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Wiener-Hopf integral operators with PC symbols on spaces with Muckenhoupt weight

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Abstract. We describe the spectrum and the essential spectrum and give an index formula for Wiener-Hopf integral operators with piecewise continuous symbols on the space $L^p(\mathbb{R}_+, \omega)$ with a Muckenhoupt weight ω. Our main result says that the essential spectrum is a set resulting from the essential range of the symbol by joining the two endpoints of each jump by a certain sickle-shaped domain, whose shape is completely determined by the value of p and the behavior of the weight ω at the origin and at infinity.

1. Introduction.

Given $p \in (1, \infty)$, let A_p denote the set of all nonnegative functions w on $\mathbb{R}$ such that the singular integral operator S,

$$(Sf)(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - x}\, dt\,, \qquad x \in \mathbb{R}\,,$$

is bounded on the space $L^p(\mathbb{R}, w)$ with the norm

$$||f||_{p,w} = \left(\int_{-\infty}^{\infty} |w(x)f(x)|^p \right)^{1/p}.$$

If $w \in A_p$, then the compression S_+ of S to the positive half-line $\mathbb{R}_+ = [0, \infty)$,

$$(S_+ f)(x) = \frac{1}{\pi i} \int_0^\infty \frac{f(t)}{t - x}\, dt\,, \qquad x \in \mathbb{R}_+\,,$$

is a bounded operator on $L^p(\mathbb{R}_+, w)$ $(= L^p(\mathbb{R}_+, w|\mathbb{R}_+))$. The operator S_+ is the archetypal example of a Wiener-Hopf integral operator with a piecewise continuous symbol: by definition, the symbol of S_+ is the function

$$\sigma(\xi) = -\operatorname{sgn} \xi = \begin{cases} 1 & \text{for } \xi \in (-\infty, 0)\,, \\ -1 & \text{for } \xi \in (0, \infty)\,. \end{cases}$$

A fairly general class of Wiener-Hopf integral operators is constituted by operators W of the form

$$(Wf)(x) = \sum_{j=1}^{m} \frac{c_j}{\pi i} \int_0^\infty \frac{e^{i\alpha_j(t-x)} f(t)}{t - x}\, dt + \int_0^\infty k(x-t) f(t)\, dt\,, \qquad x > 0\,,$$

where $c_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{R}$ are given numbers and $k \in L^1(\mathbb{R})$ is a given function. The symbol of W is defined as the function

$$a(\xi) = -\sum_{j=1}^{m} c_j \operatorname{sgn}(\xi - \alpha_j) + \hat{k}(\xi)\,, \qquad \xi \in \mathbb{R}\,,$$

where $\hat{k}$ stands for the Fourier transform of k,

$$\hat{k}(\xi) = (Fk)(\xi) = \int_{-\infty}^{\infty} k(x) e^{i\xi x} dx\,, \qquad \xi \in \mathbb{R}\,.$$

Notice that a is a piecewise continuous function with jumps at $\alpha_1, \ldots, \alpha_m$ and at infinity.

What we are interested in here is the spectrum and essential spectrum of a Wiener-Hopf integral operator with a piecewise continuous symbol on $L^p(\mathbb{R}_+, w)$. As usual, the spectrum of W is the set of all $\lambda \in \mathbb{C}$ for which $W - \lambda I$ is not invertible. An operator W on $L^p(\mathbb{R}_+, w)$ is said to be Fredholm if it is invertible modulo the compact operators or, equivalently, if its range is closed and the kernel and cokernel dimensions $\alpha(W)$ and $\beta(W)$ are finite; in that case the index of W is defined as $\alpha(W) - \beta(W)$. Finally, the essential spectrum of W is the set of all $\lambda \in \mathbb{C}$ for which $W - \lambda I$ is not Fredholm.

It has been well known for a long time that the spectrum and the essential spectrum of a Wiener-Hopf operator with a discontinuous symbol on $L^p(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+, w)$ depend very sensitively on the value of p and the behavior of the weight w.

The pioneering work in this direction is undoubtedly Harold Widom's 1960 paper [16]. He observed that the spectrum of S_+ on $L^p(\mathbb{R}_+)$ is a certain circular arc depending on the value of p, namely the circular arc between -1 and 1 containing the point $-i\cot(\pi/p)$, which enabled him to identify the spectrum, the essential spectrum and the index of Toeplitz operators with piecewise continuous symbols on the (unweighted) Hardy spaces $H^p(\mathbb{R})$; the Toeplitz operators studied by Widom are just the operators we shall define in Section 2.9 below.

The hey-day of the development was the late sixties and early seventies. During that period Gohberg, Krupnik [8], and Duduchava [4],[5], to mention only the principal figures, considered pure Wiener-Hopf operators with piecewise continuous symbols on $L^p(\mathbb{R}_+)$ (without weight) and they proved that the essential spectrum of W is the continuous closed curve resulting from the range of the symbol a by joining the two endpoints of each jump by a certain circular arc. All these arcs are similar to one another and their shape is determined by p. The results of Gohberg, Krupnik, and Duduchava were extended by Schneider [14] to weights w of the form

$$w(x) = |x+i|^{\mu} \prod_{l=1}^{n} |x - \beta_l|^{\mu_l}, \qquad x \in \mathbb{R}.$$

He showed that the essential spectrum of W is again obtained from the range of a by filling in certain circular arcs. The interesting point of Schneider's criterion is that the circular arcs between $a(\alpha_j - 0)$ and $a(\alpha_j + 0)$ are all similar to one another and that their shape is determined solely by p and the behavior of the weight at infinity (*i.e.* the value of $\mu + \mu_1 + \cdots + \mu_n$), while the shape of the arc joining $a(+\infty)$ to $a(-\infty)$ depends only on p and the behavior of the weight at the point $x = 0$.

The main result of the present paper describes the essential spectrum of W in case w is any weight belonging to A_p. We show that this spectrum is obtained from the range of a by filling in a certain sickle-shaped domain (which will be called a "horn") for each jump. A circular arc is regarded as a degenerate horn. It turns out that the horns joining $a(\alpha_j - 0)$ and $a(\alpha_j + 0)$ are again similar to one another and that their shape is given by merely the value of p and the behavior of the weight at infinity, whereas the shape of the horn between $a(+\infty)$ and $a(-\infty)$ depends on p and the behavior of the weight at $x = 0$ alone.

2. Toeplitz Operators.

2.1. Muckenhoupt weights on the circle.

Let $\mathbb{T}$ denote the complex unit circle and let ρ be a nonegative function on $\mathbb{T}$ which does not vanish identically. For $1 < p < \infty$, consider the space $L^p(\mathbb{T}, \rho)$ with the norm

$$||f||_{p,\rho} = \Big(\int_{\mathbb{T}} |f|^p \rho^p dm \Big)^{1/p},$$

where dm is Lebesgue measure on $\mathbb{T}$. If $\rho \equiv 1$, we abbreviate $L^p(\mathbb{T}, \rho)$ to $L^p(\mathbb{T})$. The weight ρ is said to be a Muckenhoupt weight, and we write $\rho \in A_p(\mathbb{T})$ in this case, if $\rho \in L^p(\mathbb{T})$, $\rho^{-1} \in L^q(\mathbb{T})$ $(1/p + 1/q = 1)$, and

$$\sup_I \Big(\frac{1}{|I|} \int_I \rho^p dm \Big)^{1/p} \Big(\frac{1}{|I|} \int_I \rho^{-q} dm \Big)^{1/q} < \infty,$$

where the supremum is over all subarcs I of $\mathbb{T}$ and $|I|$ denotes the arc length of I. Weights of this type first appeared in connection with the boundedness of the Hardy maximal function operator in Muckenhoupt's paper [11].

