.

The decomposition of f in (3.4) induces then, by scalling back, a decomposition (*cf.* [6], p. 175)

$$\tilde{a} = b + c^2 \tag{3.6}$$

such that $b \in S(H_\nu^{-2}; H_\nu e)$, $c \in S(H_\nu^{-1}; H_\nu e)$ with b only depending on $k-1$ variables, we obtain therefore by the inductive hypothesis that

$$\begin{aligned} \|[e^\omega(ea)^\omega e^\omega, e^\omega]f\|^2 &= \|[e^\omega(e\tilde{a})^\omega e^\omega, e^\omega]f\|^2 \\ &\leq C\,(\tilde{a}^\omega f, f) + C\,\|f\|^2\,, \quad f \in C_0^\infty\,. \end{aligned} \tag{3.7}$$

To see (3.7), together with the induction hypothesis, we have to use the case $a = c^2$, that has already been delt with, and the Fefferman-Phong theorem (*cf.* [6], Theorem 18.6.8) that guarantees that

$$((c^2)^\omega f, f) \leq C\,(\tilde{a}^\omega f, f) + C\,\|f\|^2\,.$$

(The other estimate: $(b^\omega f, f) \leq C\,(\tilde{a}^\omega f, f) + C\,\|f\|^2$ is clear).

The estimate (3.7) is unfortunately not quite the wanted localised estimate. Indeed, we had to cut off the symbol a, and so we end up with $\tilde{a}$ and not a on the right hand side. (That cutting off was necessary to make (3.5), (3.6) work). It may well be that with a cleverer way of building up the induction I could have avoided that "misfiring". I propose to save the day differently. Indeed observe that the localised estimate is *only* used in Section 2 (*cf.* (2.11) and (2.11)$'$) for the special functions $f = I_\nu f$. To obtain our original symbol a on the right hand side of our estimate (3.1) it suffices therefore to be able to prove

$$\Big|\sum_\nu (i_\nu^\omega(a - \tilde{a}_\nu)^\omega i_\nu^\omega f, f)\Big| \leq C\,\|f\|^2\,, \qquad f \in C_0^\infty \tag{3.8}$$

where $\tilde{a}_\nu$ is the function $\varphi_k a = \tilde{a}$ for the index ν that was fixed just above. To see (3.8) we choose for each ν the corresponding "cutting off" function φ_k to be equal to 1 on some neighbourhood of supp i_ν. This choice makes the estimate (3.8) evident. Indeed by (C), (F), $i_\nu^\omega(a-\tilde{a}_\nu)i_\nu^\omega$ is then concentrated at ν in $S(1/h^2h^N; g)$ for all ν and $N \geq 1$ (*cf.* also the considerations at the beginning of Section 2), and if we apply (E) we obtain (3.8). Observe incidentally that we do not have to prove

$\sum_\nu |(i_\nu^\omega(a-\tilde{a}_\nu)^\omega i_\nu^\omega f, f)| = O(\|f\|^2)$ and that we *can* put the modulus sign outside the summation. This is just as well because this stronger estimate would need a different proof.

4. The conjugation operators.

In this section, I shall only consider $0 \le a(x,\xi) \in S^2_{1,0}$ a nonnegative "classical symbol" and $A = a^\omega(x,D)$. I shall show that

$$\|\Lambda^\alpha[\Lambda^{-\alpha}, A]f\|^2 \le C_1\,(Af,f) + C_2\,\|f\|^2\,, \qquad f \in C_0^\infty \tag{4.1}$$

for appropriate $C_1, C_2 \ge 0$ and $\Lambda = (1+\Delta)^{1/2}$. We shall see that this follows easily from the results of sections 2 and 3.

The first step consists in a localisation of A at $\xi \sim 2^k$, $k = 1, 2, \ldots$ This is done as usual by a partition of unity of the form

$$1 \equiv \psi_0^3(\xi) + \sum_{j\ge 1} \psi^3(2^{-j}\xi)\,, \qquad \xi \in \mathbb{R}^n$$

where $\psi, \psi_0 \in C_0^\infty$ and

$$\operatorname{supp}\psi \subset \left\{\xi : \quad \frac{1}{10} < |\xi| < 10\right\} \tag{4.2}$$

If we denote by $\varphi_0 = \psi_0$, $\varphi_k(\cdot) = \psi(2^{-k}\cdot)$, $A_k = (A\varphi_k)^\omega$ we have

$$A - \sum \varphi_k^\omega\, A_k\, \varphi_k^\omega = a_1^\omega + a_2^\omega \tag{4.3}$$

where $a_1 \in S^1_{1,0}$ takes pure imaginary values and $a_2 \in S^0_{1,0}$ (*cf.* [6], bottom of p. 174). Inserting (4.3) in our commutators we obtain

$$\begin{aligned}\|\Lambda^\alpha[\Lambda^{-\alpha}, A]f\|^2 &\le \sum_k \|\Lambda^\alpha[\Lambda^{-\alpha}, \varphi_k^\omega A_k \varphi_k^\omega]f\|^2 + C\,\|f\|^2 \\ &= \sum_k \|\varphi_k^\omega \Lambda^\alpha[\Lambda^{-\alpha}, A_k]\varphi_k^\omega f\|^2 + C_1\|f\|^2\,, \quad f \in C_0^\infty\,.\end{aligned}$$

It follows therefore that it suffices to prove (4.1) for the localised symbols A_k. Indeed if that localised estimate is known to hold we obtain that

$$\|\Lambda^\alpha[\Lambda^{-\alpha}, A]f\|^2 \le C\sum(A_k\varphi_k^\omega f, \varphi_k^\omega f) + C\sum \|\varphi_k^\omega f\|_2^2 + C\|f\|_2^2$$

when $f \in C_0^\infty$, which because of (4.3) gives the global result.

We shall suppose from now onwards that a is localised as above at $\xi \sim 2^{k_0} = \xi_0$ for some fixed $k_0 = 1, 2, \ldots$ Let us decompose then

$$\Lambda^\alpha = \check{\Lambda}^\alpha + \check{\Lambda}_\alpha\,, \quad \check{\Lambda}^\alpha = \Big(\psi_1(\xi)(1+|\xi|^2)^{\alpha/2}\Big)^\omega$$

where $\psi_1(\xi) = \psi(2^{-k_0}\xi)$ and where ψ satisfies (4.2) and is such that ψ_1 is equal to 1 on some neighbourhood of supp a (this last statement should be clear but it is abusive since a depends on ξ *and* on x).

We shall insert this decomposition in $\Lambda^\alpha[\Lambda^{-\alpha}, A]$. This will give rise to four different terms that have to be delt separately. For the first term we observe that (in terms of $S(m,g)$ notations) we have

$$\check{\Lambda}^{\pm\alpha} \in OPS(\xi_0^{\pm\alpha}; g_0)\,, \quad a \in S(\xi_0^2; g_0)$$

(uniformly in k_0) where $g_0 = dx^2 + \xi_0^{-2} d\xi^2$, so it follows that

$$\check{\Lambda}^\alpha[\check{\Lambda}^{-\alpha}, A] = (\xi_0^{-\alpha}\check{\Lambda}^\alpha)[(\xi_0^\alpha \check{\Lambda}^{-\alpha}), A]$$

where $\xi_0^{\pm\alpha}\check{\Lambda}^{\mp\alpha} \in S(1, g_0)$. This reduces the estimate to the corresponding result on $[E, A]$ examined in Section 3.

The other terms are very easy to estimate. Indeed

$$\text{supp } a \cap \text{supp } (\text{symb } \check{\Lambda}_\alpha) = \emptyset\,,$$

it follows therefore that the operators $\check{\Lambda}_\alpha A, A\check{\Lambda}_\alpha \in S_{1,0}^{-n}$, $(n \geq 0)$ for arbitrary n (by (C) among other things). Our estimate (4.1) is thus established.

5. The imaginary powers and the holomorphicity.

The operator A that we shall consider in this section is $A = a^\omega(x, D) + \lambda_0$ with $0 \leq a(x,\xi) \in S_{1,0}^2$ and λ_0 some appropriately large constant for A to be a positive Hilbert space operator. It follows from Section 4 that

$$\|(A - \Lambda^{-\alpha} A \Lambda^\alpha) f\|_{L^2} \leq C\, \|A^{1/2} f\|_{L^2} \tag{5.1}$$

provided that λ_0 is large enough to ensure that $\|f\|_{L^2} \leq C\, \|A^{1/2} f\|_{L^2}$. In fact in what follows I shall maintain the convention that was adopted in [1] and drop the $\lambda_0 \geq 0$ altogether from all the formulas. (The convention is that this λ_0 is tacitly always there, and that it is large enough without necessarily appearing explicitely). An immediate consequence of the above estimate is that

$$\|e^{-tA} - \Lambda^{-\alpha} e^{-tA} \Lambda^{\alpha}\|_{L^2 \to L^2} = O(t^{1/2}) \qquad (\text{as } t \to 0)\,. \tag{5.2}$$

This was shown in [1] Section 3 and I shall not repeat the argument here since anyway these type of estimates will be examined in details in sections 6 and 7 below. I also wish to stress that from here onwards all the $O(t^a)$ notations that will appear refer to $t \to 0$ and that the "$t \to 0$" will usually be dropped. From (5.2) the boundedness of A^{is} on H_α $(s, \alpha \in \mathbb{R})$ follows as in [1, Section 3]. When A is a *differential* operator we can deduce from this the boundedness of

$$A^{is} : H^p_\alpha \longrightarrow H^p_\alpha\,; \qquad s \in \mathbb{R}, \quad 1 < p < \infty\,, \tag{5.3}$$

$H^p_\alpha = \{f : \Lambda^\alpha f \in L^p\}$. This is proved by interpolating the information between $(\alpha = 0,\ p = p_0)$ and $(\alpha = \alpha_0,\ p = 2)$. Once we have (5.1) all these facts extend to general A and the proofs of [1], Section 3 work in this general setting. In [1], Section 3 I also gave two distinct proofs of the holomorphicity of the action of e^{-tA} on the spaces H_α. The first works under very general conditions and does not use our basic estimate (5.1). The second used the action (5.3) of A^{is} on H_α. It turns out that if we make essential use of (5.1) we can give a direct proof of the fact that the operator $(1 + i\xi)A$, $(\xi \in \mathbb{R})$ is semibounded on each Hilbert space H_α, or equivalently that

$$\text{Re}\,(1 + i\xi)(\Lambda^{-\alpha} A \Lambda^\alpha f, f) \geq -C\,\|f\|^2_{L^2}\,. \tag{5.4}$$

Indeed the left hand side of (5.4) can be rewritten

$$(Af, f) + \text{Re}\,(1 + i\xi)((\Lambda^{-\alpha} A \Lambda^\alpha - A)f, f)$$

and because of (5.1) we can bound the second term by

$$C_1\,(1 + |\xi|)\,\|A^{1/2} f\|\,\|f\|\,.$$

It is therefore only a matter of choosing the C on the right hand side of (5.4) large enough. In fact the first proof of the holomorphicity of e^{-tA} (the one that does not use at all our main estimate (5.1)) will also give the above semiboundedness. Indeed if we do that proof with care it will show that

$$\|e^{-zA}\|_{H_\alpha \to H_\alpha} \le e^{\lambda|z|}, \quad |\mathrm{Arg}\, z| \le \theta \tag{5.5}$$

with any $0 \le \theta < \pi/2$ and $\lambda > 0$ (depending on α and θ) and this is equivalent to (5.4), (*cf.* [7]). The fact that we have (5.5) rather that the coarser estimate $Me^{\lambda|z|}$ is of course not of great consequence. For *differential* operators the corresponding estimate (*i.e.* $M \equiv 1$) for the $H^p_\alpha \to H^p_\alpha$ norm also holds. This is seen by standard interpolation since it is well known that the semigroup e^{-tA} is symmetric submarkovian and therefore $\|e^{-tA}\|_{L^p \to L^p} \le 1$, $(1 \le p \le \infty)$.

