

Weighted Poincaré and Sobolev
inequalities for vector fields
satisfying Hörmander's
condition and applications

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Abstract. In this paper we mainly prove weighted Poincaré inequalities for vector fields satisfying Hörmander's condition. A crucial part here is that we are able to get a pointwise estimate for any function over any metric ball controlled by a fractional integral of certain maximal function. The Sobolev type inequalities are also derived. As applications of these weighted inequalities, we will show the local regularity of weak solutions for certain classes of strongly degenerate differential operators formed by vector fields.

Introduction.

Let X_1, \dots, X_m be real C^∞ vector fields satisfying Hörmander's condition. Since Hörmander [H] proved the hypoellipticity of

$$L = X_0 + \sum_{i=1}^m X_i^2,$$

many authors have studied the behavior of solutions to the general class of differential operators

$$\mathcal{L} = \sum_{i=1}^m X_i^2 + \sum_{i,j=1}^m f_{ij}[X_i, X_j] + \sum_{j=1}^m f_j X_j + f_0$$

with some conditions of smoothness imposed on f_{ij}, f_j, f_0 . In [Fo], [FeP], [FeS], [FS], [JS], [NSW], [RS], [Sa], a complete description of solutions to several particular cases of \mathcal{L} have been studied. A simpler proof of the results of [FeS] can be found in [Ch1]. An extensive study has been made for the analytic hypoellipticity for operators formed by vector fields. We refer the reader to [Ch2] and a lot of references therein.

In [RS], Rothschild and Stein linked these vector fields with the left invariant vector fields on certain nilpotent Lie groups. Simplification of this approach was given by Hörmander-Melin [HM]. In [NSW], Nagel, Stein and Wainger studied the equivalence of several metrics and pseudometrics and obtained a volume estimate for metric balls. In [J], Jerison proved a unweighted Poincaré inequality.

This paper consists of two parts. One of the main parts of this paper is to prove weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition. As a byproduct, we also derive a unweighted Poincaré inequality with better exponent on the left side of the inequality than that in [J]. Several estimates given by Jerison in [J] play an important role here. A very crucial point in this present paper is that we are able to obtain a pointwise estimate for any smooth function over the metric ball controlled by a fractional integral of certain maximal function. The work of Jerison [J] is very helpful here in order to get this pointwise estimate. The Sawyer-Wheeden condition for weighted inequality simplifies the proof of our main theorems. All these will be presented in Part I of the paper.

Various authors have studied the weighted Poincaré and Sobolev inequalities for the special case $X_i = \partial/\partial x_i$, we refer the reader to the works by Fabes, Kenig and Serapioni [FKS] and Chanillo and Wheeden [CW1] and references therein. A related situation was also considered by Franchi and Lanconelli [FrL] and Franchi and Serapioni [FrS] for nonsmooth vector fields.

The second part of the paper is to apply the weighted Poincaré and Sobolev inequalities to study the local regularity of certain classes of degenerate differential operators formed by vector fields. Precisely speaking, we will show the Harnack inequalities of the following two

type of differential operators

$$L = \sum_{i,j=1}^m X_i^*(a_{ij}(x)X_j)$$

and

$$\mathcal{L} = - \sum_{i,j=1}^m X_i(a_{ij}(x)X_j)$$

where X_i^* is the adjoint of X_i and the coefficient matrix $A = (a_{ij}(x))$ is symmetric and satisfies the following:

$$c^{-1}w(x) |\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \leq c w(x) |\xi|^2$$

for some weight $w \in A_2(\Omega)$. Several fairly general mean value inequalities for subsolutions to L and \mathcal{L} will be derived also. As an immediate consequence, the Hölder continuity of the solutions to L and \mathcal{L} will be obtained. All these results will be presented in Part II of this paper. We also remark out here that when the matrix $A = (a_{ij})$ satisfies

$$w(x) |\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \leq v(x) |\xi|^2$$

for certain pair of weights v and w , we can also show the Harnack inequality by obvious modifications of the proof in the present work.

Part I: Weighted Poincaré-Sobolev inequalities for vector fields satisfying Hörmander’s condition.

1. The main theorems of Part I.

Let Ω be a bounded, open and path-connected domain in \mathbb{R}^d , and let X_1, X_2, \dots, X_m be a collection of C^∞ real vector fields defined in a neighbourhood of the closure $\bar{\Omega}$ of Ω . For a multi-index $\alpha = (i_1, i_2, \dots, i_k)$, denote by X_α the commutator

$$[X_{i_1}, [X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}], \dots,]$$

of length $|\alpha| = k$. Throughout this paper we assume that the vector fields satisfy Hörmander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha| \leq s}$ span the tangent space of \mathbb{R}^d at each point of Ω . Now let

$$\begin{aligned} X^{(1)} &= \{X_1, X_2, \dots, X_m\}, X^{(2)} \\ &= \{[X_1, X_2], \dots, [X_{m-1}, X_m]\}, \dots, \text{ etc.} \end{aligned}$$

so that the components of $X^{(k)}$ are the commutators of length k . Let Y_1, Y_2, \dots, Y_q be some enumeration of $X^{(1)}, \dots, X^{(s)}$. If Y_i is an element of $X^{(j)}$, we say Y_i has formal degree $d(Y_i) = j$. For simplicity, we denote $d(Y_i)$ by d_i . Now we define the metric associated to the vector fields as in [NSW].

Let $C(\delta)$ denote the class of absolutely continuous mappings $\phi : [0, 1] \rightarrow \Omega$ which almost everywhere satisfy the differential equation

$$\phi'(t) = \sum_{j=1}^q a_j(t) Y_j(\phi(t))$$

with

$$|a_j(t)| < \delta^{d_j}.$$

Then define

$$\varrho(x, y) = \inf\{\delta > 0 : \text{there exists } \phi \in C(\delta) \text{ with } \phi(0) = x, \phi(1) = y\}$$

We then define the metric ball

$$B(x, \delta) = \{y \in \Omega : \varrho(x, y) < \delta\}.$$

In [NSW], Nagel, Stein and Wainger proved the *doubling property* of Lebesgue measure with respect to the metric balls defined as above, *i.e.*

$$|B(x, 2\delta)| \leq C |B(x, \delta)|$$

where C is a constant independent of $x \in \Omega$ with $B(x, 2\delta) \subset \Omega$. Thus (Ω, ϱ) is a homogeneous metric space in the sense of Coifman and Weiss, [CoW].

Let $w(x) \geq 0$ be a locally integrable weight function defined in Ω , and $1 < p < \infty$. We say w is a doubling weight if $w(2B) \leq Cw(B)$ with C independent of the balls $B \subset \Omega$. We say $w \in A_p(\Omega)$ if

$$\left[\int_B w(x) dx \right] \left[\int_B w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C_w |B|^p,$$

for all metric balls $B \subset \Omega$. For an example of A_p weights on the Heisenberg group, see Subsection 2.3. We shall use the notation $w(E)$ to denote $\int_E w(x)dx$ and $|E|$ to denote the Lebesgue measure of set E , and $\rho(B)$ to denote the radius of the ball B throughout this paper.

Our main results are

Theorem A. 1) (Poincaré) *Assume that $E \subset\subset \Omega$, $1 < p < q < \infty$, w_1, w_2 are two doubling weights satisfying $w_1 \in A_p(\Omega)$, $w_1 \leq w_2$ and the following condition*

$$\frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \leq C_1 \left(\frac{w_1(I)}{w_1(B)} \right)^{1/p}$$

for all metric balls $I \subset cB \subset \Omega$. Then there are positive constants C, r_0 and some q_0 with $p < q_0 < q$ such that for any metric ball $B = B(x, r) \subset \Omega$, $x \in E$, and any $f \in C^\infty(\bar{B})$, the following inequality holds

$$\left(\frac{1}{w_2(B)} \int_B |f - f_B|^{q_0} w_2 \right)^{1/q_0} \leq C r \left(\frac{1}{w_1(B)} \int_B \sum_{i=1}^m |X_i f|^p w_1 \right)^{1/p}$$

provided $0 < r < r_0$, where C, r_0 depend only on the constants associated to the weights, E , and Ω , and f_B may be taken to be

$$f_B = \frac{1}{w_2(B)} \int_B f w_2 .$$

2) (Sobolev) *If we assume $f \in C_0^\infty(B)$, then we can take $f_B = 0$ in the above inequality.*

When $w_1 = w_2 = w \in A_p$, we can have precise exponent on the left side of the above inequalities.

Theorem B. 1) (Poincaré) *Let $w \in A_p(\Omega)$, $1 < p < \infty$, and $E \subset\subset \Omega$. Then there exist constants $r_0 > 0, C > 0, Q \geq 2$, such that for any metric ball $B = B(x, r) \subset \Omega$, $x \in E$, and any $f \in C^\infty(\bar{B})$, the following inequality holds*

$$\left(\frac{1}{w(B)} \int_B |f - f_B|^q w \right)^{1/q} \leq C r \left(\frac{1}{w(B)} \int_B \sum_{i=1}^m |X_i f|^p w \right)^{1/p}$$

provided $0 < r < r_0$, $q = kp$, $1 \leq k < Q/(Q - 1) + \delta_p$, $\delta_p > 0$, where C , r_0 depend only on the A_p constant of the weight w , E , Ω , and δ_p only depends on p and the A_p constant of w , f_B may be taken to be either

$$\frac{1}{|B|} \int_B f \quad \text{or} \quad \frac{1}{w(B)} \int_B fw.$$

2) (Sobolev) If we assume $f \in C_0^\infty(B)$, then f_B can be taken to be 0 in the above inequality.

In the special case $w = 1$, the above theorems can be improved further as follows

Theorem C. 1 (Poincaré) *Let $E \subset\subset \Omega$, $1 < p < Q$, then there exist constants $r_0 > 0$, $C > 0$, such that for any metric balls $B = B(x, r) \subset \Omega$, $x \in E$, and any $f \in C^\infty(\overline{B})$, the following inequality holds*

$$\left(\frac{1}{|B|} \int_B |f - f_B|^q \right)^{1/q} \leq C r \left(\frac{1}{|B|} \int_B \sum_{i=1}^m |X_i f|^p \right)^{1/p}$$

provided $0 < r < r_0$, $1 \leq q < pQ/(Q - p)$, where C , r_0 depend only on E , Ω , f_B may be taken to be

$$f_B = \frac{1}{|B|} \int_B f.$$

2) (Sobolev) If we assume $f \in C_0^\infty(B)$, then the above inequality hold for all $1 \leq q \leq pQ/(Q - p)$ and $f_B = 0$.

REMARK 1. The above Q is actually the homogeneous dimension of the graded nilpotent group generated by the left invariant vector fields corresponding to the lifted vector fields $\{\tilde{X}_i\}$ of $\{X_i\}$. We will define Q precisely later.

REMARK 2. From the proof of Theorem A, we see that other versions of the weighted inequalities are available under weaker assumptions (e.g., removing $w_1 \leq w_2$). Moreover, the Sobolev inequality in Theorem C is probably already known.

As applications of the present article, we will prove the Harnack inequality and local regularity of certain class of degenerate second

order differential operators of vector fields (see Part II of the present paper and [L1], [L2]). In Part II, we study the degenerate differential operators

$$L = \sum_{i,j=1}^m X_i^*(a_{ij}(x)X_j)$$

and

$$\mathcal{L} = - \sum_{i,j=1}^m X_i(a_{ij}(x)X_j)$$

where X_i^* is the adjoint of X_i and the coefficient matrix $A = (a_{ij}(x))$ is symmetric and satisfies the following

$$c^{-1} w(x) |\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \leq c w(x) |\xi|^2$$

for some weight $w \in A_2(\Omega)$. The existence and size estimation of fundamental solution to L will be treated (see [L1]). In [L2], we show the Harnack inequality of the degenerate Schrödinger operator of the form

$$\sum_{i,j=1}^m X_i^*(a_{ij}(x)X_j) + V$$

for certain potential V satisfying an analogue of the Kato-Stummel condition. The matrix $A = (a_{ij}(x))$ here satisfies the same degenerate condition as above.

The following is an outline of the proof of the Poincaré inequality in Theorem A. We first prove a version of the Poincaré inequalities on the homogeneous metric spaces $(\tilde{\Omega}, \tilde{\rho})$ associated to the lifted vector fields $\{\tilde{X}_i\}$ with extra variables. The important technique here is to obtain a pointwise estimate for $|f(\xi) - C_B|$ for $\xi \in B$ with some constant C_B , where $B \subset \tilde{\Omega}$ is any metric ball. It is then possible to get control $|f(\xi) - C_B|$ on the ball B by using the fractional integrals of certain maximal function. This is the essential difference between our proof and the one given by Jerison [J] in the unweighted case. In order to derive pointwise estimates, we first obtain a pointwise estimate on the graded nilpotent Lie group. The proof uses the scaling on the group, *i.e.* group translations and dilations. In the case of the homogeneous space $(\tilde{\Omega}, \tilde{\rho})$ associated to the lifted vector fields, we will adapt this estimate on the group. Once we have the pointwise estimate, we can apply the Sawyer-Wheeden condition for weighted inequality on homogeneous

spaces [SW] together with the A_p -theory introduced by Calderón in [Ca]. This leads us to a variant of the Poincaré inequality $(\tilde{\Omega}, \tilde{\rho})$, which is

$$(1.1) \quad \left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2 \right)^{1/q} \leq C r \left(\frac{1}{w_1(B)} \int_{cB} \left(\sum_{i=1}^m |\tilde{X}_i f| + |f| \right)^p w_1 \right)^{1/p}$$

for some constants $C > 0$ and $c \geq 1$ independent of f, B .

From the above inequality (1.1), we can easily obtain a variant of the Poincaré inequality on the homogeneous space (Ω, ρ) associated to the original vector fields X_1, \dots, X_m . Finally, by employing an argument based on Whitney's decomposition in [K], and also in [J] we can get the Poincaré inequality.

The proof of weighted Sobolev's inequality in Theorem A is much easier. The pointwise estimate of the function f with compact support in a ball can be easily obtained by using the size estimate for the fundamental solution to the operator defined by taking the sum of squares of vector fields (see [NSW] or [Sa]). The remainder of the proof proceeds as in the Poincaré inequality case except that we do not need Whitney's decomposition. The proofs of theorems B and C are similar to those of A and B.

We will adopt the following notations throughout this paper. Let $B = B_r = B(x, r)$, then we use cB , B_{cr} or $B(x, cr)$ as the ball centered at x , and with radius cr . C, c and C_1, C_2, \dots will denote generic constants and may differ at different occurrences.

2. Preliminaries.

In the last section, we give a definition about the metric (Ω, ρ) defined by vector fields X_1, \dots, X_m . We now give another equivalent definition of metric defined in [NSW], we will use this alternate definition whenever necessary. We assume X_1, \dots, X_m are C^∞ vector fields in $\Omega \subset \mathbb{R}^N$ satisfying Hörmander's condition and define Y_j and $d_j = d(Y_j)$ as in the introduction.

We first introduce a simplification of notation. Let $x \in E \subset \Omega$ and $I = (i_1, i_2, \dots, i_N)$ be fixed. We shall relabel the vector fields $\{Y_{i_j}\}_{1 \leq j \leq q}$ by setting $U_j = Y_{i_j}$, $1 \leq j \leq N$, and by letting V_j , $1 \leq$

$j \leq q - N$, be some enumeration of the remaining vector fields. Let $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$, we define

$$B_I(x, \delta) = \{y \in \Omega : y = \exp\left(\sum_{j=1}^N u_j \cdot U_j\right)(x), \text{ with } |u_j| < \delta^{d(U_j)}\}.$$

Thus $B_I(x, \delta)$ is exactly the image of the box $\{u \in \mathbb{R}^N : |u_j| < \delta^{d(U_j)}\} = Q(\delta)$ under the exponential map.

We denote $d(I) = d_{i_1} + \dots + d_{i_N}$ where $d_{i_j} = d(Y_{i_j})$ is as in [NSW].

2.1. Analysis on nilpotent Lie groups.

We now review some definitions and useful results associated with graded, nilpotent Lie groups following Folland [Fo], Rothschild and Stein [RS], and Jerison [J].

Let \mathcal{G} be a finite-dimensional, graded, nilpotent Lie algebra. Assume

$$\mathcal{G} = \bigoplus_{i=1}^s V_i,$$

and $[V_i, V_j] \subset V_{i+j}$ for $i + j \leq s$, $[V_i, V_j] = 0$ for $i + j > s$. Let Y_1, \dots, Y_m be a basis for V_1 and suppose that Y_1, \dots, Y_m generate \mathcal{G} as a Lie algebra. Thus we can choose a basis $\{Y_{ij}\}$, for $1 \leq j \leq s$, $1 \leq i \leq m_j$ for V_j consisting of vectors of the form Y_α for some multi-indices α of length j . In particular, $Y_{i1} = Y_i$, $i = 1, \dots, m$ and $m = m_1$.

Let G be the simply connected Lie group associated to \mathcal{G} . Since the exponential mapping is a global diffeomorphism from \mathcal{G} to G , for each $g \in G$, there is $y = (y_{ij}) \in \mathbb{R}^N$, $1 \leq i \leq m_j$, $1 \leq j \leq s$, $N = \sum_{j=1}^s m_j$, such that

$$g = \exp\left(\sum y_{ij} Y_{ij}\right).$$

Thus we define a homogeneous norm function $|\cdot|$ on G by

$$|g| = \left(\sum |y_{ij}|^{2s!/j}\right)^{1/2s!}.$$

Let δ_t be a dilation on G defined by

$$\delta_t y = (t^j y_{ij})_{1 \leq i \leq m_j, 1 \leq j \leq s}$$

for each $t > 0$. It is easy to see that δ_t is an automorphism of G for each $t > 0$. Lebesgue measure dy is the bi-invariant Haar measure of

G and the Jacobian of $\delta_t, J\delta_t$, is equal to t^Q , where $Q = \sum_{j=1}^s jm_j$ is called the *homogeneous dimension* of G , which is usually greater than $\dim G = N$. We note that the homogeneous norm $|\cdot|$ on G satisfies the following

- (i) $|u| \geq 0$ for $u \in G$ and $|u| = 0$ if and only if $u = 0$,
- (ii) $u \rightarrow |u|$ is continuous on G and smooth on $G \setminus \{0\}$,
- (iii) $|\delta_t(u)| = t|u|$,
- (iv) $|uv| \leq C(|u| + |v|)$,
- (v) $C_1\|u\| \leq |u| \leq C_2\|u\|^{1/s}$ for $|u| \leq 1$ and $\|\cdot\|$ is the usual Euclidean norm on G .

For $f \in C^\infty(G)$, and $g \in C_0^\infty(G)$, we define the convolution on G by

$$(f * g)(x) = \int f(xy^{-1})g(y)dy = \int f(y)g(y^{-1}x)dy .$$

Denote

$$I_t f(x) = t^{-Q} f(\delta_{t^{-1}} x) .$$

The following lemma is due to Jerison [J].

Lemma 2.1. (Jerison) a) *For every multi-index α , there exist differential operators $D_{i\alpha}$, $1 \leq i \leq m$, such that*

$$Y_\alpha(f * \phi) = \sum_{i=1}^m (Y_i f) * D_{i\alpha} \phi$$

for all $f \in C^\infty(G)$ and $\phi \in C_0^\infty(G)$.

b) *There exist differential operators $D^{(i)}$, $1 \leq i \leq m$, such that for any $\phi \in C_0^\infty(G)$, the function $\phi^{(i)} = D^{(i)} \phi$ satisfy*

$$\left(\frac{\partial}{\partial t}\right)(f * I_t \phi) = \sum_{i=1}^m (Y_i f) * I_t \phi^{(i)} .$$

Now let $\rho : G \times G \rightarrow \mathbb{R}^+$ be defined by $\rho(x, y) = |xy^{-1}|$, the homogeneous norm of xy^{-1} . We denote by $B(x, r) = \{y \in G : \rho(x, y) < r\}$, the ball centered at x and with radius r . We note that (G, ρ) is a homogeneous metric space in the sense of Coifman and Weiss [CoW].

2.2. Lifting of vector fields.

In this subsection we recall some results of Rothschild and Stein [RS].

Theorem 2.2. (Rothschild-Stein) *Let X_1, \dots, X_m be C^∞ vector fields on a C^∞ manifold Ω of dimension d such that the commutators of length $\leq s$ span the tangent space at $\xi \in \Omega$. Then in terms of new variables, $t_{d+1}, t_{d+2}, \dots, t_N$, there exist smooth functions $\lambda_{kl}(\eta, t)$ defined in a neighbourhood \tilde{U} of $\tilde{\xi} = (\xi, 0) \in \Omega \times \mathbb{R}^{N-d} = \tilde{\Omega}$ such that the vector fields \tilde{X}_k given by*

$$\tilde{X}_k = X_k + \sum_{l=d+1}^N \lambda_{kl}(\eta, t) \frac{\partial}{\partial t_l}$$

are free up to step s at every point in \tilde{U} , i.e. $\{\tilde{X}_i\}$ and their commutators have no linear relations except that $[\tilde{X}_i, \tilde{X}_j] = -[\tilde{X}_j, \tilde{X}_i]$ and the Jacobian identity.