The singular integral operator S_0 ,

$$(S_0 f)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t} \, d\tau, \qquad t \in \mathbb{T},$$

is bounded on $L^p(\mathbb{T})$ by a theorem of Marcel Riesz. The problem of describing all the weights ρ such that S_0 maps $L^p(\mathbb{T}) \cap L^p(\mathbb{T}, \rho)$ into itself and extends from $L^p(\mathbb{T}) \cap L^p(\mathbb{T}, \rho)$ to a bounded operator on $L^p(\mathbb{T}, \rho)$ was solved by Hunt, Muckenhoupt and Wheeden [9]: S_0 extends to a bounded operator on $L^p(\mathbb{T}, \rho)$ if and only if $\rho \in A_p(\mathbb{T})$. We remark that nice discussions of the Hunt-Muckenhoupt-Wheeden Theorem are also in [6], [7], [10] and [13]. Notice that a so-called power weight, given by

$$\rho(t) = \prod_{l=1}^{n} |t - \tau_l|^{\mu_l}, \qquad t \in \mathbb{T},$$

where $\tau_l \in \mathbb{T}$ and $\mu_l \in \mathbb{R}$, belongs to $A_p(\mathbb{T})$ if and only if $-1/p < \mu_l < 1/q$ for all l.

Troughout what follows let $\rho \in A_p(\mathbb{T})$. Then the two projections $P_0 = (I + S_0)/2$ and $Q_0 = (I - S_0)/2$ are bounded on $L^p(\mathbb{T}, \rho)$. The Hardy space $H^p(\mathbb{T}, \rho)$ is defined as the image of P_0 in $L^p(\mathbb{T}, \rho)$, *i.e.* $H^p(\mathbb{T}, \rho) = P_0 L^p(\mathbb{T}, \rho)$.

2.2. Toeplitz operators on the unit circle.

The Toeplitz operator $T_0(a)$ generated by a function $a \in L^\infty(\mathbb{T})$ is the operator on $H^p(\mathbb{T}, \rho)$ that sends f to $P_0(af)$. Since $\rho \in A_p(\mathbb{T})$, the operator $T_0(a)$ is bounded. The function a is usually referred to as the symbol of $T_0(a)$.

A well known theorem by L.A. Coburn says that $T_0(a)$ is invertible if and only if $T_0(a)$ is Fredholm with index zero (see *e.g.* [2, p. 216]). Hence, in order to study invertibility of Toeplitz operators (or, equivalently, in order to describe their spectrum), it suffices to establish a Fredholm criterion (or to describe the essential spectrum) and to have an index formula.

Given a Banach algebra $\mathfrak{A}$ with identity element, we denote by $G\mathfrak{A}$ the invertible elements in $\mathfrak{A}$. The Hartman-Wintner Theorem (again see [2, p. 216]) tells us that if $T_0(a)$ is Fredholm, then $a \in GL^\infty(\mathbb{T})$. If a is continuous, $a \in C(\mathbb{T})$, then the invertibility of a in $L^\infty(\mathbb{T})$ (and thus in $C(\mathbb{T})$) is also sufficient for $T_0(a)$ to be Fredholm, and the index of $T_0(a)$ is then minus the winding number of $a(\mathbb{T})$ about the origin. In general, however, symbols $a \in GL^\infty(\mathbb{T})$ do not induce Fredholm Toeplitz operators.

Let $PC(\mathbb{T})$ denote the C^*-algebra of all piecewise continuous functions on $\mathbb{T}$. A function $a \in PC(\mathbb{T})$ has at most countably many jumps and the limits

$$a(t \pm 0) = \lim_{\varepsilon \to 0 \pm 0} a(te^{i\varepsilon})$$

exist for each $t \in \mathbb{T}$. Under the sole assumption that $\rho \in A_p(\mathbb{T})$, a Fredholm criterion and an index formula for Toeplitz operators on

$H^p(\mathbb{T},\rho)$ with symbols in $PC(\mathbb{T})$ were only recently obtained by one of the authors [15]. Before citing this result we need a (crucial) lemma and the definition of what we call horns.

Lemma 2.3. ([15]). *Let $\rho \in A_p(\mathbb{T})$ and $\tau \in \mathbb{T}$. Then the set*

$$I_\tau(p,\rho) = \{\mu \in \mathbb{R} : |t-\tau|^\mu \rho(t) \in A_p(\mathbb{T})\}$$

is an open interval of a length not greater than 1 *containing the origin.*

REMARK 2.4. If ρ is a power weight as in Section 2.1, then clearly

$$I_{\tau_l}(p,\rho) = (-1/p - \mu_l\,, 1/q - \mu_l)\,,$$
$$I_\tau(p,\rho) = (-1/p\,, 1/q) \quad \text{for } \tau \notin \{\tau_1,\dots,\tau_n\}\,,$$

i.e. all $I_\tau(p,\rho)$ have length 1. To produce a weight $\rho \in A_p(\mathbb{T})$ such that $I_\tau(p,\rho)$ is any prescribed interval $(-\alpha,\beta) \ni 0$ of a length $\alpha+\beta < 1$, let first PQC denote the C^*-algebra of all piecewise quasicontinuous functions on $\mathbb{T}$ (see [2] or [3]). In [3], we showed that there exist $a \in PQC$ with a logarithm $\log a \in PQC$ such that $T_0(a)$ is invertible on $H^p(\mathbb{T}, |t-\tau|^\mu)$ if and only if $1/p + \mu \in (0, \alpha+\beta)$. This implies (see [15]) that if we put

$$\rho(t) = |\exp(P_0(\log a))|\, |t-\tau|^{\alpha-1/p}\,,$$

then $\rho \in A_p(\mathbb{T})$ and $I_\tau(p,\rho) = (-\alpha,\beta)$.

Definition 2.5. *For $\rho \in A_p(\mathbb{T})$ and $\tau \in \mathbb{T}$, let $I_\tau(p,\rho)$ be the interval determined by Lemma* 2.3 *and define the numbers $\nu_\tau^\pm(p,\rho)$ by*

$$(-\nu_\tau^-(p,\rho), 1-\nu_\tau^+(p,\rho)) = I_\tau(p,\rho)\,.$$

Because $I_\tau(p,\rho)$ contains the origin and is of a length not greater than 1, we have

$$0 < \nu_\tau^-(p,\rho) \le \nu_\tau^+(p,\rho) < 1\,.$$

2.6. Horns.

In what follows the argument of a nonzero complex number is always specified to belong to $[0,2\pi)$. Given two real numbers γ,δ such that $0 < \gamma \le \delta < 1$ and two distinct complex numbers z, w, we define the γ, δ horn joining z and w to be the set

$$\mathcal{H}(z, w; \gamma, \delta) = \left\{\zeta \in \mathbb{C}\backslash\{z, w\} : \arg \frac{\zeta - z}{\zeta - w} \in [2\pi\gamma, 2\pi\delta]\right\} \cup \{z, w\}.$$

Notice that for each $\phi \in (0, 1)$ the set

$$\{\zeta \in \mathbb{C}\backslash\{z, w\} : \arg \frac{\zeta - z}{\zeta - w} = 2\pi\phi\}$$

is a circular arc. If $\phi = 1/2$, this arc degenerates to the open line segment (z, w). For $\phi \in (0, 1/2)$ (respectively $\phi \in (1/2, 1)$) this arc is located on the right (respectively, left) of the straight line passing first z and then w, and it consists just of the points at which the segment $[z, w]$ is seen at the angle $2\pi\phi$ (respectively, $2\pi(1 - \phi)$). To cover the case $z = w$, we also define $\mathcal{H}(z, z; \gamma, \delta) = \{z\}$.

