

Further pseudodifferential
operators generating
Feller semigroups
and Dirichlet forms

Niels Jacob

Abstract. We prove for a large class of symmetric pseudo differential operators that they generate a Feller semigroup and therefore a Dirichlet form. Our construction uses the Yosida-Hille-Ray Theorem and a priori estimates in anisotropic Sobolev spaces. Using these a priori estimates it is possible to obtain further information about the stochastic process associated with the Dirichlet form under consideration.

Introduction.

Generators of Feller semigroups are characterized by the Yosida-Hille-Ray Theorem, see [7] or [13]. This characterization involves the notion of the positive maximum principle. It was Ph. Courrège [6] who gave a characterization of operators satisfying this maximum principle. He proved that these operators are certain integro-differential-operators and later these results had been developed further in order to continue studies started by von Waldenfels [39]-[40]. More recent results on generators of Feller semigroups can be found in [8] or [37]. However, even in the paper [6], Courrège gave also a characterization of

the operators satisfying the positive maximum principle as pseudodifferential operators. These pseudodifferential operators have a symbol $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ with the property that for any $x \in \mathbb{R}^n$ the function $\xi \mapsto a(x, \xi)$ is a continuous negative definite function, see Definition 1.1. At that time the theory of pseudodifferential operators was rather young, see [18] and [28], and it seems to us that this characterization has never been used to construct Feller semigroups or Markov processes. Only recently there are some investigations using the theory of pseudodifferential operators to construct and study Markov processes. For example T. Komatsu in [29]-[32] uses the theory of elliptic standard pseudo differential operators to study perturbations of symmetric stable processes; and in [27] A. N. Kochubei uses the theory of hypersingular integrals due to S. G. Samko, see [36], to obtain Markov processes by constructing fundamental solutions of certain parabolic pseudo differential operators.

On the other hand continuous negative definite functions do enter in a quite different way in the theory of Markov processes. They do also characterize translation invariant Dirichlet forms, see [2], [9] and as a standard reference for Dirichlet forms [14]. In [23] we pointed out that it is possible to combine Hilbert space methods, which are very convenient when working with Dirichlet forms, with the result of Courrège in order to construct a Feller semigroup starting with a pseudodifferential operator having a symbol with the properties mentioned above. In [24] a special example of an elliptic pseudodifferential operator was discussed in detail.

The purpose of this paper is to give further examples of pseudo differential operators generating Feller semigroups and to study properties of corresponding objects like the associated Dirichlet form and Markov process. Now the operators under consideration are no longer elliptic pseudodifferential operators and some classical calculus of pseudodifferential operators is not applicable. While our strategy in constructing the semigroup is just the same as in [23], it was necessary to strengthen some results of [23]. In particular the commutator estimate in Section 6 improving an earlier result, see [22], plays an essential part in proving the fundamental estimates.

1. Notations and auxiliary results.

Most of our notations are standard. The spaces $C_0^\infty(\mathbb{R}^n)$, $C^m(\mathbb{R}^n)$, $0 \leq m \leq \infty$, $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, the Schwartz space $S(\mathbb{R}^n)$ and the Sobolev space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, are defined as usual. In the case $m = 0$ we write $C(\mathbb{R}^n)$ instead of $C^m(\mathbb{R}^n)$. By $C_\infty(\mathbb{R}^n)$ the space of all continuous functions $u : \mathbb{R}^n \rightarrow \mathbb{R}$ vanishing at infinity is denoted. Further we set $H^\infty(\mathbb{R}^n) = \bigcap_{s \geq 0} H^s(\mathbb{R}^n)$. The supremum norm is denoted by $\|\cdot\|_\infty$, the norm in $H^s(\mathbb{R}^n)$ is denoted by $\|\cdot\|_s$, in particular the norm in $L^2(\mathbb{R}^n)$ is $\|\cdot\|_0$ and $(\cdot, \cdot)_0$ is the scalar product in $L^2(\mathbb{R}^n)$. The norm in $L^p(\mathbb{R}^n)$, $p \neq 2, \infty$, is denoted by $\|\cdot\|_{L^p}$. For $s \in \mathbb{R}$ we define the function $\Lambda^s : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\xi \mapsto \Lambda^s(\xi) = (1 + |\xi|^2)^{s/2}$ and we have $\|u\|_s = \|\Lambda^s(D)u\|_0$, where $\Lambda^s(D)$ is given by

$$(1.1) \quad \Lambda^s(D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \Lambda^s(\xi) \hat{u}(\xi) d\xi .$$

Here and in the following we denote by \hat{u} the Fourier transform of u . By the Sobolev embedding theorem, see p. 121 of [33], we find for $s > m + n/2$, $m \in \mathbb{N}$, that

$$(1.2) \quad \|u\|_{C^m} \leq c \|u\|_s$$

holds for all $u \in H^s(\mathbb{R}^n)$, where

$$\|u\|_{C^m} = \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha u(x)| .$$

Moreover it follows that for $|\alpha| \leq m$ we have $\partial^\alpha u \in C_\infty(\mathbb{R}^n)$. In order that the last statement makes sense we have to identify $H^s(\mathbb{R}^n)$ with a subspace of $C^m(\mathbb{R}^n)$, which is possible by Sobolev's embedding theorem for $s > m + n/2$. Thus in this case we will always regard $H^s(\mathbb{R}^n)$ as a subspace of $C^m(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for all $s \geq 0$ and since $C_0^\infty(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_\infty$, it follows that for $s > n/2$ the space $H^s(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$ with respect to the supremum norm.