6. The square root $A^{1/2}$.

In this section I shall draw the first consequences of the two estimates that have been established in sections 1 to 4:

$$\|[A, E]f\|_X \le C\,\|A^{1/2}f\|_X\,, \quad \|(c_\lambda - c_\mu)Af\|_X \le C\,\|A^{1/2}f\|_X \tag{6.1}$$

when $f \in C_0^\infty$. Here $A \in S^2_{1,0}$ is as in Section 5, $c_\lambda(T) = \Lambda^\lambda T \Lambda^{-\lambda}$ is the conjugation operator applied to any operator T and $X = L^2$. We shall also denote $\|\cdot\| = \|\cdot\|_X$. This section relies very heavily on the methods, ideas and even notations of [1] and it would be unrealistic for the reader to try to read it without being familiar with [1].

The first consequence of (6.1) that I shall draw is

$$\|[A^{1/2}, E]f\|_X \le C\,\|f\|_X\,, \quad \|(c_\lambda - c_\mu)A^{1/2}f\|_X \le C\,\|f\|_X \tag{6.2}$$

when $f \in C_0^\infty$. For the proof I shall use the scale $X_\alpha = \{f : A^{\alpha/2}f \in X\}$, $(\alpha \in \mathbb{R})$. The norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha\to\beta}$ will refer to that scale. To prove (6.2) I shall start by proving

$$\|[e^{-tA}, E]\|_{\alpha\to\beta}, \quad \|(c_\lambda - c_\mu)e^{-tA}\|_{\alpha\to\beta} = O\left(t^{1/2+(\alpha-\beta)/2}\right) \tag{6.3}$$

where $\alpha \in]-1, 1], \beta \in [0, 2[$. Once we have established (6.3) the estimate (6.2) will follow automatically by the machinery of [1]. Indeed it would then follow from (6.3), by the real interpolation method of [1] Section 1, that

$$\|[A^\sigma, E]\|_{\alpha\to\beta} \le C\,, \tag{6.4}$$

for $\alpha\in]-1,1[$, $\beta\in]0,2[$, $-\mathrm{Re}\,\sigma+\frac{1}{2}+\frac{\alpha-\beta}{2}=0$ (the analogous estimate for $(c_\lambda-c_\mu)A^\sigma$ would also follow), so in particular we obtain

$$(6.4)'\quad \|[A^\sigma,E]\|_{\alpha\to\alpha},\|(c_\lambda-c_\mu)A^\sigma\|_{\alpha\to\alpha}\le C\,,\quad \mathrm{Re}\,\sigma=\frac{1}{2},\ \alpha\in]0,1[\,.$$

By duality it follows that (6.4)′ also holds for $\alpha\in]-1,0[$ and thus, by interpolation, (6.4)′ also holds for $\alpha\in]-1,1[$. Our required estimate (6.2) is the case $\alpha=0$ of (6.4). What follows from the above is, in fact, the stronger estimate where in (6.2) we replace $A^{1/2}$ by $A^{1/2+is}$ ($s\in\mathbb{R}$).

The proof of the basic estimates (6.3) can in fact be found in sections 4 and 5 of [1]. One simply has to run through the proof and observe that in the range $\alpha\in]-1,1]$, $\beta\in[0,2[$ the proof given there works under the assumption (6.1). For the convenience of the reader I shall recall the main points of the proof and I shall start with the easiest of the two estimates (which already contains the main idea). For $\alpha,\beta,\gamma\in\mathbb{R}$ we have

$$(6.5)\quad \begin{aligned}&\|[e^{-tA},E]\|_{\alpha\to\beta}\\ &\quad\le\int_0^t\|e^{-(t-s)A}\|_{\gamma-1\to\beta}\|[E,A]\|_{\gamma\to\gamma-1}\|e^{-sA}\|_{\alpha\to\gamma}\,ds\,.\end{aligned}$$

To be able to exploit the above factorization we must have

$$\gamma\in[\alpha,\alpha+2[\,,\quad \beta\in[\gamma-1,\gamma+1[\,.$$

For the middle term we will use the following estimate

$$(6.6)\quad \|[E,A]\|_{\gamma\to\gamma-1}\le C\,,\quad \|(1-c_\lambda)A\|_{\gamma\to\gamma-1}\le C\,,\quad \gamma\in[0,1]\,.$$

Indeed the case $\gamma=1$ of (6.6) is our hypothesis, the case $\gamma=0$ is the dual statement and the values in between are obtained by interpolation. In fact in (6.5) we can just set $\gamma=1$ (which is our hypothesis) and $\alpha\in]-1,1],\beta\in[0,2[$ the integral in (6.4) gives then the required estimate as in Section 4 of [1]. The proof for $c_\lambda-c_\mu$ is more involved and is essentially contained in [1], Section 5. First for all by taking differences it is enough to consider the case $\mu=0$. Let us then use the notations of [1], Section 5 and set

$$\varphi(t)=(1-c_\lambda)e^{-tA}=e^{-tA}-R_t\,.$$

I shall ignore the refinements of Section 5 in [1] and simply show that

$$\|\varphi(t)\|_{\alpha\to\alpha} = O(t^{1/2}), \quad \alpha \in [0,1]. \tag{6.7}$$

This will suffice to give us the estimate

$$\|R_t\|_{\alpha\to\alpha} = \|c_\lambda(e^{-tA})\|_{\alpha\to\alpha} = O(1), \quad \alpha \in [0,1]. \tag{6.8}$$

Once we have (6.8), we shall use the formula

$$\varphi(t) = \int_0^t e^{-(t-s)A}((1-c_\lambda)A)R_s\, ds \tag{6.9}$$

and the factorisation

$$\|\varphi(t)\|_{\alpha\to\beta} \le \int_0^t \|e^{-(t-s)A}\|_{\alpha-1\to\beta}\|(1-c_\lambda)A\|_{\alpha\to\alpha-1}\|R_s\|_{\alpha\to\alpha}\, ds$$

which together with (6.8) and (6.6) establishes (6.3) for $\alpha \in [0,1]$, $\beta \in [0,1[$. From this by the same method as before we finish the proof of (6.2). To establish (6.7) we use the same integral inequality as in Section 5 of [1]. We rewrite (6.9)

$$\varphi(t) = \int_0^t e^{-(t-s)A}(1-c_\lambda)A\varphi(s)ds + I(t),$$
$$I(t) = \int_0^t e^{-(t-s)A}(1-c_\lambda)Ae^{-sA}ds$$

and start by estimating

$$\|I(t)\|_{\alpha\to\alpha} = O(t^{1/2}), \quad \alpha \in [0,1].$$

This is done *exactly* as for $[E, e^{-tA}]$ (with $\alpha = \beta$). The next step is to fix some $f \in C_0^\infty$ in the unit ball of X_α and set $\psi(t) = \|\varphi(t)f\|_\alpha$. The function $\psi(t) \ge 0$ is such that $\psi(t) \to 0$ (as $t \to 0$) and satisfies the integral inequality

$$\psi(t) \le C\,t^{1/2} + C\int_0^t \|e^{-(t-s)A}\|_{\alpha-1\to\alpha}\|(1-c_\lambda)A\|_{\alpha\to\alpha-1}\psi(s)\, ds$$

so that we have

$$\psi(t) \le C\,t^{1/2} + C\int_0^t \psi(t-s)s^{1/2}\, ds, \quad \alpha \in [0,1].$$

This clearly implies the required estimate (6.7) just as in Section 5 of [1].

At this stage the scale X_α will be *abandoned* for good and the only information that will ve retained is at the level $\alpha = 0$, *i.e.* on the Hilbert space $X = L^2$ itself. This is exactly what was done in Section 6 of [1]. The scale we shall use from now onwards is the classical Sobolev scale

$$H_\alpha = \{f : \quad \Lambda^\alpha f \in L^2\}$$

and *from here onwards* right through the next section the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha\to\beta}$ will refer to that scale. Let us use the same notation as in Section 6 of [1] and set $B = \Lambda^s = (1+\Delta)^{s/2}$, $(s \in \mathbb{R})$. Then the estimate (6.1) of [1]

$$C\,\|A^\sigma f\|_X \leq \|BA^\sigma B^{-1} f\|_X \leq C\,\|A^\sigma f\|_X\,, \quad f \in C_0^\infty\,, \tag{6.10}$$

holds for $\sigma = 1$ and Re $\sigma = 1/2$. The proof of this fact that we gave in Section 6 of [1] works because of (6.1) and (6.2). From (6.10) we deduce just as in Section 6 of [1] that

$$\begin{aligned} \|[A,E]f\|_\alpha &\leq C\,\|A^{1/2} f\|_\alpha \\ \|(c_\lambda - c_\mu)Af\|_\alpha &\leq C\,\|A^{1/2} f\|_\alpha \end{aligned} \tag{6.11}$$

when $f \in C_0^\infty$. We have thus generalised the estimate (6.1) to all the classical Sobolev norms. More generally just as in Section 6 of [1] we can deduce from (6.2), (6.10) and (6.11) that

$$\begin{aligned} \|S^{n_1} A^\sigma S^{n_2} f\|_m &\leq C\,\|A^\sigma f\|_p \\ \|S^{n_1}[A^\sigma, S^{n_2}]S^{n_3} f\|_m &\leq C\,\|A^{\sigma-1/2} f\|_p \end{aligned} \tag{6.12}$$

with $\sigma = 1$ or Re $\sigma = 1/2$ and $p = m + \sum n_i$ and with $S^n \in OPS^n_{1,0}$ arbitrary pseudodifferential operators.