Theorem 2.3. (Rothschild-Stein). *Let $\tilde{X}_1, \dots, \tilde{X}_m$ be vector fields on a manifold $\tilde{\Omega}$, $\tilde{\xi}_0 \in \tilde{\Omega}$ such that*

- a) *Commutators of length $\leq s$ span the tangent space.*
- b) *$\{\tilde{X}_k\}$ is free up to step s at $\tilde{\xi}_0$.*

Choose $\{\tilde{X}_{jk}\}$, commutators of length $\leq s$, determining a system of canonical coordinates (u_{jk}) around $\tilde{\xi}_0$ by

$$(u_{jk}) \leftrightarrow \exp\left(\sum u_{jk} \tilde{X}_{jk}\right)(\tilde{\xi}_0).$$

Let G be the free Lie group of step s on m generators and \mathcal{G} be its Lie algebra. Then there is a basis $\{Y_{jk}\}$ of \mathcal{G} and neighbourhoods \tilde{V} of $\tilde{\xi}_0 \in \tilde{\Omega}$ and U of $0 \in G$ with the following properties

- (i) *There exists a mapping Θ on $\tilde{V} \times \tilde{V}$ to U such that*

$$\Theta(\tilde{\xi}, \tilde{\eta}) = \exp\left(\sum u_{jk} Y_{jk}\right) \in U$$

where $\tilde{\eta} = \exp\left(\sum u_{jk} \tilde{X}_{jk}\right)(\tilde{\xi})$.

(ii) For each fixed $\tilde{\xi}$, the mapping

$$\tilde{\eta} \rightarrow \Theta_{\tilde{\xi}}(\tilde{\eta}) = \Theta(\tilde{\xi}, \tilde{\eta}) = (u_{jk})$$

is a coordinate chart for \tilde{V} centered at $\tilde{\xi}$.

(iii) Furthermore,

$$(\Theta_{\tilde{\xi}})_*(\tilde{X}_\alpha) = Y_\alpha + E_\alpha^{\tilde{\xi}}$$

where $E_\alpha^{\tilde{\xi}}$ is a differential operator of local degree $\leq |\alpha| - 1$.

Define

$$\tilde{\varrho}(\tilde{\xi}, \tilde{\eta}) = |\Theta(\tilde{\xi}, \tilde{\eta})|.$$

Thus from [RS], we can see that $\tilde{\varrho} : \tilde{\Omega} \times \tilde{\Omega} \rightarrow \mathbb{R}^+$ is a pseudometric on $\tilde{\Omega}$ (we shall shrink $\tilde{\Omega}$ if necessary). We note that the Lebesgue measure of the ball $|\tilde{B}(\tilde{\xi}, r)| \approx r^Q$, where Q is the homogeneous dimension of G , and $\tilde{B}(\tilde{\xi}, r)$ is the metric ball in $(\tilde{\Omega}, \tilde{\varrho})$. Thus $(\tilde{\Omega}, \tilde{\varrho})$ is a homogeneous space in the sense of [CoW].

2.3. A_p weights on homogeneous spaces.

In this subsection, we are going to review the theory of A_p weights on homogeneous spaces introduced by Calderón [Ca]. We will also discuss the relation between A_p weights on (Ω, ϱ) and $(\tilde{\Omega}, \tilde{\varrho})$.

Let (X, ϱ) be the homogeneous space and μ the doubling Borel measure such that

$$\varrho(x, z) \leq K (\varrho(x, y) + \varrho(y, z))$$

and

$$0 < \mu(B(x, 2r)) \leq A \mu(B(x, r)) < \infty$$

for $x, y, z \in X$, $r > 0$. A and K are called the *metric constants* of (X, ϱ) . We note that both Vitali and Whitney type covering lemmas hold as proved in [CoW]. We say the weight function $w(x) > 0$, belongs to class $A_p = A_p(X)$ if

$$\left[\int_B w(x) d\mu \right] \left[\int_B w(x)^{-1/(p-1)} d\mu \right]^{p-1} \leq C_w |B|^p, \quad \text{for } 1 < p < \infty$$

or

$$\int_B w(x)dx \leq C_w |B| \operatorname{ess\,inf}_{x \in B} w(x), \quad \text{for } p = 1$$

for all balls B , where $|B| = \mu(B)$. We call C_w the A_p constant of w . We refer to [Ca] for proofs of the theorems that follow.

Theorem 2.4. *If $w \in A_p$, $p > 1$, then $w \in A_{p_0}$ for some $1 < p_0 < p$ and p_0 depends on p , C_w , A and K .*

Theorem 2.5. *If $w \in A_p$, then $w \in A_\infty$, i.e. for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $E \subset B$ is any subset with $|E| \leq \delta|B|$, then $w(E) \leq \varepsilon w(B)$.*

We define the Hardy-Littlewood Maximal function in homogeneous spaces as follows

$$Mf(x) = \sup \frac{1}{\mu(B)} \int_B |f(y)|d\mu ,$$

the above supremum is taken over all balls B centered at x . We then have

Theorem 2.6. *If $w \in A_p$, $p > 1$, then $\|Mf\|_{L^p_w} \leq C \|f\|_{L^p_w}$.*

We now discuss the relations between A_p weights on metric spaces (Ω, ϱ) and $(\tilde{\Omega}, \tilde{\varrho})$. As before we denote $\tilde{\Omega} = \Omega \times \mathbb{R}^l$, $l = N - d$, $N = \dim G$. We also denote by $\tilde{B}_r(\tilde{\xi})$ the ball of radius r and centered at $\tilde{\xi}$ in $\tilde{\Omega}$ for the metric associated with vector fields $\tilde{X}_1, \dots, \tilde{X}_m$, where $\tilde{\xi} = (\xi, t) \in \tilde{\Omega}$, $\xi \in \Omega$, $t \in \mathbb{R}^l$.

Now let $w(\xi)$ be a function defined in Ω , we define a new function in $\tilde{\Omega}$ by

$$\tilde{w}(\tilde{\xi}) = \tilde{w}(\xi, t) \equiv w(\xi)$$

for $\tilde{\xi} = (\xi, t) \in \tilde{\Omega}$, $\xi \in \Omega$.

Lemma 2.7. *If $w(\xi) \in A_p(\Omega)$, then $\tilde{w}(\tilde{\xi}) \in A_p(\tilde{\Omega})$, for $p \geq 1$.*

PROOF. Let $\tilde{B} = \tilde{B}(\tilde{\xi}, r) \subset \tilde{\Omega}$, $\tilde{\xi} = (\xi, t)$, $\xi \in \Omega$, then $\tilde{B} \subset B \times \mathbb{R}^l$ for $B = B(\xi, r)$. We consider the case $p > 1$ first. By Lemma 4.4 in [J], we have

$$\int_{\tilde{B}} \tilde{w}(\tilde{\xi})d\tilde{\xi} = \int_{\tilde{B}} w(\xi)\chi_{\tilde{B}} d\xi dt$$

$$\begin{aligned} &\leq \int_B w(\xi) \left(\int_{\mathbb{R}^d} \chi_{\tilde{B}} dt \right) d\xi \\ &\leq C \frac{|\tilde{B}|}{|B|} \int_B w(\xi) d\xi . \end{aligned}$$

Similarly,

$$\left(\int_{\tilde{B}} \tilde{w}(\tilde{\xi})^{-1/(p-1)} d\tilde{\xi} \right)^{p-1} \leq C \left(\frac{|\tilde{B}|}{|B|} \right)^{p-1} \left(\int_B w(\xi)^{-1/(p-1)} d\xi \right)^{p-1}$$

Since $w(\xi) \in A_p(\Omega)$, then

$$\left(\int_B w(\xi) d\xi \right) \left(\int_B w(\xi)^{-1/(p-1)} d\xi \right)^{p-1} \leq C .$$

Thus,

$$\left(\int_{\tilde{B}} \tilde{w}(\tilde{\xi}) d\tilde{\xi} \right) \left(\int_{\tilde{B}} \tilde{w}(\tilde{\xi})^{-1/(p-1)} d\tilde{\xi} \right)^{p-1} \leq C |\tilde{B}|^p .$$

This shows $\tilde{w}(\tilde{\xi}) \in A_p(\tilde{\Omega})$. For the case $p = 1$, we note

$$\begin{aligned} \int_{\tilde{B}} \tilde{w}(\tilde{\xi}) d\tilde{\xi} &\leq C \left(\frac{|\tilde{B}|}{|B|} \right) \int_B w(\xi) d\xi \\ &\leq C |\tilde{B}| \operatorname{ess\,inf}_{\xi \in B} w(\xi) \\ &\leq C |\tilde{B}| \operatorname{ess\,inf}_{\tilde{\xi} \in \tilde{B}} \tilde{w}(\tilde{\xi}) . \end{aligned}$$

The last inequality is because $\tilde{\xi} \in \tilde{B}$ implies $\xi \in B$. Thus $\tilde{w} \in A_1(\tilde{\Omega})$.

We conclude this section by giving an example of A_p weights on the Heisenberg group. Let H^d be the Heisenberg group (of degree d), *i.e.* the nilpotent Lie group whose underlying manifold is $\mathbb{C}^d \times \mathbb{R}$ with coordinates $(z_1, \dots, z_d, t) = (z, t)$ and whose group law is

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2 \operatorname{Im}(z \cdot \bar{z}'))$$

where $z \cdot \bar{z}' = \sum_{j=1}^d z_j \bar{z}'_j$.

We define the group dilations on H^d by $\delta_r(z, t) = (rz, r^2t)$ which satisfy

$$\delta_r((z, t) \cdot (z', t')) = \delta_r(z, t) \delta_r(z', t') ,$$

and we also define the *homogeneous norm* ρ by

$$\rho(z, t) = (|z|^4 + |t|^2)^{1/4}, \quad \text{with } |z|^2 = z \cdot z.$$

Thus the function ρ satisfies $\rho(\delta_r(z, t)) = r\rho(z, t)$. We define the distance between two points $u = (z_1, t_1)$ and $v = (z_2, t_2)$ by $\rho(u, v) = \rho(u^{-1} \cdot v)$. Given $u_0 = (z_0, t_0) \in H^d$, the Heisenberg ball centered at u_0 with radius r is given by

$$B_r(u_0) = \{u = (z, t) \in H^d : \rho(u, u_0) < r\}.$$

Let $z = x + iy$. Then, $x_1, \dots, x_d, y_1, \dots, y_d, t$ are real coordinates on H^d . Set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, d.$$

Then $X_1, \dots, X_d, Y_1, \dots, Y_d$ generate the Lie algebra of H^d . These vector fields satisfy Hörmander's condition of step 2. The homogeneous dimension of H^d is $Q = 2d + 2$.

We now let $w(z, t) = \rho(z, t)^\alpha$, then it is easy to show that w is locally integrable on H^d for $\alpha > -Q$. We can also check that there are constants c_1, c_2 independent of $u_0 = (z_0, t_0) \in H^d$ and r such that

$$c_1 (\rho(u_0) + r)^\alpha r^Q \leq \int_{B_r(u_0)} w(u) du \leq c_2 (\rho(u_0) + r)^\alpha r^Q.$$

Thus $w(u) = w(z, t) = (|z|^4 + |t|^2)^{\alpha/4}$ is doubling if $\alpha > -Q$ and $w \in A_p(H^d)$ if and only if $-Q < \alpha < Q(p - 1)$ for $p > 1$. Furthermore, we can also show $w \in A_1(H^d)$ if and only if $-Q < \alpha \leq 0$.

3. Pointwise estimate on the homogeneous space $(\tilde{\Omega}, \tilde{\varrho})$ with extra variables.

We first prove the pointwise estimate $|f(y) - C_B|$ for $f \in C^\infty(B)$ on the graded nilpotent Lie group G defined in Subsection 2.1. We are going to use the same notation as used there.

Let (G, ϱ) be the homogeneous metric space with Lebesgue measure as its doubling measure defined in Subsection 2.1, i.e. $\varrho(x, y) = |x^{-1}y|$, where $|\cdot|$ is the homogeneous norm on G . Let $B = B(x, r) = \{y \in$

$G : \varrho(x, y) < r$ be the ball in G and Y_1, \dots, Y_m be the left invariant vector fields which generates the Lie algebra \mathcal{G} of G . Then we have the following

Lemma 3.1. *Given any ball $B = B(x, r) \subset G$, then there exist positive constants C_1, C_2 and C_3 , such that for any $f \in C^\infty(\overline{C_3 B})$, the following holds*

$$|f(y) - C_2| \leq C_1 \int_{C_3 B} \frac{\sum_{i=1}^m |Y_i f(z)|}{|z^{-1}y|^{Q-1}} dz$$

for any $y \in B$, where C_1 and C_3 are independent of y, B, f .

PROOF. *Step 1.* We first show the lemma for the special case $B = B_1 = B(0, 1) = \{y \in G : |y| \leq 1\}$.

Let $\phi \in C_0^\infty(G)$, $\phi \geq 0$, $\text{supp } \phi \subset B_1$, and denote $I_t \phi(x) = t^{-Q} \phi(\delta_{t^{-1}} x)$. Thus by Lemma 2.1 b), we have for $y \in B_1$

$$\begin{aligned} |(f * \phi)(y) - f(y)| &= \left| \int_0^1 \frac{\partial}{\partial t} (f * I_t \phi)(y) dt \right| \\ &= \left| \int_0^1 \sum_{i=1}^m [(Y_i f) * I_t \phi^{(i)}](y) dt \right| \\ &\leq C \sum_{i=1}^m \int_0^1 \int_G |(Y_i f)(z) I_t \phi^{(i)}(z^{-1}y)| dz dt \\ &\leq C \sum_{i=1}^m \int_0^1 \frac{1}{t^Q} \int_{|z^{-1}y| \leq t} |(Y_i f)(z)| dz dt \\ &\leq C \sum_{i=1}^m \int_{|z^{-1}y| \leq 1} |(Y_i f)(z)| \int_{|z^{-1}y|}^\infty \frac{1}{t^Q} dt \\ &\leq C \int_{|z^{-1}y| \leq 1} \frac{\sum_{i=1}^m |(Y_i f)(z)|}{|z^{-1}y|^{Q-1}} dz \\ &\leq C \int_{|z| \leq c} \frac{\sum_{i=1}^m |(Y_i f)(z)|}{|z^{-1}y|^{Q-1}} dz. \end{aligned}$$

The last inequality above is due to $|z| = |yy^{-1}z| \leq c(|y| + |y^{-1}z|) \leq c$.

On the other hand, we note $B_1 \subset \hat{B} = \{y : \|y\| \leq c\}$, the Euclidean ball in G centered at the origin and with radius c . We denote by ∇ the usual Euclidean gradient. We also recall $\dim G = N$. Then by Lemma

2.1 a) and the well-known result in [FKS], we have

$$\begin{aligned} & \left| (f * \phi)(y) - \frac{1}{|\tilde{B}|} \int_{\tilde{B}} (f * \phi)(z) dz \right| \\ & \leq C \int_{\tilde{B}} \frac{|\nabla(f * \phi)(z)|}{\|z - y\|^{N-1}} dz \\ & \leq C \sum_{|\alpha| \leq s} \int_{\tilde{B}} \frac{|Y_\alpha(f * \phi)(z)|}{\|z - y\|^{N-1}} dz \\ & \leq C \sum_{|\alpha| \leq s} \sum_{i=1}^m \int_{\tilde{B}} \frac{|(Y_i f) * (D_{i\alpha} \phi)(z)|}{\|z - y\|^{N-1}} dz . \end{aligned}$$

But,

$$\begin{aligned} (Y_i f) * (D_{i\alpha} \phi)(z) &= \int_G (Y_i f)(w) (D_{i\alpha} \phi)(w^{-1}z) dw \\ &\leq C \int_{|w^{-1}z| \leq 1} |(Y_i f)(w)| dw . \end{aligned}$$

Now because $|w^{-1}z| \leq 1$ and $|z| \leq c$ implies $|w| \leq c$, thus the integral above is bounded by

$$C \int_{|w| \leq c} |(Y_i f)(w)| dw .$$

Thus we obtain

$$\begin{aligned} & \left| (f * \phi)(y) - \frac{1}{|\tilde{B}|} \int_{\tilde{B}} (f * \phi)(z) dz \right| \\ & \leq C \sum_{i=1}^m \int_{\tilde{B}} \int_{|w| \leq c} \frac{|(Y_i f)(w)|}{\|z - y\|^{N-1}} dw dz \\ & = C \sum_{i=1}^m \int_{|w| \leq c} \left(\int_{\tilde{B}} \frac{dz}{\|z - y\|^{N-1}} \right) |Y_i f(w)| dw \\ & \leq C \sum_{i=1}^m \int_{|w| \leq c} |(Y_i f)(w)| dw \end{aligned}$$

since

$$\int_{\tilde{B}} \frac{dz}{\|z - y\|^{N-1}}$$

is uniformly bounded for $y \in B_1 \subset \tilde{B}$. Note further $|w^{-1}y| \leq c(|w^{-1}| + |y|) \leq c$, the integral above is bounded by

$$C \sum_{i=1}^m \int_{|w| \leq c} \frac{|(Y_i f)(w)|}{|w^{-1}y|^{Q-1}} dw .$$

Therefore, we have proved

$$\left| f(y) - \frac{1}{|\tilde{B}|} \int_{\tilde{B}} (f * \phi)(z) dz \right| \leq C \sum_{i=1}^m \int_{|w| \leq c} \frac{|(Y_i f)(w)|}{|w^{-1}y|^{Q-1}} dw$$

for $y \in B_1$.

Step 2. For the case $B = B_r = B(0, r) = \{y : |y| < r\}$, we define the function $f_r(w) = f(\delta_r w)$, thus by the result in Step 1, we will be done for this case.

Step 3. Now we consider the general case $B = B(x, r) = \{y \in G : |x^{-1}y| \leq r\}$. We define a function $f_x(z) = f(xz)$ for $|z| \leq r$. By Step 2, the lemma will follow.

Thus this completes the proof of Lemma 3.1.

We now are going to derive the pointwise estimate for $|f(\xi) - C_B|$ over the metric ball B in $(\tilde{\Omega}, \tilde{\rho})$, where $(\tilde{\Omega}, \tilde{\rho})$ is the homogeneous space with the metric associated with the lifted vector fields $\tilde{X}_1, \dots, \tilde{X}_m$. We will set Mf to be the Hardy-Littlewood maximal function for f in the space $(\tilde{\Omega}, \tilde{\rho})$ (see Section 2). Throughout this section, and Section 4, we drop the tildes from $\tilde{X}_i, \tilde{\rho}, \tilde{\Omega}, \tilde{\xi}$, etc. The main result of this section is the following

Lemma 3.2. *Given any ball $B(\xi_0, r)$, $\xi_0 \in E \subset \subset \Omega$, any function $f \in C^\infty(\overline{cB})$, there exist constants C, c and C_B such that for any $\xi \in B$*

$$|f(\xi) - C_B| \leq C \int_{cB} \frac{M[(\sum_{i=1}^m |X_i f| + |f|)\chi_{cB}](\eta)}{\varrho(\xi, \eta)^{Q-1}} d\eta$$

where C and c are independent of f, B and ξ .

Now we let ϱ_r be the metric defined by vector fields rX_1, \dots, rX_m , and denote by $B_r(t)$ the ball of radius t relative to this metric. Then

the volume of the ball $|B_r(t)| \approx (rt)^Q$. We will need the following lemma from [J].

Lemma 3.3. *Let $\phi \in C_0^\infty(G)$, $\text{supp } \phi \subset \{y : |y| < 1\}$. For $h(\xi)$, $\xi \in \mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^l$, $r > 0$, define*

$$M_t h(\xi) = \int h(\eta) I_t \phi(\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q} d\eta.$$

Then,

$$(i) \quad \frac{\partial}{\partial t} M_t h(\xi) = \sum_{i=1}^m \int r X_i h(\eta) I_t \phi^{(i)}(\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q} d\eta + \int h(\eta) K_t^r(\xi, \eta) r^{-Q} d\eta$$

and $|K_t^r(\xi, \eta)| \leq cr t^{-Q}$, $\text{supp } \{K_t^r\} \subset \{(\xi, \eta) : \varrho_r(\xi, \eta) \leq ct\}$.

$$(ii) \quad r^{|\alpha|} X_\alpha M_t h(\xi) = \sum_{i=1}^m \int r X_i h(\eta) \phi_{i\alpha}(\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q} d\eta + \int h(\eta) F_\alpha^r(\xi, \eta) r^{-Q} d\eta$$

and $|F_\alpha^r(\xi, \eta)| \leq cr$, $\text{supp } \{F_\alpha^r\} \subset \{(\xi, \eta) : \varrho_r(\xi, \eta) \leq c\}$. (The functions $\phi^{(i)} = D^{(i)}\phi$, $\phi_{i\alpha} = D_{i\alpha}\phi$ for some differential operators $D_{i\alpha}$, $D^{(i)}$ as defined in Lemma 2.1.

