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Exceptional modular form of weight 4 on an exceptional domain contained in $\mathbb{C}^{27}$

Henry H. Kim

Abstract. Resnikoff [12] proved that weights of a non trivial singular modular form should be integral multiples of 1/2, 1, 2, 4 for the Siegel, Hermitian, quaternion and exceptional cases, respectively. The θ-functions in the Siegel, Hermitian and quaternion cases provide examples of singular modular forms (Krieg [10]). Shimura [15] obtained a modular form of half-integral weight by analytically continuing an Eisenstein series. Bump and Baily suggested the possibility of applying an analogue of Shimura's method to obtain singular modular forms, *i.e.* modular forms of weight 4 and 8, on the exceptional domain of 3×3 hermitian matrices over Cayley numbers. The idea is to use Fourier expansion of a non-holomorphic Eisenstein series defined by using the factor of automorphy as in Karel [7]. The Fourier coefficients are the product of confluent hypergeometric functions as in Nagaoka [11] and certain singular series which we calculate by the method of Karel [6]. In this note we describe a modular form of weight 4 which may be viewed as an analogue of a θ zero-value and as an application, we consider its Mellin transform and prove a functional equation of the Eisenstein series which is a Nagaoka's conjecture (Nagaoka [11]).

Introduction.

As it is well-known, the classical θ-functions are modular forms of half-integral weight with respect to a congruence subgroup. They provide examples of modular forms of singular weights (or critical weights). Resnikoff [12], using differential operators, proved an interesting theorem that weights of a non trivial singular modular form should be some integral mutiples of $1/2, 1, 2, 4$ for the Siegel, Hermitian, quaternion and exceptional cases, respectively. Later he proved the converse. Krieg [10] considered θ-functions in the Hermitian and quaternion cases and showed that they are the modular forms in Resnikoff's Theorem. However, in the exceptional case, one does not know how to construct θ-functions. It has been a long-standing problem since Baily [1] initiated the study of automorphic forms on the exceptional domain, to construct a "θ-function" on the exceptional domain, that is, to construct modular forms of weight 4 and 8, if they exist.

Raghavan considered a non-holomorphic Eisenstein series of degree 3 in the Siegel case. He obtained a modular form of weight 4 by analytic continuation and showed that it is a θ-function. Shimura [15] actually obtained a modular form of half-integral weight by analytically continuing an Eisenstein series. Daniel Bump and Walter Baily suggested the possibility of applying an analogue of Shimura's method to obtain singular modular forms, *i.e.* modular forms of weight 4 and 8, on the exceptional domain. The idea is to use the Fourier expansion of a non-holomorphic Eisenstein series. We use a factor of automorphy similar to that defined by Karel [7] to define a non-holomorphic Eisenstein series; we thank him for his ideas which he shared in a private conversation. The Fourier coefficients are the product of confluent hypergeometric functions and certain singular series which are called Siegel series or Whittaker functions. The confluent hypergeometric functions were studied by Nagaoka [11]. The singular series for the full rank case were explicitly calculated by Karel [6] who used it to prove that there is a common denominator for the Fourier coefficients of a holomorphic Eisenstein series defined by Baily [1]. We calculate the singular series for the singular cases by modifying Karel's method. Thus we calculate the Fourier coefficients completely. Using an estimate on confluent hypergeometric functions, the Eisenstein series can be continued as meromorphic functions to the whole complex plane and especially by considering $s \longrightarrow 0$, we obtain an interesting result that we have a holomorphic modular form of weight k unless $k = 2, 6, 10$. Thus we obtain modular forms of weight 4 and 8. Nagaoka [11] made a conjecture on a functional equation of an Eisenstein series similar to our Eisenstein series. The constant term of the Fourier expansion of the Eisenstein series contains an "Epstein zeta function". We consider the Mellin transform of the modular form of weight 4 just like a θ-function (Riemann's trick) to get a functional equation of the "Epstein zeta function" and of the Eisenstein series.

It is noted that there is no modular form of the lowest weight $1/2$, $1, 2$ with respect to the full modular group in the Siegel, Hermitian and quaternion cases, respectively. Non-zero modular forms of these weights exist only with respect to congruence subgroups. But in our exceptional case, the modular form of lowest weight is a modular form with respect to the full modular group.

1. An exceptional modular group and its Eisenstein series.

We recall several definitions of an exceptional group of type E_7 (of index -25) and a tube domain from Baily [1].

(i) *Integral Cayley numbers and exceptional Jordan algebras.* Let $\mathfrak{C}$ be the Cayley algebra and $\mathfrak{o}$ be the integral Cayley numbers. Let $\mathfrak{o}_p = \mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $\mathfrak{C}_p = \mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

We define an exceptional Jordan algebra $\mathfrak{J}$ as the vector space of matrices $X = (x_{ij})$, $x_{ij} \in \mathfrak{C}$ satisfying $x_{ij} = \bar{x}_{ji}$, suppied with the product

$$X \circ Y = \frac{1}{2}(XY + YX),$$

where XY is the ordinary matrix product. Let e_{ij} be the 3×3 matrix with a 1 in the intersection of the i-th row and j-th column and zeros elsewhere, and let $e_i = e_{ii}$, $i = 1, 2, 3$.

If $X \in \mathfrak{J}$, then we shall write

$$X = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix}.$$

Define

$$\mathrm{tr}(X) = a + b + c$$

and an inner product $(\cdot, \cdot)$ on $\mathfrak{J}$ by

$$(X, Y) = \mathrm{tr}(X \circ Y).$$

Moreover we define

$$\det(X) = abc - aN(z) - bN(y) - cN(x) + \mathrm{tr}((xz)\bar{y}),$$

and define a symmetric trilinear form $(\cdot,\cdot,\cdot)$ on $\mathfrak{J}\times\mathfrak{J}\times\mathfrak{J}$ by letting

$$(X,X,X) = \det(X).$$

Then we define a bilinear mapping $(X,Y) \longrightarrow X\times Y$ of $\mathfrak{J}\times\mathfrak{J}$ into $\mathfrak{J}$ by requiring the identity

$$(X\times Y, Z) = 3\,(X,Y,Z)$$

to hold. We have

$$X\times X = \begin{pmatrix} bc - N(z) & y\bar{z} - cx & xz - by \\ z\bar{y} - c\bar{x} & ac - N(y) & \bar{x}y - az \\ \bar{z}\bar{x} - b\bar{y} & \bar{y}x - a\bar{z} & ab - N(x) \end{pmatrix}. \tag{1.1}$$

Since $X\circ(X\times X) = (\det X)\,\varepsilon$ for any $X\in\mathfrak{J}$, where $\varepsilon = D(1,1,1)$, $X\times X$ plays the role of a matrix adjoint for $X\in\mathfrak{J}$.

We define

$$\begin{aligned}
\mathfrak{K}_3 &= \{X\in\mathfrak{J}:\ \det X\neq 0\},\\
\mathfrak{K}_2 &= \{X\in\mathfrak{J}:\ \det X = 0,\ X\times X\neq 0\},\\
\mathfrak{K}_1 &= \{X\in\mathfrak{J}:\ X\times X = 0,\ X\neq 0\},\\
\mathfrak{K}_0 &= \{0\}.
\end{aligned}$$

Then $\mathfrak{J}$ is the disjoint union of these four sets. Finally we denote by $\mathfrak{K}$ the set of squares of elements of $\mathfrak{J}$ and put $\mathfrak{K}_i^+ = \mathfrak{K}_i\cap\mathfrak{K}$, $i=1,2,3$. Then $\mathfrak{K}_i^+$ is a cone and $\mathfrak{K}_3^+$ is open and convex in $\mathfrak{J}$. We let

$$\mathfrak{J}_{\mathfrak{o}} = \{X\in\mathfrak{J}:\ x_{ij}\in\mathfrak{o},\ i,j=1,2,3\};$$

this lattice is self-dual with respect to the inner product $(\cdot,\cdot)$. Let $\mathfrak{J}(j) = \mathfrak{J}^{(j)} = \{X = (x_{ii'})\in\mathfrak{J}:\ x_{ii'} = 0$ unless both $i,i'\leq j\}$, and $\Lambda(j) = \Lambda_j = \mathfrak{J}(j)\cap\mathfrak{J}_{\mathfrak{o}}$. We identify $\mathfrak{J}^{(j)}$ with $j\times j$ hermitian matrices over Cayley numbers.

(ii) *An exceptional group of type E_7.* Define two subgroups of $GL(\mathfrak{J})$ as follows:

$$\begin{aligned}
\mathcal{S} &= \{g:\quad g\in GL(\mathfrak{J}),\ \det(gX)\equiv\nu(g)\det(X),\ \nu(g)\neq 0\},\\
\mathcal{J} &= \{g\in\mathcal{S}:\quad \nu(g)=1\}.
\end{aligned} \tag{1.2}$$

We define $g^* \in \mathcal{S}$ so that $(gX, g^*Y) \equiv (X, Y)$. Let

$$\mathfrak{T} = \{Z = X + iY \in \mathfrak{J}_{\mathbb{C}} : \ X \in \mathfrak{J}_{\mathbb{R}}, \ Y \in \mathfrak{R}_3^+\}.$$

The group of holomorphic automorphisms of the domain $\mathfrak{T}$ has been described by Freudenthal [5] as follows (*cf.* Baily [1]). Let V and V' be two real vector spaces, each isomorphic to $\mathfrak{J}$, and let Ξ and Ξ' be copies of $\mathbb{R}$. Let $\mathbb{W} = V \oplus \Xi \oplus V \oplus \Xi'$. Define a quartic form Q on $\mathbb{W}_{\mathbb{C}}$ by

$$Q(w) = (X \times X, X' \times X') - \xi \det X - \xi' \det X' - \frac{1}{4}((X, X') - \xi\xi')^2,$$

and an alternating form $\{\cdot,\cdot\}$ by

$$\{w_1, w_2\} = (X_1, X_2') - (X_2, X_1') + \xi_1\xi_2' - \xi_2\xi_1',$$

for $w = (X, \xi, X', \xi')$. Then

$$\mathcal{G} = \{g \in GL(\mathbb{W}) : \quad Qg = Q, \ g\{\cdot,\cdot\} = \{\cdot,\cdot\}\}$$

defines a connected algebraic $\mathbb{Q}$-group of type E_7. The group $\mathcal{S}$ is embedded in $\mathcal{G}$ by operating on $\mathbb{W}$

$$(1.3) \qquad k(X, \xi, X', \xi') = (kX, \nu(k)^{-1}\xi, k^*X', \nu(k)\xi'), \quad k \in \mathcal{S}.$$

A copy, $\mathbb{P}^+$, of the additive group $\mathfrak{J}$ is embedded in $\mathcal{G}$ by letting p'_B, for $B \in \mathfrak{J}$, be defined by

$$p'_B(X, \xi, X', \xi') = (X_1, \xi_1, X_1', \xi_1'),$$

where

$$(1.4) \qquad \begin{aligned} X_1 &= X + 2B \times X' + \xi B \times B, \quad X_1' = X' + \xi B, \\ \xi_1' &= \xi' + (B, X) + (B \times B, X') + \xi \det B, \quad \xi_1 = \xi. \end{aligned}$$

Define $\iota \in \mathcal{G}$ by

$$(1.5) \qquad \iota(X, \xi, X', \xi') = (-X', -\xi', X, \xi),$$

and let $p_B = \iota^{-1} p'_{-B} \iota$.

Let $N_0 = \{g \in \mathcal{G} : \ g(0,0,0,\xi') = (0,0,0,\xi''), \quad \xi', \xi'' \in \mathbb{R}\}$. Then N_0 is a maximal $\mathbb{Q}$-parabolic subgroup of $\mathcal{G}$. Define $\iota_{e_i} = p'_{e_i} \cdot p_{-e_i} \cdot p'_{e_i}$, and ι_J be the product of all ι_j for $j \in J$ if J is a subset of $\{1,2,3\}$ and let $(j) = \{1,\ldots,j\}$. Then we have Bruhat decomposition

$$\mathcal{G}_{\mathbb{Q}} = \bigcup_{j=0}^{3} N_{0\mathbb{Q}}\iota_{(j)}N_{0\mathbb{Q}} . \tag{1.6}$$

Let $\mathbb{W}_o$ be the lattice in $\mathbb{W}_{\mathbb{R}}$ given by $\mathbb{W}_o = V_o \oplus \mathbb{Z} \oplus V_o' \oplus \mathbb{Z}$, where V_o and V_o' are identified with the lattice $\mathfrak{J}_o$. Then let

$$\Gamma = \{g \in \mathcal{G} : \quad g\mathbb{W}_o = \mathbb{W}_o\} .$$

Γ is an arithmetic subgroup of $\mathcal{G}_{\mathbb{Q}}$ and Γ shares two most important properties with $Sp_n(\mathbb{Z})$, that is, Γ is a unicuspidal and maximal discrete subgroup of Γ. Let $\Gamma_0 = \Gamma \cap N_{0\mathbb{Q}}$. Then by Baily [1, p. 531], Γ_0 is the semi-direct product of $\mathcal{S}_o$ and $\mathbb{P}_o^+$. (Here $\mathcal{S}_o = \{\pm I\}\mathfrak{J}_o$ by Baily [1, p. 524]). We note that $\mathcal{G}$ has a center $Z_2 = \{\pm I\}$, and $\mathcal{G}/Z_2 \cong \mathrm{Hol}(\mathfrak{T})$.

(iii) *The group action and Eisenstein series.* The exceptional group $\mathcal{G}_{\mathbb{R}}$ of type of E_7 (of index -25) acts on the tube domain $\mathfrak{T} = \{Z = X + iY \in \mathfrak{J}_{\mathbb{C}} : Y \in \mathfrak{K}_3^+\}$. In Baily [1], the explicit form of the action (defined by Harish-Chandra) is given by showing that for each $g \in \mathcal{G}_{\mathbb{R}}$ and $Z \in \mathfrak{T}$ there is a unique solution $Z_1 \in \mathfrak{T}$, $A \in \mathfrak{J}_{\mathbb{C}}$ and $k \in \mathcal{S}$ such that

$$p'_Z \cdot g = p_A k p'_{Z_1} . \tag{1.7}$$

Now we put

$$Z \cdot g = Z_1 , \tag{1}$$

$$\mathfrak{z}(Z,g) = k \in \mathcal{S}, \quad j(Z,g) = \nu(\mathfrak{z}(Z,g)) , \tag{2}$$

where ν is as in (1.2). Then (1) define the action of $\mathcal{G}_{\mathbb{R}}$ on $\mathfrak{T}$ and $j(Z,g)$ is the canonical factor of automorphy as in Karel [7](*cf.* Borel [3]). Then we have the following properties of $j(Z,\gamma)$:

(i) $\quad j(Z,p'_B) = 1 \quad$ for all $B \in \mathfrak{J}_{\mathbb{R}}$,

(ii) $\quad j(Z,\gamma) = 1 \quad$ for $\gamma \in \mathfrak{J}_o$,

(iii) $\quad j(Z,\iota) = \det(-Z)$,

(iv) $\quad j(Z,g_1g_2) = j(Z,g_1)j(Z \cdot g_1, g_2)$.

Moreover, if $J(Z,g)$ is the functional determinant of g at Z, then we have,

$$\text{(v)} \qquad J(Z,g) = j(Z,g)^{-18}.$$

(i)-(v) can be proved by (1.3), (1.4), (1.5) and (1.6) and Tsao [16, Theorem 5.3].

Let f be a holomorphic function on $\mathfrak{T}$ which for some integer $k > 0$ satisfies

$$(1.8) \qquad f(Z \cdot \gamma) = f(Z) j(Z,\gamma)^k, \qquad Z \in \mathfrak{T}, \quad \gamma \in \Gamma.$$

Then f is called a modular form of weight k on $\mathfrak{T}$ with respect to Γ. If we take in particular $\gamma = -I$, then we can see easily that $Z \cdot \gamma = Z$, $j(Z,\gamma) = -1$. Therefore if k is an odd integer, then $f \equiv 0$. So we consider only a modular form of even weight. From (1.8), $f(Z+B) = f(Z)$, $B \in \mathfrak{J}_o$. This implies that f has a Fourier expansion, and since the lattice $\mathfrak{J}_o$ is self-dual, this has the form

$$f(Z) = \sum_{T \in \mathfrak{J}_o} a(T) e^{2\pi i (T,Z)}.$$

Now we define an Eisenstein series as follows

$$E_{k,s}(Z) = \sum_{\gamma \in \Gamma / \Gamma_0} j(Z,\gamma)^{-k} |j(Z,\gamma)|^{-s},$$

where k is even positive integer and $s \in \mathbb{C}$ and $Z = X + iY$, $Y \in \mathfrak{K}_3^+$. Since Γ_0 is a semi-direct product of $\pm\{I\}\mathfrak{J}_o$ and $\mathbb{P}_o^+$, this is well-defined and converges absolutely and uniformly on compact subsets of $\mathfrak{T}$ if $k + \operatorname{Re} s > 18$.

The purpose of this note is to prove the following two theorems.

Theorem A. *$E_{k,s}(Z)$ can be continued as a meromorphic function in s to a whole complex plane and*

1) *$E_{k,s}(Z)$ is finite at $s = 0$ for all k,*
2) *$E_{k,0}(Z)$ is holomorphic in Z unless $k = 2, 6, 10$,*
3) *$E_{k,0}(Z)$ is a modular form of weight k with rational Fourier coefficients unless $k = 2, 6, 10$,*

4) *$E_{4,0}(Z)$ and $E_{8,0}(Z)$ are singular modular forms,*

$$E_{4,0}(Z) = 1 + 240 \sum_{T \in \mathfrak{J}_o^+,\, \mathrm{rank}\, T = 1} \sigma_3(\Delta(T)) e^{2\pi i (T,Z)},$$

where $\sigma_3(t) = \sum_{a|t} a^3$ and $\Delta(T)$ is as in Karel [6, p. 186].

Theorem B. *Let*

$$\Psi(s) = (\det Y)^{s/2} \rho(s)\rho(s-4)\rho(s-8)(s-2)(s-4)E_{0,s}(Z),$$

where $\rho(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\Psi(s)$ can be continued as a meromorphic function in s to a whole complex plane with a simple pole at $s = 0, 1, 5, 8, 10, 13, 17, 18$ and satisfies a functional equation

$$\Psi(18 - s) = \Psi(s).$$

2. Fourier expansion of the Eisenstein series.

By (1.6), we have a relative Bruhat decomposition

$$\mathcal{G}_{\mathbb{Q}} = \bigcup_{j=0}^{3} N_{0\mathbb{Q}} \iota_{(j)} N_{0\mathbb{Q}}.$$

By Tsao [16], every element of $N_{0\mathbb{Q}} \iota_{(j)} N_{0\mathbb{Q}}$ can be represented by

$$\mu p'_X \iota_{(j)} N_{0\mathbb{Q}} / N_{0\mathbb{Q}}, \qquad \mu \in \mathfrak{J}_o, X \in \mathfrak{J}_{\mathbb{Q}}^{(j)}.$$

Now we show that, for $g \in \mu p'_X \iota_{(j)} N_{0\mathbb{Q}} / N_{0\mathbb{Q}}$,

$$j(g, Z) = \pm \det((Z \cdot \mu)_j + X)\kappa(X), \tag{2.1}$$

where $(Z \cdot \mu)_j$ is the left upper corner $j \times j$ matrix of $Z \cdot \mu$ and $\kappa(X)$ is as in Baily [1, p. 522]. For $g \in \mu p'_X \iota_{(j)} N_{0\mathbb{Q}} / N_{0\mathbb{Q}}$, $g = \mu p'_X \iota_{(j)} p$ for $p \in N_{0\mathbb{Q}}$. Then

$$p'_Z \cdot g = (p'_Z \mu p'_X \iota(j))p = (p_A k p'_{Z_1}) k_1 p'_B,$$

where $p = k_1 p'_B$. Therefore we get

$$p'_Z \cdot g = p_A k k_1 p'_{\bar{Z}_1}\,.$$

Now by using the formula for $\iota_{(j)}$, we can see that $\nu(k) = \det((Z\cdot\mu)_j + X)$. Also by Tsao [16, p. 266], we have $\nu(k_1) = \kappa(X)$. Therefore we get (2.1).