To illustrate (6.12) let us denote by $Q_t = e^{itA^{1/2}}$, ($t \in \mathbb{R}$ which is a group). We have then

$$Q_t - c_\lambda(Q_t) = c\int_0^t Q_{t-s}(1-c_\lambda)A^{1/2} c_\lambda(Q_s)dt$$

using then the same argument as in the beginning of this section and the fact $\|(1-c_\lambda)A^{1/2}\|_{\alpha\to\alpha} \leq C$ (which is but a special case of (6.12)) we obtain that

$$\|Q_t - c_\lambda(Q_t)\|_{L^2\to L^2} = O(|t|)\,.$$

This in particular proves our last assertion in Section 0.1.

7. Commutators with E.

I strongly urge the reader (to help him get the idea) to read first the part of this paragraph that starts soon after relation (7.2) where two special cases are considered.

All the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha\to\beta}$ refer to the classical Sobolev scale $H_\alpha = \{f : \Lambda^\alpha f \in L^2\}$. The operator A is as in Section 6 and will be assumed subelliptic so that there exists $\delta \in [0,1[$ such that

$$\|f\|_{1-\delta} \le C\,\|A^{1/2} f\|\,, \quad f \in C_0^\infty\,.$$

The letter δ will be reserved throughout to indicate that parameter.

We shall indicate multiple commutators throughout with the usual notation

$$[X, E_1, \dots, E_k] = [\dots[X, E_1], E_2, \dots], E_k]$$

with $E, E_1, \dots \in OPS^0_{1,0}$ and, as before, the same letter E will be reserved to indicate arbitrary $OPS^0_{1,0}$ which are not necessarily identical in different places. I shall also need the specific notation

$$\varphi(0) = 0\,, \quad \varphi(1) = \frac{1}{2}\,, \quad \varphi(2) = \varphi(3) = \dots = 1\,.$$

The following assertions (P_k) will be proved in this section inductively on $k = 0, 1, 2, \dots$.

Assertions (P_k):

$$(P'_k) \qquad \|[e^{-zA}, E_1, E_2, \dots, E_k]\|_{\alpha\to\beta} = O\left(|z|^{k/2+(\alpha-\beta)/2(1-\delta)}\right)$$

$$\text{where} \quad |\mathrm{Arg}\, z| < \frac{\pi}{4}\,, \quad \alpha, \beta \in \mathbb{R}, \quad \frac{k}{2} + \frac{\alpha-\beta}{2(1-\delta)} \le \varphi(k)\,.$$

$$(P''_0) \qquad \|A^{1/2}\|_{\alpha\to\alpha-1} \le C\,, \quad \alpha \in \mathbb{R}\,.$$

$$(P''_k) \quad \|[A^{1/2}, E_1, E_2, \dots, E_k]\|_{\alpha\to\alpha+(k-1)(1-\delta)} \le C\,, \quad k \ge 1, \alpha \in \mathbb{R}\,.$$

Two more conditions will be considered in the induction (z always lies in the sector $|\mathrm{Arg}\ z| < \pi/4$)

$$(P_k''') \qquad \|[A^{1/2}e^{-zA}, E_1, E_2, \dots, E_k]\|_{\alpha\to\beta} = O\left(|z|^{k/2+(\alpha-\beta)/2(1-\delta)-1/2}\right)$$

$$\text{where} \quad \frac{k}{2}+\frac{\alpha-\beta}{2(1-\delta)} < \frac{1}{2}, \quad \frac{k}{2}+\frac{\alpha-\beta}{2(1-\delta)} \leq \varphi(k).$$

$$(P_k^{(\mathrm{iv})}) \quad \|A^{1/2}[e^{-zA}, E_1, \dots, E_k]\|_{\alpha\to\beta} = O\left(|z|^{k/2+(\alpha-\beta)/2(1-\delta)-1/2}\right)$$

$$\text{where} \quad \frac{k}{2}+\frac{\alpha-\beta}{2(1-\delta)} < \frac{1}{2}, \quad \frac{k}{2}+\frac{\alpha-\beta}{2(1-\delta)} \leq \varphi(k).$$

A few more obvious remarks are in order. (P_0'), (P_0'') are contained in [1] and [8]. (P_1'') is just our estimate (6.2). Also for $k = 0$ the two statements (P_0''') and $(P_0^{(\mathrm{iv})})$ are identical and are automatic consequences of the holomorphicity of the action of e^{-tA} on the Sobolev spaces H_α. Furthermore observe that for any $k \geq 1$ the statements (P_j'') in conjunction with the statements (P_j'''), $j \leq k$, implies $(P_k^{(\mathrm{iv})})$. Graphically

$$(P_j'') \oplus (P_j'''), \quad 0 \leq j \leq k \quad \text{implies } (P_k^{(\mathrm{iv})}).$$

This is only a matter of developping out the commutator in (P''').

The next observation is less obvious and says that for $k = 2, 3, \dots$

$$(P_k') \quad \text{implies} \quad (P_k'').$$

In fact something more general holds: for $k \geq 2$ under the assumption that (P_k') holds (only needed for $z = t \geq 0$) we have

$$(7.1) \qquad \|[A^\sigma, E_1, E_2, \dots, E_k]\|_{\alpha\to\alpha+k(1-\delta)-2\mathrm{Re}\ \sigma(1-\delta)} \leq C$$

for all $\alpha \in \mathbb{R}$ and $\mathrm{Re}\ \sigma < 1$. This follows from our interpolation theorem of Section 1 in [1] applied to the scale H_α and

$$\Phi(t) = [e^{-tA}, E_1, E_2, \dots, E_k] = [e^{-tA} - I, E_1, E_2, \dots, E_k].$$

So that for Re $\sigma < 1$, Re $\sigma \neq 0$, we have

$$[A^\sigma, E_1, \dots, E_k] = c \int_0^\infty t^{-\sigma-1} \Phi(t) dt .$$

Indeed for $\beta = \alpha + k(1-\delta) - 2\text{Re}\,\sigma(1-\delta)$ we have $k/2 + (\alpha - \beta)/2(1-\delta) = \text{Re}\,\sigma < 1$, and that last strict inequality gives us the "room" that we need to play, for the interpolation of Section 1 in [1]. The case Re $\sigma = 0$ of (7.1) has to be delt separately but it can be deduced from Re $\sigma \neq 0$ by complex interpolation (applied to an analytic family of operators).

I shall finally show that for $k = 1, 2, \dots$

$$(P'_k) \quad \text{implies} \quad (P'''_k) .$$

The proof relies on the fact that the function

$$F(z) = [e^{-zA}, E_1, E_2, \dots, E_k] , \quad |\text{Arg } z| < \frac{\pi}{4}$$

is an operator valued holomorphic fuction. It follows therefore from Cauchy's Theorem and (P'_k) that

$$\left\| \left[\frac{d}{dz} e^{-zA}, E_1, \dots, E_k \right] \right\|_{\alpha \to \beta} = O\left(|z|^{k/2+(\alpha-\beta)/2(1-\delta)-1} \right)$$

when $k/2 + (\alpha - \beta)/2(1-\delta) \leq \varphi(k)$. On the other hand we have

$$[A^{1/2} e^{-zA}, E_1, \dots, E_k] = \int_0^\infty s^{-1/2} \frac{d}{ds} [e^{-(s+z)A}, E_1, \dots, E_k] ds$$

and therefore

$$\begin{aligned} &\|[A^{1/2} e^{-zA}, E_1, \dots, E_k]\|_{\alpha \to \beta} \\ &\qquad \leq C \int_0^\infty s^{-1/2} |s+z|^{k/2+(\alpha-\beta)/2(1-\delta)-1} ds \\ &\qquad = O\left(|z|^{k/2+(\alpha-\beta)/2(1-\delta)-1/2} \right) \end{aligned}$$

Provided that we have $k/2 + (\alpha - \beta)/2(1-\delta) < 1/2$ (the *strict* inequality (less than 1/2) is needed to give uniform bounds at infinity. I do not know if this inequality has to be strict or whether it can be relaxed to less than or equal to 1/2. But on the other hand this will be of no consequence at this point).

The upshot of all the above considerations is that in the proof of the inductive step of P_k it suffices simply to prove the step

$$(P'_{k-1}) \quad \text{implies} \quad (P'_k) \tag{7.2}$$

and in the proof of that step (7.2) I am allowed to use all the information contained in $(P_j)_{0\leq j\leq k-1}$.

To simplify notations from here onwards I shall drop the complex variable z ($|\mathrm{Arg}\ z| < \pi/2-\varepsilon_0$) and consider only $z=t>0$. The proofs are identical for complex z. I also urge the reader at this point to study Section 4 of [1] since otherwise he will find it difficult to understand the considerations that follow. Let us first consider simple commutators. We have

$$\begin{aligned}
&\|[e^{-2tA}, E]\|_{\alpha\to\beta} \\
&\quad \leq \int_0^t \|e^{-(2t-s)A}A^{1/2}\|_{\alpha\to\beta}\|A^{-1/2}[E,A]\|_{\alpha\to\alpha}\|e^{-sA}\|_{\alpha\to\alpha}ds \\
&\qquad + \int_0^t \|e^{-(t-s)A}\|_{\beta\to\beta}\|[E,A]A^{-1/2}\|_{\beta\to\beta}\|A^{1/2}e^{-(t+s)A}\|_{\alpha\to\beta}ds\,.
\end{aligned}$$

The two middle "factors" inside the integrals are adjoined of each other and are bounded (*cf.* (6.11)). The other terms can be estimated by the results of [8] and the holomorphicity of the semigroup e^{-tA} on H_α. Putting everything together we obtain

$$\|[e^{-tA}, E]\|_{\alpha\to\beta} = O\left(t^{1/2+(\alpha-\beta)/2(1-\delta)}\right)\,, \quad \beta\geq\alpha\,.$$

At this stage I could embark in the proof of the general inductive step (7.2). What is involved there however hides the main idea of the proof. To help the reader understand what is going on, I propose to prove "ad hoc" (P'_2) (*i.e.* $k=2$), and then perform the general inductive step. In fact if the reader is a "believer" he could skip in a fist reading the proof of that general inductive step. Let us examine commutators of order 2 where we shall expand $[e^{-2tA}, E, E]$ into six integrals as in Section 4 of [1]

$$\int_0^t e^{\cdot\cdot A}[A,E,E,]e^{\cdot\cdot A}\,, \qquad \int [e^{\cdot\cdot A}, E][A,E]e^{\cdot\cdot A}\,, \\ \int e^{\cdot\cdot A}[A,E][e^{\cdot\cdot A},E]\,. \tag{7.3}$$

The ".." on $e^{\cdot\cdot A}$ at the two ends of the integrals indicate the two combinations $-(2t-s)$ and $-s$ or $-(t-s)$ and $-(s+t)$ and they will be needed to perform the jump $\alpha \to \beta$ at one end or the other. The above integrals give corresponding "factorisations" of the $\|[e^{-tA}, E, E]\|_{\alpha\to\beta}$ norm and will be delt one at a time. For the first we proceed as follows

$$\int_0^t \|e^{-(2t-s)A}\|_{\alpha\to\beta}\| \cdot \|_{\alpha\to\alpha}\|e^{-sA}\|_{\alpha\to\alpha}$$

$$\int_0^t \|e^{-(t-s)A}\|_{\beta\to\beta}\| \cdot \|_{\beta\to\beta}\|e^{-(t+s)A}\|_{\alpha\to\beta}$$

and since $[A, E, E,] \in OPS^0_{1,0}$, its $\| \cdot \|_{\gamma\to\gamma}$ norm is bounded and the contribution of both of these two integrals is $O\left(t^{1+(\alpha-\beta)/2(1-\delta)}\right)$ for $\beta \geq \alpha$ (the results of [8] have to be used again).