PROOF OF LEMMA 3.2. Define $h(\xi)$ by $f(\xi) = g(\xi)h(\xi)$, where $g(\xi) = \tilde{g}(\xi, 0)$, $\tilde{g} \in C^\infty$ is defined by the pull-back

$$((\delta_r^{-1} \circ \Theta_\xi)^{-1})^* (r^{-Q} d\eta) = \tilde{g}(\xi, \delta_r y) dy$$

and \tilde{g} is bounded above and also bounded below from 0 on $B(\xi, r) \times \{y : |y| < 1\}$ for ξ in a compact subset of Ω (see [J] also). Then

$$(3.4) \quad \sum_{i=1}^m |X_i h(\xi)| \leq C \left(\sum_{i=1}^m |X_i f(\xi)| + |f(\xi)| \right).$$

Since $\lim_{t \rightarrow 0} M_t h(\xi) = h(\xi)g(\xi) = f(\xi)$, we have

$$M_1 h(\xi) - f(\xi) = \int_0^1 \frac{\partial}{\partial t} M_t h(\xi) dt$$

$$\begin{aligned}
 &= \int_0^1 \int \sum_{i=1}^m r X_i h(\eta) I_t \phi^{(i)} (\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q} d\eta dt \\
 &\quad + \int_0^1 \int h(\eta) K_t^r(\xi, \eta) r^{-Q} d\eta dt \\
 &= I + II,
 \end{aligned}$$

and we have

$$\begin{aligned}
 |I| &\leq C \int_0^1 \int_{\varrho(\xi, \eta) \leq rt} \sum_{i=1}^m r |X_i h(\eta)| t^{-Q} r^{-Q} d\eta dt \\
 &\leq C \int_{\varrho(\xi, \eta) \leq r} \sum_{i=1}^m |X_i h(\eta)| \int_{t \geq \varrho(\xi, \eta)/r} (tr)^{-Q} r dt d\eta \\
 &\leq C \int_{\varrho(\xi, \eta) \leq r} \sum_{i=1}^m |X_i h(\eta)| \varrho(\xi, \eta)^{-Q+1} d\eta.
 \end{aligned}$$

By (3.4), the inequality above is

$$\begin{aligned}
 &\leq \int_{\varrho(\xi, \eta) \leq r} \frac{\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)|}{\varrho(\xi, \eta)^{Q-1}} d\eta \\
 &\leq C \int_{B(\xi_0, cr)} \frac{\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)|}{\varrho(\xi, \eta)^{Q-1}} d\eta.
 \end{aligned}$$

Since $|K_t^r(\xi, \eta)| \leq C r t^{-Q}$, $\text{supp} \{K_t^r\} \subset \{(\xi, \eta) : \varrho_r(\xi, \eta) \leq ct\}$, we obtain

$$|II| \leq C \int_0^1 \int_{\varrho(\xi, \eta) \leq crt} h(\eta) r t^{-Q} r^{-Q} d\eta dt \leq C \int_{B(\xi_0, Cr)} \frac{|f(\eta)|}{\varrho(\xi, \eta)^{Q-1}} d\eta.$$

Hence,

$$|M_1 h(\xi) - f(\xi)| \leq C \int_{B(\xi_0, Cr)} \frac{\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)|}{\varrho(\xi, \eta)^{Q-1}} d\eta$$

for $\xi \in B(\xi_0, r)$. Since any function is bounded above almost everywhere by its maximal function, thus

$$(3.5) \quad |M_1(\xi) - f(\xi)| \leq C \int_{B(\xi_0, Cr)} \frac{M [(\sum_{i=1}^m |X_i f| + |f|) \chi_{cB}](\eta)}{\varrho(\xi, \eta)^{Q-1}} d\eta.$$

Next, we want to estimate $|M_1 h(\xi) - C_B|$ over the ball $B = B(\xi_0, r)$ for some constant C_B which is independent of $\xi \in B$. The basic idea is to apply the corresponding estimate on the group obtained in the last section.

Let $F(y) = M_1 h((\delta_r^{-1} \circ \Theta_{\xi_0})^{-1}(y))$ for $|y| < 1$, where $\delta_r^{-1} \circ \Theta_{\xi_0}(\xi) = y$. It is easy to see that $\delta_r^{-1} \circ \Theta_{\xi_0}(B(\xi_0, r))$ is comparable to the unit ball $\{y : |y| < 1\} \subset G$. Set $\delta_r^{-1} \circ \Theta_{\xi_0}(\eta) = z$. Thus by Lemma 3.1, we have

$$|F(y) - C_1| \leq C \int_{|z| \leq c} \frac{\sum_{i=1}^m |Y_i F(z)|}{|z^{-1}y|^{Q-1}} dz .$$

Now we note that $|z^{-1}y| \approx r^{-1} \varrho(\xi, \eta)$ and that

$$(\delta_r^{-1} \circ \Theta_{\xi_0})_* (rX_i) = Y_i + rE_i^{\xi_0, r}$$

where $E_i^{\xi_0, r}$ is a vector field on G of weight ≤ 0 whose coefficients are C^p norm bounded for some p and $r \leq r_0$ for some r_0 (see [FeS]). Thus

$$Y_i F(z) = rX_i M_1 h(\eta) - rE_i^{\xi_0, r} F(z) .$$

We note that

$$\begin{aligned} X_i M_1 h(\eta) &= X_i \int h(\zeta) \phi(\delta_r^{-1} \circ \Theta(\zeta, \eta)) r^{-Q} d\zeta \\ &= \int X_i^* h(\zeta) \phi(\delta_r^{-1} \circ \Theta(\zeta, \eta)) r^{-Q} d\zeta , \end{aligned}$$

where X_i^* is the adjoint of X_i with respect to $d\zeta$. Since X_i^* differs of $-X_i$ by a bounded function, we then have

$$|X_i^* h(\zeta)| \leq C (|X_i h(\zeta)| + |h(\zeta)|) .$$

Now,

$$\varrho(\zeta, \xi_0) \leq K(\varrho(\zeta, \eta) + \varrho(\eta, \xi_0)) \leq cr .$$

Thus

$$\begin{aligned} |X_i M_1 h(\eta)| &\leq Cr^{-Q} \int_{\varrho(\zeta, \eta) \leq r} (|X_i h(\zeta)| + |h(\zeta)|) \chi_{cB}(\zeta) d\zeta \\ &\leq CM[(|X_i h| + |h|)\chi_{cB}](\eta) . \end{aligned}$$

On the other hand, if we note $z = \delta_r^{-1} \circ \Theta_{\xi_0}(\eta)$, then

$$E_i^{\xi_0, r} F(z) = E_i^{\xi_0, r} M_1 h(\eta) = \int h(\zeta) E_i^{\xi_0, r} \phi(\delta_r^{-1} \circ \Theta(\zeta, \eta)) r^{-Q} d\zeta .$$

Keeping in mind that the above derivatives $E_i^{\xi_0, r}$ are taken with respect to z and η is a function of z , and also recalling $E_i^{\xi_0, r}$ is a vector field on G of weight ≤ 0 , then $|E_i^{\xi_0, r} \phi(\delta_r^{-1} \circ \Theta(\zeta, \eta))|$ is bounded for all $r \leq r_0$. Thus,

$$|E_i^{\xi_0, r} M_1 h(\eta)| \leq C r^{-Q} \int_{\varrho(\zeta, \eta) \leq r} h(\zeta) d\zeta \leq C M[h\chi_{cB}](\eta).$$

Hence,

$$|Y_i F(z)| = |(rX_i - rE_i^{\xi_0, r})M_1 h(\eta)| \leq C r M[(|X_i h| + |h|)\chi_{cB}](\eta).$$

Note

$$|z^{-1}y|^{Q-1} \approx \varrho(\xi, \eta)^{Q-1} r^{-Q+1}$$

and

$$((\delta_r^{-1} \circ \Theta_{\xi_0})^{-1})^* (r^{-Q} d\eta) = \tilde{g}(\xi_0, \delta_r z) dz.$$

Further, $|z| \leq c$ implies $\varrho(\eta, \xi_0) \leq cr$. Thus

$$\begin{aligned} |M_1 h(\xi) - C_1| &\leq C \int_{\varrho(\xi_0, \eta) \leq cr} \frac{r \sum_{i=1}^m M[(|X_i h| + |h|)\chi_{cB}](\eta)}{r^{-Q+1} \varrho(\xi, \eta)^{-Q+1}} r^{-Q} d\eta \\ &\leq C \int_{\varrho(\xi_0, \eta) \leq cr} \frac{\sum_{i=1}^m M[(|X_i h| + |h|)\chi_{cB}](\eta)}{\varrho(\xi, \eta)^{Q-1}} d\eta \\ &\leq C \int_{\varrho(\xi_0, \eta) \leq cr} \frac{M[(\sum_{i=1}^m |X_i h| + |h|)\chi_{cB}](\eta)}{\varrho(\xi, \eta)^{Q-1}} d\eta. \end{aligned}$$

Thus by (3.4) again,

$$(3.6) \quad |M_1 h(\xi) - C_1| \leq C \int_{\varrho(\xi_0, \eta) \leq cr} \frac{M[(\sum_{i=1}^m |X_i f| + |f|)\chi_{cB}](\eta)}{\varrho(\xi, \eta)^{Q-1}} d\eta.$$

Now,

$$|f(\xi) - C_1| \leq |M_1 h(\xi) - f(\xi)| + |M_1 h(\xi) - C_1|$$

Applying (3.5) to the first term and (3.6) to the second, we get our lemma.

4. A variant of the Poincaré inequalities on homogeneous spaces with extra variables

In this section, we are going to prove a variant of the Poincaré inequalities on the homogeneous space $(\tilde{\Omega}, \tilde{\rho})$. The proofs adapt the Sawyer-Wheeden condition for the weighted inequality of fractional integrals on homogeneous spaces [SW]. This proof is much simpler than that in the author's thesis [L1]. As in last section, we will drop the tildes. We first state these results.

Theorem 4.1. 1) *Let $1 < p < q < \infty$ and assume $w_1 \in A_p(\Omega)$, w_2 a doubling weight satisfying the following condition*

$$(4.2) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \leq C \left(\frac{w_1(I)}{w_1(B)} \right)^{1/p}$$

for all metric balls I and B with $I \subset cB \subset \Omega$, where $\rho(B)$ is the radius of the ball B and c is as in Lemma 3.2. Then there is a positive constant C such that for any balls $B = B(\xi_0, r)$ with $cB \subset \Omega$ and $f \in C^\infty(\overline{cB})$, the following inequality holds

$$(4.3) \quad \left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2 \right)^{1/q} \leq Cr \left(\frac{1}{w_1(B)} \int_{cB} \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{1/p}$$

where f_B may be taken to be

$$f_B = \frac{1}{w_2(B)} \int_B f w_2.$$

2) *In the case $w_1 = w_2 = w \in A_p$ and $p > 1$, the q in (4.3) can be taken as $q = kp$ for $1 \leq k \leq Q/(Q-1) + \delta_p$ for some $\delta_p > 0$ and f_B can be taken as*

$$\frac{1}{|B|} \int_B f \quad \text{or} \quad \frac{1}{w(B)} \int_B f w.$$

3) *In the case $w_1 = w_2 = 1$ and $1 < p < Q$, the above q can be taken as $1 \leq q \leq pQ/(Q-p)$ and f_B can be taken as*

$$f_B = \frac{1}{|B|} \int_B f.$$

PROOF OF THEOREM 4.1. We define the fractional integral

$$Tf(\xi) = \int_{\Omega} f(\eta)K(\xi, \eta) d\eta$$

where $K(\xi, \eta) = 1/\varrho(\xi, \eta)^{Q-1}$.

Given a metric ball $B \subset \Omega$, we will show for certain pair of weights \tilde{v} and \tilde{w} , that

$$(4.4) \quad \|Tf\|_{L^q(B, \tilde{v})} \leq C \|f\|_{L^p(cB, \tilde{w})}$$

for all $f \geq 0$ and $\text{supp } f \subset cB$, where

$$\|f\|_{L^p(S, \tilde{v})} = \left(\int_S f^p \tilde{v} \right)^{1/p}$$

and $p < q$. By Theorem 3 in [SW], if \tilde{v}, \tilde{w} are doubling weights and the following condition holds

$$(4.5) \quad \phi(I) \left(\int_I \tilde{v}(\xi) d\xi \right)^{1/q} \left(\int_I \tilde{w}^{1-p'}(\xi) d\xi \right)^{1/p'} \leq C,$$

for all metric balls $I \subset cB \subset \Omega$, then (4.4) holds. $\phi(I)$ in the above expression is defined to be

$$\phi(I) = \sup\{K(\xi, \eta) : \xi, \eta \in I, \varrho(\xi, \eta) \geq \frac{1}{2}\rho(I)\} \approx \frac{1}{|I|^{1-1/Q}}$$

where $\rho(I)$ is the radius of the ball I and $|I| \approx \rho(I)^Q$.

We let

$$\tilde{v} = \frac{w_2}{w_2(B)} \quad \text{and} \quad \tilde{w} = \frac{\sigma(B)^{p/p'} \rho(B)^p}{|B|^p} w_1,$$

where w_1 and w_2 are doubling weights, $\sigma(\xi) = w_1(\xi)^{1-p'}$, $1/p + 1/p' = 1$. Then (4.5) is equivalent to

$$(4.6) \quad \frac{\rho(I)}{\rho(B)} \frac{|B|}{|I|} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \left(\frac{\sigma(I)}{\sigma(B)} \right)^{1/p'} \leq C.$$

Thus by (4.4) and Lemma 3.2, we have

$$(4.7) \quad \left(\frac{1}{w_2(B)} \int_B |f - C_B|^q w_2 \right)^{1/q} \leq C(|B|^{-1} w_1(B)^{1/p} \sigma(B)^{1/p'}) \rho(B) \cdot \left[\frac{1}{w_1(B)} \int_{cB} \left(M \left(\sum_{i=1}^m |X_i f| + |f| \right) \chi_{cB} \right)^p w_1 \right]^{1/p}$$

for all $f \in C^\infty(\overline{cB})$. If assume $w_1 \in A_p$, then $|B|^{-1} w_1(B)^{1/p} \sigma(B)^{1/p'} \leq C$. Furthermore, by Theorem 2.6 the expression on the right side of (4.7) is bounded by

$$C \rho(B) \left(\frac{1}{w_1(B)} \int_{cB} \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{1/p}.$$

We also note that (4.2) is equivalent to (4.6) when $w_1 \in A_p$. This shows the first part of Theorem 4.1.

When $w_1 = w_2 = w \in A_p(\Omega)$, we note that (4.2) is equivalent to

$$(4.8) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{w(I)}{w(B)} \right)^{1/q} \leq C \left(\frac{w(I)}{w(B)} \right)^{1/p}$$

Since $|B| \approx \rho(B)^Q$, $|I| \approx \rho(I)^Q$ and $A_p \subset D_p$ (doubling of order p), *i.e.*

$$w(B) \leq C \left(\frac{|B|}{|I|} \right)^p w(I), \quad \text{for any } I \subset cB \subset \Omega$$

then the assumption that $w \in A_p(\Omega)$ implies that (4.8) holds if q is chosen so that $(Q(1/p - 1/q))^{-1} = p$, *i.e.* $q = pQ/(Q - 1)$. Actually, since $w \in A_p$ implies $w \in A_{p_0}$ for some $p_0 < p$ by Theorem 2.4, it follows that q can be chosen slightly larger than $pQ/(Q - 1)$. Thus the second part of Theorem 4.1 follows by Hölder’s inequality.

When $w_1 = w_2 = 1$, (4.8) is further equivalent to

$$(4.9) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{|I|}{|B|} \right)^{1/q} \leq C \left(\frac{|I|}{|B|} \right)^{1/p}.$$

We note that $|B| \approx \rho(B)^Q$ and $\rho(I) \leq c\rho(B)$, thus (4.9) holds if $p < q \leq pQ/(Q - p)$, which implies the third part of Theorem 4.1 for $p < q \leq pQ/(Q - p)$. For $1 \leq q \leq p$, the result follows by Hölder’s inequality.

5. A variant of Sobolev inequalities on the original spaces (Ω, ϱ) .

The main purpose of this section is to show the variant of Sobolev inequalities on the homogeneous space (Ω, ϱ) associated to the original vector fields X_1, \dots, X_m . We start with the pointwise estimate for any function $f \in C_0^\infty(B)$, and $B \subset \Omega$.

Lemma 5.1. *Given any ball $B = B(\xi_0, r)$, $\xi_0 \in E \subset \subset \Omega$, any $f \in C_0^\infty(B)$, there exists a constant $C > 0$ independent of f , B , $\xi \in B$ such that for $\xi \in B$,*

$$|f(\xi)| \leq C \int_B \frac{\varrho(\xi, \eta)}{|B(\eta, \varrho(\xi, \eta))|} \left(\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)| \right) d\eta$$

PROOF OF LEMMA 5.1. Let us consider the second order differential operator $L = \sum_{i=1}^m X_i^2$. By the result in [NSW], there is a unique fundamental solution $G(\xi, \eta)$ to L with

$$|X_i G(\xi, \eta)| \leq C \frac{\varrho(\xi, \eta)}{|B(\eta, \varrho(\xi, \eta))|}.$$

Thus for every $f \in C_0^\infty(B)$, we have

$$f(\xi) = \int \sum_{i=1}^m X_i^2 G(\xi, \eta) f(\eta) d\eta = \int \sum_{i=1}^m X_i G(\xi, \eta) X_i^* f(\eta) d\eta.$$

Note that X_i^* is the adjoint of X_i with respect to η and that

$$|X_i^* f(\eta)| \leq C (|X_i f(\eta)| + |f(\eta)|).$$

Hence we have the following

$$|f(\xi)| \leq C \int_B \frac{\varrho(\xi, \eta)}{|B(\eta, \varrho(\xi, \eta))|} \left(\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)| \right) d\eta.$$

Then the lemma follows.

By the above lemma we can show the following theorems.

Theorem 5.2. 1) *Let $1 < p < q < \infty$ and assume $w_1 \in A_p(\Omega)$, w_2 a doubling weight satisfying the following condition*

$$(5.3) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \leq C \left(\frac{w_1(I)}{w_1(B)} \right)^{1/p}$$

for all metric balls I and B with $I \subset cB \subset \Omega$, where $\rho(B)$ is the radius of the ball B . Then there are positive constants C and r_0 such that for any ball $B = B(\xi_0, r) \subset \Omega$, $r \leq r_0$, and any $f \in C_0^\infty(B)$, the following inequality holds

$$\left(\frac{1}{w_2(B)} \int_B |f|^q w_2 \right)^{1/q} \leq C r \left(\frac{1}{w_1(B)} \int_B \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{1/p}.$$

2) *In the case that $w_1 = w_2 = w \in A_p$ and $p > 1$, the above q can be replaced by $q = kp$ with $1 \leq k \leq Q/(Q - 1) + \delta_p$ for some $\delta_p > 0$.*

3) *In the case that $w_1 = w_2 = 1$ and $1 < p < Q$, the above q ranges $1 \leq q \leq pQ/(Q - p)$.*

PROOF OF THEOREM 5.2. The proof is basically similar to the one in Section 4. Thus we adapt the same notations of \tilde{v} , \tilde{w} , I , B , $\sigma(\xi)$ and just indicates the substantial difference. We define the fractional integral

$$Tf(\xi) = \int_{\Omega} f(\eta)K(\xi, \eta) d\eta$$

where $K(\xi, \eta) = \varrho(\xi, \eta)/|B(\eta, \varrho(\xi, \eta))|$.

We note here $\phi(I)$ will be

$$\phi(I) = \sup\{K(\xi, \eta) : \xi, \eta \in I, \varrho(\xi, \eta) \geq \frac{1}{2} \rho(I)\} \approx \frac{\rho(I)}{|I|}$$

where $\rho(I)$ is the radius of the ball I .

But the Sawyer-Wheeden condition here (inequality (4.5) in Section 4) is still equivalent to

$$(5.4) \quad \frac{\rho(I)}{\rho(B)} \frac{|B|}{|I|} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \left(\frac{\sigma(I)}{\sigma(B)} \right)^{1/p'} \leq C,$$

for all balls $I \subset cB$. Under this condition we have by Lemma 5.1 and Theorem 3 in [SW],

$$(5.5) \quad \left(\frac{1}{w_2(B)} \int_B |f|^q w_2 \right)^{1/q} \leq C (|B|^{-1} w_1(B)^{1/p} \sigma(B)^{1/p'}) \rho(B) \cdot \left(\frac{1}{w_1(B)} \int_B \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{1/p}$$

for all $f \in C_0^\infty(B)$. If assume $w_1 \in A_p$, then $|B|^{-1} w_1(B)^{1/p} \sigma(B)^{1/p'} \leq C$. We also note that (5.3) is equivalent to (5.4) when $w_1 \in A_p$. This shows the first case of Theorem 5.2.