Note that $0 \notin \mathcal{H}(z, w; \gamma, \delta)$ if and only if $z \neq 0$, $w \neq 0$, and $\arg(z/w)$ does not belong to $[2\pi\gamma, 2\pi\delta]$.

For $a \in PC(\mathbb{T})$, the set

$$a_{p,\rho} = \bigcup_{\tau \in \mathbb{T}} \mathcal{H}(a(\tau - 0), a(\tau + 0); \nu_\tau^-(p, \rho), \nu_\tau^+(p, \rho))$$

results from the (essential) range of a by filling in a well-defined horn into each jump. The set $a_{p,\rho}$ is clearly connected. Any closed continuous curve obtained from the (essential) range of a by joining $a(t - 0)$ to $a(t + 0)$ by a circular arc contained in the horn between $a(t - 0)$ and $a(t + 0)$ inherits an orientation in a natural fashion. If $0 \notin a_{p,\rho}$, we denote by $\operatorname{wind} a_{p,\rho}$ the winding number of that curve about the origin.

Theorem 2.7. ([15]). *If $\rho \in A_p(\mathbb{T})$ and $a \in PC(\mathbb{T})$, then the essential spectrum of $T_0(a)$ on $H^p(\mathbb{T}, \rho)$ is the set $a_{p,\rho}$. In case $0 \notin a_{p,\rho}$, the index of $T_0(a)$ on $H^p(\mathbb{T}, \rho)$ equals $-\operatorname{wind} a_{p,\rho}$.*

2.8. Muckenhoupt weights on the real line.

Our next concern is to carry over Theorem 2.7 to Toeplitz operators on Hardy spaces of the real line.

For $p \in (1,\infty)$ and a nonnegative function w on $\mathbb{R}$ which is not identically zero, we consider the space $L^p(\mathbb{R}, w)$, whose norm is given by

$$\|f\|_{p,w} = \Big(\int_{\mathbb{R}} |w(x)f(x)|^p dx \Big)^{1/p}.$$

Again we abbreviate $L^p(\mathbb{R},1)$ to $L^p(\mathbb{R})$.

We write $w \in A_p$ and call w a Muckenhoupt weight if $w \in L^p(\mathbb{R})$, $w^{-1} \in L^q(\mathbb{R})$ $(1/p + 1/q = 1)$, and

$$\sup_I \Big(\frac{1}{|I|} \int_I w(x)^p dx \Big)^{1/p} \Big(\frac{1}{|I|} \int_I w(x)^{-q} dx \Big)^{1/q} < \infty,$$

where I ranges over all finite intervals $I \subset \mathbb{R}$ and $|I|$ stands for the length of the interval I.

The singular integral operator S (see the Introduction) is bounded on $L^p(\mathbb{R})$, and it was also Hunt, Muckenhoupt and Wheeden [9] who showed that S maps $L^p(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ into itself and extends to a bounded operator on $L^p(\mathbb{R}, w)$ if and only if $w \in A_p$.

Henceforth let always $w \in A_p$. The projections $P = (I+S)/2$ and $Q = (I-S)/2$ are bounded on $L^p(\mathbb{R}, w)$, and the image of P in $L^p(\mathbb{R}, w)$ is denoted by $H^p(\mathbb{R}, w)$ and called the p-th Hardy space of $\mathbb{R}$ with the weight function w.

2.9. Toeplitz operators on the real line.

Given $a \in L^\infty(\mathbb{R})$, define the Toeplitz operator $T(a)$ on $H^p(\mathbb{R}, w)$ by $T(a)f = P(af)$. Since $w \in A_p$, this is a bounded operator. Again the function a is called the symbol of $T(a)$.

The Coburn and Hartman-Wintner theorems extend to Toeplitz operators on $H^p(\mathbb{R}, w)$: the operator $T(a)$ is invertible if and only if it is Fredholm of index zero, and the Fredholmness of $T(a)$ implies the invertibility of a in $L^\infty(\mathbb{R})$. If $a \in C(\dot{\mathbb{R}})$, which means that $a \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and that the limits $a(\pm\infty)$ exist and are equal to each other, then for $T(a)$ to be Fredholm on $H^p(\mathbb{R}, w)$ it is necessary and sufficient that $a \in GL^\infty(\mathbb{R})$; in that case the index of $T(a)$ is minus the winding number of the range of a about the origin.

Let PC be the C^*-subalgebra of $L^\infty(\mathbb{R})$ consisting of all functions a in $L^\infty(\mathbb{R})$ which have limits $a(\xi \pm 0)$ for each $\xi \in \mathbb{R}$ and for which the limits $a(\pm\infty)$ exist. Note that functions in PC have at most countably many jumps. Also notice that PC contains $C(\bar{\mathbb{R}})$, the C^*-algebra of all functions $a \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with finite (but not necessarily equal) limits $a(\pm\infty)$.

Theorem 2.10. *Let $w \in A_p$ and $a \in PC$.*

(1) *Each of the sets*

$$I_\xi(p,w) = \{\mu \in \mathbb{R} : \ \left|\frac{x-\xi}{x-i}\right|^\mu w(x) \in A_p\}, \qquad \xi \in \mathbb{R},$$
$$I_\infty(p,w) = \{\mu \in \mathbb{R} : \ |x-i|^{-\mu} w(x) \in A_p\}$$

is an open interval of a length not greater than 1 *which contains the origin*

$$I_\xi(p,w) = (-\nu_\xi^-(p,w), 1-\nu_\xi^+(p,w)) \qquad (\xi \in \dot{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\})$$

with $0 < \nu_\xi^-(p,w) \le \nu_\xi^+(p,w) < 1$.

(2) *The essential spectrum of $T(a)$ on $H^p(\mathbb{R},w)$ equals*

$$a_{p,w} = \Big(\bigcup_{\xi\in\mathbb{R}} \mathcal{H}\big(a(\xi-0), a(\xi+0); \nu_\xi^-(p,w), \nu_\xi^+(p,w)\big)\Big)$$
$$\bigcup \mathcal{H}\big(a(+\infty), a(-\infty); \nu_\infty^-(p,w), \nu_\infty^+(p,w)\big).$$

If $0 \notin a_{p,w}$, then the index of $T(a)$ is $-\operatorname{wind} a_{p,w}$.

PROOF. We reduce the case of the real line to the situation on the circle in a standard way (see *e.g.* [8, p. 307]).

Define the weight ρ on $\mathbb{T}$ by

$$\rho(t) = w\Big(i\frac{t+1}{t-1}\Big)|t-1|^{1-2/p}, \qquad t \in \mathbb{T}.$$

Then the operator $B : L^p(\mathbb{R},w) \to L^p(\mathbb{T},\rho)$ given by

$$(B\phi)(t) = \frac{1}{t-1}\,\phi\Big(i\frac{t+1}{t-1}\Big), \qquad t \in \mathbb{T},\ \phi \in L^p(\mathbb{R},w),$$

is an isomorphism, the inverse operator being

$$(B^{-1}\psi)(x) = \frac{2i}{x-i}\,\psi\Big(\frac{x+i}{x-i}\Big), \qquad x \in \mathbb{R},\ \psi \in L^p(\mathbb{T},\rho).$$

Moreover, $B^{-1}S_0B = -S$. The latter equality in conjunction with the Hunt-Muckenhoupt-Wheeden theorems implies that $\rho \in A_p(\mathbb{T})$ if and only if $w \in A_p$ and also that, for $\xi = i(\tau+1)/(\tau-1) \in \mathbb{R}$,

$$\left|\frac{x-\xi}{x-i}\right|^{\mu} w(x) \in A_p$$

if and only if

$$\left|\frac{i\frac{t+1}{t-1} - i\frac{\tau+1}{\tau-1}}{i\frac{t+1}{t-1} - i}\right|^{\mu} w\Big(i\frac{t+1}{t-1}\Big)|t-1|^{1-2/p} = |t-\tau|^{\mu}\rho(t) \in A_p(\mathbb{T}),$$

and, analogously, that

$$|x-i|^{-\mu}w(x) \in A_p$$

exactly if

$$\left|i\frac{t+1}{t-1} - i\right|^{-\mu} w\Big(i\frac{t+1}{t-1}\Big)|t-1|^{1-2/p} = 2^{-\mu}|t-1|^{\mu}\rho(t) \in A_p(\mathbb{T}).$$

So part (1) of the present theorem is an immediate consequence of Lemma 2.3.