The second integral in (7.3) gives rise to the factorisation

$$\int \|[e^{\cdot\cdot A}, E]\|_{\gamma\to\beta}\|[A, E]A^{-1/2}\|_{\gamma\to\gamma}\|A^{1/2}e^{\cdot\cdot A}\|_{\alpha\to\gamma}$$

where the γ is either α or β depending on the combination that we have adopted, $\{-(2t-s)A; -sA\}$ or $\{-(t-s)A; -(t+s)A\}$ of $\{..;..\}$. The norm $\|[A, E]A^{-1/2}\|_{\gamma\to\gamma}$ is bounded by (6.11), and using our previous result on the simple commutator $[e^{-tA}, E]$, we obtain again the contribution $O\left(t^{1+(\alpha-\beta)/2(1-\delta)}\right)$ (observe that there is a "loss" of 1/2 at one end, $A^{1/2}e^{\cdot\cdot A}$, but a "gain" of 1/2 at the other end, $[e^{\cdot\cdot A}, E]$.

The final integral in (7.3) is of course dual to the one we just considered. We put everything together and we conclude therefore that

$$\|[e^{-tA}, E, E]\|_{\alpha\to\beta} = O\left(t^{1+(\alpha-\beta)/2(1-\delta)}\right)$$

for $\beta \geq \alpha$. This is our statement (P_2'').

To work out the proof of the general inductive step (7.2) we shall introduce the following

$$K_k(t) = [e^{-tA}, E_1, \dots, E_k], \quad k = 0, 1, \dots, \quad t > 0$$

and use the analogue of our previous formulas to decompose $K_k(t)$ into a number of integrals of the form

$$I_{p,q,r} = \int_0^t K_p(..)[A, E_1, \dots, E_q]K_r(..)ds \tag{7.4}$$

where $q \geq 1$, $p+q+r=k$ and where the combination "..,.." at the two ends of the integral is as before either $2t-s, s$ or $t-s, t+s$. We shall assume that $(P_j)_{j \leq k-1}$ holds and we shall distinguishes two cases in (7.4).

Case (i): $q \geq 2$. Let

$$a = \frac{k}{2} + \frac{\alpha - \beta}{2(1-\delta)} \leq \varphi(k) = 1 \tag{7.5}$$

since we may suppose that $k \geq 2$. I shall introduce $\alpha', \beta' \in \mathbb{R}$ such that

$$x = \frac{r}{2} + \frac{\alpha - \alpha'}{2(1-\delta)} \leq \varphi(r), \quad y = \frac{p}{2} + \frac{\beta' - \beta}{2(1-\delta)} \leq \varphi(p), \tag{7.6}$$

$$z = \frac{q}{2} + \frac{\alpha' - \beta'}{2(1-\delta)} \leq \varphi(q) = 1,$$

$$\beta' \leq \alpha' + q - 2 \quad \text{or equivalently} \quad \frac{2-q\delta}{2-2\delta} \leq z. \tag{7.7}$$

The compatibility of the above conditions will be examined shortly. It is then possible to estimate

$$\|I_{p,q,r}\|_{\alpha\to\beta} \leq \int_0^t \|K_p(..)\|_{\beta'\to\beta}\| \cdot \|_{\alpha'\to\beta'} \|K_r(..)\|_{\alpha\to\alpha'} ds$$

where the middle term is bounded (because of (7.7)) and for the other two terms we can use the inductive hypothesis. It is necessary in the above to make sure that the integral converges at the two ends. Only one of the two ends will be a problem, and which one of the two ends will give trouble depends on the choice of combination "..;..". To ensure the convergence of the integral we must impose therefore the additional condition

$$x > -1 \quad \text{(respectively, } y > -1\text{)} \tag{7.8}$$

(the "respectively" refers of course to the choice of "..;..").

Assuming that the conditions (7.6), (7.7) and (7.8) are verified, we obtain then that (*cf.* (7.5))

$$\|I_{p,q,r}\|_{\alpha\to\beta} = O\left(t^{1+x+y}\right) = O\left(t^{x+y+z}\right) = O\left(t^{a}\right)$$

provided that $z \leq 1$, which proves the statement (P'_k). To prove the compatibility of our conditions observe that it suffices to find $x, y \in \mathbb{R}$ such that

$$x + y + z = a \leq 1\,, \quad \frac{2 - q\delta}{2 - 2\delta} \leq z \leq 1\,,$$
$$x \leq \varphi(r)\,, \quad y \leq \varphi(p)\,,$$
$$-1 < x\,, \quad (\text{respectively, } -1 < y)\,.$$

For indeed α', β' can then be determined to satisfy the *three* equations (7.6) (since $p + q + r = k$).

The above conditions on (x, y, z) are clearly compatible. It suffices to set

$$z = 1 \quad \text{and} \quad (x, y) = (0, a - 1) \quad (\text{respectively } (a - 1, 0))\,.$$

Case (ii): $q = 1$. We shall also assume without loss of generality that $p \geq 2$. Indeed one of the two p or r is larger than or equal to 2, since $k \geq 3$, and we can pass from one to the other by considering the adjoint operator. We proceed then as follows

$$\|I_{p,q,r}\|_{\alpha\to\beta} \leq \int_0^t \|K_p(..)\|_{\gamma\to\beta}\|[A, E]A^{-1/2}\|_{\gamma\to\gamma}\|A^{1/2}K_r(..)\|_{\alpha\to\gamma}ds$$

with the same meaning to the notation "..;..". We shall choose the $\gamma \in \mathbb{R}$ so that

$$x = \frac{r}{2} + \frac{\gamma - \alpha}{2(1 - \delta)} \leq \varphi(r)\,, \quad x < \frac{1}{2}\,,$$
$$y = \frac{p}{2} + \frac{\beta - \gamma}{2(1 - \delta)} \leq \varphi(p) = 1\,, \tag{7.9}$$

and

$$-1 < x \quad (\text{respectively, } -1 < y)\,, \tag{7.10}$$

with the same meaning as before for the "respectively" (it depends on the choice of "..;.." which is necessary to make the integral converge). When (7.9) and (7.10) are verified we can integrate and we obtain the required inductive step

$$\|I_{p,q,r}\|_{\alpha\to\beta} = O\left(t^{x+y-1/2+1}\right) = O\left(t^{k/2+(\alpha-\beta)/2(1-\delta)}\right)\,.$$

To check the compatibility set

$$x+y=\frac{k}{2}+\frac{\alpha-\beta}{2(1-\delta)}-\frac{1}{2}=a\leq\varphi(k)-\frac{1}{2}=\frac{1}{2}\,.$$

It is enough to choose $x, y \in \mathbb{R}$ so that

$$x<\frac{1}{2},\quad x\leq\varphi(r),\quad y=a-x\leq 1$$

and also

$$-1<x \quad \text{(respectively, } -1<y\text{)}$$

for then $\gamma \in \mathbb{R}$ can be determined to satisfy (7.9).

The compatibility of the above x, y conditions is clear, indeed it suffices to set $x = 0$ (respectively $x = 0$ if $a \in [0, 1/2]$ or $x = a$ if $a < 0$).

In the following proposition we collect together some important information obtained up to now.

Proposition. *Let* $\sigma \in \mathbb{C}$ *with* $\mathrm{Re}\ \sigma \leq 0$, $\alpha \in \mathbb{R}$, $k = 0, 1, \ldots$ *Then for commutators of length* k *we have*

$$\|[A^{\sigma}, E, E, \ldots, E]\|_{\alpha\to\alpha+k(1-\delta)-2\mathrm{Re}\ \sigma(1-\delta)} \leq C\,.$$

The proof was given in (7.1) for $k \geq 2$ and $\mathrm{Re}\ \sigma < 1$. It is very easy to see that the same proof works for $k = 1$, $\mathrm{Re}\ \sigma < 1/2$ and $\mathrm{Re}\ \sigma < 0$, $k = 0$. For $\mathrm{Re}\ \sigma = 0$, $k = 0$ the Proposition holds by (5.3). Observe finally that the above estimate also holds for $\mathrm{Re}\ \sigma = 1/2$, $k = 1$ (*cf.* (6.12), and the remark a couple of lines after (6.4)) but we shall have no use of the cases $\mathrm{Re}\ \sigma > 0$ in what follows.

8. General commutators and the classes $S^m_{\rho,\delta}$.

In this section A will denote a general linear operator $A : C_0^\infty \to \mathcal{D}'$ (to avoid necessary complications I shall also suppose, when necessary, that A is "compactly supported" in the sense that there exists some compact set K such that $A(\varphi) \equiv 0$ if $\mathrm{supp}\ \varphi \cap K = \emptyset$ and $A(\varphi) \equiv 0$ outside K, $\forall \varphi \in C_0^\infty$). I shall denote as usual by $E \in S^0_{1,0}$ and also by

$$\mathfrak{a}_k=\mathfrak{a}=[A,E,E,\ldots,E]=[\ldots[[A,E],E],\ldots]$$

where k is the length of the commutators. The E's are as usual $E \in OPS^0_{1,0}$ not necessarily all the same.