Therefore we can decompose the Eisenstein series as follows

$$E_{k,s}(Z) = 1 + E^{(1)}_{k,s}(Z) + E^{(2)}_{k,s}(Z) + E^{(3)}_{k,s}(Z)\,,$$

where

$$E^{(j)}_{k,s}(Z) = \sum_{\mu\iota_{(j)}\in\mathfrak{I}_0\iota_{(j)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}}\ \sum_{X\in\mathfrak{J}^{(j)}_{\mathbb{Q}}/\Lambda_j} \kappa(X)^{-(k+s)} S_j((Z\cdot\mu)_j + X; k+\frac{s}{2}, \frac{s}{2})\,,$$

$$S_j(z;\alpha,\beta) = \sum_{a\in\Lambda_j} \det(z+a)^{-\alpha}\det(\bar{z}+a)^{-\beta}.$$

As we will see in Section 3, $S_j(z;\alpha,\beta)$ has a Fourier expansion

$$\mu(\mathfrak{J}^{(j)}_{\mathbb{R}}/\Lambda_j) S_j(z;\alpha,\beta) = \sum_{h\in\Lambda_j} e^{2\pi i(h,x)}\xi_j(y,h;\alpha,\beta)\,.$$

Therefore we have the Fourier expansion of $E^{(j)}_{k,s}$:

$$\begin{aligned}E^{(j)}_{k,s}(Z) &= \sum_{\mu\iota_{(j)}\in\mathfrak{I}_0\iota_{(j)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}}\ \sum_{T\in\Lambda_j} \frac{1}{\mu_j}\xi_j((\mu^{*-1}Y)_j, T; k+\frac{s}{2},\frac{s}{2}) \\ &\qquad\qquad \cdot S(T,k+s)e^{2\pi i(T,(\mu^{*-1}X)_j)} \\ &= \sum_{T\in\Lambda_j}\ \sum_{\mu\iota_{(j)}\in\mathfrak{I}_0\iota(j)N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} a(T,Y,s)e^{2\pi i(T,(\mu^{*-1}X)_j)},\end{aligned} \tag{2.2}$$

where $Z = X + iY$ and

$$a(T,Y,s) = \frac{1}{\mu_j}\xi_j((\mu^{*-1}Y)_j, T; k+\frac{s}{2},\frac{s}{2})S(T,k+s)\,,$$

$$\mu_j = \mu(\mathfrak{J}^{(j)}_{\mathbb{R}}/\Lambda_j)\,,$$

$$S(T,k+s) = \sum_{X\in\mathfrak{J}^{(j)}_{\mathbb{Q}}/\Lambda_j} \kappa(X)^{-(k+s)}e^{2\pi i(T,X)}\,.$$

3. Confluent hypergeometric function.

Consider the infinite series

$$S_m(z, L_m : \alpha, \beta) = \sum_{a \in L_m} \det(z+a)^{-\alpha} \det(\bar{z}+a)^{-\beta},$$

where $z \in H_m$, $z = x + iy$, $y \in \mathfrak{K}_m^+$, L_m is a lattice in $\mathfrak{J}_{\mathbb{R}}^{(m)}$. This has a Fourier expansion

$$\mu(\mathfrak{J}_{\mathbb{R}}^{(m)}/L_m) S_m(z, L_m : \alpha, \beta) = \sum_{h \in L'_m} e^{2\pi i (h,x)} \xi_m(y, h : \alpha, \beta),$$

where L'_m is the dual lattice of L_m with respect to $(\cdot, \cdot)$,

$$(x, y) = \frac{1}{2} \operatorname{tr}(xy + yx)$$

and $\mu(\mathfrak{J}_{\mathbb{R}}^{(m)}/L_m)$ is the measure of $\mathfrak{J}_{\mathbb{R}}^{(m)}/L_m$ and

$$\xi_m(g, h : \alpha, \beta) = \int_{\mathfrak{J}_{\mathbb{R}}^{(m)}} e^{-2\pi i (h,x)} \det(x + ig)^{-\alpha} \det(x - ig)^{-\beta} \, dx,$$

where $g \in \mathfrak{K}_m^+$. Consider the function

$$\eta_m(g, h : \alpha, \beta) = \int_{Q(h)} e^{-(g,x)} \det(x + h)^{\alpha - \kappa(m)} \det(x - h)^{\beta - \kappa(m)} \, dx,$$

where $Q(h) = \{x \in \mathfrak{J}_{\mathbb{R}}^{(m)} : \ x \pm h > 0\}$, $\kappa(m) = 4m - 3$.

S. Nagaoka [11] defined the following function ω_m and gave a theorem on its analytic continuation and functional equation but did not publish a proof. In this section we prove his assertion and get the additional results which are necessary for the analytic continuation of the Eisenstein series.

We denote by $V(p, q, r)$ the subset of $\mathfrak{J}_{\mathbb{R}}^{(m)}$ consisting of the elements with p positive, q negative, and r zero eigenvalues $(p + q + r = m)$. The precise definition of eigenvalues is as follows. When $m = 3$, the eigenvalues of an element $h \in \mathfrak{J}_{\mathbb{R}}^{(3)}$ are defined as the roots of a cubic equation

$$t^3 - \operatorname{tr}(h)t^2 + \operatorname{tr}(h \times h)t - \det(h) = (t\varepsilon - h, t\varepsilon - h, t\varepsilon - h) = 0,$$

where $x \times y$ denotes the crossed product of $x, y \in \mathfrak{J}_{\mathbb{R}}^{(3)}$ (*cf.* Baily [1, p. 516]). In case $m = 2$, as in Shimura [13], we define the eigenvalues of $h \in \mathfrak{J}_{\mathbb{R}}^{(2)}$ to be the roots of a quadratic equation $t^2 - \mathrm{tr}(h)t + \det(h) = 0$. Moreover, as in Shimura [13], we introduce the notion of the eigenvalues of h *relative to* g for $h \in \mathfrak{J}_{\mathbb{R}}^{(m)}$ and $g \in \mathfrak{K}_m^+$. In case of $m = 3$, we define them to be the roots of an equation

$$t^3 - (g, h)t^2 + (g \times g, h \times h)t - \det(g)\det(h) = 0\,.$$

When $m = 2$, they are defined as the roots of an equation $t^2 - (g, h)t + \det(g)\det(h) = 0$. Now we denote by $\delta_+(hg)$ (respectively, $\tau_+(hg)$) the product (respectively, the sum) of all positive eigenvalues of h relative to g. Moreover, we put

$$\delta_-(hg) = \delta_+((-h)g)\,, \quad \tau_-(hg) = \tau_+((-h)g)$$

and

$$\tau(hg) = \tau_+(hg) + \tau_-(hg)\,.$$

We also denote by $\mu(hg)$ the smallest absolute value of non zero eigenvalues of h relative to g if $h \neq 0$; $\mu(hg) = 1$ if $h = 0$. We define, as in Baily [1, p. 520], $a^* \in \mathcal{S}$ so that $(ax, a^*y) = (x, y)$ for all x and y. Then from the definitions , we can see easily that the above quantities are invariant under the map $(g, h) \mapsto (a^*g, ah)$ for all $a \in \mathcal{S}$.

Now we can show the following facts as in Shimura [13, sections 1 and 2].

1) $$\eta_m(g, 0 : \alpha, \beta) = \Gamma_m(\alpha + \beta - \kappa(m)) \det g^{\kappa(m) - \alpha - \beta}\,,$$

where $$\Gamma_m(s) = \pi^{2m(m-1)} \prod_{n=0}^{m-1} \Gamma(s - 4n) = \int_{\mathfrak{K}_m^+} e^{-\mathrm{tr}\, x} \det x^{s - \kappa(m)}\, dx\,.$$

2) $$\eta_m(g, \varepsilon : \alpha, \beta) = e^{-(g, \varepsilon)} 2^{m(\alpha + \beta - \kappa(m))} \zeta_m(2g : \alpha, \beta)\,,$$

where $$\zeta_m(g : \alpha, \beta) = \int_{\mathfrak{K}_m^+} e^{-(g, u)} \det(u + \varepsilon)^{\alpha - \kappa(m)} \det u^{\beta - \kappa(m)}\, du\,.$$

3)

$$\xi_m(g, h : \alpha, \beta) = |\sigma_m|^{-1}\, i^{m(\beta - \alpha)}\, 2^{m\kappa(m)}\, \pi^{m(\alpha + \beta)} \cdot \Gamma_m(\alpha)^{-1} \Gamma_m(\beta)^{-1} \eta_m(2\pi g, h : \alpha, \beta)\,.$$

4) If $g \in \mathfrak{K}_m^+$, $h \in V(p,q,r)$, $p+q+r=m$, then there exists $a \in \mathcal{S}$ such that a^*g is diagonal and $a\,h = \mathrm{diag}(1_p, -1_q, 0_r)$.

5) $$d(ax) = \nu(a)^{\kappa(m)}dx \qquad \text{for } a \in \mathcal{S}, \quad \nu(a^*) = \nu(a)^{-1},$$

6) $$\eta_m(g, -h : \alpha, \beta) = \eta_m(g, h : \alpha, \beta).$$

7) Let $\eta_m^*(g, h : \alpha, \beta) = \det g^{\alpha+\beta-\kappa(m)}\eta_m(g, h : \alpha, \beta)$. Then for all $a \in \mathcal{S}$,

$$\eta_m^*(g, h : \alpha, \beta) = \eta_m^*(a^*g, ah : \alpha, \beta).$$

Define for $g \in \mathfrak{K}_m^+$, $h \in V(p,q,r)$,

$$\begin{aligned}\omega_m(g, h : \alpha, \beta) = {} & 2^{-p\alpha-q\beta}\,\Gamma_p(\beta - 4(m-p))^{-1} \\ & \cdot \Gamma_q(\alpha - 4(m-q))^{-1}\Gamma_r(\alpha+\beta-\kappa(m))^{-1} \\ & \cdot \delta_+(hg)^{\kappa(m)-\alpha-2q}\delta_-(hg)^{\kappa(m)-\beta-2p} \\ & \cdot \eta_m^*(g, h : \alpha, \beta).\end{aligned} \tag{3.1}$$

Theorem. *The function ω_m can be continued as a holomorphic function in (α, β) to the whole $\mathbb{C}^2$ and satisfies the functional equation*

$$\omega_m(g, h : \alpha, \beta) = \omega_m(g, h : \kappa(m) + 4r - \beta, \kappa(m) + 4r - \alpha). \tag{3.2}$$

Moreover, for every compact set $T \subset \mathbb{C}^2$, there exist two positive constants A,B depending only on T such that

$$|\omega_m(g, h : \alpha, \beta)| \le A\, e^{-\tau(hg)}(1 + \mu(hg)^{-B}) \tag{3.3}$$

for all $(g, h) \in \mathfrak{K}_m^+ \times \mathfrak{J}_{\mathbb{R}}^{(m)}$ and $(\alpha, \beta) \in T$.

PROOF. *Case* 1. $m = 2$.

We note that H_2 is a domain of type IV as in Shimura [13]. Because of (4), (6) and (7), it is enough to consider the cases

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $0 < \lambda_1 \le \lambda_2$.

(i) $h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\eta_2(g, h : \alpha, \beta) = \int_{x \pm h \in \mathfrak{K}_2^+} e^{-(g,x)}\det(x+h)^{\alpha-\kappa(2)}\det(x-h)^{\beta-\kappa(2)}\,dx.$$

By Shimura [13], the equation preceding (4.29) with $2\lambda_1 = a, \quad 2\lambda_2 = b$ and $\kappa(2) = 5$,

$$\begin{aligned}\eta_2(g, h : \alpha, \beta) = \pi^4\, 2^{2\alpha+2\beta-10} e^{-\lambda_1}\, (2\lambda_1)^{-4}\, (2\lambda_2)^{5-\alpha-\beta} \\ \cdot \Gamma(\alpha+\beta-5)\, \zeta_1(2\lambda_1 : \alpha-4, \beta-4)\,.\end{aligned}$$

So we get

$$\omega_2(g, h : \alpha, \beta) = 2^{-5}\, \pi^4 e^{-\lambda_1}\, \omega_1(2\lambda_1 : \alpha-4, \beta-4)\,, \tag{3.4}$$

where $\omega_1(g; \alpha, \beta) = \Gamma(\beta)^{-1} g^{\beta} \zeta(g; \alpha, \beta)$ for $g > 0$. Our assertion follows from (3.4) by Shimura [13, Theorem 3.1].

(ii) $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned}\eta_2(g, h : \alpha, \beta) &= \int_{x \pm h \in \mathfrak{K}_2^+} e^{-(g,x)} \det(x+h)^{\alpha-5} \det(x-h)^{\beta-5}\, dx \\ &= e^{-(\lambda_1+\lambda_2)}\, 2^{2(\alpha+\beta-5)}\, \zeta_2(2g : \alpha, \beta)\,.\end{aligned}$$

By Theorem 3.1 in Shimura [13] with $a = 2\lambda_1$, $b = 2\lambda_2$, we get

$$\begin{aligned}\zeta_2(g : \alpha, \beta) = \int_{\mathbb{R}^8} \zeta_1(\lambda_1 + \lambda_2 W : \alpha, \beta)\, \zeta_1(\lambda_2(1+W) : \alpha-4, \beta-4) \\ \cdot e^{-\lambda_2 W} (1+W)^{\alpha+\beta-9}\, dw\,,\end{aligned}$$

where $W = \|w\|^2$. Therefore we get

$$\begin{aligned}\omega_2(g, h : \alpha, \beta) =& e^{-(\lambda_1+\lambda_2)}\, 2^{-6}\, \pi^{-4}\, \lambda_2^4 \int_{\mathbb{R}^8} e^{-2\lambda_2 W} (1 + \lambda_1^{-1}\lambda_2 W)^{-\beta} \\ & \cdot (1+W)^{\alpha-5} \omega_1(2(\lambda_1 + \lambda_2 W) : \alpha, \beta) \\ & \cdot \omega_1(2\lambda_2(1+W) : \alpha-4, \beta-4)\, dw\,.\end{aligned}$$

Then our assertion follows from this expression as in Shimura [13, Theorem 3.1, Case IV].

(iii) $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\eta_2(g, h : \alpha, \beta) = \int_{x \pm h \in \mathfrak{K}_2^+} e^{-(g,x)} \det(x+h)^{\alpha-5} \det(x-h)^{\beta-5}\, dx\,.$$

By (4.28) in Shimura [13] with $a = 2\lambda_1$, $b = 2\lambda_2$, we get

$$\eta_2(g, h : \alpha, \beta) = 2^{2\alpha+2\beta-10} e^{-(\lambda_1+\lambda_2)} \int_{\mathbb{R}^8} e^{-2(\lambda_1+\lambda_2)W}(1+W)^{\alpha+\beta-5} \cdot \zeta_1(2\lambda_1(1+W) : \alpha, \beta-4) \cdot \zeta_1(2\lambda_2(1+W) : \beta, \alpha-4)\, dw\,.$$

Therefore

$$\omega_2(g, h : \alpha, \beta) = 2^{-2}\,(\lambda_1\lambda_2)^2\, e^{-(\lambda_1+\lambda_2)} \cdot \int_{\mathbb{R}^8} e^{-2(\lambda_1+\lambda_2)W}(1+W)^3 \omega_1(2\lambda_1(1+W) : \alpha, \beta-4) \cdot \omega_1(2\lambda_2(1+W) : \beta, \alpha-4)\, dw.$$

Our assertion follows from this expression as in Shimura [13, Theorem 4.2, Case IV].

Case 2. $m = 3$.

Because of (4), (6) and (7), it is enough to consider the cases

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_3 \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

(i) $h = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$, where $k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let $x = \begin{pmatrix} z & y \\ y^* & c \end{pmatrix}$, where z is a 2×2 Hermitian matrix over the Cayley numbers. Then

$\eta_3(g, h : \alpha, \beta) =$

$$= \int_{Q(h)} e^{-(\lambda,z)-\lambda_3 c} \det \begin{pmatrix} z+k & y \\ y^* & c \end{pmatrix}^{\alpha-9} \det \begin{pmatrix} z-k & y \\ y^* & c \end{pmatrix}^{\beta-9} dzdydc .$$

Put $u = z - c^{-1}yy^*$. If $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, then

$$yy^* = \begin{pmatrix} N(y_1) & y_1\bar{y}_2 \\ y_2\bar{y}_1 & N(y_2) \end{pmatrix} .$$

Therefore

$$\begin{pmatrix} z & y \\ y^* & c \end{pmatrix} \in \mathfrak{K}_3^+ \quad \text{if and only if} \quad z - c^{-1}yy^* \in \mathfrak{K}_2^+, \quad c > 0 ,$$

and

$$\det \begin{pmatrix} z & y \\ y^* & c \end{pmatrix} = c \det(z - c^{-1}yy^*) .$$

Then $Q(h)$ is mapped into $Q' = \{(u,y,c) : \ u \pm k \in \mathfrak{K}_2^+\}$ by $(z,y,c) \longrightarrow (u,y,c)$. So

$$\eta_3(g,h:\alpha,\beta) =$$
$$\int_{Q'} e^{-(\lambda,u)-\lambda_3 c-(\lambda,c^{-1}yy^*)} \det(u+k)^{\alpha-9} \det(u-k)^{\beta-9} c^{\alpha+\beta-18}\, dudydc$$
$$= \frac{\pi^8\,\Gamma(\alpha+\beta-9)\,\lambda_3^{-(\alpha+\beta-9)}}{\det\lambda^4} \cdot \int_{u\in\mathfrak{K}_2^+} e^{-(\lambda,u)} \det(u+k)^{\alpha-9} \det(u-k)^{\beta-9}\, du$$
$$= \frac{\pi^8\Gamma(\alpha+\beta-9)\lambda_3^{-(\alpha+\beta-9)}}{\det\lambda^4}\, \eta_2(\lambda,k:\alpha-4,\beta-4) .$$

Therefore we get

$$\omega_3(g,h:\alpha,\beta) = 2^{-4p-4q}\,\pi^8\,\omega_2(\lambda,k:\alpha-4,\beta-4) \tag{3.5}$$

for $k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$$\omega_3(g,h:\alpha,\beta) = \pi^4\,2^{-4}\,\omega_2(\lambda,k:\alpha-4,\beta-4) \tag{3.6}$$

for $k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Our assertion follows from these relations by Case 1.