Next let us introduce the notion of a negative definite function which will be central in our paper.

Definition 1.1. A function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be negative definite if for all $m \in \mathbb{N}$ and (x^1, \dots, x^m) , $x^j \in \mathbb{R}^n$, $1 \leq j \leq m$, the matrix

$$\left(a(x^i) + \overline{a(x^j)} - a(x^i - x^j) \right)_{i,j=1,\dots,m}$$

is positive Hermitian, i.e. if for all m -tuples $(c_1, \dots, c_m) \in \mathbb{C}^m$

$$(1.3) \quad \sum_{i,j=1}^m \left(a(x^i) + \overline{a(x^j)} - a(x^i - x^j) \right) c_i \overline{c_j} \geq 0.$$

A standard reference for negative definite functions is the book [1]. In particular the following lemma is proved there.

Lemma 1.1. *Let $a : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function. Then a is nonnegative and there exists a constant c_a such that*

$$(1.4) \quad a(\xi) \leq c_a (1 + |\xi|^2)$$

for all $\xi \in \mathbb{R}^n$. Further $a^{1/2}$ is also a negative definite function.

We also recall Lemma 2.1 of our paper [22].

Lemma 1.2. *Let $a : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function. Then we have for all $\xi, \eta \in \mathbb{R}^n$*

$$(1.5) \quad |a(\xi) - a(\eta)| \leq 4 a^{1/2}(\xi) a^{1/2}(\xi - \eta) + a(\xi - \eta).$$

As pointed out in [19, p. 327-328], (see also Section 10 of this paper) using examples of continuous negative definite functions given in [1], continuous negative definite functions need not be differentiable nor do they in general belong to classical symbol classes such as the class $S_{\rho,\delta}^m(\mathbb{R}^n)$, see [38] for the definition.

Finally let us remark that throughout this paper c denotes a non-negative constant which may change from line to line.

2. Some function spaces.

In this section we will introduce a family of anisotropic Sobolev spaces related to a continuous negative definite function.

Definition 2.1. *Let $a^2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function and $q \geq 0$ a real number. We define the space $H^{a^2,q}(\mathbb{R}^n)$ by*

$$(2.1) \quad H^{a^2,q}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + a^2(\xi))^{2q} |\hat{u}(\xi)|^2 d\xi < \infty \right\}$$

On $H^{a^2, q}(\mathbb{R}^n)$ we have the norm

$$(2.2) \quad \|u\|_{q, a^2}^2 = \int_{\mathbb{R}^n} (1 + a^2(\xi))^{2q} |\hat{u}(\xi)|^2 d\xi .$$

With the norm (2.2) the space $H^{a^2, q}(\mathbb{R}^n)$ is a Hilbert space and $C_0^\infty(\mathbb{R}^n)$ is a dense subspace. Moreover, by Lemma 1.1 it follows that $H^{a^2, q}(\mathbb{R}^n)$ contains the space $H^{2q}(\mathbb{R}^n)$ and is contained in $L^2(\mathbb{R}^n)$. Later we will often assume that for some t , $0 < t \leq 2$, the estimate

$$(2.3) \quad c_t (1 + |\xi|^2)^{t/4} \leq (1 + a^2(\xi))^{1/2}$$

holds. Clearly this implies that $H^{a^2, q}(\mathbb{R}^n)$ is continuously embedded into the space $H^{tq}(\mathbb{R}^n)$, *i.e.* we have

$$(2.4) \quad \|u\|_{tq} \leq c'_t \|u\|_{q, a^2} .$$

Thus when (2.3) holds and q is sufficiently large we can identify $H^{a^2, q}(\mathbb{R}^n)$ with a dense subspace of $C_\infty(\mathbb{R}^n)$.

The next lemma is proved as Proposition 1.5.A in [21].

Lemma 2.1. *Suppose that $\lim_{|\xi| \rightarrow \infty} (1 + a^2(\xi)) = \infty$ holds. Further let $q \geq 0$ be given and $r > q$. Then for any $\eta > 0$ there exists a constant $c(\eta) = c(\eta; r, q, a^2)$ such that*

$$(2.5) \quad \|u\|_{q, a^2} \leq \eta \|u\|_{r, a^2} + c(\eta) \|u\|_0$$

for all $u \in H^{a^2, r}(\mathbb{R}^n)$.

We will need the following characterization of the dual space of $H^{a^2, q}(\mathbb{R}^n)$.