Let us now show that $0 \notin a_{p,w}$ if and only if $T(a)$ is Fredholm. Since the essential range of a is a subset of $a_{p,w}$ and the Fredholmness of $T(a)$ necessitates the invertibility of a in $L^\infty(\mathbb{R})$, we may without loss of generality a priori assume that $a \in GL^p(\mathbb{R})$.

It is easily seen that $T(a) = Pa|\operatorname{Im} P$ is Fredholm of index κ on $H^p(\mathbb{R}, w) = PL^p(\mathbb{R}, w)$ if and only if the operator $Q + PaP$ is Fredholm of index κ on $L^p(\mathbb{R}, w)$. Because $BSB^{-1} = -S_0$, it follows that

$$\begin{aligned} B(Q+PaP)B^{-1} &= P_0 + Q_0BaB^{-1}Q_0 \\ &= P_0 + Q_0bQ_0 \\ &= (P_0 + bQ_0)(I - P_0bQ_0) \\ &= b(b^{-1}P_0 + Q_0)(I - P_0bQ_0) \\ &= b(Q_0 + P_0b^{-1}P_0)(I + Q_0b^{-1}P_0)(I - P_0bQ_0), \end{aligned}$$

where $b(t) = a\big(i(t+1)/(t-1)\big)$ for $t \in \mathbb{T}$. The operators $I + Q_0b^{-1}P_0$ and $I - P_0bQ_0$ are invertible, the inverses being $I - Q_0b^{-1}P_0$ and $I + P_0bQ_0$, respectively. Since $a \in GL^\infty(\mathbb{R})$, the operator of multiplication by b is invertible as well. Hence, $T(a)$ is Fredholm of index κ on $H^p(\mathbb{R}, w)$ if and only if

$$T_0(b^{-1}) = P_0 b^{-1} | \text{Im } P_0$$

is Fredholm of index κ on $H^p(\mathbb{T}, \rho) = P_0 L^p(\mathbb{T}, \rho)$.

From Theorem 2.7 we infer that $T_0(b^{-1})$ is Fredholm if and only if

$$\begin{aligned} 0 \notin (b^{-1})_{p,\rho} &= \bigcup_{\tau \in \mathbb{T}} \mathcal{H}(b^{-1}(\tau - 0), b^{-1}(\tau + 0); \nu_\tau^-(p, \rho), \nu_\tau^+(p, \rho)) \\ &= \bigcup_{\tau \in \mathbb{T} \setminus \{1\}} \mathcal{H}(b^{-1}(\tau - 0), b^{-1}(\tau + 0); \nu_\tau^-(p, \rho), \nu_\tau^+(p, \rho)) \\ &\quad \bigcup \mathcal{H}(b^{-1}(1 - 0), b^{-1}(1 + 0); \nu_0^-(p, \rho), \nu_0^+(p, \rho)) \\ &= \bigcup_{\xi \in \mathbb{R}} \mathcal{H}(a^{-1}(\xi + 0), a^{-1}(\xi - 0); \nu_\xi^-(p, \rho), \nu_\xi^+(p, \rho)) \\ &\quad \bigcup \mathcal{H}(a^{-1}(-\infty), a^{-1}(+\infty); \nu_\infty^-(p, \rho), \nu_\infty^+(p, \rho)) . \end{aligned}$$

Consequently, $0 \notin (b^{-1})_{p,\rho}$ exactly if

$$\arg \frac{a^{-1}(\xi + 0)}{a^{-1}(\xi - 0)} = \arg \frac{a(\xi - 0)}{a(\xi + 0)} \notin [2\pi\nu_\xi^-(p, w), 2\pi\nu_\xi^+(p, w)]$$

for all $\xi \in \mathbb{R}$ and

$$\arg \frac{a^{-1}(-\infty)}{a^{-1}(+\infty)} = \arg \frac{a(+\infty)}{a(-\infty)} \notin [2\pi\nu_\infty^-(p, w), 2\pi\nu_\infty^+(p, w)] ,$$

which is equivalent to the condition that $0 \notin a_{p,w}$.

3. Convolution operators on the real line.

3.1. Fourier multipliers.

Again let $1 < p < \infty$ and $w \in A_p$. Denote by $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the Fourier transform and by F^{-1} its inverse. If a is any function defined on $\mathbb{R}$, the multiplication operator $f \mapsto af$ is traditionally denoted by aI, and in case aI is applied after another operator, B say, one writes aB instead of aIB .

A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^p(\mathbb{R}, w)$ if the mapping $f \mapsto F^{-1}aFf$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ into itself and extends to a bounded operator of $L^p(\mathbb{R}, w)$ into itself. The latter operator is then usually denoted by $W^0(a)$. It is well known that the set $M^p(w)$ of all Fourier multipliers on $L^p(\mathbb{R}, w)$ is a Banach algebra under the norm

$$||a||_{p,w} = ||W^0(a)||_{\mathcal{L}(L^p(\mathbb{R},w))} ,$$

where $\mathcal{L}(X)$ stands for the Banach algebra of all bounded operators on a Banach space X.

One can show (see *e.g.* [5] and [12] for power weights) that $M^p(w)$ contains all functions $a \in L^\infty(\mathbb{R})$ with finite total variation $\mathrm{Var}\,(a)$ and that for such functions the estimate

$$||a||_{p,w} \leq c_{p,w}\,(||a||_\infty + \mathrm{Var}\,(a))$$

holds, where $c_{p,w}$ is some constant independent of a.

Let $\dot{\mathbb{R}}$ be the compactification of $\mathbb{R}$ by one point at infinity. The closure in $M^p(w)$ of the set of all functions $a \in C(\dot{\mathbb{R}})$ with $\mathrm{Var}\,(a) < \infty$ is denoted by $C^p(w)$, and we let $PC^p(w)$ stand for the closure in $M^p(w)$ of the set of all piecewise continuous functions on $\mathbb{R}$ which have finite total variation and at most finitely many jumps. Clearly $C^p(w) \subset PC^p(w)$. It can be shown (see [14] and [12, Proposition 12.2] for power weights) that $PC^p(w)$ is continuously embedded into $L^\infty(\mathbb{R})$. This implies that $C^p(w) \subset C(\dot{\mathbb{R}})$ and $PC^p(w) \subset PC$, where PC refers to the C^*-subalgebra of $L^\infty(\mathbb{R})$ consisting of all functions a which possess finite one-sided limits $a(\xi \pm 0)$ at every point $\xi \in \dot{\mathbb{R}}$. Moreover, a function $a \in C^p(w)$ (respectively, $a \in PC^p(w)$) is invertible in $C^p(w)$ (respectively, $PC^p(w)$) if and only if it is invertible in $L^\infty(\mathbb{R})$ (see [12] for the case of power weights).