Our standing hypothesis in this section will be that

$$(8.1) \qquad \|\mathfrak{a}_k\|_{\alpha\to\alpha+\delta k-m} \le C\,, \quad \alpha \in \mathbb{R}, \quad k = 0, 1, \dots$$

for some fixed $0 < \delta \le 1$, $m \in \mathbb{R}$. $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha\to\beta}$ refer to the standard Sobolev norms.

To present the arguments in this section it is necessary to establish a good set of notations. The basis of our reasoning is the classical decomposition of unity

$$(8.2) \qquad 1 = \sum_{j=1}^{\infty} \psi_j(\xi)\,, \quad \psi_0 \in C_0^\infty\,, \quad \psi_j(\xi) = \psi^N(2^{-j}\xi)\,, \quad \xi \in \mathbb{R}^m$$

for some $\psi \in C_0^\infty$ with supp $\psi \subset \{\xi : 1/10 < |\xi| < 10\}$ and where the power N will be important because it will allow us to decompose the corresponding components into arbitrarily many factors. The $N \ge 1$ will be chosen at the beginning and appropriately large. The partition (8.2) will be used to decompose

$$(8.3) \qquad \Lambda^\alpha = \sum_{j\ge 1} 2^{\alpha j} E_j + E_0\,.$$

In (8.3) and in what follows, I shall reserve throughout the notation $E_j \in S^0_{1,0}$ for operators indexed by $j \ge 1$ that will satisfy several properties which will be enumerated below. It is important to understand that although all the E_j's have these properties they are not in general identical when they appear in different places. This notational convention gives us great flexibility in the arguments. Observe finally that E_0 that comes from ψ_0 is special, and will often enough be ignored since it never causes any trouble. All the properties below will be satisfied uniformly in the indices when the case arises.

(i) $E_j = e_j(D)\,,$ supp $e_j \subset \{\xi : 2^j/K < |\xi| < K\,2^j\}$

(the $K \gg 1$ can vary from place to place but does not depend on j).

(ii) $2^{\alpha j} E_j \in S^\alpha_{1,0}$,

I shall also adopte the notation $\lambda_j = 2^j$, $j \ge 0$.

(iii) $\sum \sigma_j \lambda_j^\alpha E_j \in S^\alpha_{1,0}$ for arbitrary $(\sigma_j)_{j\ge 0} \in l^\infty$.

This is an automatic consequence of (i) and (ii) and it implies that

$$\|\sum \sigma_j \lambda_j^\alpha E_j\|_{\beta+\alpha\to\beta} \le C\,, \quad \alpha,\beta\in\mathbb{R} \tag{8.4}$$

(iv) Each e_j and therefore each E_j can be factored into as many e_j's (respectively E_j's) as we need $E_j = E_j E_j \dots E_j$.

This simply come from the large exponent N in (8.2).

(v) For arbitrary $(\sigma_j)\in l^\infty$ and $\alpha,\beta\in\mathbb{R}$ we have

$$\|\sum \sigma_j \lambda_j^\alpha E_j f_j\|_\beta^2 \le C\sum \|f_j\|_{\alpha+\beta}^2\,, \quad f_j\in C_0^\infty\,. \tag{8.5}$$

Indeed let $F=\sum \sigma_j\lambda_j^\alpha E_j f_j$, $\varphi_j=\Lambda^{\alpha+\beta} f_j$ we have $\Lambda^\beta F = \sum \sigma_j E_j\varphi_j$ (the new E_j is of course different!) It suffices therefore to prove our assertion for $\alpha=\beta=0$. But then $\|F\|^2=\sum_{j,k}(E_jE_k\varphi_k,\varphi_j)$ and by (i) we can estimate this by $\sum\|E_j\varphi_j\|^2$. This gives our assertion.

(vi) Using the fact that $E_j = E_j^2$ (for a different E_j !) we can deduce from (8.5) (simply set $f_j = E_j f$) that

$$\|\sum \sigma_j \lambda_j^\alpha E_j f\|_\beta^2 \le C\sum \|E_j f\|_{\alpha+\beta}^2\,. \tag{8.6}$$

We also have

$$\sum \lambda_j^\alpha \|E_j f\|_\beta^2 \le C\,\|f\|_{\alpha+\beta}^2\,. \tag{8.7}$$

To see this, we set $F_\sigma = \sum \sigma_j\lambda_j^\alpha E_j f$ so that $\|F_\sigma\|_\beta \le C\,\|f\|_{\alpha+\beta}$, uniformly in σ, by (8.4). If we take expectations over $\sigma_j=\pm1$, (8.7) follows.

At this point let me recall that for ±1 centered independent random variables $\zeta_j,\xi_j,\eta_j,\dots$ and $h_{i,j,k,\dots}\in X$ (=some Hilbert space) we have

$$E\|\sum(\zeta_i\xi_j\eta_k\dots)h_{i,j,k,\dots}\|_X^2 \sim \sum \|h_{i,j,k,\dots}\|_X^2\,. \tag{8.8}$$

This is standard. (What is slightly less standard is that we have the one sided inequalities for $X=L^p(\Omega)$, $1\le p\le 2$. We have to make essential use of this refinement if we want to develop the L^p-theory of these operators).

The following terminology will now be used. We shall say that $T : C_0^\infty \to \mathcal{D}'$ is of smoothing order $\leq n \in \mathbb{R}$ (or simply "is of order n" if no confusion arises) if

$$\|Tf\|_\alpha \leq C \|f\|_{\alpha+n} , \quad \alpha \in \mathbb{R} . \tag{8.9}$$

In this terminology our operators $\mathfrak{a}_k = [A, E, E, \ldots, E]$, $k \geq 0$ are of order $\operatorname{ord}(\mathfrak{a}_k) = m - k\delta$. Clearly when T is of order n so is its adjoined T^*. We have then

(vii) Let $\mathfrak{a}_p = [A, E, E, \ldots, E]$ be as above. Then for every fixed $q \in \mathbb{R}$ the following two operators (adjoined of each other)

$$\sum_j \lambda_j^q E_j[E_j, \ldots, E_j, \mathfrak{a}] , \qquad \sum_j \lambda_j^q [E_j, \ldots, E_j, \mathfrak{a}] E_j \tag{8.10}$$

are of order $m + q - (p + n)\delta$. Here n is the number of E_j's inside the brackets of (8.10).

Indeed from the above observations it follows that it suffices to consider the first operator and from (8.6) it follows that it suffices to prove that for all n and α we have

$$\sum_j \|[E_j, E_j, \ldots, E_j, \mathfrak{a}_p]f\|_\alpha^2 \leq C \|f\|_{\alpha+m-(n+p)\delta}^2 \tag{8.11}$$

when $f \in C_0^\infty$, $\alpha \in \mathbb{R}$, $p = 0, 1, \ldots$. Here n indicates, as before, the number of E_j's inside the bracket. To prove this estimate we consider

$$E_\zeta = \sum \zeta_i E_i \ , \quad E_\xi = \sum \xi_i E_i \ , \ldots \in S_{1,0}^0$$

where $\zeta_i, \xi_i, \ldots$ are independent ± 1 random variables as above. Taking expectations and using (8.8) we obtain

$$\begin{aligned}\sum_{j_1, \ldots, j_n} \|[E_{j_1}, E_{j_2}, \ldots, E_{j_n}, \mathfrak{a}_p]f\|_\alpha^2 &\leq C \sup_{\zeta, \xi, \ldots} \|[E_\zeta, E_\xi, \ldots, \mathfrak{a}_p]f\|_\alpha^2 \\ &\leq C \|f\|_{\alpha-(p+n)\delta+m}^2\end{aligned}$$

where for the second inequality we use our hypothesis (8.1). The above estimate contains (8.11).

We now come to the main estimate of this section:

Let $\mathfrak{a} = \mathfrak{a}_p = [A, E, E, \dots, E]$, $(p \geq 0)$ as before and let $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ $(k \geq 0)$. We shall prove that the operator

$$[\Lambda^{\alpha_1}, \Lambda^{\alpha_2}, \dots, \Lambda^{\alpha_k}, \mathfrak{a}] = B \tag{8.12}$$

is of order $\alpha_1 + \dots + \alpha_k - (p+k)\delta + m$.

To prove this fact, I shall partition each Λ^α as in (8.3) (and also $I = \sum E_i$) and I shall write

$$B = \sum_{i, j_1, \dots, j_k} \lambda_{j_1}^{\alpha_1} \dots \lambda_{j_k}^{\alpha_k} E_i [E_{j_1}, \dots, E_{j_k}, \mathfrak{a}] .$$

I shall decompose the above summation into two parts. The first comes from terms for which $i \sim j_1 \sim \dots \sim j_k$, *i.e.* equal up to a fixed constant. The contribution we obtain then is

$$B' = \sum_i \lambda_i^{\sum \alpha_i} E_i [E_i, \dots, E_i, \mathfrak{a}]$$

and using (vii) we see that B' has the correct order. In the second summation, since the j_r's are interchangeable (all the E_k's commute !) we may suppose that $|j_1 - i| \geq C \gg 1$. This gives the following contribution (In the argument below I make essential use of the fact that the E_k's commute. On the other hand even if the E_k's did not commute we could make this argument work by considering higher commutators)

$$\begin{aligned} B'' &= \sum_{j_2, \dots, j_k} \lambda_{j_2}^{\alpha_2} \dots \lambda_{j_k}^{\alpha_k} [E_{j_2}, \dots, E_{j_k}, \sum_i E_i [\sum_{|j-i| \geq C} \lambda_j^{\alpha_1} E_j, \mathfrak{a}]] \\ &= [S_2, S_3, \dots, S_k, M] \end{aligned} \tag{8.13}$$

where $S_r \in S_{1,0}^{\alpha_r}$, $r = 2, \dots, k$ and

$$M = \sum_{|j-i| \geq C} \lambda_j^{\alpha_1} E_i [E_j, \mathfrak{a}] . \tag{8.14}$$

Using the fact that each E_j can be written as E_j^N and also the fact that $E_i E_j = 0$ for $|j - i| \geq C$ we deduce that the general term in the summation (8.14) can be replaced by

$$\lambda_j^{\alpha_1} E_i [E_j, E_j, \dots, E_j, \mathfrak{a}] E_j$$

with as many E_j's as we need inside the bracket. We conclude therefore that

$$M = \sum_{|i-j|\geq C} \lambda_j^{\alpha_1} E_i[E_j, \dots, E_j, \mathfrak{a}]E_j\,.$$

Summing first over i and observing that $\sum_{|i-j|\geq C} E_i = E - E_j$ we deduce that

$$\text{(8.15)}\quad M = E(\sum_j \lambda_j^{\alpha_1}[E_j, \dots, E_j, \mathfrak{a}]E_j) + \sum_j \lambda_j^{\alpha_1} E_j[E_j, \dots, E_j, \mathfrak{a}]E_j$$

but in the second summation we can absorb the E_j on the right by introducing an *extra commutator.*

Putting together (8.13), (8.14) and (8.15) we finally see that it is a consequence of (vii) that M and thus B'' have as low an order as we like (*i.e.* they are infinitely regularising. Indeed it is only a matter of taking the length of the brackets in (8.15) high enough). This proves our assertion.