When $w_1 = w_2 = w \in A_p(\Omega)$, (5.3) is equivalent to

$$(5.6) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{w(I)}{w(B)} \right)^{1/q} \leq C \left(\frac{w(I)}{w(B)} \right)^{1/p}.$$

We note when the ball B is small, say $\rho(B) \leq r_0$, there is some N -tuple J such that for any ball $I \subset cB \subset \Omega$ we have

$$|I| \approx \rho(I)^{d(J)}, \quad |B| \approx \rho(B)^{d(J)},$$

where $d(J) \leq Q$ is the length of the commutator X_J (see Section 2 or see [NSW]). Since $A_p \subset D_p$ (doubling of order p), *i.e.*

$$w(B) \leq C \left(\frac{|B|}{|I|} \right)^p w(I), \quad \text{for any } I \subset cB \subset \Omega,$$

then the assumption that $w \in A_p(\Omega)$ implies that (5.6) holds if q is chosen so that $(d(J)(1/p - 1/q))^{-1} = p$, *i.e.* $q = (d(J)p)/(d(J) - 1)$. Actually, since $w \in A_p$ implies $w \in A_{p_0}$ for some $p_0 < p$ by Theorem 2.4, it follows that q can be chosen slightly larger than $d(J)p/(d(J) - 1)$. By noticing that

$$\frac{Qp}{Q-1} \leq \frac{d(J)p}{d(J)-1}, \quad \text{since } d(J) \leq Q,$$

thus case 2 of the Theorem 5.2 follows.

When $w_1 = w_2 = 1$, (5.2) is further equivalent to

$$(5.7) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{|I|}{|B|} \right)^{1/q} \leq C \left(\frac{|I|}{|B|} \right)^{1/p}.$$

We note again that $|B| \approx \rho(B)^{d(J)}$, $|I| \approx \rho(I)^{d(J)}$, and $\rho(I) \leq c \rho(B)$, thus (5.7) holds if $d(J)(1/q - 1/p) + 1 \geq 0$. But

$$Q\left(\frac{1}{q} - \frac{1}{p}\right) + 1 \leq d(J)\left(\frac{1}{q} - \frac{1}{p}\right) + 1, \quad \text{since } p < q \text{ and } d(J) \leq Q,$$

and $Q(1/q - 1/p) + 1 \geq 0$ for $1 \leq q \leq pQ/(Q - p)$, thus the third case of Theorem 5.2 follows for $p < q \leq pQ/(Q - P)$. For $1 \leq q \leq p$ the result follows by Hölder's inequality.

6. Proof of the main theorems.

This section is devoted to the proof of the main theorems in this paper, theorems A, B and C. The proof of Poincaré inequalities is based on the variants of Poincaré type inequalities on the homogeneous space $(\tilde{\Omega}, \tilde{\rho})$ proved in Section 4. The basic technique is an argument by Kohn (see [K]). This method was further developed and applied to the proof of unweighted Poincaré inequality for vector fields by Jerison (see [J]). We first prove the following variants of Poincaré inequalities on the homogeneous space (Ω, ρ) associated to the original vector fields X_1, \dots, X_m .

Lemma 6.1. 1) *Let $1 < p < q < \infty$ and assume $w_1 \in A_p(\Omega)$, w_2 a doubling weight satisfying the following condition*

$$(6.2) \quad \frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \leq C \left(\frac{w_1(I)}{w_1(B)} \right)^{1/p}$$

for all metric balls I and B with $I \subset cB \subset \Omega$, where $\rho(B)$ is the radius of the ball B . Let $E \subset\subset \Omega$, then there is a positive constant C such that for any balls $B = B(\xi, r)$ with $\xi \in E$, $cB \subset \Omega$ and $f \in C^\infty(\overline{cB})$, the following inequality holds

$$\begin{aligned} & \left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2 \right)^{1/q} \\ & \leq C r \left(\frac{1}{w_1(B)} \int_{cB} \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{1/p} \end{aligned}$$

where f_B may be taken to be

$$f_B = \frac{1}{w_2(B)} \int_B f w_2.$$

2) In the case that $w_1 = w_2 = w \in A_p$, the above q in the inequality can be taken as $q = kp$ with $1 \leq k \leq Q/(Q - 1) + \delta_p$ for some $\delta_p > 0$, and f_B can be taken as

$$\frac{1}{|B|} \int_B f \quad \text{or} \quad \frac{1}{|B|} \int_B fw.$$

3) In the case that $w_1 = w_2 = 1$ and $1 < p < Q$, the q can be taken as $1 \leq q \leq pQ/(Q - p)$ and f_B can be taken as

$$f_B = \frac{1}{|B|} \int_B f.$$

PROOF OF LEMMA 6.1. Let $B = B(\xi_0, r)$ and $\tilde{B} = \tilde{B}((\xi_0, 0), r)$. Define $\tilde{w}_i(\tilde{\xi}) = \tilde{w}(\xi, t) \equiv w(\xi)$ ($i = 1, 2$) as in Subsection 2.3, then by Lemma 2.7, $\tilde{w}_i \in A_p(\tilde{\Omega})$. It is also easy to check that (4.2) holds for \tilde{w}_1 and \tilde{w}_2 . We also define $\tilde{f}(\tilde{\xi}) = \tilde{f}(\xi, t) \equiv f(\xi)$. Then \tilde{f} is a function on $B(\xi, r) \times \mathbb{R}^l$ that is independent of $t \in \mathbb{R}^l$ and $\tilde{X}_i \tilde{f} = X_i f$. Consequently, by Theorem 4.1, there are constants $C = C(\tilde{f}, \tilde{B})$ and $c \geq 1$, such that

$$\begin{aligned} & \left(\frac{1}{\tilde{w}_2(\tilde{B}((\xi_0, 0), r))} \int_{\tilde{B}((\xi_0, 0), r)} |\tilde{f}(\tilde{\xi}) - C|^q \tilde{w}_2(\tilde{\xi}) d\tilde{\xi} \right)^{1/q} \\ & \leq Cr \left(\frac{1}{\tilde{w}_1(\tilde{B}((\xi_0, 0), cr))} \right. \\ & \quad \left. \cdot \int_{\tilde{B}((\xi_0, 0), cr)} \left(\sum_{i=1}^m |\tilde{X}_i \tilde{f}(\tilde{\xi})| + |\tilde{f}(\tilde{\xi})| \right)^p \tilde{w}_1(\tilde{\xi}) d\tilde{\xi} \right)^{1/p}. \end{aligned}$$

Note $B(\xi_0, r) \times \{0\} \subset \tilde{B}((\xi_0, 0), r) \subset B(\xi_0, r) \times \mathbb{R}^l$ and $d\tilde{\xi} = d\xi dt$, also note $\tilde{w}_i(\tilde{B}((\xi_0, 0), r)) \approx |\tilde{B}|w_i(B)/|B|$ ($i = 1, 2$), thus integration with respect to t and an application of Lemma 4.4 in [J] yields the following

$$\begin{aligned} & \left(\frac{1}{w_2(B(\xi_0, r))} \int_{B(\xi_0, r)} |f(\xi) - C|^q w_2(\xi) d\xi \right)^{1/q} \\ & \leq Cr \left(\frac{1}{w_1(B(\xi_0, cr))} \right. \\ & \quad \left. \cdot \int_{B(\xi_0, cr)} \left(\sum_{i=1}^m |X_i f(\xi)| + |f(\xi)| \right)^p w_1(\xi) d\xi \right)^{1/p} \end{aligned}$$

for some constant $c \geq 1$ independent of f and B . Then the first part of the lemma follows.

By using the corresponding cases in Theorem 4.1, we can prove the other two cases of the lemma in the same way.

6.1. A Whitney decomposition.

In this subsection we will recall the Whitney decomposition in metric spaces (Ω, ϱ) . We first give an alternate definition of the metric on Ω . An admissible path γ is a Lipschitz curve $\gamma : [a, b] \rightarrow \Omega$ such that there exist functions c_i , $a \leq t \leq b$, with $\sum_{i=1}^m c_i(t)^2 \leq 1$ and $\gamma'(t) = \sum_{i=1}^m c_i(t)X_i(\gamma(t))$ for almost every $t \in [a, b]$. We define a metric associated to X_1, \dots, X_m by $\varrho(\xi, \eta) = \min\{b \geq 0 : \text{there exists } \gamma : [0, b] \rightarrow \Omega \text{ such that } \gamma(0) = \xi, \gamma(b) = \eta\}$. Obviously, this metric is equivalent to those defined before. Throughout this subsection and the remaining of this section, we always use this definition of the metric.

For a ball $B = B(\xi, r)$, we denote $B' = B(\xi, 2r), B'' = B(\xi, 4r)$ and $B^* = B(\xi, 10r)$. We shall also denote the radius of B by $\rho(B)$. We also recall the doubling condition

$$(6.3) \quad |B(\xi, 2r)| \leq A |B(\xi, r)|, \quad \xi \in E \subset \subset \Omega.$$

Now we recall some results proved by Jerison [J].

Lemma 6.4. (Whitney decomposition). *Let $E_1 = B(\xi_1, r_1)$, then there is a pairwise disjoint family of balls \mathcal{F} and a constant M depending only on the doubling constant in (6.3) such that*

- (i) $E_1 = \bigcup_{B \in \mathcal{F}} B'$,
 - (ii) $B \in \mathcal{F}$ implies $10^2 \rho(B) \leq \rho(B, \partial E_1) \leq 10^3 \rho(B)$,
- where $\rho(B, \partial E_1)$ is the distance in the metric ϱ from B to ∂E_1 .
- (iii) $\#\{B \in \mathcal{F} : \eta \in B^*\} \leq M$.
- ($\#S$ is the number of elements in the set S).

For $B \in \mathcal{F}$, define γ_B as an admissible path from the center of B to ξ_1 (the center of E_1) of length $\leq r_1$. Denote the subset of E_1 defined by the image of γ_B by γ_B as well. This path may not be unique, but will be fixed throughout this subsection and next one. Denote $\mathcal{F}(B) = \{A \in \mathcal{F} : A' \cap \gamma_B \neq \emptyset\}$. The following has been proved by Jerison [J].

Lemma 6.5. (Jerison) *Let $B \in \mathcal{F}$, then*

- (i) *There are no elements of $\mathcal{F}(B)$ of radius less than $10^{-2}\rho(B)$.*
- (ii) *For any r , $\#\{A \in \mathcal{F}(B) : r \leq \rho(A) \leq 2r\} \leq M$, a constant depending only on the metric of Ω .*
- (iii) $\#\mathcal{F}(B) \leq C \log(r_1/\rho(B))$.

Lemma 6.6. *Let w be a doubling weight in (Ω, ρ) , then there exists some $\varepsilon > 0$ such that for any $A \in \mathcal{F}$ and any $r > 0$,*

$$\sum_{\substack{B \in A(\mathcal{F}) \\ r \leq \rho(B) \leq 2r}} w(B) \leq C \left(\frac{\rho(B)}{\rho(A)} \right)^\varepsilon w(A),$$

where $A(\mathcal{F}) = \{B \in \mathcal{F} : A \in \mathcal{F}(B)\}$.

PROOF. The proof follows by using the argument for the unweighted case in [J] and in view of the doubling condition on w .

We also need the following lemma.

Lemma 6.7. *Let $w \in D_\mu(\Omega)$, i.e. doubling of order μ , and $q < 1$. For $c_1 > 0$, there exists $c_2 = c_2(c_1, w, q)$ such that for every pair of balls B_1, B_2 in the metric space (Ω, ρ) satisfying $|B_2| \leq c_1|B_1 \cap B_2|$ and any f such that $\int_{B_j} |f - f_{B_j}|^q w \leq A$ for $j = 1, 2$, we have*

$$\int_{B_1 \cup B_2} |f - f_{B_1}|^q w \leq c_2 A.$$

The proof is easy and we omit it here.

By applying Lemma 6.7 and a well-known covering lemma argument, we can show the following

Lemma 6.8. 1) *Let w_1, w_2, p, q be as in (6.2). Given any $\xi \in E \subset \subset \Omega$, $r > 0$ with $B_{4r} = B(\xi, 4r) \subset \Omega$ and $f \in C^\infty(\overline{B_{2r}})$, we have*

$$\begin{aligned} & \left(\frac{1}{w_2(B_r)} \int_{B_r} |f - f_{B_r}|^q w_2 \right)^{1/q} \\ & \leq Cr \left(\frac{1}{w_1(B_r)} \int_{B_{2r}} \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{1/p} \end{aligned}$$

- where $B_r = B(\xi, r)$, $B_{2r} = B(\xi, 2r)$ and C independent of B and f .
- 2) In the case that $w_1 = w_2 = w \in A_p$ and $p > 1$, the q in the above inequality can be replaced by $q = kp$ with $1 \leq k \leq Q/(Q-1) + \delta_p$ for some $\delta_p > 0$.
- 3) In the case that $w_1 = w_2 = 1$ and $1 < p < Q$, the above inequality holds for $1 \leq q \leq pQ/(Q-p)$.

The proof is easy and well-known and we refer the reader to the final subsection for a more crucial argument.

6.2. Proof of the main theorems.

We first like to remark out that the Sobolev inequalities in Theorem B and C follow immediately from Theorem 5.2. Indeed, we note that

$$\left(\frac{1}{w(B)} \int_B |f|^p w \right)^{1/p} \leq \left(\frac{1}{w(B)} \int_B |f|^q w \right)^{1/q} \quad \text{for } p \leq q.$$

Thus by Lemma 5.2 and Minkowski’s inequality, Sobolev’s inequality in Theorem B follows if

$$p \leq q \leq \left(\frac{Q}{Q-1} + \delta_p \right) p$$

and $r = \rho(B)$ small. For $q \leq p$, it follows from Hölder’s inequality. We can argue similarly to show Sobolev’s inequality in Theorem C. We now turn to prove Theorem A, and Poincaré’s inequalities in theorems B and C. It should be pointed out here that an analogue of Lemma 6.9 below has been proved by J. Fernández [Fer] in their context. This kind of argument seems well-known now and one may also find it in the various places in the paper by C. Gutiérrez and R. Wheeden [GW]. We need such a lemma here to show the Poincaré inequality for the two weights case. We modify the proof given in [Fer] or [GW]. We recall again that $\Omega \subset \mathbb{R}^N$.

Lemma 6.9. *Assume (6.2) holds for doubling weights w_1, w_2 and $1 < p < q < \infty$. Then there are $r_0 > 0, \alpha > 0$ and some q_0 with $p < q_0 < q$ such that*

$$\left(\frac{\rho(I)}{\rho(B)} \right)^{p-\alpha} \left(\frac{w_2(I)}{w_2(B)} \right)^{p/q_0} \leq C \frac{w_1(I)}{w_1(B)},$$

for all metric balls $I \subset cB \subset \Omega$ provided $\rho(B) \leq r_0$.

PROOF. Since $w_2 \in D_\mu$ (doubling of order μ), then $w_2 \in RD_\nu$ (reverse doubling of order ν). Thus we have for all $I \subset cB \subset \Omega$,

$$\frac{w_2(I)}{w_2(B)} \geq C \left(\frac{|I|}{|B|} \right)^\mu,$$

and

$$\frac{w_2(I)}{w_2(B)} \leq C \left(\frac{|I|}{|B|} \right)^\nu.$$

By Theorem 7 in [NSW], there is an N -tuple J , depending on B , such that $|I| \approx \rho(I)^{d(J)}$, $|B| \approx \rho(B)^{d(J)}$ for all $I \subset cB \subset \Omega$ provided $\rho(B) \leq r_0$ for some $r_0 > 0$. The above $d(J)$ is the length of the commutator X_J (see Section 2) and $N \leq d(J) \leq Q$.

If we select $\alpha < \nu(1 - p/q)N$, then $\alpha < \nu(1 - p/q)d(J)$. By (6.2), we have

$$\begin{aligned} \frac{w_2(I)}{w_2(B)} &= \left(\frac{w_2(I)}{w_2(B)} \right)^{p/q} \left(\frac{w_2(I)}{w_2(B)} \right)^{1-p/q} \\ &\leq C \left(\frac{\rho(B)}{\rho(I)} \right)^{p-\alpha} \frac{w_1(I)}{w_1(B)} \left(\frac{w_2(I)}{w_2(B)} \right)^{1-p/q} \left(\frac{\rho(B)}{\rho(I)} \right)^\alpha \\ &\leq C \left(\frac{\rho(B)}{\rho(I)} \right)^{p-\alpha} \frac{w_1(I)}{w_1(B)} \left(\frac{\rho(I)}{\rho(B)} \right)^{\nu(1-p/q)d(J)-\alpha} \\ &\leq C \left(\frac{\rho(B)}{\rho(I)} \right)^{p-\alpha} \frac{w_1(I)}{w_1(B)} \left(\frac{w_2(I)}{w_2(B)} \right)^{\nu(1-p/q)/\mu-\alpha/\mu d(J)} \end{aligned}$$

Thus,

$$\left(\frac{w_2(I)}{w_2(B)} \right)^{1-\nu(1-p/q)\mu+\alpha\mu d(J)} \leq C \left(\frac{\rho(B)}{\rho(I)} \right)^{p-\alpha} \frac{w_1(I)}{w_1(B)}.$$

Therefore, by noticing $w_2(I) \leq c w_2(B)$ and $\alpha < \nu(1 - p/q)N$, Lemma 6.9 will follow if we select $p < q_0 < q$ satisfying

$$1 - \frac{\nu}{\mu} \left(1 - \frac{p}{q} \right) + \frac{\alpha}{\mu N} < \frac{p}{q_0},$$

which implies

$$1 - \frac{\nu}{\mu} \left(1 - \frac{p}{q} \right) + \frac{\alpha}{\mu d(J)} < \frac{p}{q_0}.$$

As an easy consequence, we have

Corollary 6.10. *Assume (6.2) holds for doubling weights w_1, w_2 and $1 < p < q < \infty$. Then there are $r_0 > 0, \alpha > 0$ and some q_0 with $p < q_0 < q$ such that*

$$\left(\log \frac{\rho(B)}{\rho(I)}\right)^{q_0-1} \left(\frac{\rho(I)}{\rho(B)}\right)^{q_0} \left(\frac{w_2(I)}{w_2(B)}\right) \leq C \left(\frac{w_1(I)}{w_1(B)}\right)^{q_0/p},$$

for all metric balls $I \subset cB \subset \Omega$ provided $\rho(B) \leq r_0$.

We like to remark out that we do not know how big q_0 above can be. The only information is that q_0 is some number less than q . However, when $w_1 = w_2 = w \in A_p$ or $w_1 = w_2 = 1$, we will be able to know the precise range of q_0 . We first show the following two lemmas.

Lemma 6.11. *Assume $w \in A_p(\Omega), p > 1$. Given any two balls I and B with $I \subset cB \subset \Omega$, and $\delta_1 = \rho(I), \delta_2 = \rho(B)$, we have the following*

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \leq C$$

provided that $p \leq q \leq pQ/(Q-1) + \delta_p$, for some $\delta_p > 0$, and $\rho(B) \leq r_0$ for some $r_0 > 0$.

Lemma 6.12. *Assume $p > 1$, then for any two balls I and B as in Lemma 6.11, we have the following*

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\frac{|I|}{|B|}\right)^{1/q-1/p} \leq C$$

provided that $1 \leq q \leq pQ/(Q-p)$ and $\rho(B) \leq r_0$ for some $r_0 > 0$.

PROOF OF LEMMA 6.11. Since $w \in A_p(\Omega)$ implies $w \in A_{p_0}(\Omega)$ for some $p_0 < p$, and noting that $I \subset cB$, then we have

$$\left(\frac{w(B)}{w(I)}\right) \leq c \left(\frac{|B|}{|I|}\right)^{p_0}.$$

Thus an easy calculation shows by noting that $|I| \leq c|B|$,

$$(6.13) \quad \left(\frac{|I|}{|B|}\right)^{1/Q} \left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \leq C$$

provided that $p \leq q \leq pQ/(Q - 1) + \delta_p$. Reasoning as before, there is an N -tuple J such that for $\delta_2 < r_0$,

$$|I| \approx \delta_1^{d(J)}, \quad |B| \approx \delta_2^{d(J)}.$$

Hence,

$$\left(\frac{|I|}{|B|}\right)^{1/Q} \approx \left(\frac{\delta_1}{\delta_2}\right)^{d(J)/Q}.$$

Since $\delta_1 < c\delta_2$, $d(J) \leq Q$, we obtain

$$\left(\frac{\delta_1}{\delta_2}\right) \leq C \left(\frac{\delta_1}{\delta_2}\right)^{d(J)/Q}.$$

Therefore, by (6.13), it completes the proof.

The proof of Lemma 6.12 is similar and we do not present it here. Now we prove the following two corollaries which will be needed when we prove the Poincaré inequalities for the case of equal weights and unweighted case.

Corollary 6.14. *Let $\delta_1, \delta_2, I, B, w$ be as above, then*

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\log \frac{\delta_2}{\delta_1}\right)^{1-1/q} \left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \leq C$$

provided $p \leq q < pQ/(Q - 1) + \delta_p$ for the same δ_p in Lemma 6.11.