3.2. Wiener-Hopf integral operators.

The Wiener-Hopf operator $W(a)$ generated by a function $a \in M^p(w)$ (its so-called symbol) is the compression of $W^0(a)$ to the positive half-line $\mathbb{R}_+ = (0, \infty)$, *i.e.* $W(a)$ is the bounded operator on $L^p(\mathbb{R}_+, w)$ $(= L^p(\mathbb{R}_+, w|\mathbb{R}_+))$ acting by the rule $f \mapsto (W^0(a)f)|\mathbb{R}_+$. Let χ_+ be the characteristic function of $\mathbb{R}_+$. The space $L^p(\mathbb{R}_+, w)$ may be identified with $\chi_+ L^p(\mathbb{R}, w)$ and consequently, we may also think of $W(a)$ as the operator $\chi_+ W^0(a)|\mathrm{Im}\,\chi_+ I$.

Our aim is to describe the spectrum of $W(a)$ on $L^p(\mathbb{R}_+, w)$ if a belongs to $PC^p(w)$. The proof of Proposition 2.8 of [5] for $w \equiv 1$ along with the arguments used in the proof of Proposition 1.6 of [14] for power weights can be easily modified to show that if $a \in PC^p(w) \cap GL^\infty(\mathbb{R})$, then $W(a)$ is invertible on $L^p(\mathbb{R}_+, w)$ if and only if $W(a)$ is Fredholm of index zero on $L^p(\mathbb{R}_+, w)$. We shall prove below that $a \in GL^\infty(\mathbb{R})$ whenever $a \in PC^p(w)$ and $W(a)$ is Fredholm. Thus, in order to identify the spectrum of $W(a)$ we are again left with finding a Fredholm criterion and an index formula.

We finally remark that if $a \in C^p(w)$, then $W(a)$ is Fredholm on $L^p(\mathbb{R}_+, w)$ exactly if $a(\xi) \neq 0$ for all $\xi \in \dot{\mathbb{R}}$, in which case the index of $W(a)$ equals minus the winding number of the naturally oriented curve $a(\dot{\mathbb{R}})$ about the origin (see [2], [4], [8] and [12] for power weights).

3.3. Singular integral operators.

The connection between Toeplitz operators on $H^p(\mathbb{R}, w)$ and Wiener-Hopf operators on $L^p(\mathbb{R}_+, w)$ is established by singular integral operators on $L^p(\mathbb{R}, w)$, *i.e.* operators of the form $bI + cS = bI + cW^0(\sigma)$ or, slightly more generally, of the form

$$\lambda_\eta^{-1}(bI + cS)\lambda_\eta I = bI + cW^0(\sigma_\eta)\,,$$

where $\lambda_\eta(x) = e^{i\eta x}$ for $x, \eta \in \mathbb{R}$, $\sigma(\xi) = -\operatorname{sgn}\xi$ for $\xi \in \mathbb{R}$, and $\sigma_\eta(\xi) = -\operatorname{sgn}(\xi - \eta)$ for $\xi, \eta \in \mathbb{R}$.

Let χ_- and χ_+ be the characteristic functions of $(-\infty, 0)$ and $(0, +\infty)$, respectively. We have

$$(I + \chi_+ W^0(a)\chi_- I)(\chi_- I + \chi_+ W^0(a)\chi_+ I) = \chi_- I + \chi_+ W^0(a)\,,$$

and since $I + \chi_+ W^0(a)\chi_- I$ has the inverse $I - \chi_+ W^0(a)\chi_- I$, it follows that the Wiener-Hopf operator $W(a) = \chi_+ W^0(a)|\mathrm{Im}\ \chi_+ I$ is Fredholm on $L^p(\mathbb{R}_+, w)$ if and only if so is the operator $\chi_- I + \chi_+ W^0(a)$ on $L^p(\mathbb{R}, w)$.

In the next chapters we shall use localization techniques to reduce the study of $\chi_- I + \chi_+ W^0(a)$ to the investigation of the operators

$$\chi_- I + \chi_+ W^0(a(\eta - 0)\chi_\eta^- + a(\eta + 0)\chi_\eta^+)\,, \qquad \eta \in \mathbb{R}\,,$$

and

$$\chi_- I + \chi_+ W^0(a(-\infty)\chi_0^- + a(+\infty)\chi_0^+)\,,$$

where χ_η^- and χ_η^+ are, respectively, the characteristic functions of $(-\infty,\eta)$ and $(\eta,+\infty)$. But if $\alpha,\beta\in\mathbb{C}$, then

$$\begin{aligned}\chi_-I+\chi_+W^0(\alpha\chi_\eta^-+\beta\chi_\eta^+)&=\chi_-I+\chi_+W^0\Big(\alpha\frac{1+\sigma_\eta}{2}+\beta\frac{1-\sigma_\eta}{2}\Big)\\&=\Big(\chi_-+\frac{\alpha+\beta}{2}\chi_+\Big)I+\chi_+\frac{\alpha-\beta}{2}W^0(\sigma_\eta)\\&=bI+cW^0(\sigma_\eta)\,,\end{aligned}$$

and since $W^0(\sigma_\eta)=\lambda_\eta^{-1}S\lambda_\eta I$ and $\lambda_\eta^{\pm1}I$ are isomorphisms, we arrive at the operators

$$\begin{aligned}bI+cS&=b(P+Q)+c(P-Q)=(b+c)P+(b-c)Q\\&=(b-c)\Big(\frac{b+c}{b-c}P+Q\Big)=(b-c)(dP+Q)\,.\end{aligned}$$

Because now $(dP+Q)=(PdP+Q)(I+QdP)$ and $I+QdP$ has the inverse $I-QdP$, we are finally led to operators of the form $Pd|\mathrm{Im}\,P$ with $d\in PC$. But the latter operators are just the Toeplitz operators $T(b)$ on $\mathrm{Im}\,P=H^p(\mathbb{R},w)$ we have already studied in Chapter 2.

For further reference we summarize part of the preceding reasoning.

Lemma 3.4. *Let $b,c\in PC$ and $\eta\in\mathbb{R}$, and suppose $b-c\in GL^\infty(\mathbb{R})$. Then $bI+cW^0(\sigma_\eta)$ is Fredholm on $L^p(\mathbb{R},w)$ if and only if $T\big((b+c)/(b-c)\big)$ is Fredholm on $H^p(\mathbb{R},w)$.*

4. Local singular integral operators.

4.1. Preliminaries.

To carry out the program sketched in Section 3.3 we make use of the local principle of Gohberg and Krupnik [8] (see also [1], [2], [5] and [12]).

Let $\mathfrak{A}$ be a Banach algebra with identity element. A bounded subset $\mathfrak{M}\subset\mathfrak{A}$ is called a localizing class if $0\notin\mathfrak{M}$ and for every two elements $B_1,B_2\in\mathfrak{M}$ there exists a third element $B\in\mathfrak{M}$ such that $B_jB=BB_j=B$ for $j=1,2$. A family $\{\mathfrak{M}_\tau\}_{\tau\in T}$ of localizing classes is said to be covering if for every choice $\{B_\tau\}_{\tau\in T}$ of elements $B_\tau\in\mathfrak{M}_\tau$ there exist finitely many $\tau_1,\dots,\tau_m$ such that $B_{\tau_1}+\dots+B_{\tau_m}$ is invertible in $\mathfrak{A}$.

Let now $\{\mathfrak{M}_\tau\}_{\tau \in T}$ be a covering family of localizing classes in $\mathfrak{A}$ and put $\mathfrak{B} = \bigcup\{\mathfrak{M}_\tau : \tau \in T\}$. The commutant $\mathrm{Com}\,\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$. Fot $\tau \in T$, define

$$\mathfrak{Z}_\tau = \{A \in \mathrm{Com}\,\mathfrak{B} : \inf_{B \in \mathfrak{M}_\tau} ||AB|| = \inf_{B \in \mathfrak{M}_\tau} ||BA|| = 0\}\,.$$

One can easily show that $\mathfrak{Z}_\tau$ is a closed proper two-sided ideal of $\mathrm{Com}\,\mathfrak{B}$. Finally, for $A \in \mathrm{Com}\,\mathfrak{B}$, denote by A_τ the coset of the quotient algebra $\mathrm{Com}\,\mathfrak{B}/\mathfrak{Z}_\tau$ containing A.