Let us now consider arbitrary $S_j \in OPS^{n_j}_{1,0}$. The final claim that I will be made in this section concerns (always under the hypothesis (8.1)) the smoothing order of the following commutator

$$\text{(8.16)}\qquad C_p = [A, S_1, \dots, S_p] = [\dots[A, S_1], S_2]\dots S_p]\,,$$

$$\text{(8.17)}\qquad \text{Smoothing order } C_p \leq \sum n_j + m - \delta k\,.$$

Here of course the smoothing order is defined as in (8.9). This statement will be proved by induction on the length p. It clearly holds for $p = 0$. I shall assume it to hold up to $p-1$, and proceed to prove the inductive step.

Towards that I start by factorising each $S_j = E\Lambda^{n_j} = \Lambda^{n_j}E$ and expand the commutators (8.16). What is obtained by that expansion is a linear combination of terms

$$\text{(8.18)}\qquad P[A, T_1, T_2, \dots, T_k]Q + \dots$$

where $T_1, T_2, \dots, T_j = E$, I shall then say that ord $T_i = t_i = 0$ $(1 \leq i \leq j)$, and $T_r = \Lambda^{t_r}$ and say ord $T_r = t_r$, $(j+1 \leq r \leq k)$. Furthermore $P \in OPS^p_{1,0}$, $Q \in OPS^q_{1,0}$.

The important point is that $p + q + \sum t_i \leq \sum n_i$. This is obvious because in the various monomials that appear in the decomposition of C_p there is *no way* at all that we can increase the total order of the pseudodifferentials. The first term in (8.18) has therefore the required smoothing order by the corresponding statement on (8.12) (the reader has to make here the distinction between the smoothing order and the order of a pseudodifferential).

It remains to examine the remainder terms "..." in (8.18). These are the terms for which the "principal commutator" has length $p' < p$, and they look like

$$P[A, S'_1, \dots, S'_{p'}]Q$$

with $p' < p$. It should be clear what is ment by "principal commutator": it is the commutator that contain A. All the other commutators contract to P and Q which are ordinary pseudodifferential operators. We have again $\sum \text{ord } S'_j + \text{ord } P + \text{ord } Q \leq \sum n_i$. But more can in fact be asserted, we have

$$\sum \text{ord } S'_j + \text{ord } P + \text{ord } Q + (p - p') \leq \sum n_i \,. \tag{8.19}$$

After a moment reflexion the reason for this should be clear. Indeed if we have decreased the length of the "principal commutators", say, be one unit, this is because somewhere in the product we have bracketed two S's, $[S_j, S_k]$. But this bracket makes us gain one unit in the total order (I mean here the order in the sense of pseudodifferential calculus) and so on.

From (8.19) it follows that the inductive step applies. Indeed in the conclusion (8.17) we gain $p - p'$ and lose $-\delta(p - p')$ and since $\delta \leq 1$ the inductive hypothesis gives us, if anything, a stroger estimate. This completes the proof.

If we put together everything that was done in this section we see that we can reduce our criterion at the beginning of Section 0.2 to the Beals criterion [3]. Since A is assumed to be "compactly supported" we can, in fact, use the form of the Beals criterion given in [5], Chapter III.

The assertion (0.4) follows by the same criterion and the estimates (P_k) of Section 7; the proof is therefore, if anything, easier. The assertion (0.5) also follows from the criterion of Section 0.2. To see this let us call $\mathcal{C}^m_\rho$, $(m, \rho \in \mathbb{R})$ the class of operators T as in (0.2) that satisfy the condition (0.2). It is then a formal verification to see that $T_i \in \mathcal{C}^{m_i}_{\rho_i}$, $(i = 1, 2)$ implies that $T_1 T_2 \in \mathcal{C}^{m_1+m_2}_{\min\{\rho_1, \rho_2\}}$. If we combine therefore our result $A^{is} \in \mathcal{C}^0_{1-\delta}$, $(s \in \mathbb{R})$ of Section 6, together with the fact $A^n \in S^{2n}_{1,0} \subset \mathcal{C}^{2n}_1$ $(n = 1, 2 \dots)$, we deduce that $A^\sigma \in \mathcal{C}^0_{1-\delta}$ for $\text{Re } \sigma = 0, 1, 2, \dots$. Complex interpolation gives then that $A^\sigma \in \mathcal{C}^{2\text{Re } \sigma}_{1-\delta}$, $(\text{Re } \sigma \geq 0,\ 0 \leq \delta < 1)$. Our criterion does the rest.

9. An application of a theorem of R. Beals.

We shall place ourselves here in the context of Theorem 5.4 of [3] (*cf.* also [4], [9] for the general setup). We set $P = p^{\omega}(x, D)$ with $p \in S^2_{phg}$ a polyhomogeneous symbol (*cf.* Definition 18.1.5 in Hörmander, vol. III). More general symbols in S^m_{phg}, $m = 2, 3\ldots$ can also be treated but we shall restrict ourselves to $m = 2$ for simplicity.

Following Beals we must impose on the principal symbol p_2 the same conditions as in [3] (*e.g.* $p_m(x,\xi)$ belongs to the sector $|\text{Arg}\, z| \le \pi/2 - \varepsilon_0$ or the even less restrictive condition [3]) and also that P is subelliptic with a loss of 1 derivative, *i.e.* that

$$\|u\|_1 \le C\,(\|Pu\| + \|u\|)\,, \quad u \in C_0^\infty \tag{9.1}$$

for the usual Sobolev norms $\|\cdot\|_\alpha$ and $\|\cdot\| = \|\cdot\|_0$. We shall suppose also that the complex powers P^σ, $(\sigma \in \mathbb{C})$ can be defined by say, a ray of minimal growth (*cf.* [9], p. 153). For simplicity we shall in fact assume here that the symbol of P is $p(x,\xi) + \lambda_0$ with some large λ_0 and $p(x,\xi) \ge 0$ and then all the above conditions are verified.

I shall show in this section how the results of R. Beals in [3], [4] and [9] imply our basic estimate (0.1) very easily and in full generality, for $A = P$ as above.

To do this we introduce the (φ, Φ) functions of p. 56 in [3] (with $m = 1$) and consider the corresponding metric

$$g_{x,\xi}(y,\eta) = \frac{|y|^2}{\varphi^2(x,\xi)} + \frac{|\eta|^2}{\Phi^2(x,\xi)} = m\left(|y|^2 + \frac{|\eta|^2}{1+|\xi|^2}\right) = m g_0(y,\eta)$$

(*cf.* [10], Example 3, p. 378) with $m = \langle\xi\rangle^2\Phi^{-2}(x,\xi) \ge 1$ and the uncertainty parameter $h = (\varphi\Phi)^{-1} \approx \langle\xi\rangle(|p_2| + \langle\xi\rangle)^{-1} \le 1$. What counts of course is that the symbol of P lies in the class $S(\Phi^2; g)$ (This is proved in [3] and here I swich freely from Beals to Hörmander's notations).

The additional observation that we need is the fact that $\langle\xi\rangle^m \in O(\Phi, \varphi)$ (with the notations of [3] and [9]) since $R = \Phi/\varphi = \langle\xi\rangle$, in other words $\langle\xi\rangle^m$ is an admissible weight function (in Hörmander's terminology [6], sections 18.4 and 18.5 for the classes $S(m; g)$) for the metric g. This will allow us to exploit the "mixed symbolic" calculus

$$S(m_1; g_1) \times S(m_2; g_2) \quad \longrightarrow \quad S(m_1 m_2; g_1 + g_2)$$

of Theorem [6], 18.5.5 and make a gain on the "order" of the commutators. Le us be more explicit. We shall apply this Theorem 18.5.5 with

$$g_1 = g_0\,, \quad g_2 = g\,, \quad \frac{g_1 + g_2}{2} \approx g$$

$m_1 = \langle \xi \rangle^m$, ($m \in \mathbb{R}$) and m_2 any weight function of g. As we just saw both m_1, m_2 are then continuous $\sigma - (g_1 + g_2)/2$ temperate weight functions. As for the condition [6], (18.5.13) on g_1, g_2 it is guarantied here by [6], Proposition 18.5.7, which also gives us that

$$H = (h_1 h_2)^{1/2} = (|p_2| + \langle \xi \rangle)^{-1/2}\,.$$

The application of Beals theory ([3], [4], *cf.* Appendix at the end of the paper) gives then that for all $\sigma \in \mathbb{C}$ we have

$$P^\sigma = q_\sigma^\omega(x, D)\,, \quad q_\sigma \in S((|p_2| + \langle \xi \rangle)^{\mathrm{Re}\ \sigma}; g)\,.$$

From this and [6], Theorem 18.5.5 we deduce that

$$[P^\sigma, S] = a^\omega(x, D)\,, \quad a \in S(\langle \xi \rangle^n (|p_2| + \langle \xi \rangle)^{\mathrm{Re}\ \sigma - 1/2}; g)$$

for any $S \in OPS^n_{1,0}$. The application of [6], Theorem 18.5.5 can clearly be iterated and we obtain

$$S_0[\dots[P^\sigma, S_1]\dots]S_k]S_{k+1} = b^\omega(x, D)\,, \qquad b \in S(\langle \xi \rangle^{\sum n_j} (|p_2| + \langle \xi \rangle)^{\mathrm{Re}\ \sigma - k/2}; g) \tag{9.2}$$

for arbitrary pseudodifferentials $S_j \in OPS^{n_j}_{1,0}$, $(0 \leq j \leq k+1)$.

To obtain our basic estimate (0.1) from (9.2) we must find a way to prove that

$$\|f\| = \|f\|_{L^2} + \|\Lambda^n P^m f\|_{L^2}\,, \quad f \in C_0^\infty \tag{9.3}$$

is an "admissible norm" (in the sense of [4]) for the space $H(\langle \xi \rangle^n (|p_2| + \langle \xi \rangle)^m; g)$, $(n, m \in \mathbb{R})$, (with Beals notations in [4]). That this is the case for $n = 0$ is proved in Beals [3], [4] and the key to that is Theorem 3.7 of [4] (one easily sees that the same argument gives $n \in \mathbb{R}$, $m = 1, 2, \dots$).