PROOF. It is sufficient to show that

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\frac{w(I)}{w(B)}\right)^{1/q_0-1/p} \leq C_1$$

implies

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\log \frac{\delta_2}{\delta_1}\right)^{1-1/q} \left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \leq C_2$$

for $q < q_0$.

Since $q < q_0$, there is some $\varepsilon > 0$ such that

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{p} + \varepsilon.$$

We note since $w \in A_p \subset A_\infty$, then by Theorem 2.5 there is $\eta > 0$ such that

$$\frac{w(I)}{w(B)} \leq C \left(\frac{|I|}{|B|} \right)^\eta \leq C \left(\frac{\delta_1}{\delta_2} \right)^{d(J)\eta}.$$

Then,

$$\begin{aligned} & \left(\frac{\delta_1}{\delta_2} \right) \left(\log \frac{\delta_2}{\delta_1} \right)^{1-1/q} \left(\frac{w(I)}{w(B)} \right)^{1/q-1/p} \\ &= \left(\frac{\delta_1}{\delta_2} \right) \left(\log \frac{\delta_2}{\delta_1} \right)^{1-1/q} \left(\frac{w(I)}{w(B)} \right)^{1/q_0-1/p} \left(\frac{w(I)}{w(B)} \right)^\varepsilon \\ &\leq C_1 \left(\log \frac{\delta_2}{\delta_1} \right)^{1-1/q} \left(\frac{\delta_1}{\delta_2} \right)^{\varepsilon\eta d(J)} \leq C_2 \end{aligned}$$

since $\delta_1 < c\delta_2$.

Corollary 6.15. *Let $\delta_1, \delta_2, I, B, w$ be as above, then*

$$\left(\frac{\delta_1}{\delta_2} \right) \left(\log \frac{\delta_2}{\delta_1} \right)^{1-1/q} \left(\frac{|I|}{|B|} \right)^{1/q-1/p} \leq C$$

provided $1 \leq q < pQ/(Q-p)$.

The proof of Corollary 6.15 is similar to that of Corollary 6.14.

We recall that the inequality in Lemma 6.8 is equivalent to

$$(6.16) \quad \int_B |f - f_B|^q w_2 \leq C \rho(B)^q w_2(B) w_1(B)^{-q/p} \left(\int_{2B} \left(\sum_{i=1}^m |X_i f| + |f| \right)^p w_1 \right)^{q/p}$$

for $p < q$ and w_1, w_2 satisfying (6.2).

Now we are ready to prove the following Poincaré inequality for two weights (*i.e.* Theorem A).

Theorem 6.17. *Let $1 < p < q$, w_1, w_2 satisfy (6.2), $w_1 \in A_p$, and $E_1 = B(\xi_1, r_1)$ be as in Lemma 6.4. We also assume here that $w_1 \leq w_2$.*

Then there are constants $C > 0$, $r_0 > 0$ and some q_0 as in Lemma 6.9 such that for any $f \in C^\infty(\overline{E_1})$,

$$\begin{aligned} & \left(\frac{1}{w_2(E_1)} \int_{E_1} |f - f_{E_1}|^{q_0} w_2 \right)^{1/q_0} \\ & \leq C \rho(E_1) \left(\frac{1}{w_1(E_1)} \int_{E_1} \sum_{i=1}^m |X_i f|^p w_1 \right)^{1/p} \end{aligned}$$

provided $\rho(E_1) \leq r_0$, where C and r_0 are independent of the ball E_1 and the function f .

PROOF. Choose $B_0 \in \mathcal{F}$ such that $\xi_1 \in B'_0$. Write $f_0 = f_{B'_0}$, and order elements of $\mathcal{F}(B) = \{A_1, A_2, \dots, A_l\}$ such that $A_1 = B, \dots, A_l = B_0$ and $A'_k \cap A'_{k+1} \neq \emptyset$, all k . Note that $2A''_k \subset A^*_k$ and (6.16) also holds by replacing q by q_0 , then apply (6.16) to $f - f_0$ yields

$$\begin{aligned} & \int_{A''_k} |f - f_{A''_k}|^{q_0} w_2 \\ & \leq C \rho(A_k)^{q_0} w_2(A_k) w_1(A_k)^{-q_0/p} \left(\int_{A^*_k} \left(\sum_{i=1}^m |X_i f| + |f - f_0|^p \right) w_1 \right)^{q_0/p} \end{aligned}$$

Moreover, A_k, A_{k+1} have comparable radii and volumes and $A'_k \cap A'_{k+1}$ contains a ball of comparable radius to those of A_k and A_{k+1} . Thus,

$$\begin{aligned} & \int_{A''_k \cup A''_{k+1}} |f_{A''_k} - f_{A''_{k+1}}|^{q_0} w_2 \\ & \leq C \int_{A''_k \cup A''_{k+1}} |f - f_{A''_k}|^{q_0} w_2 + \int_{A''_k \cup A''_{k+1}} |f - f_{A''_{k+1}}|^{q_0} w_2 \\ & \leq C \rho(A_k)^{q_0} w_2(A_k) w_1(A_k)^{-q_0/p} \\ & \quad \cdot \left(\int_{A^*_k \cup A^*_{k+1}} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q_0/p}. \end{aligned}$$

Hence,

$$\int_{B'} |f - f_0|^{q_0} w_2 = \int_{B'} |f - f_{B''} + \sum_{k=1}^{l-1} (f_{A''_k} - f_{A''_{k+1}})|^{q_0} w_2$$

$$\begin{aligned}
 &\leq l^{q_0-1} \left(\int_{B'} |f - f_{B''}|^{q_0} w_2 + \sum_{k=1}^{l-1} \int_{B'} |f_{A_k''} - f_{A_{k+1}''}|^{q_0} w_2 \right) \\
 &\leq C l^{q_0-1} \sum_{k=1}^l \rho(A_k)^{q_0} w_2(A_k) w_1(A_k)^{-q_0/p} \frac{w_2(B)}{w_2(A_k)} \\
 &\quad \cdot \left(\int_{A_k^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q/p}.
 \end{aligned}$$

Summing over $B \in \mathcal{F}$,

$$\begin{aligned}
 \int_{E_1} |f - f_0|^{q_0} w_2 &\leq \sum_{B \in \mathcal{F}} \int_{B'} |f - f_0|^{q_0} w_2 \\
 &\leq C \sum_{B \in \mathcal{F}} \sum_{A \in \mathcal{F}(B)} (\#\mathcal{F}(B))^{q_0-1} \rho(A)^{q_0} \frac{w_2(B)}{w_2(A)} w_2(A) w_1(A)^{-q_0/p} \\
 &\quad \cdot \left(\int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q_0/p}.
 \end{aligned}$$

By Lemma 6.5 and 6.6,

$$\begin{aligned}
 &\sum_{B \in A(\mathcal{F})} (\#\mathcal{F}(B))^{q_0-1} w_2(B) \\
 &= \sum_{k=1}^{\infty} \sum_{\substack{B \in A(\mathcal{F}) \\ 2^{-k} \rho(A) \leq \rho(B) / 100 \leq 2^{-k+1} \rho(A)}} (\#\mathcal{F}(B))^{q_0-1} w_2(B) \\
 &\leq C \sum_{k=1}^{\infty} \left(k + \log \left(\frac{r_1}{\rho(A)} \right) \right)^{q_0-1} 2^{-\varepsilon k} w_2(A) \\
 &\leq C \left(\log \frac{r_1}{\rho(A)} \right)^{q_0-1} w_2(A).
 \end{aligned}$$

By Corollary 6.10

$$\begin{aligned}
 &\rho(A)^{q_0} w_2(A) w_1(A)^{-q_0/p} \left(\log \frac{r_1}{\rho(A)} \right)^{q_0-1} \\
 &= \left(\log \frac{r_1}{\rho(A)} \right)^{q_0-1} \frac{w_2(A)}{w_2(E_1)} \left(\frac{w_1(A)}{w_1(E_1)} \right)^{-q_0/p}.
 \end{aligned}$$

$$\begin{aligned} & \cdot \left(\frac{\rho(A)}{\rho(E_1)} \right)^{q_0} \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p} \\ & \leq C \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p}. \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{E_1} |f - f_0|^{q_0} w_2 \\ & \leq \sum_{A \in \mathcal{F}} \rho(A)^{q_0} w_2(A) w_1(A)^{-q_0/p} \left(\log \frac{r_1}{\rho(A)} \right)^{q_0-1} \\ & \quad \cdot \left(\int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q_0/p} \\ & \leq C \sum_{A \in \mathcal{F}} \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p} \\ & \quad \cdot \left(\int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q_0/p} \\ & \leq C \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p} \\ & \quad \cdot \left(\sum_{A \in \mathcal{F}} \int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q_0/p}, \end{aligned}$$

since $q_0/p \geq 1$

$$\begin{aligned} & \leq C \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p} \\ & \quad \cdot \left(\int_{E_1} \left(\sum_{i=1}^m |X_i f| + |f - f_0| \right)^p w_1 \right)^{q_0/p} \end{aligned}$$

The last inequality above is because all A^* , for $A \in \mathcal{F}$, have bounded overlap by Lemma 6.4. Thus by Minkowski's inequality

$$\begin{aligned} & \left(\frac{1}{w_2(E_1)} \int_{E_1} |f - f_0|^{q_0} w_2 \right)^{1/q_0} \\ & \leq C \rho(E_1) \left(\frac{1}{w_1(E_1)} \left(\sum_{i=1}^m |X_i f| \right)^p w_1 \right)^{1/p} \\ & \quad + C \rho(E_1) \left(\frac{1}{w_1(E_1)} \int_{E_1} |f - f_0|^p w_1 \right)^{1/p}. \end{aligned}$$

We recall that (6.9) implies

$$\left(\frac{\rho(E_1)}{\rho(D)}\right)^{1-\alpha/p} \left(\frac{w_2(E_1)}{w_2(D)}\right)^{1/q_0} \leq C \left(\frac{w_1(E_1)}{w_1(D)}\right)^{1/p} \leq C \left(\frac{w_1(E_1)}{w_1(D)}\right)^{1/q_0},$$

for any $E_1 \subset cD \subset \Omega$. Since the center ξ_1 of the ball $E_1 = B(\xi_1, r_1)$ is in $E \subset \subset \Omega$, we can always pick the ball $D \subset \Omega$ such that $\rho(D) \geq c_1$. We note $w_1 \in A_p(\Omega)$, then

$$\frac{1}{|D|^p} \int_D w_1 \approx \left(\int_D w_1^{-1/(p-1)} \right)^{-(p-1)} \geq C,$$

since

$$\left(\int_D w_1^{-1/(p-1)} \right)^{p-1} \leq \left(\int_\Omega w_1^{-1/(p-1)} \right)^{p-1}.$$

Thus $w_1(D) \geq c_2$. The above c_1 and c_2 are independent of the ball E_1 but only dependent on E , Ω and w_1 . We also note that

$$\rho(D) \leq c_3, \quad w_2(D) \leq w_2(\Omega).$$

Thus

$$\rho(E_1)^{1-\alpha/p} w_2(E_1)^{1/q_0} \leq C w_1(E_1)^{1/q_0},$$

this is equivalent to

$$\rho(E_1)^{1-\alpha/p} \left(\frac{w_2(E_1)}{w_1(E_1)}\right)^{1/q_0} \leq C,$$

and this further implies that

$$\rho(E_1) \left(\frac{w_2(E_1)}{w_1(E_1)}\right)^{1/q_0} \leq C \rho(E_1)^{\alpha/p}.$$

By noting that $w_1 \leq w_2$ and $p \leq q_0$, and applying the Hölder inequality, we have

$$\begin{aligned} & \left(\frac{1}{w_1(E_1)} \int_{E_1} |f - f_0|^p w_1 \right)^{1/p} \\ & \leq \left(\frac{1}{w_1(E_1)} \int_{E_1} |f - f_0|^{q_0} w_1 \right)^{1/q_0}. \end{aligned}$$

$$\leq \left(\frac{1}{w_2(E_1)} \int_{E_1} |f - f_0|^{q_0} w_2 \right)^{1/q_0} \left(\frac{w_2(E_1)}{w_1(E_1)} \right)^{1/q_0}.$$

Therefore,

$$\begin{aligned} \rho(E_1) & \left(\frac{1}{w_1(E_1)} \int_{E_1} |f - f_0|^p w_1 \right)^{1/q_0} \\ & \leq C \rho(E_1)^{\alpha/p} \left(\frac{1}{w_2(E_1)} \int_{E_1} |f - f_0|^{q_0} w_2 \right)^{1/q_0}. \end{aligned}$$

Thus if $\rho(E_1)$ is small enough, say $C\rho(E_1)^{\alpha/p} \leq 1/2$, we have

$$\begin{aligned} & \left(\frac{1}{w_2(E_1)} \int_{E_1} |f - f_0|^q w_2 \right)^{1/q} \\ & \leq C \rho(E_1) \left(\frac{1}{w_1(E_1)} \int_{E_1} \left(\sum_{i=1}^m |X_i f| \right)^p w_1 \right)^{1/p} \end{aligned}$$

The proof of Sobolev's inequalities is even simpler since the integrals on both sides of the inequality in Lemma 5.2 are over the equal ball and thus we do not need the above covering lemma argument. The proof of Poincaré's inequalities in theorems B and C is similar to the proof given above if we apply corollaries 6.14 and 6.15. The detailed proof can be found in [L3].

REMARK 3. Since Jerison-Kohn's argument only works for metrics, the Poincaré inequalities in theorems A, B and C only hold for metric balls B . If the balls B are pseudometric balls, such as $B_I(x, \delta)$, the Poincaré inequality will have the following form

$$\begin{aligned} & \left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2 \right)^{1/q} \\ & \leq C \rho(B) \left(\frac{1}{w_1(B)} \int_{cB} \left(\sum_{i=1}^m |X_i f| \right)^p w_1 \right)^{1/p} \end{aligned}$$

for some constant $c \geq 1$.

On the other hand, the Sobolev inequalities hold for both metrics and pseudometrics.

REMARK 4. The reason for not having the Poincaré inequality for $q = pQ/(Q - p)$ in Theorem C is due to Corollary 6.15. We can see this when we do the above covering lemma argument. It is hoped that this inequality should be true for $q = pQ/(Q - p)$.

Part II: The local regularity of solutions to certain classes of degenerate differential operators formed by vector fields.

This part studies Harnack's inequalities and several general mean-value inequalities for the solutions to the following degenerate differential equations

$$L = \sum_{i,j=1}^m X_i^*(a_{ij}(x)X_j)$$

and

$$\mathcal{L} = - \sum_{i,j=1}^m X_i(a_{ij}(x)X_j)$$

where X_1, \dots, X_m are vector fields satisfying Hörmander's condition, and X_i^* is the adjoint of X_i . The coefficient matrix $A = (a_{ij}(x))$ is symmetric and satisfies

$$c^{-1}w(x)|\xi|^2 \leq \langle A\xi, \xi \rangle \leq c w(x)|\xi|^2, \quad \xi \in \mathbb{R}^m$$

where $w(x) \in A_2(\Omega)$ is as defined in Part I. The Hölder continuity of the solutions will also be derived.

7. The main theorems of Part II.

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected domain, and let X_1, \dots, X_m be C^∞ real vector fields, in a neighbourhood of $\bar{\Omega}$, satisfying Hörmander condition. Let X_i^* be the adjoint of X_i , we will consider the differential operators

$$L = \sum_{i,j=1}^m X_i^*(x)(a_{ij}(x)X_j(x)),$$

and

$$\mathcal{L} = - \sum_{i,j=1}^m X_i(x)(a_{ij}(x)X_j(x))$$

where the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

$$(7.1) \quad c^{-1}w(x)|\xi|^2 \leq \langle A\xi, \xi \rangle \leq c w(x)|\xi|^2,$$

$\langle \cdot, \cdot \rangle$ denotes the usual dot product, and $w \in A_2(\Omega)$ is a nonnegative weight function as defined in Part I.

Our main aim here will be to derive the mean value and Harnack inequalities for suitably defined weak solutions of L and \mathcal{L} . Many authors have studied such equations, we refer the reader to the following work: [De], [EP], [K], [M], [N], [GT], [FKS], [CW2], [St]. The situation considered there was for the case when $X_i = \partial/\partial x_i$ and $m = N$. The Harnack inequality for differential operator \mathcal{L} when the matrix $A = (a_{ij}(x))$ is the identity one follows from the work of fundamental solutions by Sánchez-Calle [Sa], Nagel-Stein-Wainger [NSW] or Sánchez-Calle and Jerison [JS]. A related situation was considered in [Bon], [FS] and [FrL].

We now recall the Poincaré and Sobolev inequalities in the particular case $p = 2$. By Theorem B and the Remark 3 in Part I for the metric ball $B_I \subset \Omega$ we have

Poincaré Inequality. *Let $w \in A_2(\Omega)$ and $E \subset \subset \Omega$, then there exist constants $r_0 > 0$, $C > 0$, $\delta > 0$, $a > 1$, such that for any metric ball $B = B_I(x, r)$ with $aB = B_I(x, ar) \subset \Omega$, $x \in E$, and function $f \in C^\infty(B)$, the following inequality holds*

$$(7.2) \quad \left(\frac{1}{w(B)} \int_B |f - f_B|^q w(y) dy \right)^{1/q} \leq Cr \left(\frac{1}{w(B)} \int_{aB} \sum_{i=1}^m |X_i f|^2 w(y) dy \right)^{1/2}$$

provided $0 < r < r_0$, $q = 2k$, $1 \leq k \leq Q/(Q - 1) + \delta$, where C, r_0 depend only on the A_2 constant of the weight w , E , Ω , and f_B may be taken to be either

$$\frac{1}{|B|} \int_B f(y) dy \quad \text{or} \quad \frac{1}{w(B)} \int_B f(y) w(y) dy.$$

By Theorem B in Part I again, for any metric ball $B \in \Omega$ we also have

Sobolev Inequality. *Let $w \in A_2(\Omega), E \subset\subset \Omega$, then there exist constants $r_0 > 0, C > 0, \delta > 0$, such that for any metric ball $B = B(x, r) \subset \Omega, x \in E$ as above, and any function $f \in C_0^\infty(B)$, the following inequality holds*

$$(7.3) \quad \left(\frac{1}{w(B)} \int_B |f|^q w(y) dy \right)^{1/q} \leq Cr \left(\frac{1}{w(B)} \int_B \sum_{i=1}^m |X_i f|^2 w(y) dy \right)^{1/2}$$

provided $q = 2k, 1 \leq k \leq Q/(Q - 1) + \delta, 0 < r < r_0$, where $C > 0, r_0 > 0$ only depend on the A_2 constant of the weight w, E, Ω .

REMARK 5. The above constant $Q \geq 2$ is the homogeneous dimension of the graded nilpotent group G corresponding to the lifted vector fields $\{\tilde{X}_i\}$ of $\{X_i\}$ (see Part I).

Throughout this part, w is always assumed to be in $A_2(\Omega)$, and all integrals are Lebesgue integrals. We also use the notations $Xf = (X_1 f, \dots, X_m f), |Xf| = (\sum_{i=1}^m |X_i f|^2)^{1/2}$. We also fix $q > 2$ on the left sides of inequalities (7.2) and (7.3) and denote $q = 2\sigma$ for some $\sigma > 1$. For the definitions of solutions, subsolutions and supersolutions, and the associated function \tilde{u} to the function u , see Section 8.

Now we state the main theorems in this part.

Theorem 7.4. (Mean-Value inequality) *Given $B = B_I(x_0, h) \subset \Omega$. Let $u \in H(B)$ be a nonnegative solution of $Lu = 0$, and \tilde{u} be the function in L_w^2 associated to u . Then there exist positive constants c and d depending only on the parameters in (7.1) and (7.3) such that for any ball $B = B_I(x, h) \subset \Omega$, and for $1/2 \leq \alpha < 1$ and $-\infty < p < \infty$, we have*

$$\operatorname{ess\,sup}_{\alpha B} \tilde{u}^p \leq \frac{c}{(1 - \alpha)^d} (1 + |p|)^{2\sigma/(\sigma-1)} \frac{1}{w(B)} \int_B \tilde{u}^p w.$$

Theorem 7.5. (Mean-Value inequality) *Let u be a subsolution of $Lu = 0$ belonging to $H(B)$. Let \tilde{u} be the function in L_w^2 associated with u .*

Then there exist constants c and d such that for $1/2 \leq \alpha < 1$ and $0 < p < \infty$,

$$(\operatorname{ess\,sup}_{\alpha B} \tilde{u}^+)^p \leq C \frac{1}{w(B)} \int_B (\tilde{u}^+)^p w,$$

where $C \leq c/(1 - \alpha)^d$ if $p \geq 2$ and $\leq c^{\log(3/p)}/(1 - \alpha)^d$ if $0 < p < 2$, and c, d are independent of u and B .

Corollary 7.6. *If in the theorem above u is a solution, then the conclusion holds with \tilde{u}^+ replaced by $|\tilde{u}|$, i.e. for $0 < p < \infty$, $0 < \alpha < 1$,*

$$\operatorname{ess\,sup}_{\alpha B} |\tilde{u}|^p \leq C \frac{1}{w(B)} \int_B |\tilde{u}|^p w,$$

where C is as in the Theorem 7.5 above.