Theorem 4.2. (Local principle of Gohberg and Krupnik, [8]). *With the notation introduced in Section* 4.1, *an element* $A \in \mathrm{Com}\,\mathfrak{B}$ *is invertible in* $\mathfrak{A}$ *if and only if* A_τ *is invertible in* $\mathrm{Com}\,\mathfrak{B}/\mathfrak{Z}_\tau$ *for every* $\tau \in T$.

4.3. The algebra $\mathfrak{A}$.

We apply Theorem 4.2 to the Calkin algebra $\mathfrak{A} = \mathcal{L}/\mathcal{K}$, where $\mathcal{L}$ is the Banach algebra of all bounded operators on $L^p(\mathbb{R}_+, w)$ and $\mathcal{K}$ stands for the ideal of all compact operators on $L^p(\mathbb{R}_+, w)$.

4.4. Localizing classes in $\mathfrak{A}$.

We are interested in "localizing" operators of the form $bI + cW^0(a)$, where $b, c \in PC$ and $a \in PC^p(w)$. In order to "localize" the coefficients b and c at $y \in \dot{\mathbb{R}}$ and the symbol a at $\eta \in \dot{\mathbb{R}}$, we consider the following candidates $\mathfrak{M}_{y,\eta}$ for localizing classes in $\mathfrak{A} = \mathcal{L}/\mathcal{K}$: the set $\mathfrak{M}_{y,\eta}$ consists of all cosets of the form $vW^0(u) + \mathcal{K}$ such that $v, u \in C(\dot{\mathbb{R}})$ are piecewise linear with finite total variation and v (respectively, u) is identically 1 in some open neighborhood of y (respectively, η) and identically 0 outside some other open neighborhood of y (respectively, η). One can show as in [5] (for $w \equiv 1$) or in [12] (for power weights) that $\mathfrak{M}_{y,\eta}$ coincides with $\mathcal{K}$ and thus the zero in $\mathcal{L}/\mathcal{K}$ whenever $y \in \mathbb{R}$ and $\eta \in \mathbb{R}$. However, if

$$(y, \eta) \in (\dot{\mathbb{R}} \times \dot{\mathbb{R}}) \backslash (\mathbb{R} \times \mathbb{R}) = (\mathbb{R} \times \{\infty\}) \cup (\{\infty\} \times \mathbb{R}) \cup \{(\infty, \infty)\} = T\,,$$

then $\mathfrak{M}_{y,\eta}$ is indeed a localizing class in $\mathcal{L}/\mathcal{K}$ (again see [5] or [12] for power weights).

To check whether $\{\mathfrak{M}_{(y,\eta)}\}_{(y,\eta)\in T}$ is a covering family, *i.e.* whether $\sum_{j=1}^{m} u_j W^0(v_j)$ is Fredholm provided $\sum u_j \geq 1$ and $\sum v_j \geq 1$, and to decide whether the cosets we are interested in, namely

$$bI + cW^0(a) + \mathcal{K}\,, \qquad b, c \in PC,\ a \in PC^p(w)\,,$$

belong to Com $\mathfrak{B}$ a good piece of work must be done. For $w \equiv 1$ all this is done in [5], and for power weights a detailed exposition of these things is in [12] (see also [14]). It is not difficult to convince oneself that the arguments of [5] and [12] extend to arbitrary Muckenhoupt weights and thus show that $\{\mathfrak{M}_{(y,\eta)}\}_{(y,\eta)\in T}$ is a covering family of localizing classes in $\mathcal{L}/\mathcal{K}$ and that all the cosets mentioned above are in Com $\mathfrak{B}$.

4.5. Localization in $\mathfrak{A}$.

For $b, c \in PC, a \in PC^p(w)$ and $(y, \eta) \in T$ we put

$$[bI + cW^0(a)]^\pi_{y,\eta} = [bI + cW^0(a) + \mathcal{K}]_{(y,\eta)}.$$

One can show (as in [4] and [12]) that

$$[bI + cW^0(a)]^\pi_{y,\eta} = [b_y I + c_y W^0(a_\eta)]^\pi_{y,\eta}$$

if $b_y,\ c_y \in PC$ and $a_\eta \in PC^p(w)$ are any function such that $b-b_y, c-c_y$ are continuous at y and $a - a_\eta$ is continuous at η. Hence, instead with an operator $bI + cW^0(a)$, one has to deal with the in general simpler operators $b_y I + c_y W^0(a_\eta)$; the price for this reduction is that invertibility in $\mathcal{L}/\mathcal{K}$ is replaced by invertibility in Com $\mathfrak{B}/\mathfrak{Z}_{(y,\eta)}$.

Our main concern in this chapter is the invertibility of the elements ("local singular integral operators") $[bI+cW^0(\sigma_\zeta)]^\pi_{y,\eta}$ in Com $\mathfrak{B}/\mathfrak{Z}_{(y,\eta)}$.

Lemma 4.6. *Let $b, c \in PC$, assume that $b - c$ and $b + c$ are both invertible in $L^\infty(\mathbb{R})$ (and thus in PC), and put $d = d_{b,c} = (b+c)/(b-c)$. Suppose further that $y, \eta, \zeta \in \mathbb{R}$. Then*

(1) $[bI + cW^0(\sigma_\zeta)]^\pi_{y,\infty}$ *is invertible if and only if*

$$0 \notin \mathcal{H}(d(y-0), d(y+0); \nu_y^-(p,w), \nu_y^+(p,w))\,;$$

(2) $[bI + cW^0(\sigma_\zeta)]^\pi_{\infty,\infty}$ *is invertible*;

(3) $[bI+cW^0(\sigma_\zeta)]^\pi_{\infty,\eta}$ *is invertible if* $\eta\neq\zeta$;

(4) $[bI+cW^0(\sigma_\zeta)]^\pi_{\infty,\zeta}$ *is invertible if and only if*

$$0\notin\mathcal{H}(d(+\infty),d(-\infty);\nu^-_\infty(p,w),\nu^+_\infty(p,w)).$$

PROOF. (1) We have

$$[bI+cW^0(\sigma_\zeta)]^\pi_{y,\infty}=[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{y,\infty}\ ,$$

where $b_y,c_y\in PC$ are any functions such that $b_y(y\pm0)=b(y\pm0)$ and $c_y(y\pm0)=c(y\pm0)$. The functions b_y and c_y may be chosen to be continuous on $\dot{\mathbb{R}}\backslash\{y\}$ and to satisfy $b_y\pm c_y\in GPC$ and $d_{b_y,c_y}(x)\neq0$ for all $x\in\dot{\mathbb{R}}\backslash\{y\}$.

Suppose first that 0 does not belong to the horn $\mathcal{H}=\mathcal{H}(\cdots)$. We then infer from Lemma 3.4 and Theorem 2.10(1) that $b_yI+c_yW^0(\sigma_\zeta)$ is Fredholm, which, by Theorem 4.2, implies that $[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{y,\infty}$ is all the more invertible.