No doubt one can generalise Beals theory to obtain the above more general result for arbitrary $n, m \in \mathbb{R}$. This will not be necessary here however. Indeed from (9.2) and the above results of Beals we certainly have the special case (since then $n = \sum n_i = 0$)

$$\|\mathcal{L}_1 \mathcal{L}_2 \dots \mathcal{L}_k (P^\sigma) f\|_X \leq C\, \|P^{\mathrm{Re}\ \sigma - k/2} f\|_X\,, \tag{9.4}$$

$k = 0, 1, \dots,\ f \in C_0^\infty$, where I denote by $\mathcal{L}_j$, $(j = 1, \dots, k)$

$$\mathcal{L}_j(T) = [E, T], \quad \text{or } (c_\lambda - c_\mu)(T)$$

with $E \in S^0_{1,0}$ as usual, and $c_\lambda(T) = \Lambda^{-\lambda} T \Lambda^\lambda$, $X = L^2$ (T indicates an arbitrary operator). At this stage we have to go back to Section 6 of [1] where it was shown that (9.4) implies our estimate (0.1). (This was done in Section 6 of [1] only for $k = 2$ but the proof is clearly general. Observe also that this is essentially the same argument that is used at the end of Section 8 to deal with the general commutator (8.16). The reader should have no difficulty to adapt the argument here). Our proof is complete.

10. The generalisation of the geometric theorem.

This section relies very heavily on the methods, ideas and notations of sections 7,8 and 9 of [1]. What I shall do is to use the results of the previous section to give the generalisation of the main geometric Theorem of [1], Section 0 as was promised in Section 0.1.

Let $M = a^\omega(x, D) + \lambda_0$ with $0 \le a(x, \xi) \in S^2_{phg}$ and large $\lambda_0 > 0$ satisfying the conditions of Section 9, and let $L = \sum X_j^* X_j$ a subelliptic Hörmander operator or more generally an operator of the form $L = \sum X_j^* X_j + \Lambda^\alpha$, $(0 \le \alpha \le 2)$ where $\sum X_j^* X_j$ is again assumed to be subelliptic. These operators were denoted by $\tilde{L} = \sum Y_j^* Y_j$ in [1], Section 9, and the Y's that we shall consider below are the Y's defined there. We shall further assume that the two operators L, M satisfy the subellipticity estimates of [1], Section 7,

$$(10.1)\quad \|f\|_{1-\delta} \le C\, \|(I + L)^{1/2} f\|_X\,, \quad \|f\|_{1-d} \le C\, \|(I + M)^{1/2} f\|_X\,.$$

Our conditions on M imply that $d \le 1/2$. We shall also assume that $d + \delta \le 1$ and we shall extend our Proposition in Section 7 of [1] in the present setting. More specifically we shall prove [1], equality (7.2)

$$\|(I + L)^{j/2} e^{-tM} (I + L)^{-j/2}\|_{\alpha \to \beta} = O\left(t^{(\alpha - \beta)/2(1-d)}\right), \quad \beta \ge \alpha,$$

for the above operators L and M. The norms $\|\cdot\|_\alpha$, $\|\cdot\|_{\alpha \to \beta}$ refer throughout to the classical Sobolev norms H_α. The proof of [1], (7.2), that I shall give below is very close in spirit to the proof given in [1], Section 7. Indeed it is in some sense *dual* to the proof there. Once the [1], (7.2) has been generalised for our present operators we can obtain the generalisation of the geometric theorem that was announced in Section 0.1 exactly as in [1].

Before we start the proof of the estimate, we shall need to note an easy *algebraic* identity

$$\begin{aligned}(10.2)\qquad &[m, y_1 y_2 \dots y_k] \\ &= \sum_{\sigma,j} p_{\sigma,j}[m, y_{\sigma(1)}, \dots, y_{\sigma(j)}] y_{\sigma(j+1)} \cdots y_{\sigma(k)}\,, \quad k \geq 1\end{aligned}$$

for arbitrary indeterminates m; $y_1, \dots, y_k$ and $p_{\sigma,j} \in \mathbb{Z}$, where σ runs through the permutations of $1, 2, \dots, k$. This is easily proved by induction on k.

We shall also need to introduce the following notation: for $Y_1, Y_2, \dots \in S^1_{1,0}$ determined by the operator L (or rather $\tilde{L}$ as in Section 9 of [1]) and $k = 0, 1, \dots$ I shall denote by

$$R_k(t) = Y_{i_1} Y_{i_2} \dots Y_{i_k} e^{-tM} (I + L)^{-k/2}\,.$$

There are of course several R_k's for a fixed k and they depend on the choice of $i_1, \dots, i_k$.

Our first step is to prove by induction on k that for all $\beta \geq \alpha$ and $k = 0, 1, \dots$, we have

$$(10.3)\qquad \|R_k(t)\|_{\alpha \to \beta} = O\left(t^{(\alpha-\beta)/2(1-d)}\right)\,.$$

This statement for $k = 0$ is contained in [8] (*cf.* also Section 3 of [1]).

Our aim is therefore to assume that (10.3) holds for $0, 1, \dots, k$ and prove it for $k+1$. Towards that we fix $(i_1, \dots, i_{k+1})$ which to simplify notations we shall *rename* $1, 2, \dots, k+1$. We then develop in our usual way

$$\begin{aligned}[e^{-tM}, Y_1 \dots Y_{k+1}] &= I_1 + I_2 \\ &= \int_0^t e^{-(2t-s)M} [M, Y_1 \dots Y_{k+1}] e^{-sM} ds \\ &\quad + \int_0^t e^{-(t-s)M} [M, Y_1 \dots Y_{k+1}] e^{-(t+s)M} ds\,.\end{aligned}$$

This together with our identity (10.2) gives us a decomposition

$$[e^{-tM}, Y_1 \dots Y_{k+1}](I+L)^{-(k+1)/2} = \sum_j p_j(I_j^{(1)} + I_j^{(2)})$$

$$I_j^{(1)} = \int_0^t e^{-(2t-s)M}[M, Y_1 \dots Y_j] R_{k+1-j}(s)\, ds\, (I+L)^{-j/2}$$

and the analogous expression with the usual switch $(2t-s) \to t-s$, $s \to t+s$ for $I_j^{(2)}$. The Y_i's in the above formula have, of course, undergone one more renaming (they really are $Y_{\sigma(i)}$'s for an appropriate permutation σ).

We shall factor $\|I_j^{(1)}\|_{\alpha\to\beta}$, $(\beta \geq \alpha)$ and estimate it by

$$\begin{aligned}(10.4)\qquad \int_0^t \|e^{-(2t-s)M} M^{1-j/2}\|\, \|M^{-1+j/2}[M, Y_1, \dots, Y_j]\| \\ \cdot \|R_{k+1-j}(s)\|\, ds\, \|(I+L)^{-j/2}\|\end{aligned}$$

which is an in Section 7 of [1] an appropriate cascade of $\|\cdot\|_{r\to s}$ norms, that unfolds as follows

$$\|(I+L)^{-j/2}\|_{\alpha\to\alpha+j(1-\delta)} \leq C\,,$$
$$\|R_{k+1-j}(s)\|_{\alpha+j(1-\delta)\to\alpha+j(1-\delta)} = O(1)\,,$$
$$\|M^{-1+j/2}[M, Y_1, \dots, Y_j]\|_{\alpha+j(1-\delta)\to\alpha-j\delta} \leq C$$

for the first estimate (*cf.* [1], [8]). The second follows from our inductive hypothesis and to see the third we use the result of Section 9 together with the fact that each $Y_j \in OPS^1_{1,0}$. To estimate the first term in the integral (10.4) we recall that for $\lambda \geq 0$ we have for $\beta \geq \alpha$

$$\begin{aligned}(10.5)\qquad \|M^\lambda e^{-tM}\|_{\alpha\to\beta} &\leq \|M^\lambda e^{-t/2M}\|_{\beta\to\beta} \|e^{-t/2M}\|_{\alpha\to\beta} \\ &= O\left(t^{-\lambda+(\alpha-\beta)/2(1-d)}\right)\end{aligned}$$

and

$$\begin{aligned}(10.6)\qquad \|e^{-tM}M^{-\lambda}\|_{\alpha\to\beta} &\leq \|e^{-tM}\|_{\alpha+2\lambda(1-d)\to\beta} \|M^{-\lambda}\|_{\alpha\to\alpha+2\lambda(1-d)} \\ &= O\left(t^{\lambda+(\alpha-\beta)/2(1-d)}\right)\end{aligned}$$

provided that $\beta \geq \alpha + 2\lambda(1-d)$ (since the factor $\|M^{-\lambda}\|$ is bounded (*cf.* [1], [8])). We apply this to the first factor inside the integral of

(10.4) and distinguish two cases $j = 1,2$ and $j > 2$. In the first case (10.5) gives us

$$\|e^{-(2t-s)M} M^{1-j/2}\|_{\alpha - j\delta \to \beta} = O\left((2t-s)^{-1+j/2+(\alpha-j\delta-\beta)/2(1-d)}\right) . \tag{10.7}$$

We multiply out and integrate and obtain

$$\|L_j^{(1)}\|_{\alpha\to\beta} = O\left(t^{(j(1-d-\delta))/2(1-d)+(\alpha-\beta)/2(1-d)}\right) . \tag{10.8}$$

If $j > 2$ we use the estimate (10.6) to obtain (10.7) again and we obtain also exactly the same estimate (10.8) for $I_j^{(1)}$ as long as the *exponent* of t in $O(t^{\text{ exponent}})$ of (10.7) is ≤ 0. This however is always te case since $\beta \geq \alpha$ and $0 \leq d < 1$.