Theorem 7.7. (Harnack inequality) *Let u be nonnegative solution of $Lu = 0$ belonging to $H(2B)$ for a ball $B = B(x, r)$ with $2B \subset \Omega, r \leq r_0$ for some $r_0 > 0$. Let \tilde{u} be the function in L^2_w associated with u , then*

$$\operatorname{ess\,sup}_B \tilde{u} \leq c \operatorname{ess\,inf}_B \tilde{u},$$

with c independent of u and B .

As an easy consequence of Theorem 7.7 and following the proof of continuity of solutions given by Moser [M], we can also obtain the Hölder continuity of solutions to L .

Theorem 7.8. (Hölder continuity of solutions) *Let u be a weak solution to L . Let $B = B(x_0, r)$, $r \leq r_0$ with $B \subset \Omega$. Then for \tilde{u} , and some $\alpha > 0$, and $x, y \in B(x_0, r/2)$, and any $p > 0$,*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c \left(\frac{\varrho(x, y)}{r} \right)^\alpha \left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |\tilde{u}|^p w \right)^{1/p}$$

with $c = c(p)$ independent of u and B .

The Mean-value and Harnack inequality for the operator \mathcal{L} will also be proved in this part. Since the statement is the same, we do not present here (see Section 10).

We conclude this section by proving the existence of a cut-off function relative to certain metric balls. Before we do so, we first review some basic results in [NSW].

Let Y_1, \dots, Y_q be an enumeration of the vector fields X_1, \dots, X_m and its commutators. Given any N -tuple $I = (i_1, \dots, i_N)$, let $U_j = Y_{i_j}, 1 \leq j \leq N$ and $\{V_j\}_{j=1}^{q-N}$ be the remaining $q - N$ vector fields. Let $d(U_j), d(V_j)$ be the degrees of vector fields U_j and V_j respectively, *i.e.* the length of their commutators. For $u \in \mathbb{R}^N, v \in \mathbb{R}^{q-N}$, set

$$z = \exp(vV) = \exp\left(\sum_{j=1}^{q-N} v_j V_j\right)(x)$$

$$\Phi_v(u) = \exp(uU + vV)(x)$$

and define

$$B_I(x, z, \delta) = \{y \in \Omega : y = \exp(uU + vV)(x) \text{ with } |u_j| < \delta^{d(U_j)}\}.$$

In the sequel, we denote $d_j = d(U_j), 1 \leq j \leq N$. We also denote $B_I(x, z, \delta)$ by $B_I(x, \delta)$ when $v = 0 \in \mathbb{R}^{q-N}$. Then by [NSW], one has the following theorem.

Theorem 7.9. (Nagel-Stein-Wainger) *Let $E \subset\subset \Omega$ be compact. Then there exist $0 < \eta_2 < \eta_1 < 1, \delta_0 > 0$ such that if $x \in E$, then there exists an N -tuple $I = (i_1, \dots, i_N)$ such that for any $\delta < \delta_0$, the following properties hold*

(i) $B(x, \eta_2 \delta) \subset B_I(x, z, \eta_1 \delta) \subset B(x, \delta)$.

(ii) *There is an open neighbourhood of $y = \Phi_v(u)$ on which Φ_v has an inverse map $\Psi = (\Psi_1, \dots, \Psi_N)$ so that locally $\Psi_j(\Phi_v(u)) = u_j$. Then we can regard Ψ_1, \dots, Ψ_N as coordinate functions near y . Moreover,*

$$(7.10) \quad |U_l(\Psi_k)| \leq c \delta^{d(U_k) - d(U_l)} = c \delta^{d_k - d_l}$$

for $y \in B_I(x, z, \delta)$, where $U_j = Y_{i_j}$.

(iii) *Let $X_i = \sum_{l=1}^N a_i^l U_l$, then*

$$(7.11) \quad |a_i^l| \leq c \delta^{d_i - 1}$$

for $y \in B(x, \eta_2 \delta)$ and $0 < \delta < \delta_0$.

The inclusions in (i) were proved in Theorem 7 in [NSW], while (ii) and (iii) were proved in Theorem 6 in [NSW].

We now prove the following lemma concerning the existence of cut-off functions.

Lemma 7.12. *For any $x \in E \subset\subset \Omega$, $0 < r_1 < r_2 < \delta_0$, and I as chosen in the above theorem, then there is a $\phi \in C_0^\infty(B_I(x, r_2))$ such that*

$$\phi \equiv 1 \quad \text{on } B_I(x, r_1),$$

and

$$|X\phi| = \sum_{i=1}^N |X_i\phi| \leq \frac{c}{r_2 - r_1}$$

where c is independent of the balls B but only depends on $E \subset\subset \Omega$.

PROOF. Let $\phi_j(t)$ be 1 for $|t| < (r_1/r_2)^{d_j}$, and 0 for $|t| > 1$, $\phi_j \in C_0^\infty$ and $|\phi_j'| \leq c/(1 - (r_1/r_2)^{d_j})$, $1 \leq j \leq N$, where $d_j = d(U_j)$, $U_j = Y_{i_j}$, $I = (i_1, \dots, i_N)$ as before. Now let

$$\phi(y) = \prod_{j=1}^N \phi_j\left(\frac{\Psi_j(y)}{r_2^{d_j}}\right)$$

where $y = \exp(\sum_{j=1}^N u_j U_j)(x)$, $u_j = \Psi_j(y)$. Then by (7.10), $|U_l(\Psi_j)| \leq C r_2^{d_j - d_l}$ on $B_I(x, r_2)$. Let $X_i = \sum_{l=1}^N a_i^l U_l$, then by (7.11),

$$|a_i^l| \leq C r_2^{d_l - 1} \quad \text{on } B_I(x, r_2)$$

Now it is easy to verify $\phi(y)$ satisfying the following

(i) $\phi(y) \equiv 1$ on $B_I(x, r_1) = \{y \in \Omega : y = \exp(\sum_{j=1}^N u_j U_j)(x), |u_j| < r_1^{d(U_j)}\}$.

(ii) $\phi \equiv 0$ outside $B_I(x, r_2) = \{y \in \Omega : y = \exp(\sum_{j=1}^N u_j U_j)(x), |u_j| < r_2^{d(U_j)}\}$.

(iii) $|X_i\phi(y)| \leq \frac{C}{r_2 - r_1}$.

The assertions (i) and (ii) are trivial. To prove (iii), note

$$U_l\phi(y) = \sum_{k=1}^N \left(\prod_{j \neq k} \phi_j\right)(U_l\phi_k) = \sum_{k=1}^N \left(\prod_{j \neq k} \phi_j\right) \phi_k' \left(\frac{\Psi_k}{r_2^{d_k}}\right) \frac{1}{r_2^{d_k}} U_l(\Psi_k).$$

Thus,

$$|U_l \phi(y)| \leq \sum_{k=1}^N \frac{1}{1 - (r_1/r_2)^{d_k}} \frac{1}{r_2^{d_k}} r_2^{d_k - d_l} = C \sum_{k=1}^N \frac{r_2^{d_k}}{r_2^{d_k} - r_1^{d_k}} r_2^{-d_l}.$$

Hence

$$\begin{aligned} |X_i \phi| &\leq \sum_{l=1}^N |a_i^l| |U_l \phi| \\ (7.13) \quad &\leq C \sum_{l=1}^N r_2^{d_l - 1} \sum_{k=1}^N \frac{r_2^{d_k}}{r_2^{d_k} - r_1^{d_k}} r_2^{-d_l} \\ &= C N \sum_{k=1}^N \frac{r_2^{d_k - 1}}{r_2^{d_k} - r_1^{d_k}}. \end{aligned}$$

We note,

$$r_2^{d_k} - r_1^{d_k} = (r_2 - r_1) \sum_{j=1}^{d_k - 1} r_2^j r_1^{d_k - j - 1} \geq (r_2 - r_1) r_2^{d_k - 1}.$$

Substituting this inequality into the expression (7.13), we see that (7.13) is bounded by $C(r_2 - r_1)^{-1}$. This proves the lemma.

The rest of this part is organized as follows: In Section 8, we first define the Hilbert spaces with respect to the vector fields and also define the weak solutions to $Lu = 0$. Harnack's inequality is proved for nonnegative solutions for the operator L . In Section 9, Mean-value inequalities will be stated for general solutions and subsolutions to $Lu = 0$ and Hölder's continuity will be derived. In Section 10, the Harnack inequality for the operator \mathcal{L} is proved.

8. Harnack's inequality for solutions to $L = \sum_{i,j=1}^m X_i^*(a_{ij} X_j)$.

We begin this section by introducing the space $H(\Omega)$. For $u \in \text{Lip}_1(\bar{\Omega})$, define

$$(8.1) \quad \|u\|^2 = \int_{\Omega} \langle AXu, Xu \rangle + \int_{\Omega} u^2 w$$

where $Xu = (X_1u, \dots, X_mu)$, $w \in A_2(\Omega, \rho)$. Note

$$(8.2) \quad \|u\|^2 \sim \int_{\Omega} |Xu|^2 w + \int_{\Omega} u^2 w$$

where $|Xu| = (\sum_{i=1}^m |X_i u|)^{1/2}$. We also define

$$(8.3) \quad a(u, \phi) = \int_{\Omega} \langle AXu, X\phi \rangle + \int_{\Omega} u \phi w.$$

for $u, \phi \in \text{Lip}_1(\overline{\Omega})$.

Using the fact that

$$|\langle Ax, y \rangle| \leq \langle Ax, x \rangle^{1/2} \langle Ay, y \rangle^{1/2}$$

we can show that $a(u, \phi)$ is an inner product on $\text{Lip}_1(\overline{\Omega})$, so that $\|\cdot\|$ is a norm on $\text{Lip}_1(\overline{\Omega})$. We also note for any $\varepsilon > 0$,

$$(8.4) \quad |\langle Ax, y \rangle| \leq \frac{\varepsilon}{2} \langle Ax, x \rangle + \frac{1}{2\varepsilon} \langle Ay, y \rangle$$

Let $H = H(\Omega)$ be the completion of $\text{Lip}_1(\overline{\Omega})$ under $\|\cdot\|$, *i.e.* H is formed by adjoining to $\text{Lip}_1(\overline{\Omega})$ elements $u = \{u_k\}$, $u_k \in \text{Lip}_1(\overline{\Omega})$, which are Cauchy sequences with respect to $\|\cdot\|$. $\text{Lip}_1(\overline{\Omega})$ is contained in H by considering $\{u_k\}$ with all $u_k = u \in \text{Lip}_1(\overline{\Omega})$. If given $u = \{u_k\}$ and $\phi = \{\phi_k\}$ in H , it is easy to see that $a(u_k, \phi_k)$ is convergent, and we can define

$$a(u, \phi) = \lim_{k \rightarrow \infty} a(u_k, \phi_k), \quad \|u\| = \lim_{k \rightarrow \infty} \|u_k\|.$$

Thus, H becomes a Hilbert space with inner product $a(u, \phi)$ and norm $\|u\| = a(u, u)^{1/2}$.

We note from (8.2) that if $u \in H$, $u = \{u_k\}$, then both $\{Xu_k\}$ and $\{u_k\}$ are Cauchy sequences in $L_w^2(\Omega)$. Consequently, there exist $\tilde{u} \in L_w^2$ and a vector $\vec{\alpha} \in L_w^2$ such that $u_k \rightarrow \tilde{u}$ in L_w^2 and $Xu_k \rightarrow \vec{\alpha}$ in L_w^2 . We also note that any equivalent sequence gives rise to the same \tilde{u} and $\vec{\alpha}$ by (8.2). We shall refer to \tilde{u} as the function in L_w^2 associated with u .

Now define

$$a_0(u, \phi) = \int_{\Omega} \langle AXu, X\phi \rangle \quad \text{for } u, \phi \in \text{Lip}_1(\overline{\Omega}),$$

$$\|u\|_0 = \int_{\Omega} \langle AXu, Xu \rangle \quad \text{on } \text{Lip}_1(\overline{\Omega}) .$$

Then

$$\|u\|_0 \sim \int_{\Omega} |Xu|^2 w .$$

If $u = \{u_k\}$ and $\phi = \{\phi_k\}$, $u, \phi \in H$, let $a_0(u, \phi) = \lim_{k \rightarrow \infty} a_0(u_k, \phi_k)$. Then

$$|a_0(u, \phi)| \leq a_0(u, u)^{1/2} a_0(\phi, \phi)^{1/2}$$

and

$$a(u, \phi) = a_0(u, \phi) + \int_{\Omega} \tilde{u} \tilde{\phi} w .$$

Although $\|\cdot\|_0$ is not a norm on $\text{Lip}_1(\overline{\Omega})$, it is still easy to check $\|\cdot\|_0$ is a norm on $\text{Lip}_0(\Omega)$ since X_1, \dots, X_m satisfy Hörmander condition. ($X_i u = 0$ implies $X_{\alpha} u \equiv 0$ for any commutator X_{α} and thus $\partial u / \partial x_i = 0$ for each i , thus u is constant and thus $u \equiv 0$.)

Let $H_0 = H_0(\Omega)$ be the completion of $\text{Lip}_0(\Omega)$ under $\|\cdot\|_0$. Thus $a_0(u, u)^{1/2}$ is a norm on H_0 , and we still denote $\|u\|_0 = a_0(u, u)^{1/2}$ for $u \in H_0$.

Let $u \in H$, u is said to be $u \geq 0$, if $u_k \geq 0$ for all k for some $\{u_k\}$ representing u . Note that if \tilde{u} is the function in L^2_w associated with u , then $\tilde{u} \geq 0$ a.e. if $u \geq 0$. Now we can give the definition of solution to L .

Definition 8.5. $u \in H(\Omega)$ is said to be a solution of $Lu = \sum_{i=1}^m X_i^* (a_{ij} X_j) = 0$ if for any $\phi \in H_0(\Omega)$, we have $a_0(u, \phi) = 0$. $u \in H(\Omega)$ is said to be a subsolution if

$$a_0(u, \phi) \leq 0, \quad \text{for all } \phi \geq 0, \quad \phi \in H_0(\Omega) .$$

u is said to be a supersolution if

$$a_0(u, \phi) \geq 0, \quad \text{for all } \phi \geq 0, \quad \phi \in H_0(\Omega) .$$

For fixed $\Psi \in H$, $-a_0(\Psi, \cdot)$ is a continuous linear functional on H_0 . Thus by Riesz representation theorem, there is a unique $F \in H_0$ such that $a_0(F, \phi) = -a_0(\Psi, \phi)$ for all $\phi \in H_0$. Thus, $u = F + \Psi$ will solve uniquely the Dirichlet problem $Lu = 0$ with $u = \Psi$ on $\partial\Omega$ in the sense that $u - \Psi \in H_0$ and $a_0(u, \phi) = 0$ for all $\phi \in H_0$.

Lemma 8.6. *Given $B = B_I(x_0, h) \subset \Omega$. Let $u \in H(B)$ be a non-negative subsolution of $Lu = 0$, and \tilde{u} be the function in L_w^2 associated to u . Then there exist positive constants c and d depending only the parameters in (7.1) and (7.3) such that for $1/2 \leq \alpha < 1$ and $p \geq 2$,*

$$(\operatorname{ess\,sup}_{\alpha B} \tilde{u})^p \leq \frac{c}{(1-\alpha)^d} \frac{1}{w(B)} \int_B \tilde{u}^p w.$$

PROOF. For $\beta \geq 1$ and $0 < M < \infty$, define $H_M(t) = t^\beta$ for $t \in [0, M]$ and $H_M(t) = M^\beta + \beta M^{\beta-1}(t-M)$ for $t > M$. Note $H'_M(t)$ exists for all t and bounded for each fixed M since $H'_M(t) = \beta t^{\beta-1}$ for $t \leq M$ and equals to $\beta M^{\beta-1}$ for $t \geq M$. Let $u = \{u_k\}$, $u_k \in \operatorname{Lip}_1(\overline{B})$, $u_k \geq 0$, $\|u_k - u_j\| \rightarrow 0$, and for each fixed M , define

$$\phi_k(x) = \eta^2(x) \int_0^{u_k} H'_M(t)^2 dt,$$

for $\eta \in C_0^\infty(B)$ to be chosen later.

Now $\phi_k \in \operatorname{Lip}_0(B)$ and $\phi_k \geq 0$. Further,

$$(8.7) \quad X\phi_k = \eta^2 H'_M(u_k)^2 Xu_k + 2\eta X\eta \int_0^{u_k} H'_M(t)^2 dt.$$

Computation now shows that,

$$\begin{aligned} \|\phi_k\|_0^2 &= \int \langle AX\phi_k, X\phi_k \rangle \\ &= \int (\eta^2 H'_M(u_k))^2 \langle AXu_k, Xu_k \rangle \\ &\quad + 4 \int \eta^3 H'_M(u_k)^2 \left(\int_0^{u_k} H'_M(t)^2 dt \right) \langle AXu_k, X\eta \rangle \\ &\quad + 4 \int \eta^2 \left(\int_0^{u_k} H'_M(t)^2 dt \right)^2 \langle AX\eta, X\eta \rangle \\ &= I + II + III. \end{aligned}$$

Then we can show as in [CW2] that each of I , II and III is bounded and thus $\|\phi_k\|_0$ is bounded. Then there exists a weakly convergent sequence, $\phi_{k_j} \rightarrow \phi \in H$. Since $u \in H$, and $|a_0(u, \phi)| \leq a_0(u, u)^{1/2} a_0(\phi, \phi)^{1/2} \leq \|u\| \|\phi\|_0$, it follows that $a_0(u, \cdot)$ is a continuous linear functional on H_0 . Thus

$$a_0(u, \phi) = \lim a_0(u, \phi_{k_j}) = \lim a_0(u_{k_j}, \phi_{k_j}).$$

Since u is a subsolution, $a_0(u, \phi) \leq 0$, and to continue further, we distinguish the following two cases $\lim a_0(u_{k_j}, \phi_{k_j}) = 0$ and $\lim a_0(u_{k_j}, \phi_{k_j}) < 0$. We will set $a_0(u_{k_j}, \phi_{k_j}) = \delta_{k_j}$. For simplicity, we drop the subscripts for u_{k_j}, ϕ_{k_j} and δ_{k_j} . From (8.7) and noticing that $H'_M(u)Xu = XH_M(u)$, we obtain

$$\begin{aligned}
 \delta = a(u, \phi) &= \int \langle AXu, Xu \rangle \eta^2 H'_M(u)^2 \\
 &+ \int \langle AXu, X\eta \rangle 2\eta \int_0^u H'_M(t)^2 dt \\
 (8.8) \qquad &= \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \\
 &+ 2 \int \langle AXu, X\eta \rangle \eta \int_0^u H'_M(t)^2 dt.
 \end{aligned}$$

Since $\int_0^u H'_M(t)^2 dt \leq u(H'_M(u))^2$, thus

$$\begin{aligned}
 &\int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \\
 &= -2 \int \langle AXu, X\eta \rangle \eta \int_0^u H'_M(t)^2 dt + \delta \\
 &\leq 2 \int |\langle AXH_M(u), u(H'_M(u))^2 X\eta \rangle| + \delta.
 \end{aligned}$$

By (8.4), the expression above is

$$\leq \varepsilon \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 + \frac{1}{\varepsilon} \int \langle AX\eta, X\eta \rangle u^2 H'_M(u)^2 + \delta.$$

Taking $\varepsilon = 1/2$, we have

$$\int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \leq 4 \int \langle AX\eta, X\eta \rangle u^2 H'_M(u)^2 + 2\delta.$$

Thus

$$(8.9) \qquad \int |XH_M(u)|^2 \eta^2 w \leq c \int |X\eta|^2 u^2 H'_M(u)^2 w + c\delta.$$

In the case $\lim \delta_{k_j} < 0$, we simply drop the term δ above. Proceeding further note,

$$\begin{aligned}
 (8.10) \qquad |X(\eta H_M(u))|^2 w &= |(X\eta)H_M(u) + \eta XH_M(u)|^2 w \\
 &\leq 2(|X\eta|^2 H_M(u)^2 w + \eta^2 |XH_M(u)|^2 w).
 \end{aligned}$$

Hence by (8.9) and (8.10) and using the estimate $H_M(u) \leq uH'_M(u)$, we have

$$\begin{aligned}
 \int |X(\eta H_M(u))|^2 w &\leq c \int |X\eta|^2 u^2 H'_M(u)^2 w \\
 (8.11) \qquad \qquad \qquad &+ c \int |X\eta|^2 H_M(u)^2 w + c|\delta| \\
 &\leq c \int |X\eta|^2 u^2 H'_M(u)^2 w + c|\delta|.
 \end{aligned}$$