(ii) $h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By (2),

$$\eta_3(g, \varepsilon : \alpha, \beta) = e^{-(g,\varepsilon)}\, 2^{3(\alpha+\beta-9)} \zeta_3(2g : \alpha, \beta)\,, \tag{3.7}$$

$$\zeta_3(g : \alpha, \beta) = \int_{\mathfrak{K}_3^+} e^{-(g,x)} \det(x+\varepsilon)^{\alpha-9} \det x^{\beta-9}\, dx. \tag{3.8}$$

As in Shimura [13, p. 283], we make the following substitution

$$x = \begin{pmatrix} z & y \\ y^* & c \end{pmatrix} \longrightarrow \begin{pmatrix} u & v \\ v^* & w \end{pmatrix},$$

$$u = z, \quad y = (u^2+u)^{1/2} v, \quad r\,w\,r = c - (z+1)[v], \quad r = (1+v\,v^*)^{1/2}.$$

NOTE. Here we define

$$v^* v = N(v_1) + N(v_2), \quad v\,v^* = \begin{pmatrix} N(v_1) & v_1 \bar{v}_2 \\ v_2 \bar{v}_1 & N(v_2) \end{pmatrix} \quad \text{if } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

$$Y[\xi] = y_1 N(\xi_1) + y_2 N(\xi_2) + (y_{12}\,\xi_2, \xi_1)$$

if $Y = \begin{pmatrix} y_1 & y_{12} \\ \bar{y}_{12} & y_2 \end{pmatrix}$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

Then we can prove that

$$A[B\xi] = B^* A B\,[\xi],$$

where A, B are 2×2 Hermitian matrices over Cayley numbers. (Here B^*AB is well-defined since $B(AB) = (BA)B$ which follows from the identity $(ax)a = a(xa)$ for a, x Cayley numbers.) And also

$$\mathrm{tr}((xy)z) = \mathrm{tr}(x(yz)) = \mathrm{tr}((yz)x) = \mathrm{tr}(y(zx))\,.$$

Therefore we can show that

$$(z+1)[v] = z^{-1}[y]\,, \quad (z+1)^{-1}[y] = z[v]\,.$$

Here z^{-1} represents Jordan algebra inverse of z and we can show by direct calculation that $\det(z(z+1)) = \det z \det(z+1)$. Hence

$$r\,w\,r = c - z^{-1}[y]\,,$$
$$c + 1 - (z+1)^{-1}[y] = c + 1 - z[v] = r\,w\,r + r^2 = r(1+w)r\,.$$

Therefore

$$\begin{aligned}\det(x+\varepsilon) &= \det(z+1)(c+1-(z+1)^{-1}[y]) \\ &= \det(1+u)(1+w)(1+v^*v)\,, \\ \det x &= \det z(c - z^{-1}[y]) = \det u(1+v^*v)w\,.\end{aligned}$$

By the above substitution, $\mathfrak{K}_3^+$ is mapped bijectively into $\{(u,v,w) : u > 0, w > 0, v \in \mathfrak{C}^2\} = R$. Then

$$\frac{\partial(z,y,c)}{\partial(u,v,w)} = r^2 \det(u^2+u)^4 = (1+v^*v)\det u^4 \det(1+u)^4\,.$$

Therefore

$$\begin{aligned}\zeta_3(g:\alpha,\beta) &= \int_R e^{-(\lambda,u)-\lambda_3(rwr+(u+1)[v])}\det(u+1)^{\alpha-5} \\ &\qquad \cdot \det u^{\beta-5}(1+w)^{\alpha-9}w^{\beta-9}(1+v^*v)^{\alpha+\beta-17}\,dudvdw \\ &= \int_{v\in F}\left(\int_{u>0} e^{-(\lambda+\lambda_3 vv^*,u)}\det(u+1)^{\alpha-5}\det u^{\beta-5}\,du\right) \\ &\qquad \cdot\left(\int_{w>0} e^{-\lambda_3 r^2 w}(1+w)^{\alpha-9}w^{\beta-9}\,dw\right) \\ &\qquad \cdot e^{-\lambda_3(N(v_1)+N(v_2))}(1+v^*v)^{\alpha+\beta-17}\,dv \\ &= \int_{v\in F}\zeta_2(\lambda+\lambda_3 vv^*:\alpha,\beta)\,\zeta_1(\lambda_3(1+v^*v):\alpha-8,\beta-8) \\ &\qquad \cdot e^{-\lambda_3(N(v_1)+N(v_2))}(1+v^*v)^{\alpha+\beta-17}\,dv.\end{aligned}$$

where $F = \mathfrak{C}^2$. Therefore we get

$$\begin{aligned}\omega_3(g,\varepsilon:\alpha,\beta) &= 2^{-9}\pi^{-8} \\ &\quad\cdot\int_F \omega_2(\lambda+\lambda_3 vv^*,\varepsilon:\alpha,\beta)\,\omega_1(2\lambda_3(1+v^*v):\alpha-8,\beta-8) \\ &\qquad \cdot e^{-\lambda_3(1+v^*v)}(\lambda_1\lambda_2)^{\beta}\lambda_3^{8}(1+v^*v)^{\alpha-9}\det(\lambda+\lambda_3 vv^*)^{-\beta}\,dv\,.\end{aligned}$$

This integral expression provides the analytic continuation of ω_3. Now we can assume $\lambda_1,\ \lambda_2 \geq \lambda_3$. Then we have

$$\det(\lambda + \lambda_3 vv^*) \leq \lambda_1\lambda_2(1 + v^*v)\,.$$

By induction, we get

$$|\,\omega_3(g, \varepsilon : \alpha, \beta)| \leq A\, e^{-\tau(hg)}(1 + \mu(hg)^{-B})\,.$$

(Use Lemma 2.8 in Shimura [13, p. 278]).
To prove the functional equation, consider

$$\begin{aligned}
&\Gamma_3(\beta)\zeta_3(g : 9 - \beta, \alpha)\\
&\quad = \int_{\mathfrak{R}_3^+} e^{-(g,x)}\Gamma_3(\beta)\det(x+\varepsilon)^{-\beta}\det x^{\alpha-9}\,dx\\
&\quad = \int_{\mathfrak{R}_3^+} e^{-(g,x)}\int_{\mathfrak{R}_3^+} e^{-(u,x+\varepsilon)}\delta(u)^{\beta-9}\,du\,\det x^{\alpha-9}\,dx\\
&\quad = \int_{\mathfrak{R}_3^+} e^{-(u,\varepsilon)}\delta(u)^{\beta-9}\int_{\mathfrak{R}_3^+} e^{-(g+u,x)}\det x^{\alpha-9}\,dxdu\\
&\quad = \Gamma_3(\alpha)\int_{\mathfrak{R}_3^+} e^{-(ay,\varepsilon)}\det a(y+\varepsilon)^{-\alpha}\det ay^{\beta-9}\nu(a)^9\,dy\\
&\quad = \Gamma_3(\alpha)\int_{\mathfrak{R}_3^+} e^{-(y,a^{*-1}\varepsilon)}\det(y+\varepsilon)^{-\alpha}\det y^{\beta-9}\nu(a)^{\beta-\alpha}\,dy\\
&\quad = \Gamma_3(\alpha)\,\nu(a)^{\beta-\alpha}\zeta_3(a^{*-1}\varepsilon : 9-\alpha, \beta)\,.
\end{aligned}$$

(Here we have taken $a \in \mathbb{S}$ such that $g = a\,\varepsilon$ and let $ay = u$.)
Here $\nu(a) = \det g$, a^{*-1} and g have the same eigenvalues, *i.e.* there exists $a, b \in GL(\mathfrak{J}_{\mathbb{R}}^{(3)})$ such that $b\,g = a^{*-1}\varepsilon$. So we have $\zeta_3(a^{*-1}\varepsilon : 9-\alpha,\beta) = \zeta_3(g : 9-\alpha,\beta)$. Therefore

$$\Gamma_3(\beta)\zeta_3(g : 9-\beta, \alpha) = \Gamma_3(\alpha)\det g^{\beta-\alpha}\zeta_3(g : 9-\alpha, \beta)\,.$$

This proves the functional equation (3.2) from (3.1) and (3.8).

(iii) $h = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$. Then

$$\eta_3(g, h : \alpha, \beta) = \int_{Q(h)} e^{-(g,x)}\det(x+h)^{\alpha-9}\det(x-h)^{\beta-9}\,dx\,.$$

Let $x = \begin{pmatrix} z & y \\ y^* & c \end{pmatrix}$. Then

$$x + h = \begin{pmatrix} z+1 & y \\ y^* & c-1 \end{pmatrix}, \quad x - h = \begin{pmatrix} z-1 & y \\ y^* & c+1 \end{pmatrix}.$$

As in Shimura [13, p. 289], we make the following substitution

$$\begin{gathered}(z, y, c) \longrightarrow (u, v, w), \\ u = z - 1 - 2\,v\,v^*, \qquad w = c - 1 - 2\,v^* v, \\ v = (1 - r\,r^*)^{-1/2} r, \qquad r = (z+1)^{-1/2}(c+1)^{-1/2} y.\end{gathered}$$

Here

$$\begin{aligned}1 - r\,r^* &= 1 - (c+1)^{-1}(z+1)^{-1/2} y y^* (z+1)^{-1/2} \\ &= (z+1)^{-1/2}(z + 1 - (c+1)^{-1} y y^*)(z+1)^{-1/2} \in \mathfrak{K}_2^+ .\end{aligned}$$

(Here we use the fact that $(ax)(ya) = a(xy)a$, for $a, x, y \in \mathfrak{C}$ in order to justify the associativity.) And

$$\begin{aligned}1 + v\,v^* &= (1 - r\,r^*)^{-1}, \\ r &= (1 + v\,v^*)^{-1/2} v, \\ \det(1 + v\,v^*) &= 1 + v^* v, \\ c - 1 - (z+1)^{-1}[y] &= c - 1 - (c+1) r^* r \\ &= (c+1)(1 - r^* r) - 2 \\ &= (c+1)(1 + v^* v)^{-1}\left(1 - \frac{2(1 + v^* v)}{c+1}\right) \\ &= (1 + v^* v)^{-1} w, \\ z - 1 - (c+1)^{-1} y\,y^* &= z - 1 - (z+1)^{1/2} r\,r^* (z+1)^{1/2} \\ &= (z+1)^{1/2}(1 - r\,r^*)(z+1)^{1/2} - 2 \\ &= (z+1)^{1/2}(1 + v\,v^*)^{-1} \\ &\quad \cdot ((z + 1 - 2(1 + vv^*))(z+1)^{-1/2}) \\ &= (z+1)^{1/2}(1 + v\,v^*)^{-1}(u(z+1)^{-1/2}).\end{aligned}$$

(Here we can show by direct calculation that $zz^{-1/2} = z^{1/2}$ for $z \in \mathfrak{K}_2^+$.) Now we can check by direct calculation that

$$\det(zwz - 2) = \det w \, \det(z^2 - 2w^{-1}).$$

(Both are well-defined.) Therefore

$$\det(z-1-(c+1)^{-1}y\,y^*) = \det((z+1)^{1/2}(1-r\,r^*)(z+1)^{1/2}-2)$$
$$= \det(1+v\,v^*)^{-1}\det u\,.$$

The substitution maps $Q(h)$ into $Q' = \{(u,v,w): \ u>0, w>0, v \in \mathfrak{C}^2\}$. Here the inverse map is given by

$$\begin{aligned} z &= u+1+2\,v\,v^*\,,\\ c &= w+1+2\,v^*v\,,\\ y &= (z+1)^{1/2}(c+1)^{1/2}r\,,\\ r &= (1+v\,v^*)^{-1/2}v\,, \end{aligned}$$

and the Jacobian determinant is

$$\begin{aligned} \frac{\partial(u,v,w)}{\partial(z,r,c)} &= \det\left(\frac{\partial v}{\partial r}\right),\\ \frac{\partial(z,r,c)}{\partial(z,y,c)} &= \frac{\partial r}{\partial y}\\ &= \det((z+1)^{-1/2}(c+1)^{-1/2})^8\\ &= (c+1)^{-8}\det(z+1)^{-4}\,. \end{aligned}$$

As in Shimura [13, p. 278], we can show

$$\det\left(\frac{\partial v}{\partial r}\right) = \det(1-r\,r^*)^{-9} = \det(1+v\,v^*)^9\,.$$

Therefore

$$\frac{\partial(z,y,c)}{\partial(u,v,w)} = \det(z+1)^4(c+1)^8\det(1+v\,v^*)^{-9}\,.$$

So we have

$$\begin{aligned} &\eta_3(g,h:\alpha,\beta)\\ &\quad= \int_{Q'} e^{-(\lambda,u+1+2v\,v^*)-\lambda_3(w+1+2v^*v)}\det(u+2(1+v\,v^*))^{\alpha-9}\\ &\qquad\cdot(1+v^*v)^{-(\alpha-9)}w^{\alpha-9}(w+2(1+v^*v))^{\beta-9}\det(1+v\,v^*)^{-(\beta-9)}\\ &\qquad\cdot\det u^{\beta-9}\det(u+2(1+v\,v^*)^4(w+2(1+v^*v))^8 \end{aligned}$$

$$\cdot \det(1+v\,v^*)^{-9}\,dudvdw\,.$$

Let $K = v\,v^*$, $K' = v^*v$. And we note that $1+v^*v = \det(1+v\,v^*)$. Therefore we get

$$\eta_3(g,h:\alpha,\beta) = e^{-\operatorname{tr} g}\int_{Q'} e^{-(\lambda,u)}\det(u+2(1+K))^{\alpha-5}\det u^{\beta-9}$$
$$\cdot (w+2(1+K'))^{\beta-1}w^{\alpha-9}e^{-\lambda_3 w}$$
$$\cdot e^{-2(\lambda,K)-2\lambda_3 K'}(1+K')^{9-\alpha-\beta}\,dudvdw\,.$$

Here

$$\int_{u>0} e^{-(\lambda,u)}\det(u+2(1+K))^{\alpha-5}\det u^{\beta-9}$$
$$= (\det(2(1+K)))^{\alpha+\beta-9}\zeta_2(a^{*-1}\lambda:\alpha,\beta-4)\,.$$

(Take $a\in\mathcal{S}$ such that $2(1+K) = a\,\varepsilon$. Actually, $a : x \longrightarrow (2(1+K))^{1/2}x(2(1+K))^{1/2}$.)

$$\int_{w>0} e^{-\lambda_3 w}(w+2(1+K'))^{\beta-1}w^{\alpha-9}\,dw$$
$$= \int_{w>0} e^{-2\lambda_3(1+K')w}(1+w)^{\beta-1}w^{\alpha-9}\,2^{\alpha+\beta-9}(1+K')^{\alpha+\beta-9}\,dw$$
$$= 2^{\alpha+\beta-9}(1+K')^{\alpha+\beta-9}\zeta_1(2\lambda_3(1+K'):\beta,\alpha-8)\,.$$

Therefore

$$\eta_3(g,h:\alpha,\beta) = e^{-\operatorname{tr} g}\,2^{3(\alpha+\beta-9)}$$
$$\cdot\int_F \zeta_2(a^{*-1}\lambda:\alpha,\beta-4)\,\zeta_1(2\lambda_3(1+K'):\beta,\alpha-8)$$
$$\cdot\, e^{-2(\lambda,K)-2\lambda_3 K'}(1+K')^{\alpha+\beta-9}\,dv\,,$$

where $F = \mathfrak{C}^2$. So

$$\omega_3(g,h:\alpha,\beta) = 2^{-1}e^{-\operatorname{tr} g}\int_F e^{-2(\lambda,K)-2\lambda_3 K'}e^{(\lambda,1+K)}(1+K')^3$$
$$\cdot\,(\det\lambda)^2\,\lambda_3^4\,\omega_2(\frac{1}{2}a^{*-1}\lambda,\varepsilon:\alpha,\beta-4)$$
$$\cdot\,\omega_1(2\lambda_3(1+K'):\beta,\alpha-8)\,dv\,.$$

Now we have

$$|\omega_2(\frac{1}{2}a^{*-1}\lambda, \varepsilon : \alpha, \beta - 4)| \le A(1 + \mu(hg)^{-B})e^{-(\lambda, 1+K)},$$
$$|\omega_1(2\lambda_3(1+K') : \beta, \alpha - 8)| \le A(1 + \mu(hg)^{-B'}).$$

(Use $\mu(\lambda) \le \mu(\frac{1}{2}a^{*-1}\lambda)$.) Therefore

$$\begin{aligned}|\omega_3(g, h : \alpha, \beta)| \le & A(1 + \mu(hg)^{-B})(\det \lambda)^2 \lambda_3^4 e^{-\tau(hg)} \\ & \cdot \int_F e^{-2(\lambda, K) - 2\lambda_3 K'}(1+K')^3 \, dv,\end{aligned}$$

if (α, β) stays in a compact set T in $\mathbb{C}^2$ and $A, B > 0$ are constants depending only on T.

By Holder's inequality,

$$\begin{aligned}&\int_F e^{-2(\lambda, K) - 2\lambda_3 K'}(1+K')^3 \, dv \\ &\le \left(\int_F e^{-4(\lambda, K)} \det(1+K)^3 \, dv \int_F e^{-4\lambda_3 K'}(1+K')^3 \, dv\right)^{1/2} \\ &\le A(\det \lambda)^{-2}\lambda_3^{-4}(1 + \mu(hg)^{-B}).\end{aligned}$$

(Here we use Lemma 2.8 in Shimura [13, p. 278] in the second inequality.) Therefore this proves the analytic continuation as well as the inequality. On the other hand,

$$\begin{aligned}&\omega_3(g, h : 9 - \beta, 9 - \alpha) \\ &= 2^{-1}e^{-\operatorname{tr} g} \int_F e^{-2(\lambda, K) - 2\lambda_3 K'} e^{(\lambda, 1+K)}(1+K')^3 (\det \lambda)^2 \lambda_3^4 \\ &\quad \cdot \omega_2(\frac{1}{2}a^{*-1}\lambda, \varepsilon : 9 - \beta, 5 - \alpha)\omega_1(2\lambda_3(1+K') : 9 - \alpha, 1 - \beta) \, dv \\ &= \omega_3(g, h : \alpha, \beta).\end{aligned}$$

This completes the proof of the Theorem.

From (3) and (3.1), we have

$$\begin{aligned}\xi_m(g, h; \alpha, \beta) = & |\sigma_m|^{-1} i^{m(\beta - \alpha)} 2^{\varphi}\pi^{\psi} \Gamma_r(\alpha + \beta - \kappa(m)) \\ & \cdot \Gamma_{m-q}(\alpha)^{-1}\Gamma_{m-p}(\beta)^{-1} \\ & \cdot \det g^{\kappa(m) - \alpha - \beta} \delta_+(hg)^{\alpha + 2q - \kappa(m)} \\ & \cdot \delta_-(hg)^{\beta + 2p - \kappa(m)} \omega_m(2\pi g, h; \alpha, \beta),\end{aligned} \tag{3.9}$$

if $h \in V(p,q,r)$, where $|\sigma_m| = 2^{m(\kappa(m)-1)}$ and

$$\varphi = (2p-m)\alpha + (2q-m)\beta + (m+r)\kappa(m) + 4pq\,,$$
$$\psi = p\alpha + q\beta + r + 4(r(r-1) - pq)\,.$$

Furthermore, we have

Corollary. *If $h \in V(p,0,r)$,*

$$\omega_m(g,h;\alpha,4r) = \omega_m(g,h;\kappa(m),\beta) = 2^{-p\kappa(m)}\pi^{4pr}e^{-(g,h)}. \tag{3.10}$$

PROOF. The case when $r=0$, *i.e.* $h \in \mathfrak{K}_m^+$, follows from (2) and (3.1) by observing that

$$\zeta_m(g;\kappa(m),\beta) = \int_{\mathfrak{K}_m^+} e^{-(g,u)} \det u^{\beta-\kappa(m)}\,du = \Gamma_m(\beta)\det g^{-\beta}\,.$$

If $h \in V(p,0,r)$ with $r > 0$, the formula follows from (3.4), (3.5) and (3.6) by noting that $\omega_1(g;1,\beta) = 1$.

4. The singular series $S(T,s)$.

$$S(T,s) = \sum_{X \in \mathfrak{Z}_{\mathbb{Q}}^j/\Lambda_j} e^{2\pi i(T,X)}\kappa(X)^{-s}. \tag{4.1}$$

By Baily [1], we can assume $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ where T_1 is a $j \times j$ nonsingular matrix if T has rank equal to j. In this section we calculate $S(T,s)$ for T a singular matrix by Karel's method (*cf.* Karel [6]). For T nonsingular, Karel [6] gave explicit formulas. We modify his method to get the results for the case when T is singular. The idea is to represent $S(T,s)$ in terms of $S(T_1, s-8(3-j))$.