The integrals $I_j^{(2)}$ are estimated by the analogue of the integral (10.4) where we replace $(2t - s)$ by $(t - s)$ and s by $(t + s)$ on the exponentials and $R_{k+1-j}(\cdot)$. The cascade of $\|\cdot\|_{r\to s}$ norms runs now as follows

$$\|(I+L)^{-j/2}\|_{\alpha\to\alpha+j(1-\delta)} \leq C\,,$$
$$\|R_{k+1-j}(t+s)\|_{\alpha+j(1-\delta)\to\gamma+j(1-\delta)} = O(1)\,,$$

by the induction hypothesis provided that $\gamma = \alpha + \varepsilon \geq \alpha$. We also have, just as before,

$$\|M^{-1+j/2}[M, Y_1, \ldots, Y_j]\|_{\gamma+j(1-\delta)\to\gamma-j\delta} \leq C\,.$$

To estimate the first term we have to distinguish again the two cases $j = 1,2$ and $j > 2$. In the first case we have

$$\|e^{-(t-s)M} M^{1-j/2}\|_{\gamma-j\delta\to\beta} = O\left((t-s)^{-1+j/2+(\gamma-\beta-j\delta)/2(1-d)}\right) \tag{10.9}$$

as long as $\beta \geq \gamma - j\delta$. Then since $\beta \geq \alpha$ we can choose $\gamma = \beta$ and after integration we obtain

$$\|I_j^{(2)}\|_{\alpha\to\beta} = O\left(t^{(j(1-d-\delta))/2(1-d)+(\alpha-\beta)/2(1-d)}\right) . \tag{10.10}$$

In the second case $j > 2$ we obtain the same estimates (10.9) and (10.10) provided that (the left inequality below is to make the integral converge)

$$(10.11)\qquad \begin{aligned} -1 < \frac{j}{2} - 1 + \frac{\gamma - \beta - j\delta}{2(1-d)} \\ = -1 + \frac{j(1-d-\delta)}{2(1-d)} + \frac{\alpha-\beta}{2(1-d)} + \frac{\varepsilon}{2(1-d)} \leq 0 . \end{aligned}$$

At first sight it looks as if here we are in trouble. Indeed for $1-d-\delta > 0$ and $j \gg 0$, (10.11) is incompatible for $\varepsilon \geq 0$. But of course we can get round that difficulty simply by *assuming* that $1 - d - \delta = 0$. This is no loss of generality since we can allways increase the d and δ in the definition of subellipticity of L and M without altering the validity of the conditions (10.1). In that case $d + \delta = 1$ the inequalities are then always compatible for some $\varepsilon \geq 0$ since by our hypothesis $\beta \geq \alpha$.

All in all we have therefore established that, under the inductive hypothesis, we have

$$(10.12)\quad \|[e^{-tM}, Y_1 \dots Y_{k+1}](I+L)^{-(k+1)/2}\|_{\alpha\to\beta} = O\left(t^{(\alpha-\beta)/2(1-d)}\right)$$

(provided that $d + \delta \leq 1$).

At this stage we shall invoque the estimate (9.1) of [1], (where the subellipticity of $\sum X_j^* X_j$ is apparently needed). This together with (10.12) and the (standard by now, I hope) fact that

$$\|e^{-tM}\|_{\alpha\to\beta} = O\left(t^{(\alpha-\beta)/2(1-d)}\right)$$

establishes the inductive step and complete the proof of (10.3) in all generality.

We shall now finish the proof of [1], (7.2). Assume that $j = 2k$ is an even integer, then

$$(I+L)^{j/2} = \sum_{p\leq k} \lambda_i Y_{i_1} Y_{i_2} \dots Y_{i_{2p}}\,, \quad \lambda_j \in \mathbb{Z}$$

and our estimate [1], (7.2) for $\alpha = \beta$ follows from (10.3). Equivalently what we have proved is

$$e^{-tA} : X_{2j} \longrightarrow X_{2j}\,, \quad j = 1, 2, \dots$$

with our old notation $X_\alpha = \{f : (1+L)^{\alpha/2} f \in L^2\}$. Duality and interpolation completes the proof of [1], (7.2) for $\alpha = \beta$.

This is good enough for our purposes and proves the analogue of the proposition in Section 7 of [1]. We can however also prove [1], (7.2) in full generality $\beta \geq \alpha$ by a slightly more sophisticated variant of complex interpolation. This was explained in Section 7 of [1].

REMARK. One of the facts that was used in [1], Section 8, is that $e^{-t\widetilde{L}}$ acts on the spaces X_α ($\alpha \in \mathbb{R}$). This fact when $\alpha = 2n$ ($n = 1, 2, \dots$) is a consequence of the semiboundedness of Δ on X_{2n}, and this was proved in [1], Section 10. The general fact follows then by duality and interpolation.

Contrary to what was asserted in [1], Section 10, on the other hand, this actual semiboundedness of Δ on each X_α (for some appropriate scalar product) does *not* seem to follow by interpolation. This semiboundedness is however never used anywhere else so we do not need to prove it.

Appendix on the Beals theory.

In this appendix, using the Beals theory [3], [4] and [9], I shall outline a proof of the fact that the norm $\|f\|$ in (9.3) with $n = 0$ and $m = N/2$, $N = 1, 2, \dots$) a half integer is an admissible norm for the space $H((|p_2| + \langle\xi\rangle)^m; g)$. This fact is explicitely proved in the papers of Beals. The point is however, that the direct proof that I give here, only uses the basic definitions of the Beals theory and none of the more sophisticated machinery developed by Beals. On the other hand this special case ($n = 0$, $m = N/2$) is all that is needed for the proof of our basic estimate (0.1). In other words we only need (9.4) for $\sigma = N/2$ (a half integer) and then if we inject that information in Section 6 of [1] we can make everything work.

The first thing to observe towards that goal is that our basic hypothesis (9.1) implies that

$$C\,\|(P+I)^\alpha f\| \geq \|(P+a\Lambda)^\alpha f\|, \quad f \in C_0^\infty \tag{A.1}$$

for all a, $\alpha \geq 0$. Indeed it suffices to prove (A.1) for $\alpha = 1, 2, \dots$ we can develop then $(P + a\Lambda)^\alpha$ and we reduce the problem to proving that

$$\|L_1 L_2 \dots L_k f\| \leq C\,\|(P+I)^\alpha f\| \tag{A.2}$$

where L_j is either $A = P+I$ or $L_j \in S^{n_j}_{1,0}$ and where if s is a the number of A's then $\sum n_j + s \leq \alpha$. For $s = 0$ (A.2) is a consequence of (9.1) (*cf.* [8]). We can thus use induction on s. The inductive hypothesis and the fact that $[A, S^n_{1,0}] \subset S^{n+1}_{1,0}$ allow us then to commute and bring all the A's at the beginning of the product. (A.2) is thus reduced to

$$\|TA^s f\| \leq C\,\|A^\alpha f\|\,, \quad f \in C_0^\infty$$

with $T \in S^{\alpha-s}_{1,0}$, $0 \leq s \leq \alpha$. This is clearly a consequence of [8] (set $\varphi = A^s f$).

Having proved (A.1) let us denote $P_N = (P + a\Lambda)^N$ (for some large $a \geq 0$, $N = 1, 2, \dots$). Our problem is to show that $\|f\|_N = (P_N f, f)^{1/2}$ is a norm for the space $H(m^{N/2}; g)$ where we denote by

$$m = p + C\langle\xi\rangle \approx p_2 + C\langle\xi\rangle.$$

For simplicity we shall suppose here that the symbol of P is nonnegative, $p(x,\xi) \geq 0$.

The proof of this fact is an easy consequence of the existence of the following two "parametrices"

$$\text{(A.3)} \qquad \begin{cases} Q_\pm = q_\pm^\omega(x, D)\,, & q_\pm \in S(m^{\pm N/2}; g) \\ P_N \equiv Q_+^* Q_+ \text{ mod-}OPS(m^N h^s; g)\,, & \\ Q_- Q_+ \equiv I \text{ mod-}OPS(h^s; g) & \end{cases}$$

where $s \geq 1$ can be chosen in advance and arbitrarily large. To construct these parametrices let us denote by $R_N = (m^N)^\omega(x, D)$, $(N \in \mathbb{R})$ and let us observe that by standard symbolic calculus we have

$$R_{-N/2} P_N R_{-N/2} \equiv 1 \quad \text{mod-}OPS(h; g)\,.$$

This allows us to use the binomial $(1+z)^{1/2} = 1 + z/2 + \dots$ and write

$$R_{-N/2} P_N R_{-N/2} \equiv Y^2 \quad \text{mod-}OPS(h^s; g)\,,\ Y = Y^* \in OPS(1; g)$$

with arbitrary high $s \geq 1$. Similarly we have

$$R_{-N/2} R_{N/2} \equiv 1 \quad \text{mod-}OPS(h; g)$$

and the Neumann series $1 - z + z^2 - \dots$ allows us to construct a parametrix $\tilde{R}_{N/2} \in OPS(m^{N/2}; g)$

$$R_{-N/2} \tilde{R}_{N/2} \equiv 1 \quad \text{mod-}OPS(h^s; g)$$

with arbitrarily high $s \geq 1$. Combining these two facts we obtain

$$P_N \equiv \tilde{R}^*_{N/2} Y^2 \tilde{R}_{N/2} \quad \text{mod-}OPS(m^N h^s; g).$$

It follows thus that we can set $Q_+ = Y\tilde{R}_{N/2} = q^{\omega}_{+}(x, D)$ in (A.3).

Observe now that $Y \equiv 1 \quad \text{mod-}OPS(h; g)$ and so $R_{-N/2}Q_+ = R_{-N/2}Y\tilde{R}_{N/2} \equiv 1 \quad \text{mod-}OPS(h; g)$. The same Neumann series $1-z+z^2-\ldots$ allows us therefore to construct in (A.3) the required parametrix Q_- of Q_+.

Once we have (A.3) we can write (with $s \geq N/2$)

$$(P_N f, f) - \|Q_+ f\|^2 = (Tf, f),$$
$$T \in OPS(m^{N/2} h^s; g) \subseteq OPS(\langle\xi\rangle^{N/2}; g).$$

It follows that

$$|(Tf, f)| \leq \|R\Lambda^{N/4} f\| \, \|\Lambda^{N/4} f\|, \quad R = \Lambda^{-N/4} T \Lambda^{-N/4} \in OPS(1; g).$$

On the other hand (with obvious notations !) we have

$$\|f\|_{\tilde{m}} \leq C \, \|Q_- Q_+ f\|_{\tilde{m}} + C \, \|f\|_{\tilde{m}h^s}, \quad f \in C_0^\infty$$

(for any arbitrary weight function $\tilde{m}$). If we set $\tilde{m} = m^{N/2}$ and $s \geq 1$ large enough we obtain

$$\text{(A.5)} \qquad \|f\|_{m^{N/2}} \leq C \left(\|Q_+ f\| + \|f\|_{\langle\xi\rangle^{N/2}} \right).$$

But clearly also

$$\text{(A.6)} \qquad \|f\|_{\langle\xi\rangle^{N/2}} \leq \|\Lambda^{-N/2}\Lambda^{N/2} f\|_{\langle\xi\rangle^{N/2}} \leq C \, \|\Lambda^{N/2} f\|$$

since $\Lambda^{-N/2} \in OPS(\langle\xi\rangle^{N/2}; g)$. Putting together (A.4), (A.5) and (A.6) we deduce that

$$\|f\|_{m^{N/2}} \leq C \, \|f\|_N, \quad f \in C_0^\infty,$$

which is the desired estimate.

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Recibido: 5 de diciembre de 1.991

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