Now recall that h is the radius of the ball B , given $1/2 \leq s < t < 1$, taking $0 \leq \eta \leq 1$, $\eta = 1$ on sB , $\eta = 0$ outside tB , and $|X\eta| \leq c/(t-s)h$. Thus by the Sobolev inequality (7.3) and together with (8.11),

$$\begin{aligned}
 &\left(\frac{1}{w(tB)} \int_{tB} |\eta H_M(u)|^q w \right)^{1/q} \\
 &\leq cth \left(\frac{1}{w(tB)} \int_{tB} |X(\eta H_M(u))|^2 w \right)^{1/2} \\
 &\leq cth \frac{1}{w(tB)^{1/2}} \left(\int_{tB} |X\eta|^2 u^2 H'_M(u)^2 w + |\delta| \right)^{1/2} \\
 (8.12) \qquad &\leq cth \frac{1}{w(tB)^{1/2}} \left(\int_{tB} |X\eta|^2 u^2 H'_M(u)^2 w \right)^{1/2} \\
 &\quad + cth \frac{1}{w(tB)^{1/2}} |\delta|^{1/2} \\
 &\leq \frac{cth}{w(tB)^{1/2}} \frac{1}{(t-s)h} \left(\int_{tB} u^2 H'_M(u)^2 w \right)^{1/2} \\
 &\quad + \frac{cth}{w(tB)^{1/2}} |\delta|^{1/2}.
 \end{aligned}$$

So,

$$\begin{aligned}
 (8.13) \qquad &\left(\frac{1}{w(sB)} \int_{sB} |H_M(u)|^q w \right)^{1/q} \\
 &\leq \frac{ct}{t-s} \left(\frac{1}{w(tB)} \int_{tB} u^2 H'_M(u)^2 w \right)^{1/2} + \frac{cth}{w(tB)^{1/2}} |\delta|^{1/2}.
 \end{aligned}$$

Recalling that $u = u_{k_j} \rightarrow \tilde{u} \in L^2_w$ and $\delta = \delta_{k_j} \rightarrow 0$, and using a further subsequence if necessary, we may assume that $u_{k_j} \rightarrow \tilde{u}$ a.e. Now

$$|uH'_M(u) - \tilde{u}H'_M(\tilde{u})|^2 \leq 2(|u - \tilde{u}|^2 |H'_M(u)|^2 + |\tilde{u}|^2 |H'_M(u) - H'_M(\tilde{u})|^2).$$

Thus $u_k, H'_M u_k \rightarrow \tilde{u} H'_M(\tilde{u})$ in L^2_w . Consequently, as $j \rightarrow \infty$ the right side of (8.13) converges to

$$\frac{ct}{t-s} \left(\frac{1}{w(tB)} \int_{tB} \tilde{u}^2 H'_M(\tilde{u})^2 w \right)^{1/2}$$

Since $H_M(u_{k_j}) \rightarrow H_M(\tilde{u})$ a.e., by applying Fatou's lemma on the left side of (8.13), we obtain

$$(8.14) \quad \begin{aligned} & \left(\frac{1}{w(sB)} \int_{sB} |H_M(\tilde{u})|^q w \right)^{1/q} \\ & \leq \frac{ct}{t-s} \left(\frac{1}{w(tB)} \int_{tB} \tilde{u}^2 H'_M(\tilde{u})^2 w \right)^{1/2}. \end{aligned}$$

Now letting $M \rightarrow \infty$, using $\tilde{u}^\beta \chi_{\{\tilde{u} \leq M\}} \leq H_M(\tilde{u})$ for the left hand of (8.14), and $\tilde{u} H'_M(\tilde{u}) \leq \tilde{u} \beta \tilde{u}^{\beta-1} = \beta \tilde{u}^\beta$ for the right hand of (8.14), we obtain

$$\left(\frac{1}{w(sB)} \int_{sB} \tilde{u}^{q\beta} w \right)^{1/q} \leq \frac{ct\beta}{t-s} \left(\frac{1}{w(tB)} \int_{tB} (\tilde{u}^{2\beta}) w \right)^{1/2}.$$

Applying Moser's iteration argument to the inequality above, we get

$$\operatorname{ess\,sup}_{\alpha B} \tilde{u} \leq \frac{c}{(1-\alpha)^d} \left(\frac{1}{w(B)} \int_B \tilde{u} w \right)^{1/p}$$

for $p \geq 2$ (see [CW2]).

We now show the following mean-value inequality for nonnegative solutions when $-\infty < p < \infty$, i.e. Theorem 7.4 stated in the introduction.

Theorem 8.15. *With the same notation and hypothesis as in Lemma 8.6, except that now u is a nonnegative solution and $-\infty < p < \infty$, we have*

$$\operatorname{ess\,sup}_{\alpha B} \tilde{u}^p \leq \frac{c}{(1-\alpha)^d} (1+|p|)^{2\sigma/(\sigma-1)} \frac{1}{w(B)} \int_B \tilde{u}^p w.$$

PROOF. The case $p \geq 2$ follows from Lemma 8.6. So it is enough to consider $-\infty < p < 2$. Let $u = \{u_k\}$, $u_k \in \operatorname{Lip}_1(\bar{B})$, $u_k \geq 0$, $\|u_k - u_j\| \rightarrow 0$. By considering $u_k + \varepsilon_0$, $\varepsilon_0 > 0$, and letting $\varepsilon_0 \rightarrow 0$ at the end, we

may assume that $u_k \geq \varepsilon_0 > 0$. Let η be as in Lemma 8.6, $\eta \geq 0$, $\text{supp } \eta \subset B$, $-\infty < \beta \leq 1$, and define $\phi_k = \eta^2 u_k^\beta$, then as checked in [CW2], we can show that $\|\phi_k\|_0$ is bounded for all k . Now pick $\phi_{k_j} \rightarrow \phi \in H_0$, then $a_0(u_{k_j}, \phi_{k_j}) \rightarrow a_0(u, \phi) = 0$, that is,

$$\int \langle AX u_{k_j}, X \phi_{k_j} \rangle = \delta_{k_j} \rightarrow 0.$$

Note that

$$X \phi_{k_j} = \eta^2 \beta u_{k_j}^{\beta-1} X u_{k_j} + 2 \eta (X \eta) u_{k_j}^\beta$$

and for $\beta \neq -1$,

$$X(u_{k_j}^{(\beta+1)/2}) = \frac{\beta+1}{2} u_{k_j}^{(\beta-1)/2} X u_{k_j}$$

If we drop the subscripts, we have

$$\begin{aligned} \delta &= \int \langle AX u, X \phi \rangle \\ &= \int \langle AX u, \eta^2 \beta u^{\beta-1} X u \rangle + \int \langle AX u, 2 \eta (X \eta) u^\beta \rangle \\ &= \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \frac{2\beta}{\beta+1} \eta^2 \\ &\quad + \int \langle AX(u^{(\beta+1)/2}), X \eta \rangle \left(\frac{2}{\beta+1} \right) 2 \eta u^{(\beta+1)/2}. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{\beta}{\beta+1} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 \\ &= \frac{\beta+1}{4} \delta - \int \langle AX(u^{(\beta+1)/2}), X \eta \rangle u^{(\beta+1)/2} \eta. \end{aligned}$$

Applying (8.4) with $\varepsilon = |\beta/(\beta+1)|$, $-\infty < \beta \leq 1$, $\beta \neq 0, -1$, we have

$$\begin{aligned} &\frac{|\beta|}{|\beta+1|} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 \\ &\leq \frac{|\beta+1|}{4} |\delta| + \frac{\varepsilon}{2} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 \\ &\quad + \frac{1}{2\varepsilon} \int \langle AX \eta, X \eta \rangle u^{\beta+1} \end{aligned}$$

$$\begin{aligned}
 &= \frac{|\beta + 1|}{4} |\delta| + \frac{|\beta|}{2|\beta + 1|} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 \\
 &\quad + \frac{|\beta + 1|}{2|\beta|} \int \langle AX\eta, X\eta \rangle u^{\beta+1}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (8.16) \quad &\frac{|\beta|}{|\beta + 1|} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 \\
 &\leq c \frac{|\beta + 1|}{|\beta|} \int |X\eta|^2 u^{\beta+1} w + \frac{|\beta + 1|}{4} |\delta|.
 \end{aligned}$$

Now note

$$(8.17) \quad |X(\eta u^{(\beta+1)/2})|^2 w \leq 2(|X\eta|^2 u^{\beta+1} + \eta^2 |Xu^{(\beta+1)/2}|^2) w.$$

Thus from (8.16) and (8.17) we get

$$\int |X(\eta u^{(\beta+1)/2})|^2 w \leq c \left[\left(\frac{|\beta + 1|}{|\beta|} \right)^2 + 1 \right] \int |X\eta|^2 u^{\beta+1} w + \frac{|\beta + 1|}{4} |\delta|.$$

Taking $\eta = 1$ on sB , $0 \leq \eta \leq 1$, $\text{supp } \eta \subset tB$, $|X\eta| \leq c/((t - s)h)$, $1/2 \leq s < t < 1$, we have

$$\int_{tB} |X(\eta u^{(\beta+1)/2})|^2 w \leq c \left(\frac{|\beta + 1|}{|\beta|} + 1 \right) \int_{tB} |X\eta|^2 u^{\beta+1} w + \frac{|\beta + 1|}{4} |\delta|.$$

By Sobolev's inequality (7.3),

$$\begin{aligned}
 &\left(\frac{1}{w(tB)} \int_{tB} |\eta u^{(\beta+1)/2}|^q w \right)^{1/q} \\
 &\leq ct h \left(\frac{1}{w(tB)} \int_{tB} |X(\eta u^{(\beta+1)/2})|^2 w \right)^{1/2} \\
 &\leq \frac{ct h}{w(tB)^{1/2}} \left\{ \left[\left(\frac{|\beta + 1|}{|\beta|} \right)^2 + 1 \right] \right. \\
 &\quad \left. \cdot \frac{1}{(t - s)^2 h^2} \int_{tB} u^{\beta+1} w + \frac{|\beta + 1|}{4} \delta \right\}^{1/2}
 \end{aligned}$$

Letting $k_j \rightarrow \infty$,

$$\begin{aligned} & \left(\frac{1}{w(sB)} \int_{sB} |\tilde{u}|^{(\beta+1)/2} q w \right)^{1/q} \\ & \leq \frac{ct}{t-s} \left(\frac{|\beta+1|}{|\beta|} + 1 \right) \left(\frac{1}{w(tB)} \int_{tB} |\tilde{u}|^{\beta+1} w \right)^{1/2}. \end{aligned}$$

Using Moser's iteration argument (see [GT] or [CW2]), the lemma follows.

Lemma 8.18. *Let u be a nonnegative solution such that $u \geq \varepsilon$ for some $\varepsilon > 0$. For $1/2 \leq \alpha < 1$, define $k = k(\alpha, \tilde{u})$ by*

$$\log k = \frac{1}{w(\alpha B)} \int_{\alpha B} (\log \tilde{u}) w$$

then for $\lambda > 0$,

$$w(\{x \in \alpha B : |\log \frac{\tilde{u}}{k}| > \lambda\}) \leq \frac{c}{(1-\alpha)\lambda} w(\alpha B).$$

PROOF. Let $\eta \equiv 1$ on αB_a and $\text{supp } \eta \subset B_a, |X\eta| \leq c/a(1-\alpha)h$, h is the radius of B , $B_a = aB$, and a is the constant given in (7.2). Let $u = \{u_k\}, u_k \geq \varepsilon > 0$, and let $\phi_k = \eta^2/u_k$, then

$$X\phi_k = -\eta^2 u_k^{-2} X u_k + 2\eta u_k^{-1} X \eta.$$

It is easy to check that $\|\phi\|_0 = \int \langle AX\phi_k, X\phi_k \rangle$ is bounded for all k . Thus there is a subsequence $\phi_{k_j} \rightarrow \phi$ weakly in H_0 , i.e.

$$\delta_{k_j} = \int \langle AX u_{k_j}, X\phi_{k_j} \rangle \rightarrow 0.$$

If we drop the subscripts,

$$\begin{aligned} \delta &= \int \langle AX u, X\phi \rangle = - \int \langle AX u, X u \rangle u^{-2} \eta^2 \\ & \quad + 2 \int \langle AX u, X \eta \rangle u^{-1} \eta. \end{aligned}$$

Thus,

$$\int \langle AX u, X u \rangle u^{-2} \eta^2 = 2 \int \langle AX u, X \eta \rangle u^{-1} \eta - \delta.$$

From this we get

$$\begin{aligned} & \int \langle AX(\log u), X(\log u) \rangle \eta^2 \\ & \leq 2 \int \langle AX(\log u), X(\log u) \rangle^{1/2} \langle AX\eta, X\eta \rangle^{1/2} \eta + |\delta| \\ & \leq \varepsilon \int \langle AX(\log u), X(\log u) \rangle \eta^2 + \frac{1}{\varepsilon} \int \langle AX\eta, X\eta \rangle + |\delta|. \end{aligned}$$

Taking $\varepsilon = 1/2$,

$$\int \langle AX(\log u), X(\log u) \rangle \eta^2 \leq 4 \int \langle AX\eta, X\eta \rangle + 2|\delta|.$$

Thus,

$$\int |X(\log u)|^2 w \eta^2 \leq c \int |X\eta|^2 w + c|\delta|.$$

Then,

$$(8.19) \quad \int_{\alpha a B} |X(\log u)|^2 w \leq \frac{c}{a^2(1-\alpha)^2 h^2} \int_{aB} w + c|\delta|.$$

By Poincaré’s inequality (7.2), for q replaced by 2, and together with (8.19),

$$\begin{aligned} & \left(\frac{1}{w(\alpha B)} \int_{\alpha B} |\log u - av_{\alpha B} \log u|^2 w \right)^{1/2} \\ & \leq \frac{c\alpha h}{w(a\alpha B)^{1/2}} \left[\frac{1}{(1-\alpha)ah} \left(\int_{a\alpha B} w \right)^{1/2} + |\delta|^{1/2} \right]. \end{aligned}$$

Note $a > 1$ is a constant, $1 > \alpha > 1/2$, by the doubling property of w ,

$$\int_{\alpha B} |\log u - av_{\alpha B} \log u|^2 w \leq \left(\frac{c}{1-\alpha} + c|\delta| \right) w(a\alpha B)^{1/2}.$$

Recalling $u = u_{k_j}$, $\delta = \delta_{k_j}$, taking limits and noticing that $u_k \geq \varepsilon > 0$ implies $\tilde{u} \geq \varepsilon > 0$.

Moreover, $|\log u_{k_j} - \log \tilde{u}| \leq (|u_{k_j} - \tilde{u}|)/\varepsilon$. Thus $\log u_{k_j} \rightarrow \log \tilde{u}$ in L^2_w since $u_{k_j} \rightarrow \tilde{u}$ in L^2_w . Therefore,

$$a v_{\alpha B}(\log u_{k_j}) = \frac{1}{w(\alpha B)} \int_{\alpha B} (\log u_{k_j}) w \rightarrow \log k$$

and

$$\frac{1}{w(\alpha B)} \int_{\alpha B} |\log \tilde{u} - \log k|^2 w \leq \frac{c}{(1-\alpha)^2} \frac{1}{w(B)^{1/2}}.$$

Thus by Chebishev's inequality, for $\lambda > 0$,

$$\begin{aligned} w\left(\{x \in \alpha B : \left|\log \frac{\tilde{u}}{k}\right| > \lambda\}\right) &\leq \frac{1}{\lambda} \int_{\alpha B} \left|\log \frac{\tilde{u}}{k}\right| w \\ &\leq \frac{1}{\lambda} \left(\int_{\alpha B} \left|\log \frac{\tilde{u}}{k}\right|^2 w\right)^{1/2} w(\alpha B)^{1/2} \leq \frac{c}{\lambda(1-\alpha)} w(\alpha B). \end{aligned}$$

The lemma that follows is a variant of Bombieri's lemma for homogeneous spaces.

Lemma 8.20. *If $f \geq 0$, w a doubling measure, $B = B_I(x, r)$ and there are constants c, d such that*

- (i) $\operatorname{ess\,sup}_{sB} f^p \leq \frac{c}{(t-s)^d} \frac{1}{w(tB)} \int_{tB} f^p w$ for $0 < p < \frac{1}{\mu}$,
- (ii) $w(\{\alpha B : \log f > \lambda\}) \leq \frac{c\mu}{\lambda} w(B)$.

Then there exist constants C, D such that

$$\operatorname{ess\,sup}_B f \leq \exp\left(\frac{C}{(1-\alpha)^d}\right).$$

The proof of the above in the ball $B = B_I$ is exactly as the original Bombieri's lemma (see [Bom]).

Finally we can derive the following Harnack's inequality, i.e. Theorem 7.7, by applying Theorem 8.15, Lemma 8.18 and Lemma 8.20. Since the proof is easy, we omit it here.

Theorem 8.21. (Harnack inequality) *Let $w \in A_2$, and let u be non-negative solution of $Lu = 0$ belonging to $H(2B)$ for a ball $B = B(x, r)$ with $2B \subset \Omega$, $r \leq r_0$ for some $r_0 > 0$. Let \tilde{u} be the function in L^2_w associated with u , then*

$$\operatorname{ess\,sup}_B \tilde{u} \leq c \operatorname{ess\,inf}_B \tilde{u},$$

with c independent of u and B .

9. Mean value inequalities for general solutions and subsolutions, and Hölder’s continuity.

In the last section we proved the mean value inequalities for non-negative subsolutions to L . The mean value inequalities for general solutions and subsolutions can also be obtained by the similar method. Since the proof is arguing as in [CW2] for the case $X_i = \partial/\partial x_i$ and $m = N$, we do not present the proof. Thus in this section, we state the following results first.

Theorem 9.1. (Mean-value inequality) *Let $w \in A_2$ and u be a subsolution of $Lu = 0$ belonging to $H(B)$. Let \tilde{u} be the function in L^2_w associated with u . Then there exist constants c and d depending only the parameters in (7.1) and (7.3) such that for $1/2 \leq \alpha < 1$ and $0 < p < \infty$,*

$$(\text{ess sup}_{\alpha B} \tilde{u}^+)^p \leq C \frac{1}{w(B)} \int_B (\tilde{u}^+)^p w ,$$

where $C \leq c/(1 - \alpha)^d$ if $p \geq 2$ and $\leq c^{\log(3/p)}/(1 - \alpha)^d$ if $0 < p < 2$.

Corollary 9.2. *If in Theorem 9.1 u is a solution, then the conclusion holds with \tilde{u}^+ replaced by $|\tilde{u}|$, i.e. for $0 < p < \infty$, $0 < \alpha < 1$,*

$$\text{ess sup}_{\alpha B} |\tilde{u}|^p \leq C \frac{1}{w(B)} \int_B |\tilde{u}|^p w ,$$

where C as before.

Theorem 9.1, which is stated in Section 7 as Theorem 7.5, can be proved as in [CW2] or [GT].

Corollary 9.2, which is stated in Section 7 as Corollary 7.6, is just an easy consequence.

We now make the following additional assumption

- (*) If u is an element of $H(B)$ whose associated function \tilde{u} satisfies $\tilde{u} \geq 0$ a.e. in B , then $u \geq 0$, i.e. u is the limit in $H(B)$ of a sequence $\{u_k\}$ with $u_k \in \text{Lip}(B)$ and $u_k \geq 0$ in B .

We define the oscillation over $B(x, h)$ of a bounded \tilde{u} by

$$\text{osc}(x, h) = \underset{\tilde{u}}{\text{osc}}(x, h) = \text{ess sup}_{B(x, h)} \tilde{u} - \text{ess inf}_{B(x, h)} \tilde{u} .$$

As a consequence of Corollary 9.2 we will obtain the Hölder continuity of solutions to L , i.e. Theorem 7.8. We will follow the method of Moser [M].

Theorem 9.3. (Hölder continuity of solutions) *Let u be a weak solution to L . Let $B = B(x_0, r)$, $r \leq r_0$ with $B \subset \Omega$. Then for \tilde{u} , and some $\alpha > 0$, we have for any $x, y \in B(x_0, r/2)$, and any $p > 0$,*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c \left(\frac{\varrho(x, y)}{r} \right)^\alpha \left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |\tilde{u}|^p w \right)^{1/p}$$

with $c = c(p)$ independent of u and B .

PROOF. By Corollary 9.2, the function \tilde{u} associated to u is essentially bounded on each ball B with $2B \subset \Omega$. Let $B_h = B(x_0, h)$ and let M and m be respectively the essential supremum and infimum of \tilde{u} over $B_{h/2}$, and let M' and m' be these bounds over B_h . Since $M' - \tilde{u}$ and $\tilde{u} - m'$ are nonnegative a.e. in B_h , it follows from the assumption (*) that $M' - u \geq 0$ and $u - m' \geq 0$. Since $M' - u$ and $u - m'$ are solutions in $H(B)$, Harnack's inequality (8.12) implies that

$$M' - m \leq c (M' - M), \quad M - m' \leq c (m - m').$$

Adding the above two inequalities gives rise to

$$(9.4) \quad \text{osc}(x_0, \frac{h}{2}) \leq \frac{c-1}{c+1} \text{osc}(x, h).$$

Since $c - 1 < c + 1$, by iterating (9), and applying (9.2) and in view of the metric ϱ , the Hölder inequality of \tilde{u} will follow.