Now suppose $0\in\mathcal{H}$ and, contrary to what we want, assume $[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{y,\infty}$ is invertible. For $x\in\dot{\mathbb{R}}\backslash\{y\}$ we have

$$[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{x,\infty}=[a_y(x)I+b_y(x)W^0(\sigma_\zeta)]^\pi_{x,\infty}\ ,$$

and since the operator $a_y(x)I+b_y(x)W^0(\sigma_\zeta)$ (having constant coefficients) is Fredholm by Lemma 3.4 and Theorem 2.10(1) (for constant symbols), $[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{x,\infty}$ must be invertible due to Theorem 4.2. Finally, if η is any point of $\mathbb{R}$, then

$$[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{\infty,\eta}=[b_y(\infty)I+c_y(\infty)W^0(\sigma_\zeta)]^\pi_{\infty,\eta}\ ,$$

and combining Lemma 3.4, Theorem 2.10(1) (for constant symbols) and the "only if" part of Theorem 4.2 we conclude again that $[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{\infty,\eta}$ is invertible

Hence it turns out that $[b_yI+c_yW^0(\sigma_\zeta)]^\pi_{x,\eta}$ is invertible for all $(x,\eta)\in T$, and so the "if" portion of Theorem 4.2 gives that $b_yI+c_yW^0(\sigma_\zeta)$ is Fredholm. This however contradicts Lemma 3.4 and Theorem 2.10(1), since $0\in\mathcal{H}$.

(2) The element $[bI+cW^0(\sigma_\zeta)]^\pi_{\infty,\infty}$ is equal to

$$\begin{aligned}g=[&b(-\infty)\chi_-I+b(+\infty)\chi_+I\\&+(c(-\infty)\chi_-+c(+\infty)\chi_+)W^0(\chi_--\chi_+)]^\pi_{\infty,\infty}\end{aligned}$$

and hence belongs to the closed subalgebra $\mathfrak{C}$ of $\mathrm{Com}\,\mathfrak{B}/\mathfrak{J}_{(\infty,\infty)}$ generated by

$$e = [I]^{\pi}_{\infty,\infty}, \quad r = [\chi_+ I]^{\pi}_{\infty,\infty}, \quad s = [W^0(\chi_+)]^{\pi}_{\infty,\infty}.$$

(notice that $\bar{\chi}_+ = 1 - \chi_+$). Because $\phi W^0(\psi) - W^0(\psi)\phi I$ is compact on $L^p(\mathbb{R}, w)$ whenever $\phi \in C(\bar{\mathbb{R}})$ and $\psi \in C(\dot{\mathbb{R}}) \cap PC^p(w)$ (see *e.g.* [12, p. 93] for power weights) and there are such ϕ and ψ with

$$r = [\phi I]^{\pi}_{\infty,\infty}, \qquad s = [W^0(\psi)]^{\pi}_{\infty,\infty},$$

it follows that $\mathfrak{C}$ is commutative and that $r^2 = r$ and $s^2 = s$. Let M denote the maximal ideal space of $\mathfrak{C}$ and let $\Gamma : \mathfrak{C} \to C(M)$ stand for the Gelfand transform. The spectra of the idempotents r and s are subsets of $\{0,1\}$. For $j, k \in \{0,1\}$, put

$$M_{jk} = \{m \in M : \ (\Gamma r)(m) = j, \ (\Gamma s)(m) = k\}.$$

So $M = M_{00} \cup M_{01} \cup M_{10} \cup M_{11}$, and if $m \in M_{jk}$, then

$$(\Gamma g)(m) = b(-\infty)(1-j) + b(+\infty)j + (c(-\infty)(1-j) + c(+\infty)j)(1-k-k),$$

which is one of the four numbers

$$b(-\infty) \pm c(-\infty), \qquad b(+\infty) \pm c(+\infty).$$

Since $b \pm c \in GL^\infty(\mathbb{R})$, we obtain that g is invertible in $\mathfrak{C}$ and thus all the more in $\mathrm{Com}\,\mathfrak{B}/\mathfrak{J}_{(\infty,\infty)}$.

(3) We now have

$$[bI + cW^0(\sigma_\zeta)]^{\pi}_{\infty,\eta} = [bI + \sigma_\zeta(\eta)cI]^{\pi}_{\infty,\eta},$$

and since $b + \sigma_\zeta(\eta)c$ is either $b - c$ or $b + c$, the multiplication operator $(b + \sigma_\zeta(\eta)c)I$ is invertible on $L^p(\mathbb{R}, w)$.

(4) Because

$$[bI + cW^0(\sigma_\zeta)]^{\pi}_{\infty,\zeta} = [b_\infty I + c_\infty W^0(\sigma_\zeta)]^{\pi}_{\infty,\zeta}$$

for any functions $b_\infty, c_\infty \in C(\bar{\mathbb{R}})$ such that

$$b_\infty(\pm\infty) = b(\pm\infty), \ c_\infty(\pm\infty) = c(\pm\infty), \ b_\infty \pm c_\infty \in GPC, \ d_{b_\infty,c_\infty} \neq 0,$$

for all $x \in \mathbb{R}$, it suffices to prove that $[b_\infty I + c_\infty W^0(\sigma_\zeta)]^\pi_{\infty,\zeta}$ is invertible if and only if 0 is not in the horn $\mathcal{H} \overset{\text{def}}{=} \mathcal{H}(\cdots)$.

If $0 \notin \mathcal{H}$, then $b_\infty I + c_\infty W^0(\sigma_\zeta)$ is Fredholm by Lemma 3.4 and Theorem 2.10(1) and hence $[b_\infty I + c_\infty W^0(\sigma_\zeta)]^\pi_{\infty,\zeta}$ is invertible by Theorem 4.2.

Conversely, assume $0 \in \mathcal{H}$ but $[b_\infty I + c_\infty W^0(\sigma_\zeta)]^\pi_{\infty,\zeta}$ is invertible. If $y \in \mathbb{R}$, then

$$[b_\infty I + c_\infty W^0(\sigma_\zeta)]^\pi_{y,\infty} = [b_\infty(y) + c_\infty(y) W^0(\sigma_\zeta)]^\pi_{y,\infty} ,$$

which is invertible by Lemma 3.4, Theorem 2.10(1) (with constant symbols), and Theorem 4.2. In case $\eta \in \dot{\mathbb{R}} \backslash \{\zeta\}$, we know that $[b_\infty I + c_\infty W^0(\sigma_\eta)]^\pi_{\infty,\eta}$ is invertible from the parts (2) and (3) we have already proved. Thus, $[b_\infty I + C_\infty W^0(\sigma_\zeta)]^\pi_{\infty,\eta}$ is invertible for all $(y, \eta) \in T$. From Theorem 4.2 we so infer that $b_\infty I + c_\infty W^0(\sigma_\zeta)$ is Fredholm, which contradicts Theorem 2.10(1), since $0 \in \mathcal{H}$.

5. Wiener-Hopf integral operators.

Lemma 5.1. *If $a \in PC^p(\omega) \cap GL^\infty(\mathbb{R})$ and $y, \eta \in \mathbb{R}$, then:*

(1) $[\chi_- I + \chi_+ W^0(a)]^\pi_{y,\infty}$ *is invertible if* $y \neq 0$;

(2) $[\chi_- I + \chi_+ W^0(a)]^\pi_{0,\infty}$ *is invertible if and only if*

$$0 \notin \mathcal{H}(a(+\infty), a(-\infty); \nu_0^-(p, w), \nu_0^+(p, w)) ;$$

(3) $[\chi_- I + \chi_+ W^0(a)]^\pi_{\infty,\infty}$ *is invertible*;

(4) $[\chi_- I + \chi_+ W^0(a)]^\pi_{\infty,\eta}$ *is invertible if and only if*

$$0 \notin \mathcal{H}(a(\eta - 0), a(\eta + 0); \nu_\infty^-(p, w), \nu_\infty^+(p, w)) .$$

PROOF. (1) The element $[\chi_- I + \chi_+ W^0(a)]^\pi_{y,\infty}$ is equal to $[I]^\pi_{y,\infty}$ for $y < 0$ and equal to

$$\begin{aligned}
&[W^0(a(-\infty)\chi_- + a(+\infty)\chi_+)]^\pi_{y,\infty} \\
&= \Big[\frac{a(-\infty) + a(+\infty)}{2} I + \frac{a(-\infty) - a(+\infty)}{2} W^0(\sigma)\Big]^\pi_{y,\infty} \\
&= [bI + cW^0(\sigma)]^\pi_{y,\infty}
\end{aligned}$$

for $y > 0$. It is clear that $[I]^{\pi}_{y,\infty}$ is invertible, and since

$$\frac{b+c}{b-c} = \frac{a(-\infty)}{a(+\infty)} \neq 0,$$

we deduce the invertility of $[\chi_- I + \chi_+ W^0(a)]^{\pi}_{y,\infty}$ from Lemma 4.6(1) (with constant b and c).