Theorem. (i) $j=1$,

$$S(t,s) = \begin{cases} \dfrac{\zeta(s-1)}{\zeta(s)}\,, & \textit{if } t = 0\,, \\ \dfrac{1}{\zeta(s)} \displaystyle\sum_{a|t,a>0} a^{1-s}\,, & \textit{if } t \neq 0\,. \end{cases} \tag{4.2}$$

(ii) $j = 2$,

$$S(T,s) = \begin{cases} \dfrac{\zeta(s-5)\zeta(s-9)}{\zeta(s)\zeta(s-4)}, & \text{if } T = 0, \\ \dfrac{\zeta(s-5)\zeta(s-8)}{\zeta(s)\zeta(s-4)} S(t, s-8), & \text{if } T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \\ \dfrac{1}{\zeta(s)\zeta(s-4)} \prod_{p \mid \det T} f_T^p(p^{5-s}), & \text{if } \det T \neq 0. \end{cases} \tag{4.3}$$

(iii) $j = 3$,

$$S(T,s) = \frac{\zeta(s-9)\zeta(s-13)\zeta(s-17)}{\zeta(s)\zeta(s-4)\zeta(s-8)}, \qquad \text{if } T = 0. \tag{4.4.a}$$

$$S(T,s) = \frac{\zeta(s-9)\zeta(s-13)\zeta(s-16)}{\zeta(s)\zeta(s-4)\zeta(s-8)} S(t, s-16), \tag{4.4.b}$$

if $T = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$S(T,s) = \frac{\zeta(s-9)\zeta(s-12)}{\zeta(s)\zeta(s-4)} S(T_1, s-8), \tag{4.4.c}$$

if $T = \begin{pmatrix} T_1^{2\times 2} & 0 \\ 0 & 0 \end{pmatrix}$.

$$S(T,s) = \frac{1}{\zeta(s)\zeta(s-4)\zeta(s-8)} \prod_{p \mid \det T} f_T^p(p^{9-s}), \tag{4.4.d}$$

if $\det T \neq 0$, *where* f_T^p *is a polynomial.* (See Karel [6])

By Karel [6, pp. 186-187], we have

$$S(T,s) = \prod_p S_p(T,s),$$

$$S_p(T,s) = \sum_{X \in \mathfrak{Z}(j)_p / \Lambda(j)_p} \varepsilon_p((T,X)) \kappa_p(X)^{-s}, \tag{4.5}$$

$$\begin{aligned} (1-p^{-s})^{-1} S_p(T,s) &= \sum_{m=0}^{\infty} \alpha_m(T) p^{-ms}, \\ \alpha_m(T) &= \sum_X \omega_m^{(T,X)}, \end{aligned} \tag{4.6}$$

where $X \in \Lambda(j)_p/p^m\Lambda(j)_p$ and $\tau_i(X) \equiv 0 \mod\text{-}p^{m(i-1)}$ for $2 \le i \le j$.

When $j = 1$, the results are well-known (See Karel [6]). So we calculate the cases $j = 2$ and $j = 3$.

1) *The case* $j = 2$. By Karel [6], we can assume $T = D(t, t')$, where $t|t'$ in $\mathbb{Z}_p$. Let $q = p^m$, $\omega = \omega_m$. By (4.6),

$$S_p(T, s) = (1 - p^{-s}) \sum_{m=0}^{\infty} \alpha_m(T) p^{-ms}$$
$$\alpha_m(T) = \sum_Z \omega_m^{(T,Z)}$$

where $Z \in \Lambda(2)_p/p^m\Lambda(2)_p$, $Z^* \equiv 0 \mod\text{-}p^m$. By Karel [6, (6.2)],

$$\alpha_m(T) = q^4 \sum_{b=1}^{q} \omega^{bt'} \sum_{h=1}^{q} [h]_m^4 \,, \quad t + hb \equiv 0 \mod\text{-}q\,. \tag{4.7}$$

(i) $T = 0$. Here $t = t' = 0$. So we have, by using the fact that

$$\sum_{a,b=1}^{q} \omega^{hab} = [h]_m \,,$$

$$\begin{aligned}\alpha_m(0) &= q^4 \sum_{b=1}^{q} \sum_{h=1}^{q} [h]_m^4 \,, \quad (hb \equiv 0 \mod\text{-}q) \\ &= q^4 \sum_{h=1}^{q} [h]_m^4 \sum_{a,b=1}^{q} \omega^{hab} = q^4 \sum_{h=1}^{q} [h]_m^5 = q^4 \sum_{k=0}^{m} p^{5k} M_k \,,\end{aligned}$$

where M_k is the number of $h \mod\text{-}p^m$ with $v^m(h) = k$;

$$M_k = \begin{cases} (p-1)p^{m-k-1}\,, & \text{if } k < m\,, \\ p^{m-m} = 1\,, & \text{if } k = m\,. \end{cases}$$

Therefore

$$\alpha_m(0) = p^{4m} \frac{(p^m - p^{m-1})(1 - p^{4m}) + p^{5m}(1 - p^4)}{1 - p^4}\,.$$

So we have

$$(1-p^{-s})^{-1}S_p(0,s) = \frac{1-p^{4-s}}{(1-p^{5-s})(1-p^{9-s})}$$
$$S_p(0,s) = \frac{(1-p^{-s})(1-p^{4-s})}{(1-p^{5-s})(1-p^{9-s})}\,.$$

(ii) $T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$, $t \neq 0$. By (4.7),

$$\alpha_m(T) = q^4 \sum_{b=1}^{q} \sum_{h=1}^{q} [h]_m^4\,, \quad t + hb \equiv 0 \mod\text{-}q\,.$$

By Karel [6, (6.5)], we have

$$S_p(T,s) = (1-p^{-s})(1-p^{4-s})r_p(T,s)$$
$$r_p(T,s) = \sum_{m=0}^{\infty} \alpha'_m(T)p^{m(4-s)}$$
$$\alpha_m(T) = p^{4m}(\alpha'_m(T) - \alpha'_{m-1}(T))$$
$$\alpha'_m(T) = [t]_m^4 \sum_{b=1}^{p^m} \sum_{k=0}^{v^m(t;b)} p^{-3k} = [t]_m^4 \sum_{k=0}^{v^m(t)} p^{-3k} \sum_{\substack{b \mod\text{-}p^m \\ b\equiv 0 \mod\text{-}p^k}} 1\,.$$

Let $v = v^m(t) = \min\{m, v(t)\}$, $\tau = v(t)$. Then we have

$$\alpha'_m(T) = p^{4v} \sum_{k=0}^{v} p^{-3k} p^{m-k} = p^m \sum_{k=0}^{v} p^{4k}.$$

So

$$r_p(T,s) = \sum_{m=0}^{\infty} p^{(4-s)m} p^m \sum_{k=0}^{v} p^{4k} = \sum_{m=0}^{\infty} p^{(5-s)m} \sum_{k=0}^{v} p^{4k}$$
$$= \sum_{k=0}^{\tau} p^{4k} \sum_{m=k}^{\infty} p^{m(5-s)} = \sum_{k=0}^{\tau} p^{4k} \frac{(p^{5-s})^k}{1-p^{5-s}}$$
$$= \frac{1}{1-p^{5-s}} \sum_{k=0}^{\tau} (p^{9-s})^k\,.$$

Therefore

$$S_p(T,s) = \frac{(1-p^{-s})(1-p^{4-s})}{1-p^{5-s}} \sum_{k=0}^{\tau} (p^{9-s})^k.$$

Since $S_p(t,s) = (1-p^{-s})\sum_{k=0}^{\tau} p^{k(1-s)}$, we have

$$S_p(T,s) = \frac{(1-p^{-s})(1-p^{4-s})}{(1-p^{5-s})(1-p^{8-s})} S_p(t, s-8) \quad \text{for} \quad T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}.$$

2) $j = 3$. By Karel [6], we can assume $T = D(t, t', t'')$, where $t|t'$, $t'|t''$ in $\mathbb{Z}_p$. As in Karel [6, p. 191], let $U = D(0,0,1)$ and we use the letters j, k, l, m, n to denote rational integers; a, b, c, h, ξ denote p-adic integers; u, v, w, x, y, z denote elements of $\mathfrak{o}_p$. The letters W, Z will be used for elements of Λ_p of the respective forms

$$W = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ \bar{y} & \bar{z} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} a & x & 0 \\ \bar{x} & b & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We let $X = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix} = cU + W + Z$. The letters H will denote any elements in $p^m\Lambda_p$ with

$$H = \begin{pmatrix} h' & u & 0 \\ \bar{u} & h'' & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the letter K will denote the sum $H + hU$. Primed and subscripted variables will always have the same generic meaning as the corresponding variables without primes and subscripts. Let

$$\tilde{Z} = 2U \times Z = \begin{pmatrix} b & -x & 0 \\ -\bar{x} & a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $X^* = c'U + W' + Z'$, where $c' = ab - N(x) = Q(Z)$, $W' = -2W \circ \tilde{Z}$, $Z' = W^* + c\tilde{Z}$, *i.e.*

$$W' = \begin{pmatrix} 0 & 0 & -by + xz \\ 0 & 0 & \bar{x}y - az \\ * & * & 0 \end{pmatrix}, \quad Z' = \begin{pmatrix} bc - N(z) & y\bar{z} - cx & 0 \\ z\bar{y} - c\bar{x} & ac - N(y) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\det X = c\,Q(Z) + (W^*, Z)$ and by (4.6)

$$\alpha_m(T) = \sum_{X \ \text{mod-}q} \omega^{(T,X)}, \quad (X^* \equiv 0 \ \text{mod-}q,\ \det X \equiv 0 \ \text{mod-}q^2). \tag{4.8}$$

Using the block decomposition $X = c\,U + W + Z$, we get

$$\alpha_m(T) = \sum_{Z(q)} \sum_{c, W(q)} \omega^{(T, c\,U+W+Z)}, \tag{4.9}$$

where Z, c, W satisfy: (i) $Q(Z) \equiv 0 \ \text{mod-}q$, (ii) $W^* + c\tilde{Z} \equiv 0 \ \text{mod-}q$, (iii) $W\tilde{Z} \equiv 0 \ \text{mod-}q$, (iv) $c\,Q(Z) + (W^*, Z) \equiv 0 \ \text{mod-}q^2$. This may be rewritten

$$\alpha_m(T) = \sum_{Z(q)} \omega^{(T,Z)} \beta_m(T; Z), \quad (Q(Z) \equiv 0 \ \text{mod-}q),$$

with

$$\beta_m(T; Z) = \sum_{c, W(q)} \omega^{(T, c\,U+W)},$$

where c, W are summed under the restrictions (ii), (iii), (iv) of (4.9). By Karel [6, p. 192], we can assume $Z = D(a, b, 0)$ in order to calculate $\beta_m(T; Z)$ and $a|b$ in $\mathbb{Z}_p$, *i.e.* $v_p(a) \le v_p(b) \le m$. Then by Karel [6, (8.4), p. 193], we get

$$q^{27} \beta_m(T; Z) = \sum_K \sum_W \omega_{2m}^{(H+hZ, W^*)}, \tag{4.10}$$

where K, W are summed (mod-q^2) with the restrictions $W\tilde{Z} \equiv 0$ mod-q and

$$q\,t'' + (H, \tilde{Z}) + hQ(Z) \equiv 0 \ \text{mod-}q^2.$$

Now we calculate $S_p(T, s)$ for $T = D(t, t', 0)$, $t|t'$ (t, t' might be zero.) in terms of $S_p(T', s)$ for $T' = D(t, t')$. From (4.10), we have

$$q^{27} \beta_m(T; Z) = \sum_K \sum_W \omega_{2m}^{(H+hZ, W^*)}, \tag{4.11}$$

where $K = H + hU$, $H \in p^m \Lambda_p$, K and W are summed (mod-q^2) with the restrictions

$$W\tilde{Z} \equiv 0 \ \text{mod-}q, \quad (H, \tilde{Z}) + hQ(Z) \equiv 0 \ \text{mod-}q^2.$$

Rearranging (4.11),

$$q^{27}\beta_m(T;Z) = \sum_W \sum_K \omega_{2m}^{(H+hZ,W^*)},$$

where W, K are summed (mod-q^2) with

$$W\tilde{Z} \equiv 0 \mod\text{-}q, \quad (H,\tilde{Z}) + hQ(Z) \equiv 0 \mod\text{-}q^2. \tag{4.12}$$

CLAIM: we may add the restriction on W that $N(y) \equiv 0 \mod\text{-}p^{\zeta}$, where $\zeta = v_p(a)$.

PROOF OF CLAIM. Suppose h_0 and $H_0 = \begin{pmatrix} h_0' & u_0 & 0 \\ \bar{u}_0 & h_0'' & 0 \\ 0 & 0 & 0 \end{pmatrix}$ satisfy (4.12), *i.e.*

$$h_0 ab + bh_0' + ah_0'' \equiv 0 \mod\text{-}q, \tag{4.13}$$

Then h_0 and $H = \begin{pmatrix} h_0' & u_0 & 0 \\ \bar{u}_0 & h_0'' + h'' & 0 \\ 0 & 0 & 0 \end{pmatrix}$ also satisfy (4.12) provided $h''a \equiv 0 \mod\text{-}q^2$. Since $(H + hZ, W^*) = -h''N(y) + \sigma$, where σ does not depend on h'', so

$$\sum_K \cdots = \sum_{h,h',u} \cdots \left(\sum_{h''} \omega_{2m}^{-h''N(y)}\right).$$

But by setting $h'' = h_0'' + \alpha$, we see that

$$\begin{aligned} \sum_{h''} \omega_{2m}^{-h''N(y)} &= \Bigg(\sum_{\substack{\alpha \mod\text{-}q^2 \\ \alpha a \equiv 0 \mod\text{-}q^2}} \omega_{2m}^{-\alpha N(y)}\Bigg)\omega_{2m}^{-h_0''N(y)} \\ &= 0, \qquad \text{unless } N(y) \equiv 0 \mod\text{-}a. \end{aligned}$$

In this case, multiplying (4.12) by $N(y)$ gives

$$-(h'' + hb)aN(y) \equiv h'N(y) \mod\text{-}p^{2m+\zeta}.$$

Given $h \in \mathbb{Z}_p$, $h' \equiv 0 \mod\text{-}q$, the number of h'' mod-q^2 satisfying (4.13) and $h'' \equiv 0 \mod\text{-}q$ is p^{ζ} if $h \equiv 0 \mod\text{-}p^{\zeta-k}$, *i.e.* $h \equiv 0 \mod\text{-}p^{m-v_p(b)}$

and is zero otherwise. ($k = v^m(p^{-m}Q(Z)) = \min\{m, v_p(p^{-m}Q(Z))\} = -m + v_p(a) + v_p(b)$. So $k \le \zeta = v_p(a)$.) Since $Z = D(a, b, 0)$,

$$a(H + hZ, W^*) \equiv h'(bN(y) - aN(z)) - ha^2N(z) + a\,\mathrm{tr}(uz\bar{y}) \mod\text{-}p^{2m+\zeta}. \tag{4.14}$$

Note that h'' does not appear on the right side of (4.14). Therefore

$$\begin{aligned}\sum_K \omega_{2m}^{(H+hZ,W^*)} &= \sum_{h,H} \omega_{2m+\zeta}^{a(H+hZ,W^*)} \\ &= \sum_{h \mod\text{-}q^2} \omega_{2m+\zeta}^{-ha^2N(z)} \sum_{h' \mod\text{-}q^2} \omega_{2m+\zeta}^{h'(bN(y)-aN(z))} \\ &\quad \cdot \sum_{u \mod\text{-}q^2} \omega_{2m+\zeta}^{a\,\mathrm{tr}(uz\bar{y})} \sum_{h''} 1,\end{aligned}$$

where $h \equiv 0 \mod\text{-}p^{m-v_p(b)}$, $h' \equiv 0 \mod\text{-}q$ and $u \equiv 0 \mod\text{-}q$. Here $\sum_{h''} 1 = p^\zeta$ by the above argument. So we have

$$q^{27}\beta_m(T; Z) = p^{10m+\zeta+v_p(b)} \sum_{y,z} 1, \tag{4.15}$$

where y and z are summed (mod-q^2) and satisfy $by \equiv az \equiv 0 \mod\text{-}q$, $a^2N(z) \equiv 0 \mod\text{-}p^{m+\zeta+v_p(b)}$, $bN(y) \equiv aN(z) \mod\text{-}p^{m+\zeta}$ and $z\bar{y} \equiv 0 \mod\text{-}q$. Equivalently, y and z satisfy that $q^{-1}by$, $q^{-1}az$ are in $\mathfrak{o}_p$, $aN(z) \equiv 0 \mod\text{-}p^{m+v_p(b)}$, $bN(y) \equiv 0 \mod\text{-}p^{m+\zeta}$, and $z\bar{y} \equiv 0 \mod\text{-}p^m$. Thus, $N(q^{-1}by) \equiv N(q^{-1}az) \equiv 0 \mod\text{-}p^k$. Replacing y by $q^{-1}by$ and z by $q^{-1}az$, (4.15) becomes

$$\beta_m(T; Z) = p^{-16m+k} \sum_y A(y), \tag{4.16}$$

where $y \in \mathfrak{o}_p/p^{2m+k-\zeta}\mathfrak{o}_p$ satisfies $N(y) \equiv 0 \mod\text{-}p^k$ and where $A(y)$ is the number of $z \in \mathfrak{o}_p/p^{m+\zeta}\mathfrak{o}_p$ satisfying $z\bar{y} \equiv N(z) \equiv 0 \mod\text{-}p^k$. (Here $z\bar{y} \equiv 0 \mod\text{-}p^k$ implies $\bar{y}z \equiv 0 \mod\text{-}p^k$.) By Karel [6, p. 181, Lemma 2.4],

$$A(y) = p^{8(m+\zeta-k)}p^{4k}\left(\sum_{\nu=0}^{f} p^{3\nu} - \sum_{\nu=0}^{f-1} p^{3\nu-1}\right),$$

where $f = f(y) = v^k(y; p^{-k}N(y))$. So

$$A(y) = p^{8m+8\zeta-4k}\left(\sum_{\nu=0}^{f} p^{3\nu} - \sum_{\nu=1}^{f} p^{3\nu-4}\right).$$

Then from (4.16),

$$
\begin{aligned}
p^{8m-8\zeta+3k}\beta_m(T;Z) &= \sum_{\substack{y \mod\text{-}p^{2m+k-\zeta} \\ N(y)\equiv 0 \mod\text{-}p^k}} \Big(\sum_{\nu=0}^{f} p^{3\nu} - \sum_{\nu=1}^{f} p^{3\nu-4}\Big) \\
&= \sum_{f=0}^{k} \Big(\sum_{\nu=0}^{f} p^{3\nu} - \sum_{\nu=1}^{f} p^{3\nu-4}\Big)\Big(\sum_{\substack{y \mod\text{-}p^{2m+k-\zeta} \\ f(y)=f}} 1\Big) \qquad (4.17) \\
&= \sum_{f=0}^{k}\sum_{\nu=0}^{f} p^{3\nu}\sigma_f - \sum_{f=0}^{k}\sum_{\nu=1}^{f} p^{3\nu-4}\sigma_f \\
&= \sum_{\nu=0}^{k} p^{3\nu}\sigma'_\nu - \sum_{\nu=1}^{k} p^{3\nu-4}\sigma'_\nu \,,
\end{aligned}
$$

where

$$
\sigma_f = \sum_{\substack{y \mod\text{-}p^{2m+k-\zeta} \\ f(y)=f}} 1 \quad \text{and} \quad \sigma'_\nu = \sum_{f=\nu}^{k} \sigma_f \,.
$$

Here $\sigma'_\nu = \sum_y 1$, where $y \in \mathfrak{o}_p/p^{2m+k-\zeta}\mathfrak{o}_p$ satisfies $v^k(y;p^{-k}N(y)) \geq \nu$, *i.e.* $y \equiv 0 \mod\text{-}p^\nu$ and $N(y) \equiv 0 \mod\text{-}p^{k+\nu}$. If we write $y = p^\nu y_0$, then $N(y) \equiv 0 \mod\text{-}p^{k+\nu}$ is equivalent to $N(y_0) \equiv 0 \mod\text{-}p^{k-\nu}$; hence,

$$
\sigma'_\nu = \sum_{\substack{y \mod\text{-}p^{2m+k-\zeta-\nu} \\ N(y)\equiv 0 \mod\text{-}p^{k-\nu}}} 1 \,.
$$

By Karel [6, p. 182, Corollary of Lemma 2.4], if $\nu < k$, (the case $\nu = k$ is obvious),

$$
\sigma'_\nu = p^{8(2m-\zeta)} p^{4(k-\nu)} \Big(\sum_{i=0}^{k-\nu} p^{3i} - \sum_{i=0}^{k-\nu-1} p^{3i-1}\Big) .
$$