10. Harnack's inequality for solutions to $\mathcal{L} = -\sum_{i,j=1}^m X_i(a_{ij}X_j)$.

In this section, we will prove a Harnack inequality for the operator $\mathcal{L} = -\sum_{i,j=1}^m X_i(a_{ij}X_j)$. The difference between \mathcal{L} and L is that there are lower order terms in \mathcal{L} . So it is a little more complicated to treat the operator \mathcal{L} . However, the method used above does apply to \mathcal{L} . Hence we will be as brief as possible and just point out the essential difference. Throughout this section, we make the following hypothesis: Ω is small

enough so that the Poincaré and Sobolev inequalities hold on some ball B_0 containing Ω .

We note that any $u \in \text{Lip}_0(\Omega)$ can be extended to a function in $\text{Lip}_0(B_0)$ by setting $u = 0$ outside Ω . From the Sobolev inequality on the ball B_0 , it follows that $H_0 \subset H$, and it is also possible to associate with each $u \in H_0$ a pair $(\tilde{u}, \vec{\alpha})$ such that if $u = \{u_k\}$, then $u_k \rightarrow \tilde{u}$ in L^2_w (even L^q_w) and $Xu_k \rightarrow \vec{\alpha}$ in L^2_w .

We now introduce several notations and definitions. Throughout this section, we will denote by X_i^* the adjoint of $-X_i$, $X^*u = (X_1^*u, \dots, X_m^*u)$, and $X_i^* = X_i + b_i$, where

$$b_i = \sum_{j=1}^n \frac{\partial c_{ij}}{\partial x_j} \quad \text{if } X_i = \sum_{j=1}^n c_{ij} \frac{\partial}{\partial x_j}.$$

We also let $\vec{b} = (b_1, \dots, b_m)$. Then there is a constant $\Lambda > 0$ such that $\sum_{j=1}^m b_j^2 \leq \Lambda$.

Definition. $u \in H(\Omega)$ is said to be a solution to \mathcal{L} if

$$\int_{\Omega} \langle AXu, X^*\phi \rangle = 0, \quad \text{for every } \phi \in H_0(\Omega);$$

$u \in H(\Omega)$ is said to be a subsolution to \mathcal{L} if

$$\int_{\Omega} \langle AXu, X^*\phi \rangle \leq 0, \quad \text{for every } \phi \geq 0, \phi \in H_0(\Omega);$$

$u \in H(\Omega)$ is said to be a supersolution if $-u$ is a subsolution. We also define

$$a_0^*(u, \phi) = \int_{\Omega} \langle AXu, X^*\phi \rangle.$$

We note that

$$a_0^*(u, \phi) = a_0(u, \phi) + \int_{\Omega} \langle AXu, \vec{b}\phi \rangle.$$

Since $a(u, \phi) \leq a(u, u)^{1/2} a(\phi, \phi)^{1/2}$,

$$\left| \int_{\Omega} \langle AXu, \vec{b}\phi \rangle \right| \leq \int_{\Omega} \langle AXu, Xu \rangle^{1/2} \langle A\vec{b}\phi, \vec{b}\phi \rangle^{1/2}$$

$$\begin{aligned} &\leq \|u\| C(\Lambda) \left(\int_{\Omega} \phi^2 w \right)^{1/2} \\ &\leq C(\Lambda) \|u\|_0 \left(\int_{\Omega} |X\phi|^2 w \right) \\ &\leq C(\Lambda) \|u\| \|\phi\|_0 . \end{aligned}$$

Then we can see that $a_0^*(u, \cdot)$ is a continuous linear functional on H_0 for fixed $u \in H$.

Lemma 10.1. *Given $B = B_I(x_0, h) \subset \Omega$. Let $u \in H(B)$ be non-negative subsolution to $\mathcal{L}u=0$, and \tilde{u} be a function in L_w^2 associated to u . Then there exist positive constants c and d depending only on the parameters in (7.1) and (7.3) such that for $1/2 \leq \alpha < 1$ and $p \geq 2$,*

$$(\text{ess sup}_{\alpha B} \tilde{u})^p \leq \frac{c}{(1-\alpha)^d} \frac{1}{w(B)} \int_B \tilde{u}^p w .$$

PROOF. For $\beta \geq 1$ and $0 < M < \infty$, define $H_M(t)$ as in Section 8. Let $u = u_k$, $u_k \in \text{Lip}_1(\bar{B})$, $u_k \geq 0$, $\|u_k - u_j\| \rightarrow 0$, define also

$$\phi_k = \eta^2 \int_0^{u_k} H'_M(t)^2 dt, \quad \text{for } \eta \in C_0^\infty(B) \text{ to be chosen later,}$$

then $\phi_k \in \text{Lip}_0(B)$ and $\phi_k \geq 0$. As in Section 8, $\|\phi_k\|_0$ is bounded for all k and fixed M . Then there is a subsequence ϕ_{j_k} weakly convergent to ϕ in H_0 . Thus $a_0^*(u, \phi_{j_k}) \rightarrow a_0(u, \phi)$. Let $\delta_j = a_0^*(u_{j_k}, \phi_{j_k})$ and drop the subscripts, then

$$\begin{aligned} \delta = a_0^*(u, \phi) &= \int \langle AXu, Xu \rangle \eta^2 H'_M(u)^2 \\ &\quad + 2 \int \langle AXu, X\eta \rangle \eta \left(\int_0^u H'_M(t)^2 dt \right) \\ &\quad + \int \langle AXu, \vec{b} \rangle \eta^2 \left(\int_0^u H'_M(t)^2 dt \right) . \end{aligned}$$

So,

$$\begin{aligned} 0 &\leq \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \\ &= -2 \int \langle AXu, X\eta \rangle \eta \left(\int_0^u H'_M(t)^2 dt \right) \end{aligned}$$

$$\begin{aligned}
 & - \int \langle AXu, \vec{b} \rangle \eta^2 \left(\int_0^u H'_M(t)^2 dt \right) + \delta \\
 \leq & 2 \int |\langle AXu, X\eta \rangle \eta u H'_M(u)^2 \\
 & + \int |\langle AXu, \vec{b} \rangle| \eta^2 u H'_M(u)^2 + |\delta| \\
 = & 2 \int |\langle AXH_M(u), u H'_M(u) X\eta \rangle| \\
 & + \int |\langle AXH_M(u), u H'_M(u) \vec{b} \rangle| \eta^2 + |\delta| \\
 \leq & \varepsilon \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \\
 & + \frac{1}{\varepsilon} \int \langle AX\eta, X\eta \rangle u^2 H'_M(u)^2 \\
 & + \frac{\varepsilon}{2} \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \\
 & + \frac{1}{2\varepsilon} \int \langle A\vec{b}, \vec{b} \rangle u^2 H'_M(u)^2 \eta^2 + |\delta|.
 \end{aligned}$$

Taking $\varepsilon = 1/2$,

$$\begin{aligned}
 \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \leq & C \int |X\eta|^2 u^2 H'_M(u)^2 w \\
 & + C \int u^2 H'_M(u)^2 \eta^2 w + |\delta|.
 \end{aligned}$$

Note that

$$|X(\eta H_M(u))|^2 \leq 2 (|X\eta|^2 H_M(u)^2 w + \eta^2 |XH_M(u)|^2 w).$$

Thus by noticing that $H_M(u) \leq u H'_M(u)$, we have

$$\int |X(\eta H_M(u))|^2 w \leq C \int |X\eta|^2 u^2 H'_M(u)^2 w + C \int u^2 H'_M(u)^2 \eta^2 w + |\delta|.$$

Taking $\eta \geq 0, \eta \equiv 1$ on sB , $|X\eta| \leq c/(t-s)h$, $\text{supp } \eta \subset tB$, for $1/2 \leq s < t < 1$, then by Sobolev inequality (7.3)

$$\left(\frac{1}{w(tB)} \int_{tB} |\eta H_M(u)|^q w \right)^{1/q}$$

$$\leq C t h \left(\frac{1}{w(tB)} \int_{tB} |X(\eta H_M(u))|^2 w \right)^{1/2}.$$

Therefore

$$\begin{aligned} & \left(\frac{1}{w(sB)} \int_{sB} |H_M(u)|^q \right)^{1/q} \\ & \leq \frac{C t h}{w(sB)^{1/2}} \left(\frac{1}{(t-s)^2 h^2} \int_{tB} u^2 H'_M(u)^2 w \right. \\ & \quad \left. + \int_{tB} u^2 H'_M(u)^2 w + C |\delta| \right)^{1/2} \\ & \leq \frac{C}{t-s} \left(\frac{1}{w(sB)} \int_{tB} u^2 H'_M(u)^2 w \right)^{1/2} \\ & \quad + \left(\frac{C h}{w(tB)} \int_{tB} u^2 H'_M(u)^2 w \right)^{1/2} + \frac{C h}{w(sB)^{1/2}} |\delta|^{1/2}. \end{aligned}$$

Letting $k_j \rightarrow \infty$, and noticing $h \leq r_0, t - s \leq 1/2$, we obtain

$$\left(\frac{1}{w(sB)} \int_{sB} |H_M(\tilde{u})|^q \right)^{1/q} \leq \frac{C}{t-s} \left(\frac{1}{w(tB)} \int_{tB} (\tilde{u} H'_M(\tilde{u}))^2 w \right)^{1/2}.$$

The remaining details of the proof are the same as that in Section 8.

Lemma 10.2. *Let u be a nonnegative solution and $-\infty < p < \infty$, then*

$$(\text{ess sup}_{\alpha B} \tilde{u})^p \leq \frac{c}{(1-\alpha)^d} (1+|p|)^{2\sigma/(\sigma-1)} \frac{1}{w(B)} \int_B \tilde{u}^p w$$

where c and d are two constants only depending on the parameters in (7.1) and (7.3).

PROOF. The case $p \geq 2$ follows by Lemma 10.1. Now consider $-\infty < p < 2$. Let $u = \{u_k\}$, $u_k \in \text{Lip}_1(\overline{B})$, $u_k \geq 0$, $\|u_k - u_j\| \rightarrow 0$. By considering $u_k + \varepsilon_0$, $\varepsilon_0 > 0$ and letting $\varepsilon_0 \rightarrow 0$ at the end, we may assume $u_k \geq \varepsilon_0 > 0$. Define $\phi_k = \eta^2 u_k^\beta$, $-\infty < \beta \leq 1$, $\text{supp } \eta \subset B$, then $\|\phi_k\|_0$ is bounded in k . So we can pick a subsequence $\phi_{k_j} \rightarrow \phi$ in H_0 , and then

$$a_0^*(u_{K_j}, \phi_{k_j}) \rightarrow a_0^*(u, \phi) = 0.$$

Note if we drop the subscripts,

$$X^* \phi_{k_j} = X \phi_{k_j} + \vec{b} \phi_{k_j} = \eta^2 \beta u^{\beta-1} X \eta + 2 \eta X \eta u^\beta + \vec{b} \eta^2 u^\beta$$

and

$$X(u^{(\beta+1)/2}) = \frac{\beta+1}{2} u^{(\beta-1)/2} X u.$$

Then for $\beta \neq -1$, we have

$$\begin{aligned} \delta &= \int \langle AXu, X^* \phi \rangle \\ &= \int \langle AXu, X \phi \rangle + \int \langle AXu, \vec{b} \phi \rangle \\ &= \int \langle AXu, Xu \rangle \eta^2 \beta u^{\beta-1} + 2 \int \langle AXu, X \eta \rangle \eta u^\beta + \int \langle AXu, \vec{b} \rangle \eta^2 u^\beta \\ &= \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \left(\frac{2}{\beta+1} \right)^2 \beta \eta^2 \\ &\quad + 2 \int \langle AX(u^{(\beta+1)/2}), X \eta \rangle \frac{2}{\beta+1} \eta u^{(\beta+1)/2} \\ &\quad + \int \langle AX(u^{(\beta+1)/2}), \vec{b} \rangle \frac{2}{\beta+1} u^{(\beta+1)/2} \eta^2. \end{aligned}$$

Then

$$\begin{aligned} \frac{\beta}{\beta+1} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 &= \frac{\beta+1}{4} \delta \\ &\quad - \int \langle AX(u^{(\beta+1)/2}), X \eta \rangle u^{(\beta+1)/2} \eta \\ &\quad - \frac{1}{2} \int \langle AX(u^{(\beta+1)/2}), \vec{b} \rangle u^{(\beta+1)/2} \eta^2. \end{aligned}$$

Applying (8.4) with $\varepsilon = |\beta/(\beta+1)|$, $-\infty < \beta \leq 1$, $\beta \neq 0, -1$, then

$$\begin{aligned} &\int |\langle AX(u^{(\beta+1)/2}), \vec{b} \rangle| u^{(\beta+1)/2} \eta^2 \\ &\leq \frac{\varepsilon}{2} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 + \frac{1}{2\varepsilon} \int \langle A\vec{b}, \vec{b} \rangle u^{\beta+1} \eta^2 \end{aligned}$$

and

$$\int |\langle AX(u^{(\beta+1)/2}), X \eta \rangle| u^{(\beta+1)/2} \eta \leq \frac{\varepsilon}{2} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2$$

$$+ \frac{1}{2\varepsilon} \int \langle AX\eta, X\eta \rangle u^{\beta+1}.$$

Thus

$$\begin{aligned} & \frac{1}{4} \frac{|\beta|}{|\beta+1|} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2}) \rangle \eta^2 \\ & \leq C \frac{|\beta+1|}{|\beta|} \left(\int |X\eta|^2 u^{\beta+1} w + \int u^{\beta+1} \eta^2 w \right) \end{aligned}$$

by noticing that $|\langle A\vec{b}, \vec{b} \rangle|$ is bounded above. For $1/2 \leq s < t < 1$, we pick $\eta \in C_0^\infty(tB)$ as before, then

$$\begin{aligned} \frac{|\beta|}{|\beta+1|} \int |Xu^{(\beta+1)/2}|^2 \eta^2 w & \leq C \frac{|\beta+1|}{|\beta|} \frac{1}{(t-s)^2 h^2} \int_{tB} u^{\beta+1} w \\ & + C \frac{|\beta+1|}{|\beta|} \int_{tB} u^{\beta+1} w + C \frac{|\beta+1|^2}{|\beta|} |\delta|. \end{aligned}$$

If we note that $h \leq r_0, t-s \leq 1/2$, then

$$\int |X(u^{(\beta+1)/2})|^2 \eta^2 w \leq \frac{C}{(t-s)^2 h^2} \frac{|\beta+1|^2}{|\beta|^2} \int_{tB} u^{\beta+1} w + C_\beta |\delta|.$$

Note that

$$|X(u^{(\beta+1)/2})\eta|^2 w \leq 2(|X\eta|^2 u^{\beta+1} + \eta^2 |X(u^{(\beta+1)/2})|^2) w$$

and by Sobolev's inequality

$$\begin{aligned} & \left(\frac{1}{w(sB)} \int_{sB} |X(u^{(\beta+1)/2})|^2 w \right)^{1/2} \\ & \leq \left(\frac{1}{w(sB)} \int_{tB} |\eta u^{(\beta+1)/2}|^q \right)^{1/q} \\ & \leq t h \left(\frac{1}{w(tB)} \int_{tB} |X(\eta u^{(\beta+1)/2})|^2 w \right)^{1/2} \\ & \leq t h \frac{1}{w(tB)^{1/2}} \left[\left(\frac{|\beta+1|}{|\beta|} \right)^2 \frac{1}{(t-s)^2 h^2} \int_{tB} u^{\beta+1} w + C_\beta |\delta| \right]^{1/2} \\ & \leq \frac{C}{t-s} \frac{1}{w(sB)^{1/2}} \frac{|\beta+1|}{|\beta|} \left(\int_{tB} u^{\beta+1} w \right)^{1/2} + \frac{C_\beta}{w(tB)^{1/2}} |\delta|^{1/2}. \end{aligned}$$

Recall that $u = u_{k_j}$, $\delta = \delta_j$, $\delta_j \rightarrow 0$, let $k_j \rightarrow \infty$ and proceeding as in the proof of Theorem 8.15, we are done.

Lemma 10.3. *Let u be a nonnegative solution to \mathcal{L} such that $u \geq \varepsilon$ for some $\varepsilon > 0$. For $1/2 \leq \alpha < 1$, define $k = k(\alpha, \tilde{u})$ by*

$$\log k = \frac{1}{w(\alpha B)} \int_{\alpha B} (\log \tilde{u}) w.$$

Then for $\lambda > 0$,

$$w(\{x \in \alpha B : |\log \frac{\tilde{u}}{k}| > \lambda\}) \leq \frac{c}{(1-\alpha)\lambda} w(\alpha B).$$

PROOF. Let η, ϕ_k as in Lemma 8.18. Then

$$X^* \phi_k = -\eta^2 u_k^{-1} X u_k + 2 \eta u_k^{-1} X \eta + \vec{b} \phi_k.$$

As in Lemma 8.18, $\|\phi_k\|_0$ is bounded for all k , and there is $\phi \in H_0$ such that $\phi_{k_j} \rightarrow \phi$ in H_0 . Thus

$$\delta_{k_j} = a_0^*(u_{k_j}, \phi_{k_j}) \rightarrow a_0^*(u, \phi) = 0$$

we drop the subscripts again,

$$\begin{aligned} \int \langle AXu, Xu \rangle u^{-2} \eta^2 &= 2 \int \langle AXu, X\eta \rangle u^{-1} \eta \\ &\quad + \int \langle AXu, \vec{b} u^{-1} \eta^2 \rangle - \delta. \end{aligned}$$

Thus

$$\begin{aligned} &\int \langle AX(\log u), X(\log u) \rangle \eta^2 \\ &\leq 2 \int \langle AX(\log u), X(\log u) \rangle^{1/2} \langle AX\eta, X\eta \rangle^{1/2} \eta \\ &\quad + \int \langle AX(\log u), X(\log u) \rangle^{1/2} \langle AX\vec{b}, \vec{b} \rangle^{1/2} \eta^2 + |\delta| \\ &\leq \varepsilon \int \langle AX(\log u), X(\log u) \rangle \eta^2 + \frac{1}{\varepsilon} \int \langle AX\eta, X\eta \rangle \end{aligned}$$

$$+ \frac{\varepsilon}{2} \int \langle AX(\log u), X(\log u) \rangle \eta^2 + \frac{1}{2\varepsilon} \int \langle A\vec{b}, \vec{b} \rangle \eta^2 + |\delta|.$$

Taking $\varepsilon = 1/2$, we obtain

$$\begin{aligned} \frac{1}{4} \int \langle AX(\log u), X(\log u) \rangle \eta^2 &\leq 2 \int \langle AX\eta, X\eta \rangle \\ &\quad + \frac{1}{2} \int \langle A\vec{b}, \vec{b} \rangle \eta^2 + |\delta|. \end{aligned}$$

Then

$$\begin{aligned} \int |X(\log u)|^2 w \eta^2 &\leq C \int |X\eta|^2 w + C \int \eta^2 w + C |\delta| \\ &\leq \frac{C}{a^2(t-s)^2 h^2} \int_B w + C \int_B w + C |\delta|. \end{aligned}$$

This follows that

$$\int_{\alpha B} |X(\log u)|^2 w \leq \frac{C}{(1-\alpha)^2 h^2} w(B) + |\delta|$$

by noticing that $h \leq r_0$ and $t - s \leq 1/2$. By Poincaré’s inequality with q replaced by 2,

$$\begin{aligned} &\left(\frac{1}{w(\alpha B)} \int_{\alpha \alpha B} |\log u - av_{\alpha B} \log u|^2 w \right)^{1/2} \\ &\leq C\alpha h \left(\frac{1}{w(\alpha B)} \int_{\alpha B} |X(\log u)|^2 w \right)^{1/2}. \end{aligned}$$

The remaining details of the proof are the same as that in the proof of Lemma 8.18.

We now derive Harnack’s inequality by applying Bombieri’s lemma.

Theorem 10.4. (Harnack inequality) *Let u be a nonnegative solution to $\mathcal{L}u = 0$ belonging to $H(2B)$ for a ball $B = B_I(x_0, r)$ with $x_0 \in E \subset \subset \Omega$ and $2B \subset \Omega, r \leq r_0$. Let \tilde{u} be the function in L^2_w associated to u . Then there are constants C depending only on the parameters in (7.1), (7.2) and (7.3) such that*

$$\operatorname{ess\,sup}_B \tilde{u} \leq C \operatorname{ess\,inf}_B \tilde{u}.$$

REMARK. It is easy to see the mean value inequalities for the general solutions and subsolutions to \mathcal{L} also hold. Since the proof is also a modification of that in the case of the operator L , we do not present the details.

Acknowledgement. The core part of this paper has been included in the author's thesis at Rutgers University. He would like to thank Professor S. Chanillo, his thesis adviser, for the constant encouragement and many helpful suggestions. He also wishes to thank Professor R. Wheeden for a helpful conversation.

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Recibido: 9 de octubre de 1.991

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