(2) The coset $[\chi_- I + \chi_+ W^0(a)]^{\pi}_{0,\infty}$ equals

$$\begin{aligned}&[\chi_- I + \chi_+ W^0(a(-\infty)\chi_- + a(+\infty)\chi_+)]^{\pi}_{0,\infty}\\ &\quad = \Big[\chi_- I + \frac{a(-\infty)+a(+\infty)}{2} I + \frac{a(-\infty)-a(+\infty)}{2} W^0(\sigma)\Big]^{\pi}_{0,\infty}\\ &\quad = [bI + cW^0(\sigma)]^{\pi}_{0,\infty}\,.\end{aligned}$$

We have

$$\Big(\frac{b+c}{b-c}\Big)(x) = 1 \quad \text{for } x < 0$$

and

$$\Big(\frac{b+c}{b-c}\Big)(x) = \frac{a(-\infty)}{a(+\infty)} \neq 0 \quad \text{for } x > 0\,.$$

Hence, Lemma 4.6(1) implies that $[\chi_- I + \chi_+ W^0(a)]^{\pi}_{0,\infty}$ is invertible if and only if

$$0 \notin \mathcal{H}\Big(1, \frac{a(-\infty)}{a(+\infty)}; \nu_0^-(p,w), \nu_0^+(p,w)\Big),$$

which happens if and only if

$$0 \notin \mathcal{H}(a(+\infty), a(-\infty); \nu_0^-(p,w), \nu_0^+(p,w))\,.$$

(3) Because $[\chi_- I + \chi_+ W^0(a)]^{\pi}_{\infty,\infty}$ equals

$$[\chi_- I + \chi_+ W^0(a(-\infty)\chi_- + a(+\infty)\chi_+)]^{\pi}_{\infty,\infty} = [bI + cW^0(\sigma)]^{\pi}_{\infty,\infty}\,,$$

the assertion is immediate from Lemma 4.6(2).

(4) As in Section 3.3, let χ_η^- and χ_η^+ be the characteristic functions of $(-\infty,\eta)$ and $(\eta,+\infty)$, respectively. Then

$$\begin{aligned}&[\chi_- I + \chi_+ W^0(a)]^{\pi}_{\infty,\eta} = [\chi_- I + \chi_+ W^0(a(\eta-0)\chi_\eta^- + a(\eta+0)\chi_\eta^+)]^{\pi}_{\infty,\eta}\\ &\quad = \Big[\chi_- I + \frac{a(\eta-0)+a(\eta+0)}{2} I + \frac{a(\eta-0)-a(\eta+0)}{2} W^0(\sigma_\eta)\Big]^{\pi}_{\infty,\eta}\end{aligned}$$

$$= [bI + cW^0(\sigma_\eta)]^\pi_{\infty,\eta} ,$$

and since

$$\frac{b+c}{b-c}(-\infty) = 1 , \qquad \frac{b+c}{b-c}(+\infty) = \frac{a(\eta-0)}{a(\eta+0)} ,$$

we obtain from Lemma 4.6(4) that $[\chi_- I + \chi_+ W^0(a)]^\pi_{\infty,\eta}$ is invertible if and only if

$$0 \notin \mathcal{H}\Big(\frac{a(\eta-0)}{a(\eta+0)}, 1; \nu^-_\infty(p,w), \nu^+_\infty(p,w)\Big) ,$$

which is equivalent to the condition that

$$0 \notin \mathcal{H}(a(\eta-0), a(\eta+0); \nu^-_\infty(p,w), \nu^+_\infty(p,w)) .$$

Here now is our main result

Theorem 5.2. *Let $w \in A_p$, $a \in PC^p(w)$, and define $\nu_0^\pm(p,w)$, $\nu_\infty^\pm(p, w)$ by Theorem 2.10(1). Then the essential spectrum of $W(a)$ on $L^p(\mathbb{R}_+, w)$ is*

$$a^{p,w} = \Big(\bigcup_{\eta\in\mathbb{R}} \mathcal{H}\big(a(\eta-0), a(\eta+0); \nu^-_\infty(p,w), \nu^+_\infty(p,w)\big)\Big) \bigcup \mathcal{H}(a(+\infty), a(-\infty); \nu^-_0(p,w), \nu^+_0(p,w)) .$$

If $0 \notin a^{p,w}$ then the index of $W(a)$ on $L^p(\mathbb{R}_+, w)$ equals $-\operatorname{wind} a^{p,w}$.

PROOF. We first show that the essential range of a is a subset of the essential spectrum of $W(a)$.

Let $\eta \in \dot{\mathbb{R}}$ be a point at which a is continuous and assume

$$W(a) - a(\eta)I = W(a - a(\eta))$$

is Fredholm on $L^p(\mathbb{R}_+, w)$. Then, by Section 3.3, the operator $\chi_- I + \chi_+ W^0(a - a(\eta))$ is also Fredholm on $L^p(\mathbb{R}, w)$ and consequently, by the "only if" part of Theorem 4.2,

$$[\chi_- I + \chi_+ W^0(a - a(\eta))]^\pi_{\infty,\eta} = [\chi_- I]^\pi_{\infty,\eta}$$

is invertible. This, however, is impossible because

$$[\chi_+ I]^{\pi}_{\infty,\eta}[\chi_- I]^{\pi}_{\infty,\eta} = [0]^{\pi}_{\infty,\eta}\,, \quad [\chi_+ I]^{\pi}_{\infty,\eta} \neq [0]^{\pi}_{\infty,\eta}\,.$$

Thus, the essential spectrum of $W(a)$ contains the values of a at all points at which it is continuous. Since these values are dense in the essential range of a and the essential spectrum of $W(a)$ is closed, it follows that the whole essential range is a subset of the essential spectrum.

We are now left with showing that if $a \in PC^p(w) \cap GL^\infty(\mathbb{R})$, then $\chi_- I + \chi_+ W^0(a)$ is Fredholm on $L^p(\mathbb{R}, w)$ if and only if $0 \notin a^{p,w}$; the index formula then follows by a standard homotopy argument from the case of continuous symbols.

But Theorem 4.2 in conjunction with Lemma 5.1 implies at once that if $a \in PC^p(w) \cap GL^\infty(\mathbb{R})$, then the Fredholmness of $W(a)$ is equivalent to the condition that $0 \notin a^{p,w}$.

5.3. CONCLUDING REMARK. Lemma 4.6 can also be used to gain interesting information about the Fredholmness of operators of the form

$$A = \sum_{j=1}^{m} b_j W^0(a_j)\,,$$

i.e. pseudodifferential operators with symbols

$$\sum_{j=1}^{m} b_j(x) a_j(\xi) \qquad b_j \in PC, a_j \in PC^p(w)$$

on $L^p(\mathbb{R}, w)$ (for the case $w \equiv 1$ see [4] and for power weights see [12] and [14]). We shall devote more space to this question in a forthcoming paper.

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