Substituting this into (4.17) yields

$$
p^{-8m-k}\beta_m(T;Z) = \sum_{\nu=0}^{k} p^{3\nu} p^{-4\nu} \Big(\sum_{i=0}^{k-\nu} p^{3i} - \sum_{i=0}^{k-\nu-1} p^{3i-1}\Big)
$$

$$
\begin{aligned}
&-\sum_{\nu=1}^{k} p^{3\nu-4}p^{-4\nu}\Big(\sum_{i=0}^{k-\nu} p^{3i} - \sum_{i=0}^{k-\nu-1} p^{3i-1}\Big) \\
&= \sum_{\nu=0}^{k} p^{-\nu}\sum_{i=0}^{k-\nu} p^{3i} - \sum_{\nu=0}^{k-1} p^{-\nu}\sum_{i=0}^{k-\nu-1} p^{3i-1} \\
&\quad - \sum_{\nu=1}^{k} p^{-\nu-4}\sum_{i=0}^{k-\nu} p^{3i} + \sum_{\nu=1}^{k-1} p^{-\nu-4}\sum_{i=0}^{k-\nu-1} p^{3i-1}\,.
\end{aligned}
$$

Here

$$
\sum_{\nu=0}^{k-1} p^{-\nu}\sum_{i=0}^{k-\nu-1} p^{3i-1} = \sum_{\mu=1}^{k} p^{-\mu+1}\sum_{i=0}^{k-\mu} p^{3i-1} = \sum_{\mu=1}^{k} p^{-\mu}\sum_{i=0}^{k-\mu} p^{3i}\,,
$$

$$
\sum_{\nu=1}^{k-1} p^{-\nu-4}\sum_{i=0}^{k-\nu-1} p^{3i-1} = \sum_{\nu=2}^{k} p^{-\nu-4}\sum_{i=0}^{k-\nu} p^{3i}\,.
$$

Therefore we have

$$
p^{-8m-k}\beta_m(T;Z) = \sum_{i=0}^{k} p^{3i} - \sum_{i=0}^{k-1} p^{3i-5}\,.
$$

Because

$$
v^{m-1}(p^{-1}Z; p^{-(m-1)}Q(p^{-1}Z)) = v^m(Z; p^{-m}Q(Z)) - 1 = k-1\,,
$$

we can write

$$
\beta_m(T;Z) = p^{4m}(\beta'_m(T;Z) - \beta'_{m-1}(T;p^{-1}Z))\,,
$$

where

$$
\beta'_m(T;Z) = \begin{cases} p^{4m+4k}\sum_{i=0}^{k} p^{-3i}\,, & \text{if } Z \in \Lambda_p\,, \\ 0\,, & \text{otherwise}\,. \end{cases}
$$

So

$$
\begin{aligned}
\alpha_m(T) &= \sum_{Z \bmod\text{-}p^m} \omega_m^{(T,Z)}\beta_m(T;Z) \\
&= \sum_{Z \bmod\text{-}p^m} \omega_m^{(T,Z)} p^{4m}\beta'_m(T;Z)
\end{aligned}
$$

$$-\sum_{Z \mod\text{-}p^m} \omega_m^{(T,Z)} p^{4m} \beta'_{m-1}(T;p^{-1}Z).$$

Here $\beta'_m(T;p^{-1}Z)=0$ unless $Z\equiv 0 \mod\text{-}p$. So

$$\sum_{Z \mod\text{-}p^m} \omega_m^{(T,Z)} p^{4m} \beta'_{m-1}(T;p^{-1}Z)$$
$$= \sum_{Z \mod\text{-}p^{m-1}} \omega_{m-1}^{(T,Z)} p^{4m} \beta'_{m-1}(T;Z).$$

Therefore we write

$$\alpha_m(T)=p^{4m}(\alpha'_m(T)-\alpha'_{m-1}(T)),$$

where

$$\alpha'_m(T)=\sum_{Z \mod\text{-}p^m} \omega_m^{(T,Z)}\beta'_m(T;Z).$$

Hence

$$S_p(T,s)=(1-p^{-s})(1-p^{4-s})\sum_{m=0}^{\infty}\alpha'_m(T)p^{(4-s)m} \tag{4.18}$$

$$\alpha'_m(T)=\sum_{Z \mod\text{-}p^m}\omega_m^{(T,Z)}\beta'_m(T;Z)=p^{4m}\sum_{k=0}^{m}c_k\psi_k\,,$$

where $c_k=p^k\sum_{i=0}^k p^{3i}$ and

$$\psi_k=\sum_{\substack{Z \mod\text{-}p^m\\ v^m(Z;p^{-m}Q(Z))=k}}\omega_m^{(T,Z)}.$$

In particular, any such Z satisfies $v^m(Z)\geq k$, so we may replace Z by p^kZ. Since $v^m(p^kZ;p^{2k-m}Q(Z))=k$ is equivalent to

$$v^{m-k}(Z;p^{k-m}Q(Z))=0,$$

$\psi_k=\sum_Z\omega_{m-k}^{(T,Z)}$, where $Z \mod\text{-}p^{m-k}$ satisfies $Q(Z)\equiv 0 \mod\text{-}p^{m-k}$ and either $Z\not\equiv 0 \mod\text{-}p$ or $Q(Z)\not\equiv 0 \mod\text{-}p^{m+1-k}$. Thus, $\psi_k=\psi'_{m-k}-\psi'_{m-1-k}$ $(k<m)$, $\psi_m=\psi'_0$, where

$$\psi_\nu=\sum_{\substack{Z \mod\text{-}p^\nu\\ Q(Z)\equiv 0 \mod\text{-}p^\nu}}\omega_\nu^{(T,Z)}.$$

Hence

$$\alpha'_m(T) = p^{4m}(\alpha''_m(T) - \alpha''_{m-1}(T)), \tag{4.19}$$

where

$$\alpha''_m(T) = \sum_{k=0}^{m} c_k\, \psi'_{m-k}\,.$$

Let $T = \begin{pmatrix} T' & 0 \\ 0 & 0 \end{pmatrix}$, $T' = \begin{pmatrix} t' & 0 \\ 0 & t'' \end{pmatrix}$. Then $\psi'_\nu(T) = \alpha_\nu(T')$, ($2 \times 2$ case). By Karel [6, p. 189, (6.4) (2×2 case)],

$$\alpha_\nu(T') = p^{4\nu}(\alpha'_\nu(T') - \alpha'_{\nu-1}(T'))\,.$$

So

$$\alpha''_m(T) = \sum_{k=0}^{m} c_k\, p^{4(m-k)}(\alpha'_{m-k}(T') - \alpha'_{m-k-1}(T'))\,.$$

Let

$$c_k = p^{4k}\, b_k\,, \quad b_k = \sum_{i=0}^{k} p^{-3i} \quad \text{and} \quad b_{-1} = 0\,.$$

Then

$$\begin{aligned} \alpha''_m(T) &= \sum_{k=0}^{m} p^{4m}\, b_k\, \alpha'_{m-k}(T') - \sum_{k=0}^{m-1} p^{4m}\, b_k\, \alpha'_{m-k-1}(T') \\ &= p^{4m} \sum_{j=0}^{m} \alpha'_{m-j}(T')(b_j - b_{j-1}) \\ &= p^{4m} \sum_{j=0}^{m} \alpha'_{m-j}(T')\, p^{-3j}\,. \end{aligned}$$

Therefore by (4.18), (4.19) and the fact that

$$\sum_{\mu=0}^{\infty} p^{(12-s)\mu}\, \alpha'_\mu(T') = r_p(T', s-8)$$

by Karel [6, p. 189, (6.5)],

$$(1-p^{-s})^{-1}(1-p^{4-s})^{-1} S_p(T,s) = \sum_{m=0}^{\infty} \alpha'_m(T)\, p^{(4-s)m}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} p^{(8-s)m}(\alpha''_m(T) - \alpha''_{m-1}(T)) \\
&= (1-p^{8-s}) \sum_{m=0}^{\infty} p^{(8-s)m} \alpha''_m(T) \\
&= (1-p^{8-s}) \sum_{m=0}^{\infty} p^{(8-s)m} p^{4m} \sum_{j=0}^{m} p^{-3j} \alpha'_{m-j}(T') \\
&= (1-p^{8-s}) \sum_{j=0}^{\infty} p^{-3j} \sum_{m=j}^{\infty} p^{(12-s)m} \alpha'_{m-j}(T')
\end{aligned}
$$

(set $m-j=\mu$)

$$
\begin{aligned}
&= (1-p^{8-s}) \sum_{j=0}^{\infty} p^{-3j} p^{(12-s)j} \sum_{\mu=0}^{\infty} p^{(12-s)\mu} \alpha'_{\mu}(T') \\
&= (1-p^{8-s})\, r_p(T', s-8) \sum_{j=0}^{\infty} (p^{9-s})^j \\
&= \frac{1-p^{8-s}}{1-p^{9-s}} \frac{S_p(T', s-8)}{(1-p^{8-s})(1-p^{12-s})} .
\end{aligned}
$$

Therefore

$$
S_p(T,s) = \frac{(1-p^{-s})(1-p^{4-s})}{(1-p^{9-s})(1-p^{12-s})}\, S_p(T', s-8) .
$$

(i) $T = 0$.

$$
\begin{aligned}
S_p(0,s) &= \frac{(1-p^{-s})(1-p^{4-s})}{(1-p^{9-s})(1-p^{12-s})}\, S_p(0^{2\times 2}, s-8) \\
&= \frac{(1-p^{-s})(1-p^{4-s})(1-p^{8-s})}{(1-p^{9-s})(1-p^{13-s})(1-p^{17-s})} .
\end{aligned}
$$

(ii) $T = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t \neq 0 .$

$$
\begin{aligned}
S_p(T,s) &= \frac{(1-p^{-s})(1-p^{4-s})}{(1-p^{9-s})(1-p^{12-s})}\, S_p(T', s-8) \\
&= \frac{(1-p^{-s})(1-p^{4-s})(1-p^{8-s})}{(1-p^{9-s})(1-p^{13-s})(1-p^{16-s})}\, S_p(t, s-16) .
\end{aligned}
$$

(iii) $T = \begin{pmatrix} t & 0 & 0 \\ 0 & t' & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$
\begin{aligned}
S_p(T,s) &= \frac{(1-p^{-s})(1-p^{4-s})}{(1-p^{9-s})(1-p^{12-s})}\, S_p(T', s-8) \\
&= \frac{(1-p^{-s})(1-p^{4-s})(1-p^{8-s})}{(1-p^{9-s})}\, f_T^p(p^{13-s}),
\end{aligned}
$$

where $f_T^p(p^{13-s})$ is a polynomial. This completes the proof of the theorem.

5. Proof of Theorem A.

Because of the inequality (3.3), the Fourier expansion (2.2) converges uniformly on compact subsets of $\mathbb{C}$. Since each term in the series can be continued as a meromorphic function in s, it follows that $E_{k,s}(Z)$ can be continued as a meromorphic function in s to the whole s-plane. And in particular we can take the limit $s \longrightarrow 0$ term by term.

1) $E_{k,s}^{(1)}(Z)$. By (2.2), (3.9) and (4.2) and the fact that

$$\mu(\mathfrak{J}_{\mathbb{R}}^{(1)}/\Lambda_1) = \mu(\mathbb{R}/\mathbb{Z}) = 1,$$

we have

$$
\begin{aligned}
a(t,y,s) &= i^{-k}\, 2^{k+1}\, \pi^{k+s/2}\, \Gamma(k+\frac{s}{2})^{-1}\, t^{k+s/2-1}\, y^{-s/2} \\
&\quad \cdot \omega_1(2\pi y, t; k+\frac{s}{2}, \frac{s}{2}) \frac{1}{\zeta(k+s)} \Big(\sum_{a|t} a^{1-k-s}\Big),
\end{aligned}
\tag{5.1.a}
$$

if $t > 0$,

$$
\begin{aligned}
a(t,y,s) &= i^{-k}\, 2^{1-k}\, \pi^{s/2}\, \Gamma(\frac{s}{2})^{-1}\, y^{-k-s/2}\, |t|^{s/2-1} \\
&\quad \cdot \omega_1(2\pi y, t; k+\frac{s}{2}, \frac{s}{2}) \frac{1}{\zeta(k+s)} \Big(\sum_{a|\,|t|} a^{1-k-s}\Big),
\end{aligned}
\tag{5.1.b}
$$

if $t < 0$, and

$$
\begin{aligned}
a(t,y,s) &= i^{-k}\, 2^{2-k-s}\, \pi\, y^{-1}\, \Gamma(k+s-1)\, \Gamma(k+\frac{s}{2})^{-1} \\
&\quad \cdot \Gamma(\frac{s}{2})^{-1} \frac{\zeta(k+s-1)}{\zeta(k+s)},
\end{aligned}
\tag{5.1.c}
$$

if $t=0$. Here we use the fact that $\Gamma(s)$ function has simple poles only at $s=0,-1,-2,\ldots$ and the residue at $s=-k$ is $1/((-1)^k k!)$ and we have

$$\zeta(2k)=(2\pi)^{2k}B_{2k}/(2(2k)!)\,,\qquad \zeta(-2k)=0\,,$$
$$\zeta(-(2k-1))=(-1)^k B_{2k}/(2k)\,,\qquad \zeta(0)=-1/2\,,$$

where B_k are Bernoulli numbers. Also from (3.10), we have

$$\omega_1(2\pi y,t;k,0)=2^{-1}e^{-2\pi yt}\qquad \text{if } t>0\,.$$

Therefore, letting $s\longrightarrow 0$, we have

$$a(t,y,s)=\begin{cases}\dfrac{i^{-k}\,2k}{B_k}t^{k-1}e^{-2\pi yt}\displaystyle\sum_{a|t}a^{1-k}\,, & \text{if } t>0\,,\\ 0\,, & \text{if } t<0\,,\\ 0\,, & \text{if } t=0 \text{ and } k>2\,,\\ -\dfrac{1}{\pi B_2 y}\,, & \text{if } t=0 \text{ and } k=2\,.\end{cases}$$

2) $E^{(2)}_{k,s}(Z)$. Because of Γ-factors in ξ_2, we can easily see that if $T\in V(p,q,r)$, $q>0$, then $a(T,Y,s)\longrightarrow 0$ as $s\longrightarrow 0$ for all k. Therefore it suffices to consider the cases $q=0$. By (2.2), (3.9) and (4.3) and the fact that $\mu(\mathfrak{J}^{(2)}_{\mathbb{R}}/\Lambda_2)=\mu(\mathfrak{C}/\mathfrak{o})=2^{-4}$, we have the following three cases:

$$\begin{aligned}a(T,Y,s)=&\,i^{-2k}\,2^{2+2k}\,\pi^{2k+s-4}\,\Gamma(k+\frac{s}{2})^{-1}\,\Gamma(k+\frac{s}{2}-4)^{-1}\\ &\cdot\det T^{k+s/2-5}\det Y^{-s/2}\,\omega_2(2\pi Y,T;k+\frac{s}{2},\frac{s}{2})\\ &\cdot\frac{1}{\zeta(k+s)\zeta(k+s-4)}\prod_{p|\det T}f^p_T(p^{5-k-s})\,,\end{aligned}\tag{5.2.a}$$

if $T>0$;

$$\begin{aligned}a(T,Y,s)=&\,i^{-2k}\,2^{7-s}\,\pi^{k+\frac{s}{2}-3}\,\Gamma(k+s-5)\\ &\cdot\Gamma(k+s/2)^{-1}\,\Gamma(k+\frac{s}{2}-4)^{-1}\,\Gamma(\frac{s}{2})^{-1}\end{aligned}$$

$$(5.2.b) \qquad \cdot \det Y^{5-k-s}\, \delta_+(TY)^{k+s/2-5} \cdot \omega_2(2\pi Y, T; k+\frac{s}{2}, \frac{s}{2}) \cdot \frac{\zeta(k+s-5)}{\zeta(k+s)\zeta(k+s-4)} \Big(\sum_{a|t} a^{9-k-s}\Big),$$

if $T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$, $t > 0$; and

$$(5.2.c) \qquad \begin{aligned} a(T,Y,s) &= i^{-2k}\, 2^{12-2k-2s}\, \pi^6\, \Gamma(k+s-5)\,\Gamma(k+s-9) \\ &\cdot \Gamma(k+\frac{s}{2})^{-1}\Gamma(k+\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1} \\ &\cdot \det Y^{5-k-s}\, \frac{\zeta(k+s-5)\zeta(k+s-9)}{\zeta(k+s)\zeta(k+s-4)}, \end{aligned}$$

if $T = 0$.

Now let $s \longrightarrow 0$. Then we have:
If $T = 0$,

$$a(T,Y,s) = \begin{cases} \frac{(*)}{\pi^2}(\det Y)^{-1}, & \text{if } k = 6, \\ 0, & \text{otherwise}. \end{cases}$$

If $T > 0$,

$$a(T,Y,s) = \begin{cases} (*)(\det T)^{k-5} \prod_{p|\det T} f_T^p(p^{5-k})\, e^{-2\pi(T,Y)}, & \text{if } k \geq 6, \\ 0, & \text{if } k = 2,4. \end{cases}$$

If $T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$, $t > 0$,

$$a(T,Y,s) = \begin{cases} \frac{(*)}{\pi^5}(\det Y)^{-1}\, \delta_+(TY)\, \omega_2(2\pi Y, T; 6, 0) \\ \quad \cdot \Big(\sum_{a|t} a^3\Big), & \text{if } k = 6, \\ 0, & \text{otherwise}. \end{cases}$$

(Here $(*)$ means that it is a rational number.)

3) $E^{(3)}_{k,s}(Z)$. Again because of Γ-factors in ξ_3, we can easily show that if $T \in V(p,q,r)$, $q > 0$, then $a(T,Y,s) \longrightarrow 0$ as $s \longrightarrow 0$ for all k. Therefore it suffices to consider the cases $q = 0$. By (2.2), (3.9), (4.4) and the fact that

$$\mu(\mathfrak{J}^{(3)}_{\mathbb{R}}/\Lambda_3) = \mu(\mathfrak{C}^3/\mathfrak{o}^3) = 2^{-12},$$

we have the following four cases

$$\begin{aligned} a(T,Y,s) =& i^{-3k}\, 2^{3+3k}\, \pi^{3k+s/2-12}\, \Gamma(k+\frac{s}{2})^{-1} \\ &\cdot \Gamma(k+\frac{s}{2}-4)^{-1}\Gamma(k+\frac{s}{2}-8)^{-1}(\det Y)^{-s/2} \\ &\cdot (\det T)^{k+s/2-9}\, \omega_3(2\pi Y, T; k+\frac{s}{2}, \frac{s}{2}) \\ &\cdot \frac{1}{\zeta(k+s)\zeta(k+s-4)\zeta(k+s-8)} \\ &\cdot \prod_{p\mid\det T} f^p_T(p^{9-k-s}), \end{aligned} \tag{5.3.a}$$

if $T > 0$;

$$\begin{aligned} a(T,Y,s) =& i^{-3k}\, 2^{k-s+12}\, \pi^{2k+s-11}\, \Gamma(k+\frac{s}{2})^{-1}\, \Gamma(k+\frac{s}{2}-4)^{-1} \\ &\cdot \Gamma(k+\frac{s}{2}-8)^{-1}\Gamma(\frac{s}{2})^{-1}\Gamma(k+s-9)(\det Y)^{9-k-s} \\ &\cdot \delta_+(TY)^{k+s/2-9}\, \omega_3(2\pi Y, T; k+\frac{s}{2}, \frac{s}{2}) \\ &\cdot \frac{\zeta(k+s-9)}{\zeta(k+s)\zeta(k+s-4)\zeta(k+s-8)} \\ &\cdot \prod_{p\mid\det T_1} f^p_{T_1}(p^{13-k-s}), \end{aligned} \tag{5.3.b}$$

if $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $T_1 > 0$;

$$\begin{aligned} a(T,Y,s) =& i^{-3k}\, 2^{-k-2s+21}\, \pi^{k+s/2-2}\, \Gamma(k+\frac{s}{2})^{-1} \\ &\cdot \Gamma(k+\frac{s}{2}-4)^{-1}\Gamma(k+\frac{s}{2}-8)^{-1}\Gamma(\frac{s}{2})^{-1} \\ &\cdot \Gamma(\frac{s}{2}-4)^{-1}\Gamma(k+s-9)\Gamma(k+s-13) \end{aligned}$$

$$\cdot\,(\det Y)^{9-k-s}\,\delta_+(TY)^{k+s/2-9}\,\omega_3(2\pi Y,T;k+\frac{s}{2},\frac{s}{2})$$
$$\cdot\,\frac{\zeta(k+s-9)\zeta(k+s-13)}{\zeta(k+s)\zeta(k+s-4)\zeta(k+s-8)} \tag{5.3.c}$$
$$\cdot\,\Big(\sum_{a|t} a^{17-k-s}\Big),$$

if $T=\begin{pmatrix} t & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $t>0$; and also

$$\begin{aligned} a(T,Y,s) = {} & i^{-3k}\,2^{-3k-3s+30}\,\pi^{15}\,\Gamma(k+\frac{s}{2})^{-1}\\ & \cdot\Gamma(k+\frac{s}{2}-4)^{-1}\Gamma(k+\frac{s}{2}-8)^{-1}\Gamma(\frac{s}{2})^{-1}\\ & \cdot\Gamma(\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2}-8)^{-1}\Gamma(k+s-9)\\ & \cdot\Gamma(k+s-13)\Gamma(k+s-17)\,(\det Y)^{9-k-s}\\ & \cdot\frac{\zeta(k+s-9)\zeta(k+s-13)\zeta(k+s-17)}{\zeta(k+s)\zeta(k+s-4)\zeta(k+s-8)}, \end{aligned} \tag{5.3.d}$$

if $T=0$.

Now let $s\longrightarrow 0$. Then we have:
If $T=0$,

$$a(T,Y,s)=\begin{cases} \dfrac{(*)}{\pi^3}(\det Y)^{-1}, & \text{if } k=10,\\ 0, & \text{otherwise}. \end{cases}$$

If $T=\begin{pmatrix} t & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$,

$$a(T,Y,s)=\begin{cases} \dfrac{(*)}{\pi^{10}}(\det Y)^{-1}\,\delta_+(TY)\,\omega_3(2\pi Y,T;10,0) & \\ \quad\cdot\Big(\sum_{a|t}a^7\Big), & \text{if } k=10,\\ 0, & \text{otherwise}. \end{cases}$$

If $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $T_1 > 0$,

$$a(T,Y,s) = \begin{cases} \frac{(*)}{\pi^9}(\det Y)^{-1}\,\delta_+(TY)\,\omega_3(2\pi Y, T; 10, 0) \\ \quad \cdot \prod_{p \mid \det T_1} f^p_{T_1}(p^3)\,, & \text{if } k = 10\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

If $T > 0$,

$$a(T,Y,s) = \begin{cases} (*)(\det T)^{k-9} \prod_{p \mid \det T} f^p_T(p^{9-k})\,e^{-2\pi(T,Y)}\,, & \text{if } k \geq 10\,, \\ 0\,, & \text{if } k < 10\,. \end{cases}$$

(Here $(*)$ means that it is a rational number.)

Therefore we can summarize our results as follows:

1) $E_{k,s}(Z)$ is finite at $s = 0$ for all k,
2) $E_{k,0}(Z)$ is holomorphic in Z unless $k = 2, 6, 10$,
3) $E_{k,0}(Z)$ is a modular form of weight k with rational Fourier coefficients unless $k = 2, 6, 10$,
4) $E_{4,0}(Z)$ and $E_{8,0}(Z)$ are singular modular forms,

$$E_{4,0}(Z) = 1 + 240 \sum_{\mu\iota_{(1)} \in \mathfrak{J}_o\iota_{(1)} N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{\substack{t \in \mathbb{Z} \\ t > 0}} \sigma_3(t)\,e^{2\pi i t (Z\cdot\mu)_1}\,,$$

where $\sigma_3(t) = \sum_{a|t} a^3$. In Section 6, we show that the summation $\mu\iota_{(1)} \in \mathfrak{J}_o\iota(1)N_{0\mathbb{Q}}/N_{0\mathbb{Q}}$ is equivalent to $\mu \in \mathfrak{J}_o/(\mathcal{P}_1)_o$. Therefore

$$E_{4,0}(Z) = 1 + 240 \sum_{T \in \mathfrak{J}_o^+,\, \mathrm{rank}\, T = 1} \sigma_3(\Delta(T))\,e^{2\pi i (T,Z)}\,,$$

where $\Delta(T)$ is as in Karel [6, p. 186]. We also consider the Mellin transform (see Section 6) of $E_{4,0}(Z)$ just like θ-function in order to obtain a functional equation of "Epstein zeta function".

6. Proof of Theorem B.

In this section we prove Theorem B which is a Nagaoka's conjecture on the functional equation of the Eisenstein series. But we have a slightly different functional equation. In the case of the group $Sp_2(\mathbb{Z})$ acting on the Siegel upper half-space of degree 2, Kaufhold [9] obtained a functional equation of an Eisenstein series. We follow his procedure.

From the Fourier expansion of $E_{0,s}(Z)$, we can decompose $E_{0,s}(Z)$ as follows:

$$E_{0,s}(Z) = \Phi_0(s,Z) + \Phi_1(s,Z) + \Phi_2(s,Z) + \Phi_3(s,Z)\,,$$

where

$$\begin{aligned}
\Phi_0(s,Z) = 1 \;&+ \sum_{\mu\iota_{(1)}\in\mathfrak{I}_o\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \frac{1}{\mu_1}\xi_1((\mu^{*-1}Y)_1, 0; \frac{s}{2}, \frac{s}{2})\, S(0^{1\times1}, s) \\
&+ \sum_{\mu\iota_{(2)}\in\mathfrak{I}_o\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \frac{1}{\mu_2}\xi_2((\mu^{*-1}Y)_2, 0; \frac{s}{2}, \frac{s}{2})\, S(0^{2\times2}, s) \\
&+ \frac{1}{\mu_3}\xi_3(Y, 0; \frac{s}{2}, \frac{s}{2})\, S(0^{3\times3}, s)\,,
\end{aligned}$$

$$\begin{aligned}
\Phi_1(s,Z) = &\sum_{\mu\iota_{(1)}\in\mathfrak{I}_o\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{t\in\mathbb{Z}-0} \frac{1}{\mu_1}\xi_1((\mu^{*-1}Y)_1, t; \frac{s}{2}, \frac{s}{2}) \\
&\qquad\cdot S(t,s)\, e^{2\pi i t(\mu^{*-1}X)_1} \\
&+ \sum_{\mu\iota_{(2)}\in\mathfrak{I}_o\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{\substack{T\in\Lambda_2 \\ \operatorname{rank} T=1}} \frac{1}{\mu_2}\xi((\mu^{*-1}Y)_2, T; \frac{s}{2}, \frac{s}{2}) \\
&\qquad\cdot S(T,s)\, e^{2\pi i((\mu^{*-1}X)_2, T)} \\
&+ \sum_{\substack{T\in\Lambda_3 \\ \operatorname{rank} T=1}} \frac{1}{\mu_3}\xi_3(Y, T; \frac{s}{2}, \frac{s}{2})\, S(T,s)\, e^{2\pi i(X,T)}\,,
\end{aligned}$$

$$\begin{aligned}
\Phi_2(s,Z) = &\sum_{\mu\iota_{(2)}\in\mathfrak{I}_o\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{\substack{T\in\Lambda_2 \\ \det T\neq 0}} \frac{1}{\mu_2}\xi_2((\mu^{*-1}Y)_2, T; \frac{s}{2}, \frac{s}{2}) \\
&\qquad\cdot S(T,s)\, e^{2\pi i((\mu^{*-1}X)_2, T)}
\end{aligned}$$

$$+ \sum_{\substack{T\in\Lambda_3 \\ \mathrm{rank}\, T=2}} \frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\, S(T,s)\, e^{2\pi i(X,T)}\,,$$

$$\Phi_3(s,Z) = \sum_{\substack{T\in\Lambda_3 \\ \det T\neq 0}} \frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\, S(T,s)\, e^{2\pi i(X,T)}\,.$$

Here $\mu_1 = 1$, $\mu_2 = 2^{-4}$, $\mu_3 = 2^{-12}$.

1) $\Phi_0(s,Z)$. By (3.9), (4.2), (4.3) and (4.4), we have

$$\begin{aligned}\Phi_0(s,Z) = 1 &+ 2^{2-s}\,\pi\,\Gamma(s-1)\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2})^{-1}\frac{\zeta(s-1)}{\zeta(s)} \\ &\cdot \sum_{\mu\iota_{(1)}\in\mathfrak{J}_o\iota_{(1)}N_{o\mathbb{Q}}/N_{o\mathbb{Q}}} (\mu^{*-1}Y)_1^{1-s} \\ &+ 2^{16-2s}\,\pi^{10}\,\Gamma_2(\frac{s}{2})^{-1}\Gamma_2(s-5)\Gamma_2(\frac{s}{2})^{-1}\frac{\zeta(s-5)\zeta(s-9)}{\zeta(s)\zeta(s-4)} \\ &\cdot \sum_{\mu\iota_{(2)}\in\mathfrak{J}_o\iota_{(2)}N_{o\mathbb{Q}}/N_{o\mathbb{Q}}} \det(\mu^{*-1}Y)_2^{5-s} \\ &+ (\det Y)^{9-s}\,2^{42-3s}\,\pi^{27}\,\Gamma_3(\frac{s}{2})^{-1}\Gamma_3(\frac{s}{2})^{-1}\Gamma_3(s-9) \\ &\cdot \frac{\zeta(s-9)\zeta(s-13)\zeta(s-17)}{\zeta(s)\zeta(s-4)\zeta(s-8)}\,.\end{aligned}$$

Now we use the identities

$$\Gamma(s) = 2^{s-1}\,\pi^{-1/2}\,\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})\,,\quad \rho(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)\,.$$

Then we have

$$\begin{aligned}\Phi_0(s,Z) = 1 &+ \sum_{\mu\iota_{(1)}\in\mathfrak{J}_o\iota_{(1)}N_{o\mathbb{Q}}/N_{o\mathbb{Q}}} (\mu^{*-1}Y)_1^{1-s}\,\frac{\rho(s-1)}{\rho(s)} \\ &+ \sum_{\mu\iota_{(2)}\in\mathfrak{J}_o\iota_{(2)}N_{o\mathbb{Q}}/N_{o\mathbb{Q}}} \det(\mu^{*-1}Y)_2^{5-s}\,\frac{\rho(s-5)\rho(s-9)(s-6)(s-8)}{\rho(s)\rho(s-4)(s-2)(s-4)} \\ &+ (\det Y)^{9-s}\,\frac{\rho(s-9)\rho(s-13)\rho(s-17)(s-14)(s-16)}{\rho(s)\rho(s-4)\rho(s-8)(s-2)(s-4)}\,.\end{aligned}$$

Now let us look at the following series

$$\varphi_1(Y,s) = \sum_{\mu\iota_{(1)}\in\mathfrak{J}_o\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} (\mu^{*-1}Y)_1^{-s},$$

$$\varphi_2(Y,s) = \sum_{\mu\iota_{(2)}\in\mathfrak{J}_o\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \det(\mu^{*-1}Y)_2^{-s}.$$

By Baily [1, p. 528], $N_{0\mathbb{Q}} = \{g \in \mathcal{G}_{\mathbb{Q}} : g(0,0,0,\xi') = (0,0,0,\xi''),\ \xi',\xi'' \in \mathbb{Q}\}$. Since $\iota_{(1)} = \iota_{e_1}$, (Here we take $(j) = \{1,2,\dots,j\}$ while in Baily $(j) = \{3-j+1,\dots,3\}$.) we can show, by direct calculation,

$$\iota_{(1)}(0,0,0,\xi') = (-\xi' e_1, 0,0,0),$$

where $e_1 = \operatorname{diag}(1,0,0)$. Therefore if $\mu_1\iota_{(1)} = \mu_2\iota_{(1)}p$ for $\mu_1,\mu_2 \in \mathfrak{J}_o$ and $p \in N_{0\mathbb{Q}}$, then $\mu_1(e_1,0,0,0) = \mu_2(\xi e_1,0,0,0)$ for some $\xi \in \mathbb{Q}$. *i.e.* $\mu_1 e_1 = \xi\mu_2 e_1$. But $\mu_1,\mu_2 \in \mathfrak{J}_o$ and $\mathfrak{K}_1^+$ is stable under $\mathfrak{J}$, so $\xi = 1$. In the notation of Baily [1, p. 520], $\mu_1^{-1}\mu_2 \in (\mathcal{P}_1)_o$. Therefore

$$\varphi_1(Y,s) = \sum_{\substack{\mu\in\mathfrak{J}_o/\sim \\ \mu_1\sim\mu_2 \text{ iff } \mu_1 e_1 = \mu_2 e_1}} (\mu^{*-1}Y)_1^{-s}.$$

Here $(\mu^{*-1}Y)_1 = (\mu^{*-1}Y, e_1) = (Y, \mu e_1)$ and by Baily [1, Lemma 3.2], we have 1-1 correspondence

$$\begin{aligned} \mathfrak{J}_o/(\mathcal{P}_1)_o &\longrightarrow \mathfrak{K}_1^+ \cap \mathfrak{J}_o, \quad \text{primitive}. \\ [\mu] &\longmapsto \mu e_1 . \end{aligned}$$

Therefore

$$\varphi_1(Y,s) = \sum_{\mu\in\mathfrak{J}_o/\sim} (Y,\mu e_1)^{-s} = \sum_{\substack{X\in\mathfrak{K}_1^+\cap\mathfrak{J}_o \\ \text{primitive}}} (Y,X)^{-s}.$$

On the other hand, $\iota_{(2)} = \iota_{e_1}\iota_{e_2}$, and so $\iota_{(2)}(0,0,0,\xi') = (0,0,\xi' e_3,0)$. (use the fact that $e_1 \times e_2 = \frac{1}{2}e_3$, $e_1 \times e_3 = \frac{1}{2}e_2$.) So if $\mu_1\iota_{(2)} = \mu_2\iota_{(2)}p$ for $\mu_1,\mu_2 \in \mathfrak{J}_o$ and $p \in N_{0\mathbb{Q}}$, then $\mu_1(0,0,e_3,0) = \mu_2(0,0,\xi e_3,0)$ for some $\xi \in \mathbb{Q}$. *i.e.* $\mu_1^* e_3 = \xi\mu_2^* e_3$. But $(\mu_1^{-1}\mu_2)^* \in \mathfrak{J}_o$, and so $\xi = 1$. In the notation of Baily [1, p. 520], $\mu_1^{-1}\mu_2 \in (\mathcal{P}_3^-)_o$. Therefore

$$\varphi_2(Y,s) = \sum_{\substack{\mu\in\mathfrak{J}_o/\sim \\ \mu_1\sim\mu_2 \text{ iff } \mu_1^* e_3 = \mu_2^* e_3}} \det(\mu^{*-1}Y)_2^{-s}.$$

But $\det(Y)_2 = (Y \times Y)_{33} = (Y \times Y, e_3)$, and so we have

$$\begin{aligned}\varphi_2(Y,s) &= \sum_{\mu\in\mathfrak{J}_o/\sim} (\mu^{*-1}Y \times \mu^{*-1}Y, e_3)^{-s} \\ &= \sum_{\mu\in\mathfrak{J}_o/\sim} (Y\times Y, \mu^* e_3)^{-s} \\ &= \sum_{\substack{X\in\mathfrak{K}_1^+\cap\mathfrak{J}_o \\ \text{primitive}}} (Y\times Y, X)^{-s} = \varphi_1(Y\times Y, s)\,.\end{aligned}$$

Here

$$Y^{-1} = \frac{1}{\det Y}\, Y \times Y\,.$$

Therefore

$$\varphi_2(Y,s) = \varphi_1((\det Y)Y^{-1}, s) = (\det Y)^{-s}\,\varphi_1(Y^{-1}, s)\,.$$

Now consider $E_{4,0}(Z)$

$$E_{4,0}(Z) = 1 + 240 \sum_{\mu\iota_{(1)}\in\mathfrak{J}_o\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{\substack{t\in\mathbb{Z}\\ t>0}} \sigma_3(t)\, e^{2\pi i t (Z\cdot\mu)_1}\,,$$

where $\sigma_3(t) = \sum_{a|t} a^3$. As we saw in Section 5, $E_{4,0}(Z)$ is a modular form of weight 4, and so

$$E_{4,0}(-Z^{-1}) = E_{4,0}(Z)\det(-Z)^4\,.$$

Take $Z = i\,rY$, $r>0$. Then we have

$$E_{4,0}(\frac{i}{r}\,Y^{-1}) = r^{12}(\det Y)^4\, E_{4,0}(irY).$$

We consider the Mellin transform of $E_{4,0}(i\,rY)$: By Fubini's Theorem, we have

$$\begin{aligned}&\int_0^\infty (E_{4,0}(i\,rY)-1)r^{s-1}\,dr \\ &\qquad = 240 \sum_{\mu\iota_{(1)}\in\mathfrak{J}_o\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{\substack{t\in\mathbb{Z}\\ t>0}} \sigma_3(t) \int_0^\infty e^{-2\pi \operatorname{tr}(\mu^{*-1}Y)_1}\, r^{s-1}\,dr\end{aligned}$$

$$= 240\,(2\pi)^{-s}\,\Gamma(s)\Bigl(\sum_{\substack{t\in\mathbb{Z}\\ t>0}} \sigma_3(t)t^{-s}\Bigr)\varphi_1(Y,s) = Z(Y,s)\,.$$

Here by the well-known identity,

$$\sum_{\substack{t\in\mathbb{Z}\\ t>0}} \sigma_3(t)t^{-s} = \zeta(s)\zeta(s-3)\,, \quad \text{for } \mathrm{Re}(s) > 4\,.$$

Let $Z(Y,s) = \int_0^1 + \int_1^\infty$. Then

$$\begin{aligned}
&\int_0^1 (E_{4,0}(i\,rY) - 1)\,r^{s-1}\,dr \\
&\quad = \int_0^1 (r^{-12}(\det Y)^{-4}E_{4,0}(\frac{i}{r}Y^{-1}) - 1)r^{s-1}\,dr \\
&\quad = \int_1^\infty ((\det Y)^{-4}u^{12}E_{4,0}(i\,uY^{-1}) - 1)\,u^{-s-1}\,du \quad (\text{set } \frac{1}{r} = u) \\
&\quad = \int_1^\infty \Bigl[(\det Y)^{-4}u^{12}(E_{4,0}(i\,uY^{-1}) - 1) \\
&\qquad\qquad + (\det Y)^{-4}u^{12} - 1\Bigr]u^{-s-1}\,du \\
&\quad = \int_1^\infty (\det Y)^{-4}(E_{4,0}(i\,uY^{-1}) - 1)\,u^{11-s}\,du \\
&\qquad + \int_1^\infty \Bigl[(\det Y)^{-4}u^{11-s} - u^{-s-1}\Bigr]\,du\,.
\end{aligned}$$

Here

$$\int_1^\infty \Bigl[(\det Y)^{-4}u^{11-s} - u^{-s-1}\Bigr]\,du = -\frac{1}{s} - \frac{(\det Y)^{-4}}{12-s}\,,$$

for $\mathrm{Re}(s) > 12$. Therefore

$$\begin{aligned}
&Z(Y,s) = -\frac{1}{s} - \frac{(\det Y)^{-4}}{12-s} \\
&+ \int_1^\infty \Bigl[(E_{4,0}(i\,rY) - 1)r^{s-1} + (\det Y)^{-4}(E_{4,0}(i\,rY^{-1}) - 1)r^{11-s}\Bigr]\,dr\,.
\end{aligned}$$

Here the integral inside $Z(Y,s)$ is holomorphic in s, and so $Z(Y,s)$ is continued as a meromorphic function in s and satisfies the functional equation:

$$Z(Y^{-1}, 12-s) = Z(Y,s)\,(\det Y)^4.$$

Therefore

$$\begin{aligned}
&\Psi_0(s,Z)\\
&= (\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\Phi_0(s,Z)\\
&= (\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\\
&\quad + (\det Y)^{9-s/2}\rho(s-9)\rho(s-13)\rho(s-17)\,(s-14)(s-16)\\
&\quad + \frac{2^3\pi^2}{240}\,Z(Y,s-1)(\det Y)^{s/2}\rho(s-8)\\
&\quad + Z(Y^{-1},s-5)(\det Y)^{5-s/2}\rho(s-9).
\end{aligned} \tag{6.1}$$

So $\Psi_0(s,Z)$ is continued as a meromorphic function in s which has pole of order 1 at $s=0,1,5,8,10,13,17,18$ and by the well-known identity, $\rho(s)=\rho(1-s)$, we have

$$\Psi_0(18-s,Z)=\Psi_0(s,Z)\,.$$

REMARK. $\Psi_0(s,Z)$ has at most a pole of order 1 at $s=9$, but because of the functional equation, $\Psi_0(s,Z)$ cannot have a pole of order 1. So $\Psi_0(s,Z)$ has no pole at $s=9$.

2) $\Phi_3(s,Z)$.

$$\Phi_3(s,Z)=\sum_{\substack{T\in\Lambda_3\\ \det T\neq 0}}\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\,S(T,s)\,e^{2\pi i(X,T)}\,.$$

We prove that for each T, $\det T\neq 0$,

$$\begin{aligned}
\chi(s) = (\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)(s-2)(s-4)&\\
\cdot\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\,S(T,s)&,
\end{aligned}$$

satisfies the functional equation

$$\chi(18-s)=\chi(s)\,.$$

First, we prove that $f_T^p(X)$ satisfies a functional equation

$$X^d f_T^p(X^{-1})=f_T^p(X)\,,$$

where

$$d = \deg (f_T^p) = v_p (\det T)$$

and

$$S_p(T,s) = (1-p^{-s})(1-p^{4-s})(1-p^{8-s})\, f_T^p(p^{9-s})\,.$$

It suffices to consider the case $T = \mathrm{diag}(t_1, t_2, t_3)$, $t_1|t_2$, $t_2|t_3$ in $\mathbb{Z}_p$. Let $\tau_i = v_p(t_i)$. Then $\tau_1 \le \tau_2 \le \tau_3$.

If $\tau_1 = 0$, then by Karel [8, p. 553],

$$f_T^p(X) = \sum_{k=0}^{\tau_2} (p^4 X)^k \frac{1 - X^{\tau_3+\tau_2+1-2k}}{1-X}\,.$$

From this expression, we get

$$X^{\tau_2+\tau_3} f_T^p(X^{-1}) = f_T^p(X)\,.$$

Again by Karel [8, p. 553], if $\tau_1 = 0$, we have the formula

$$\begin{aligned}\frac{S(p^m T, s)}{S(T,s)} &= C_0(X^{-1})\, X^{3m} + C_1(X^{-1})\, q^{2m} X^{2m} \\ &\quad + C_1(X)\, q^{2m} X^m + C_0(X)\,,\end{aligned}$$

where $q = p^4$, $X = p^{9-s}$.(see Karel [8, p. 553] for the definition of C_0 and C_1.) Then

$$\begin{aligned}f_{p^m T}^p(X) = f_T^p(X)(&C_0(X^{-1})\, X^{3m} + C_1(X^{-1})\, q^{2m} X^{2m} \\ &+ C_1(X)\, q^{2m} X^m + C_0(X))\,.\end{aligned}$$

Therefore $X^{3m} X^{\tau_2+\tau_3} f_{p^m T}^p(X^{-1}) = f_{p^m T}^p(X)$ for T with $\tau_1 = 0$. So we proved that

$$X^d f_T^p(X^{-1}) = f_T^p(X) \qquad \text{for all } T.$$

From this functional equation, we have

$$\begin{aligned}\prod_{p|\det T} f_T^p(p^{s-9}) &= \prod_{p|\det T} f_T^p(p^{9-s})(p^{s-9})^{d_p} \\ &= |\det T|^{s-9} \prod_{p|\det T} f_T^p(p^{9-s})\,,\end{aligned}$$

where

$$|\det T| = \prod_{p|\det T} p^{d_p}\,.$$

Now in order to prove the functional equation of $\chi(s)$, it suffices to consider the cases $T \in V(3,0,0)$ and $T \in V(2,1,0)$ by Section 3, (6).

(i) $T \in V(3,0,0)$. By (3.9) and (4.4),

$$\begin{aligned}\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})S(T,s) = {} & 2^3\,\pi^{3s/2-12}\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2}-8)^{-1}\\ & \cdot(\det Y)^{-s/2}(\det T)^{s/2-9}\,\omega_3(2\pi Y,T;\frac{s}{2},\frac{s}{2})\\ & \cdot\frac{\prod_{p|\det T} f_T^p(p^{9-s})}{\zeta(s)\zeta(s-4)\zeta(s-8)}\,.\end{aligned}$$

So we have

$$\begin{aligned}\chi(s) = {} & 2^{21}\,\pi^{-6}\,(\det T)^{s/2-9}\,\omega_3(2\pi Y,T;\frac{s}{2},\frac{s}{2})\\ & \cdot\prod_{p|\det T} f_T^p(p^{9-s})(s-2)(s-4)\ldots(s-16)\,.\end{aligned}$$

Therefore from the functional equation of ω_3 and f_T^p, we have

$$\chi(18-s) = \chi(s)\,.$$

(ii) $T \in V(2,1,0)$. By (3.9) and (4.4),

$$\begin{aligned}\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})S(T,s) = {} & 2^{35}\,\pi^{3s/2-12}\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2})^{-1}\\ & \cdot(\det Y)^{9-s}\,\delta_+(TY)^{s/2-7}\,\delta_-(TY)^{s/2-5}\\ & \cdot\omega_3(2\pi Y,T;\frac{s}{2},\frac{s}{2})\,S(T,s)\,.\end{aligned}$$

So

$$\begin{aligned}\chi(s) = {} & 2^{33}\,\pi^{-6}\,(\det Y)^{9-s/2}\,\delta_+(TY)^{s/2-7}\,\delta_-(TY)^{s/2-5}\\ & \cdot\omega_3(2\pi Y,T;\frac{s}{2},\frac{s}{2})\prod_{p|\det T} f_T^p(p^{9-s})\,.\end{aligned}$$

Therefore from the functional equation of ω_3 and and the fact that

$$\delta_+(TY)\,\delta_-(TY) = \det Y\ |\det T|\,,$$

we have again $\chi(18-s)=\chi(s)$. Here $\chi(s)$ is holomorphic for all T and because of the inequality of ω_3 in the Theorem in Section 3,

$$(6.2)\quad \Psi_3(s,Z)=(\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)(s-2)(s-4)\,\Phi_3(s,Z),$$

converges and so is holomorphic in s.

3) $\Phi_2(s,Z)$.

$$\Phi_2(s,Z)=\sum_{\mu\iota_{(2)}\in\mathfrak{J}_{\circ}\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}}\ \sum_{\substack{T\in\Lambda_2\\ \det T\neq 0}}\frac{1}{\mu_2}\xi_2((\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})$$
$$\cdot S(T,s)\,e^{2\pi i((\mu^{*-1}X)_2,T)}$$
$$+\sum_{\substack{T\in\Lambda_3\\ \operatorname{rank}T=2}}\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\,S(T,s)\,e^{2\pi i(X,T)}.$$

By Baily [1, Lemma 3.2],

$$T=\mu\begin{pmatrix}T_1&0\\0&0\end{pmatrix}$$

runs over all $T\in\Lambda_3$, $\operatorname{rank}T=2$ if T_1 runs over all $T_1\in\Lambda_2$, $\det T\neq 0$ and $\mu\in\mathfrak{J}_{\circ}/(P_3^-)_{\circ}$. But as we saw in Section 6, (1),

$$\mu\iota_{(2)}\in\mathfrak{J}_{\circ}\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}\ \text{ if and only if }\ \mu\in\mathfrak{J}_{\circ}/(P_3^-)_{\circ}\,.$$

Therefore

$$\sum_{\substack{T\in\Lambda_3\\ \operatorname{rank}T=2}}\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\,S(T,s)\,e^{2\pi i(X,T)}$$
$$=\sum_{\mu}\sum_{\substack{T\in\Lambda_2\\ \det T\neq 0}}\frac{1}{\mu_3}\xi_3(\begin{pmatrix}(\mu^{*-1}Y)_2&*\\ *&*\end{pmatrix},\begin{pmatrix}T&0\\0&0\end{pmatrix};\frac{s}{2},\frac{s}{2})$$
$$\cdot S(\begin{pmatrix}T&0\\0&0\end{pmatrix},s)\,e^{2\pi i((\mu^{*-1}X)_2,T)},$$

where $\mu\iota_{(2)}\in\mathfrak{J}_{\circ}\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}$. But there exists $\mu_0\in\mathfrak{J}_{\mathbb{R}}$ such that

$$\mu_0^*\begin{pmatrix}(\mu^{*-1}Y)_2&*\\ *&*\end{pmatrix}=\begin{pmatrix}(\mu^{*-1}Y)_2&0\\0&*\end{pmatrix},\quad \mu_0\begin{pmatrix}T&0\\0&0\end{pmatrix}=\begin{pmatrix}T&0\\0&0\end{pmatrix}.$$

So

$$\xi_3\left(\begin{pmatrix}(\mu^{*-1}Y)_2 & * \\ * & *\end{pmatrix}, \begin{pmatrix}T & 0\\ 0 & 0\end{pmatrix}; \frac{s}{2}, \frac{s}{2}\right) = \xi_3\left(\begin{pmatrix}(\mu^{*-1}Y)_2 & 0 \\ 0 & *\end{pmatrix}, \begin{pmatrix}T & 0\\ 0 & 0\end{pmatrix}; \frac{s}{2}, \frac{s}{2}\right).$$

Therefore we have

$$\Phi_2(s, Z) = \sum_{\mu\iota_{(2)}\in\mathfrak{I}_0\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \sum_{\substack{T\in\Lambda_2\\ \det T\neq 0}} a(T,s)\, e^{2\pi i((\mu^{*-1}X)_2, T)},$$

where

$$a(T,s) = \frac{1}{\mu_2}\xi_2((\mu^{*-1}Y)_2, T; \frac{s}{2}, \frac{s}{2})\, S(T,s) + \frac{1}{\mu_3}\xi_3\left(\begin{pmatrix}(\mu^{*-1}Y)_2 & 0 \\ 0 & *\end{pmatrix}, \begin{pmatrix}T & 0\\ 0 & 0\end{pmatrix}; \frac{s}{2}, \frac{s}{2}\right) S\left(\begin{pmatrix}T & 0\\ 0 & 0\end{pmatrix}, s\right).$$

We show that, for all $T \in \Lambda_2$, $\det T \neq 0$,

$$\chi(s) = (\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\,a(T,s) = \chi(18-s).$$

It suffices to consider the cases $T \in V(2,0)$ and $T \in V(1,1)$ by Section 3, (6).

(i) $T \in V(2,0)$. By (3.5), (3.9), (4.3) and (4.4), we have

$$\begin{aligned} a(T,s) &= 2^6\,\pi^{s-4}\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}\det(\mu^{*-1}Y)_2^{-s/2}(\det T)^{s/2-5} \\ &\quad\cdot \omega_2(2\pi(\mu^{*-1}Y)_2, T; \frac{s}{2}, \frac{s}{2})\frac{1}{\zeta(s)\zeta(s-4)}\prod_{p\mid\det T} f_T^p(p^{5-s}) \\ &\quad + 2^{16-s}\,\pi^{s-3}\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2}-8)^{-1} \\ &\quad\cdot \Gamma(s-9)\,(\det Y)^{9-s}\det(\mu^{*-1}Y)_2^{s/2-9}(\det T)^{s/2-9} \\ &\quad\cdot \omega_2(2\pi(\mu^{*-1}Y)_2, T; \frac{s}{2}-4, \frac{s}{2}-4) \\ &\quad\cdot \frac{\zeta(s-9)}{\zeta(s)\zeta(s-4)\zeta(s-8)}\prod_{p\mid\det T} f_T^p(p^{13-s}). \end{aligned}$$

By using the identities,

$$\Gamma(s) = 2^{s-1}\,\pi^{-1/2}\,\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}),$$
$$\rho(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s),$$

we can write

$$\begin{aligned} a(T,s) =& 2^4\,\pi^{-2}\det(\mu^{*-1}Y)_2^{-s/2}(\det T)^{s/2-5}\,\omega_2(2\pi(\mu^{*-1}Y)_2, T; \frac{s}{2}, \frac{s}{2}) \\ &\cdot \frac{(s-6)(s-8)}{\rho(s)\rho(s-4)} \prod_{p|\det T} f_T^p(p^{5-s}) \\ &+ 2^4\,\pi^{-4}\,(\det Y)^{9-s}(\det T)^{s/2-9}\det(\mu^{*-1}Y)_2^{s/2-9} \\ &\cdot \omega_2(2\pi(\mu^{*-1}Y)_2, T; \frac{s}{2}-4, \frac{s}{2}-4)\,\frac{\rho(s-9)}{\rho(s)\rho(s-4)\rho(s-8)} \\ &\cdot \frac{(s-10)(s-12)(s-14)(s-16)}{(s-2)(s-4)} \prod_{p|\det T} f_T^p(p^{13-s}). \end{aligned}$$

Therefore

$$\begin{aligned} \chi(s) =& 2^4\,\pi^{-2}\,(\det Y)^{s/2}\det(\mu^{*-1}Y)_2^{-s/2} \\ &\cdot \omega_2(2\pi(\mu^{*-1}Y)_2, T; \frac{s}{2}, \frac{s}{2})\,\rho(s-8) \\ &\cdot (s-2)(s-4)(s-6)(s-8) \\ &\cdot (\det T)^{s/2-5} \prod_{p|\det T} f_T^p(p^{5-s}) \\ &+ 2^4\,\pi^{-2}\,(\det Y)^{9-s/2}\det(\mu^{*-1}Y)_2^{s/2-9} \\ &\cdot \omega_2(2\pi(\mu^{*-1}Y)_2, T; \frac{s}{2}-4, \frac{s}{2}-4)\,\rho(s-9) \\ &\cdot (s-10)(s-12)(s-14)(s-16) \\ &\cdot (\det T)^{s/2-9} \prod_{p|\det T} f_T^p(p^{13-s}). \end{aligned}$$

By using the functional equation of ω_2, ρ and

$$(\det T)^{s-5} \prod_{p|\det T} f_T^p(p^{5-s}) = \prod_{p|\det T} f_T^p(p^{s-5}),$$

we have $\chi(18-s)=\chi(s)$.

(ii) $T\in V(1,1)$. By (3.5), (3.9), (4.3) and (4.4), we have

$$\begin{aligned}
a(T,s) = {} & 2^{10}\,\pi^{s-4}\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2})^{-1}\det(\mu^{*-1}Y)_2^{2-s/2}|\det T|^{s/2-3}\\
&\cdot\omega_2(2\pi(\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})\,S(T,s)\\
&+2^{20-2s}\,\pi^{s-3}\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}\Gamma(s-9)\\
&\cdot(\det Y)^{9-s}\det(\mu^{*-1}Y)_2^{s/2-7}\,|\det T|^{s/2-7}\\
&\cdot\omega_2(2\pi(\mu^{*-1}Y)_2,T,\frac{s}{2}-4,\frac{s}{2}-4)\,S\left(\begin{pmatrix}T&0\\0&0\end{pmatrix},s\right)\\
= {} & 2^{12}\,\pi^{-2}\det(\mu^{*-1}Y)_2^{2-s/2}\omega_2(2\pi(\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})\\
&\cdot\frac{(\det T)^{s/2-3}\prod_{p|\det T}f_T^p(p^{5-s})}{\rho(s)\rho(s-4)(s-2)(s-4)}\\
&+2^{12}\pi^{-2}(\det Y)^{9-s}\det(\mu^{*-1}Y)_2^{s/2-7}\\
&\cdot\omega_2(2\pi(\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})\\
&\cdot\frac{\rho(s-9)\,|\det T|^{s/2-7}\prod_{p|\det T}f_T^p(p^{13-s})}{\rho(s)\rho(s-4)\rho(s-8)(s-2)(s-4)}\;.
\end{aligned}$$

Therefore

$$\begin{aligned}
\chi(s) = {} & 2^{12}\,\pi^{-2}\,(\det Y)^{s/2}\det(\mu^{*-1}Y)_2^{2-s/2}\,\omega_2(2\pi(\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})\\
&\cdot\rho(s-8)\,|\det T|^{s/2-3}\prod_{p|\det T}f_T^p(p^{5-s})\\
&+2^{12}\,\pi^{-2}\,(\det Y)^{9-s/2}\det(\mu^{*-1}Y)_2^{s/2-7}\\
&\cdot\omega_2(2\pi(\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})\,\rho(s-9)\\
&\cdot|\det T|^{s/2-7}\prod_{p|\det T}f_T^p(p^{13-s})\,.
\end{aligned}$$

So again we have $\chi(18-s)=\chi(s)$. Here $\chi(s)$ is a meromorphic function for all T which has a pole of order 1 at $s=8,10$ and because of the inequality of ω_2 in the Theorem in Section 3,

(6.3) $\Psi_2(s,Z)=(\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\,\Phi_2(s,Z)\,,$

converges and so is meromorphic in s.

4) $\Phi_1(s,Z)$. We write

$$\Phi_1(s,Z)=\Phi_1'(s,Z)+\Phi_1''(s,Z),$$

where

$$\Phi_1'(s,Z)=\sum_{\mu\iota_{(1)}\in\mathfrak{J}_0\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}}\ \sum_{t\in\mathbb{Z}-0}\frac{1}{\mu_1}\xi_1((\mu^{*-1}Y)_1,t;\frac{s}{2},\frac{s}{2})\cdot S(t,s)\,e^{2\pi it(\mu^{*-1}X)_1}$$

$$+\sum_{\substack{T\in\Lambda_3\\ \text{rank}\,T=1}}\frac{1}{\mu_3}\xi_3(Y,T;\frac{s}{2},\frac{s}{2})\,S(T,s)\,e^{2\pi i(X,T)}$$

$$\Phi_1''(s,Z)=\sum_{\mu\iota_{(2)}\in\mathfrak{J}_0\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}}\ \sum_{T\in\Lambda_2\ \text{rank}\,T=1}\frac{1}{\mu_2}\xi_2((\mu^{*-1}Y)_2,T;\frac{s}{2},\frac{s}{2})\cdot S(T,s)\,e^{2\pi i((\mu^{*-1}X)_2,T)}.$$

(i) $\Phi_1'(s,Z)$. We have the 1-1 correspondence

$$\{T\in\Lambda_3:\quad \text{rank}\,T=1\}\longleftrightarrow\{\mu\iota_{(1)}\in\mathfrak{J}_0\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}\}\times\{t\in\mathbb{Z}-0\},$$

by $T=\mu\begin{pmatrix}t&0&0\\0&0&0\\0&0&0\end{pmatrix}$. Also there exists $\mu_0\in\mathfrak{J}_{\mathbb{R}}$ such that

$$\mu_0^*\begin{pmatrix}(\mu^{*-1}Y)_1&*\\ *&*\end{pmatrix}=\begin{pmatrix}(\mu^{*-1}Y)_1&0\\0&*\end{pmatrix},$$

$$\mu_0\begin{pmatrix}t&0\\0&0^{2\times2}\end{pmatrix}=\begin{pmatrix}t&0\\0&0^{2\times2}\end{pmatrix}.$$

By (3.4), (3.5),

$$\omega_3(\begin{pmatrix}y_1&0\\0&*\end{pmatrix},\begin{pmatrix}t&0\\0&0^{2\times2}\end{pmatrix};\frac{s}{2},\frac{s}{2})=2^{-9}\,\pi^8\,e^{-|t|y_1}\,\omega_1(2y_1;\frac{s}{2}-8,\frac{s}{2}-8).$$

Therefore by (3.9), (4.2) and (4.4), we have

$$\Phi_1'(s,Z)=\sum_{\mu\iota_{(1)}\in\mathfrak{J}_0\iota_{(1)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}}\ \sum_{t\in\mathbb{Z}-0}\Big(\frac{1}{\mu_1}\xi_1((\mu^{*-1}Y)_1,t;\frac{s}{2},\frac{s}{2})\,S(t,s)$$

$$+\frac{1}{\mu_3}\xi_3\Big(\begin{pmatrix}(\mu^{*-1}Y)_1 & 0\\ 0 & *\end{pmatrix},\begin{pmatrix}t & 0\\ 0 & 0^{2\times 2}\end{pmatrix};\frac{s}{2},\frac{s}{2}\Big)S\Big(\begin{pmatrix}t & 0\\ 0 & 0^{2\times 2}\end{pmatrix},s\Big)\Big)$$
$$\cdot\, e^{2\pi i t(\mu^{*-1}X)_1}$$

$$=\sum_{\mu}\sum_{t\in\mathbb{Z}-0} e^{2\pi i|t|(Z\cdot\mu)_1}\,|t|^{-1}\Big(\pi^{s/2}(\mu^{*-1}Y)_1^{-s/2}|t|^{s/2}S(t,s)$$
$$\cdot\,\omega_1(4\pi|t|(\mu^{*-1}Y)_1;\frac{s}{2},\frac{s}{2})$$
$$+\,2^{24-2s}\,\pi^{s/2+6}\,(\det Y)^{9-s}$$
$$\cdot\,(\mu^{*-1}Y)_1^{s/2-9}\,|t|^{s/2-8}$$
$$\cdot\,\omega_1(4\pi|t|(\mu^{*-1}Y)_1;\frac{s}{2}-8,\frac{s}{2}-8)$$
$$\cdot\,\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)^{-1}$$
$$\cdot\,\Gamma(\frac{s}{2}-4)^{-1}\Gamma(\frac{s}{2}-8)^{-1}\Gamma(s-9)$$
$$\cdot\,\Gamma(s-13)\,S(t,s-16)$$
$$\cdot\,\frac{\zeta(s-9)\zeta(s-13)\zeta(s-16)}{\zeta(s)\zeta(s-4)\zeta(s-8)}\Big).$$

Here we use the identity

$$\beta(s)=|t|^{s/2}\,S(t,s)\,\zeta(s)=\beta(2-s)\,.$$

(*cf.* Kaufhold [9]) Therefore we can write

$$\Phi_1'(s,Z)=\sum_{\mu}\sum_{t\in\mathbb{Z}-0} e^{-2\pi|t|(Z\cdot\mu)_1}\,|t|^{-1}\Big((\mu^{*-1}Y)^{-s/2}\frac{\beta(s)}{\rho(s)}$$
$$\cdot\,\omega_1(4\pi|t|(\mu^{*-1}Y)_1;\frac{s}{2},\frac{s}{2})$$
$$+\,(\mu^{*-1}Y)_1^{s/2-9}(\det Y)^{9-s}\,\omega_1(4\pi|t|(\mu^{*-1}Y)_1;\frac{s}{2}-8,\frac{s}{2}-8)$$
$$\cdot\,\frac{\beta(s-16)\rho(s-9)\rho(s-13)(s-14)(s-16)}{\rho(s)\rho(s-4)\rho(s-8)(s-2)(s-4)}\Big).$$

From the functional equation of ω_3, $\rho(s)$ and $\beta(s)$,

(6.4) $(\det Y)^{s/2}\rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\,\Phi_1'(s,Z)=\Psi_1'(s,Z)$,

satisfies the functional equation

$$\Psi_1'(s,Z)=\Psi_1'(18-s,Z)\,.$$

On the other hand, because of the inequality of ω_1 in the Theorem in Section 3, $\Psi_1'(s,Z)$ converges and defines a meromorphic function in s which has a pole of order 1 at $s = 5, 8, 10, 13$.

(ii) $\Phi_1''(s,Z)$. In order to prove the functional equation of $\Phi_1''(s,Z)$, we need to look at an Eisenstein series on the tube domain

$$\mathfrak{T}^{(2)} = \{Z = X + iY : \quad X \in \mathfrak{J}_{\mathbb{R}}^{(2)}, \quad Y > 0\},$$

which is the boundary component of $\mathfrak{T}$. For an element $X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathfrak{J}^{(2)}$, we let $\det X = ab - N(x)$ and $\operatorname{tr} X = a + b$. We also define

$$\mathcal{J}^{(2)} = \{g \in \mathfrak{J}^{(2)} : \quad \det(gX) \equiv \det(X)\}.$$

Tsao studied an action of a subgroup $\mathcal{G}^{(2)}$ of $\mathcal{G}$ on the boundary component $\mathfrak{T}^{(2)}$ in (Tsao [16, p .254]). $\mathcal{G}^{(2)}$ is isogeneous to SO(10,2). We can define an Eisenstein series $E_{k,s}^{(2)}(Z)$ on $\mathfrak{T}^{(2)}$ in the exactly same way as $E_{k,s}(Z)$ and can get $E_{4,0}^{(2)}(Z)$ which is a holomorphic modular form of weight 4 on $\mathfrak{T}^{(2)}$ and which is given by

$$E_{4,0}^{(2)}(Z) = 1 + 240 \sum_{\mu \in \mathcal{J}_{\mathfrak{o}}^{(2)}/(\mathcal{P}_0)_{\mathfrak{o}}} \sum_{\substack{t \in \mathbb{Z} \\ t > 0}} \sigma_3(t)\, e^{2\pi i t (Z\cdot\mu)_1},$$

where $\mathcal{P}_0$ is the minimal pabolic subgroup of $\mathcal{J}^{(2)}$ which is the stability group of the line $\mathbb{R}e_1'$ where $e_1' = \operatorname{diag}(1,0)$.

REMARK. Recently Eie and Krieg [4] studied modular forms on $\mathfrak{T}^{(2)}$ using Fourier-Jacobi expansion. Especially they obtained $E_{4,0}^{(2)}(Z)$ in terms of theta series

$$E_{4,0}^{(2)}(Z) = \sum_{h \in \mathfrak{o}^2} e^{2\pi i \tau(Z, h\bar{h}^t)}.$$

The equivalence of two expressions come from the well-known formula

$$\#\{a \in \mathfrak{o} : \quad n = N(a)\} = 240\ \sigma_3(a) \qquad \text{for all } n \geq 1.$$

Now consider the following series which is an "Epstein zeta function"

$$\varphi^{(2)}(Y,s) = \sum_{\mu \in \mathcal{J}_{\mathfrak{o}}^{(2)}/(\mathcal{P}_0)_{\mathfrak{o}}} (\mu^{*-1}Y)_1^{-s} = \sum_{\mu \in \mathcal{J}_{\mathfrak{o}}^{(2)}/(\mathcal{P}_0)_{\mathfrak{o}}} (Y, \mu e_1')^{-s},$$

for $Y > 0$ and $s \in \mathbb{C}$.
Now we apply the Mellin transform of $E_{4,0}^{(2)}(Z)$ in the exactly same way as $E_{4,0}(Z)$ to get the analytic continuation and a functional equation of $\varphi^{(2)}(Y,s)$: If we let

$$Z^{(2)}(Y,s) = \int_0^\infty r^{s-1}(E_{4,0}^{(2)}(i\,rY) - 1)\,dr\,,$$

then

$$Z^{(2)}(Y,s) = 240\,(2\pi)^{-s}\,\Gamma(s)\,\zeta(s)\,\zeta(s-3)\,\varphi^{(2)}(Y,s)\,,$$
$$Z^{(2)}(Y^{-1}, 8-s) = (\det Y)^4\, Z^{(2)}(Y,s)\,.$$

$Z^{(2)}(Y,s)$ has a pole of order 1 at $s = 0, 8$. Here if $\det Y = 1$, then $Y^{-1} = \begin{pmatrix} b & -x \\ -\bar{x} & a \end{pmatrix}$ for $Y = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix}$. The transformation

$$X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \longmapsto X^* = \begin{pmatrix} b & -x \\ -\bar{x} & a \end{pmatrix},$$

is in $\mathfrak{J}_{\mathfrak{o}}^{(2)}$ since it preserves the determinant and the lattice $\mathfrak{J}_{\mathfrak{o}}$. Therefore

$$\varphi^{(2)}(Y^*, s) = \varphi^{(2)}(Y,s)\,.$$

So if $\det Y = 1$, we have

$$Z^{(2)}(Y, 8-s) = Z^{(2)}(Y,s)\,.$$

Now let us come back to $\Phi_1''(s,Z)$. We have 1-1 correspondence

$$\{T \in \Lambda_2 : \quad \operatorname{rank} T = 1\} \longleftrightarrow \{\nu \in \mathfrak{J}_{\mathfrak{o}}^{(2)}/(\mathcal{P}_0)_{\mathfrak{o}}\} \times \{t \in \mathbb{Z} - 0\}\,,$$

by $T = \nu \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$ and there exists $\nu' \in \mathfrak{J}_{\mathbb{R}}^{(2)}$ such that

$$\nu'^* \begin{pmatrix} y_1 & * \\ * & * \end{pmatrix} = \begin{pmatrix} y_1 & 0 \\ 0 & * \end{pmatrix}, \quad \nu' \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore by (3.4) and (4.3), we have (set $(\mu^{*-1}Y)_2 = y$)

$$\frac{1}{\mu_2}\xi_2(\nu^{*-1}y, \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}; \frac{s}{2}, \frac{s}{2})\; S(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, s)$$

$$= 2^{6-s}\,\pi^{9+s/2}\,\Gamma_2(\frac{s}{2})^{-1}\Gamma_2(\frac{s}{2})^{-1}\Gamma(\frac{s}{2}-4)\Gamma(s-5)$$
$$\cdot (\det y)^{5-s}\,(\nu^{*-1}y)_1^{s/2-5}\,|t|^{s/2-5}\,e^{-2\pi|t|(\nu^{*-1}y)_1}$$
$$\cdot \omega_1(4\pi|t|(\nu^{*-1}y)_1;\frac{s}{2}-4,\frac{s}{2}-4)$$
$$\cdot \frac{\zeta(s-5)\zeta(s-8)}{\zeta(s)\zeta(s-4)}S(t,s-8)\,.$$

Therefore by using the identities

$$\beta(s) = |t|^{s/2}\,S(t,s)\,\zeta(s)\,,\quad \rho(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)\,,$$

we get

$$\chi(s) = (\det Y)^{s/2}\,\rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\,\Phi_1''(s,Z)$$
$$= \sum_{\mu\in\mathfrak{J}_o/(\mathcal{P}_3^-)_o}\ \sum_{\nu\in\mathfrak{J}_o^{(2)}/(\mathcal{P}_0)_o}\ \sum_{t\in\mathbb{Z}-0} (\det Y)^{s/2}\,\det(\mu^{*-1}Y)_2^{5-s}$$
$$\cdot (\nu^{*-1}(\mu^{*-1}Y)_2)_1^{s/2-5}\,e^{-2\pi t(\nu^{*-1}(\mu^{*-1}Y)_2)_1}\,|t|^{-1}$$
$$\cdot \omega_1(4\pi t(\nu^{*-1}(\mu^{*-1}Y)_2)_1;\frac{s}{2}-4,\frac{s}{2}-4)$$
$$\cdot \beta(s-8)\,\rho(s-5)\rho(s-8)\,(s-6)(s-8)\,.$$

Now any element of $\gamma \in \mathfrak{J}_o/(\mathcal{P}_*)_o$ can be written uniquely

$$\gamma = \mu\cdot\nu \quad \mu \in \mathfrak{J}_o/(\mathcal{P}_3^-)_o\,,\quad \nu \in (\mathcal{P}_3^-)_o/(\mathcal{P}_*)_o\,,$$

where $\mathcal{P}_*$ is the minimal parabolic subgroup of $\mathfrak{J}$ which is the stability group of the "flag" $(\mathfrak{J}^{\{1\}},\mathfrak{J}^{\{1,2\}})$ (see Baily [1, p. 520]). But $(\mathcal{P}_3^-)_o/(\mathcal{P}_*)_o$ is identified in a natural way with $\mathfrak{J}_o^{(2)}/(\mathcal{P}_0)_o$. Therefore we can write

$$\chi(s) = \sum_{\gamma\in\mathfrak{J}_o/(\mathcal{P}_*)_o}\ \sum_{t\in\mathbb{Z}-0} (\det Y)^{s/2}\,\det(\gamma^{*-1}Y)_2^{5-s}\,(\gamma^{*-1}Y)_1^{s/2-5}$$
$$\cdot e^{-2\pi|t|(\gamma^{*-1}Y)_1}\,|t|^{-1}\,\omega_1(4\pi|t|(\gamma^{*-1}Y)_1;\frac{s}{2}-4,\frac{s}{2}-4)$$
$$\cdot \beta(s-8)\,\rho(s-5)\rho(s-8)\,(s-6)(s-8)\,.$$

On the other hand, any element of $\gamma \in \mathfrak{J}_o/(\mathcal{P}_*)_o$ can be written uniquely

$$\gamma = \mu\cdot\nu\,,\quad \mu \in \mathfrak{J}_o/(\mathcal{P}_1)_o\,,\quad \nu \in (\mathcal{P}_1)_o/(\mathcal{P}_*)_o\,.$$

Then $(\gamma^{*-1}Y)_1 = (Y, \gamma e_1) = (Y, \mu\nu e_1) = (Y, \mu e_1)$ since $\nu \in (\mathcal{P}_1)_o$. Therefore

$$\chi(s) = \sum_{\mu \in \mathfrak{J}_o/(\mathcal{P}_1)_o} (\det Y)^{s/2} (\mu^{*-1}Y)^{s/2-5}\, e^{-2\pi|t|(\mu^{*-1}Y)_1}\, |t|^{-1}$$
$$\cdot\, \omega_1(4\pi|t|(\mu^{*-1}Y)_1; \frac{s}{2}-4, \frac{s}{2}-4)\, \beta(s-8)$$
$$\cdot\, \rho(s-5)\rho(s-8)\,(s-6)(s-8)$$
$$\cdot \Big(\sum_{\nu \in (\mathcal{P}_1)_o/(\mathcal{P}_*)_o} \det((\mu\nu)^{*-1}Y)_2^{5-s} \Big).$$

Here $\det((\mu\nu)^{*-1}Y)_2 = (\mu^{*-1}Y \times \mu^{*-1}Y, \nu^* e_3)$ and we have 1-1 correspondence

$$\nu \in (\mathcal{P}_1)_o/(\mathcal{P}_*)_o \longleftrightarrow \nu^* \in (\mathcal{P}_1^-)_o/(\mathcal{P}_*)_o\,.$$

But $(\mathcal{P}_1^-)_o/(\mathcal{P}_*)_o$ can be identified in a natually way with $\mathfrak{J}_o^{(2)}/(\mathcal{P}_0)_o$ and note that

$$\mu^{*-1}Y \times \mu^{*-1}Y = \det(\mu^{*-1}Y)(\mu^{*-1}Y)^{-1} = (\det Y)(\mu^{*-1}Y)^{-1}.$$

Therefore

$$\sum_{\nu \in (\mathcal{P}_1)_o/(\mathcal{P}_*)_o} \det((\mu\nu)^{*-1}Y)_2^{5-s}$$
$$= \sum_{\nu \in \mathfrak{J}_o^{(2)}/(\mathcal{P}_0)_o} \Big((\det Y)(\mu^{*-1}Y)^{-1}_{\{2,3\}}, \nu e_3\Big)^{5-s}$$
$$= \varphi^{(2)}\Big((\det Y)(\mu^{*-1}Y)^{-1}_{\{2,3\}}, s-5\Big)\,,$$

where $(\mu^{*-1}Y)^{-1}_{\{2,3\}}$ is the right lower corner 2×2 submatrix of $(\mu^{*-1}Y)^{-1}$. But

$$\det(\mu^{*-1}Y)^{-1}_{\{2,3\}} = ((\mu^{*-1}Y)^{-1} \times (\mu^{*-1}Y)^{-1}, e_1)$$
$$= ((\det Y)^{-1}\mu^{*-1}Y, e_1) = (\det Y)^{-1}(\mu^{*-1}Y)_1\,.$$

Since

$$Z^{(2)}(Y,s) = 240\,(2\pi)^{-s}\,\Gamma(s)\,\zeta(s)\zeta(s-3)\,\varphi^{(2)}(Y,s)$$
$$= (240)\,2^{-3}\,\pi^{-2}\,(s-1)(s-3)\,\rho(s)\rho(s-3)\,\varphi^{(2)}(Y,s)\,,$$

we have

$$\chi(s) = \sum_{\mu\in\mathfrak{I}_o/(\mathcal{P}_1)_o} \Big(\sum_{t\in\mathbb{Z}-0} \frac{2^3\pi^2}{240} (\det Y)^{5/2} (\mu^{*-1}Y)_1^{-5/2}$$

$$\cdot\, e^{-2\pi|t|(\mu^{*-1}Y)_1}\, |t|^{-1}$$

$$(6.5) \qquad \cdot\, \omega_1(4\pi|t|(\mu^{*-1}Y)_1; \frac{s}{2}-4, \frac{s}{2}-4)$$

$$\cdot\, \beta(s-8)$$

$$\cdot\, Z^{(2)}\left((\det Y)^{1/2}(\mu^{*-1}Y)_1^{-1/2}(\mu^{*-1}Y)_{\{2,3\}}^{-1}, s-5\right)\Big).$$

Note that $\det((\det Y)^{1/2}(\mu^{*-1}Y)_1^{-1/2}(\mu^{*-1}Y)_{\{2,3\}}^{-1}) = 1$. Therefore by the functional equation of ρ, β, ω_1 and $Z^{(2)}$, we have the functional equation: $\chi(18-s) = \chi(s)$. Also because of the inequality of ω_1 in the theorem in Section 3, $\chi(s)$ converges and defines a meromorphic function in s which has a pole of order 1 at $s = 5, 13$. Therefore by (6.1), (6.2), (6.3), (6.4) and (6.5),

$$\begin{aligned}\Psi(s) &= (\det Y)^{s/2}\, \rho(s)\rho(s-4)\rho(s-8)\,(s-2)(s-4)\, E_{0,s}(Z) \\ &= \Psi_0(s,Z) + \Psi_1(s,Z) + \Psi_2(s,Z) + \Psi_0(s,Z),\end{aligned}$$

can be continued as a meromorphic function in s to a whole complex plane which has a pole of order 1 at $s = 0, 1, 5, 8, 10, 13, 17, 18$ and satisfies the functional equation:

$$\Psi(18-s) = \Psi(s).$$

This completes the proof of Theorem B.

REMARK. As in Shimura [14], considering residues $E_{k,s}(Z)$ at $s = 2$, we get the following result: (i) If $k = 4$, $E_{4,s}(Z)$ has a simple pole at $s = 2$ and the residue at $s = 2$ is

$$\frac{1}{\pi^2} \sum_{\mu\iota_{(2)}\in\mathfrak{I}_o\iota_{(2)}N_{0\mathbb{Q}}/N_{0\mathbb{Q}}} \det(\mu^{*-1}Y)_2^{-1} \sum_{\substack{T\in\Lambda_2\cap\mathfrak{K} \\ \det_{(2)}T=0}} b(T)\, e^{2\pi i(T,(Z\cdot\mu)_2)},$$

where $b(T) \in \mathbb{Q}$.

(ii) If $k = 8$, $E_{8,s}(Z)$ has a simple pole at $s = 2$ and the residue is $\big((\det Y)^{-1}/\pi^3\big)\times$ (singular modular form of weight 8 with rational

Fourier coefficients). Here we note that for $\mu \in \mathfrak{J}$, $Z \cdot \mu = \mu^{*-1} Z$ and so $\mathrm{Im}(Z \cdot \mu) = \mu^{*-1} Y$.

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Henry H. Kim*
Department of Mathematics
The University of Chicago
Chicago, IL 60637, USA

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