Proposition 2.1. *Let a^2 and q be as in Definition 2.1. Then the dual space of $H^{a^2, q}(\mathbb{R}^n)$ is the completion of $L^2(\mathbb{R}^n)$ with respect to the norm*

$$(2.6) \quad \|u\|_{-q, a^2} = \sup_{0 \neq v \in H^{a^2, q}(\mathbb{R}^n)} \frac{|(u, v)_0|}{\|v\|_{q, a^2}} .$$

Moreover, for $u \in L^2(\mathbb{R}^n)$ we have

$$(2.7) \quad \|u\|_{-q, a^2}^2 = \int_{\mathbb{R}^n} (1 + a^2(\xi))^{-2q} |\hat{u}(\xi)|^2 d\xi .$$

Since $L^2(\mathbb{R}^n)$ is dense in $[H^{a^2, q}(\mathbb{R}^n)]^*$ with respect to the norm $\|\cdot\|_{-q, a^2}$ we have $[H^{a^2, q}(\mathbb{R}^n)]^* = H^{a^2, -q}(\mathbb{R}^n)$, where the space $H^{a^2, -s}(\mathbb{R}^n)$ is defined by (2.1) taking $-q$ instead of q and u to be a Schwartz distribution.

In the case of the usual Sobolev spaces this result can be found in [34, p. 31]. The proof of our proposition follows essentially the lines of the considerations in [11, p. 201-203], and is left to the reader. We also refer to Proposition 1.2 in [21].

3. The operator $L(x, D)$.

As pointed out in the introduction we want to construct a Feller semigroup and therefore a Dirichlet form by starting with a pseudodifferential operator $L(x, D)$. This operator will be introduced now. For $1 \leq j \leq n$ let $a_j^2: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous negative definite functions. Further, for $1 \leq j \leq n$ we assume that functions $b_j: \mathbb{R}^n \rightarrow \mathbb{R}$ are given. The operator $L(x, D)$ is defined by

$$(3.1) \quad L(x, D) = \sum_{j=1}^n b_j(x) a_j^2(D_j) ,$$

where $a_j^2(D_j)$, $1 \leq j \leq n$, is the operator

$$(3.2) \quad a_j^2(D_j) u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} a_j^2(\xi_j) \hat{u}(\xi) d\xi .$$

Clearly by Lemma 1.1 this operator is well defined on $C_0^\infty(\mathbb{R}^n)$. Later we will introduce a larger domain for $L(x, D)$. Let $a^2: \mathbb{R}^n \rightarrow \mathbb{R}$ be the function

$$(3.3) \quad a^2(\xi) = \sum_{j=1}^n a_j^2(\xi_j) .$$

Since each of the functions a_j^2 is a continuous negative definite function it is clear that a^2 and $1 + a^2$ are continuous negative definite functions (see Section 10). Obviously we have with suitable constants for any $s \geq 0$

$$(3.4) \quad c_{n,s} \leq \frac{1 + \sum_{j=1}^n a_j^{2s}(\xi_j)}{(1 + a^2(\xi))^s} \leq \tilde{c}_{n,s} .$$

Now take $r, t \in (0, 2]$, $r \geq t > 0$ and let m_0 be the smallest even integer such that

$$(3.5) \quad t(m_0 + 1) \geq 3 + \left[\frac{n}{2} \right]$$

holds. Further suppose

$$0 < 1 - \frac{r}{2t} - \frac{r(r-t)m_0}{t^2}.$$

Let $\delta \in (0, 1 - r/(2t) - r(r-t)m_0/t^2)$ be fixed and define s by

$$(3.6) \quad s = (1 - \delta) \frac{t}{r} - \frac{1}{2}.$$

It follows that

$$(3.7) \quad s - \frac{(r-t)m_0}{t} > 0.$$

Taking t and r as above we assume

$$(3.8) \quad c_t (1 + |\xi|^2)^{t/2} \leq 1 + a^2(\xi) \leq c_r (1 + |\xi|^2)^{r/2}.$$

Now we can state our assumptions on the coefficients b_j .

B.1. It is assumed that b_j is bounded and continuous.

B.2. We suppose that $b_j = d_j + c_j$, where c_j is a real number and $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function satisfying

$$(3.9) \quad |\hat{d}_j(\xi)| \leq c_q (1 + |\xi|^2)^{-q}.$$

where $q = n + r(s + 1/2) + tm_0 + 1$.

B.3. For all $x \in \mathbb{R}^n$ we require

$$(3.10) \quad b_j(x) \geq \delta_1 > 0$$

to hold.

B.4. For some $x_0 \in \mathbb{R}^n$ we assume

$$(3.11) \quad \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \leq \frac{\delta_1}{2n} .$$

In order to prove commutator estimates for the operators $a_j(D_j)$ we have in addition to assume that

$$(3.12) \quad c_j (1 + |\xi_j|^2)^{t_j/2} \leq 1 + a_j^2(\xi_j) \leq \tilde{c}_j (1 + |\xi_j|^2)^{r_j/2} ,$$

where $0 < t \leq t_j \leq r_j \leq r \leq 2$.

In Section 10 we will give examples of operators satisfying all the conditions stated above. Note that our assumptions are far from being the most general or the sharpest. In particular when considering special operators as it was done in [24] or [16] much of our assumptions could be relaxed. Beside the operator $L(x, D)$ we also will often consider the operator $L^\lambda(x, D) = L(x, D) + \lambda$, $\lambda \in \mathbb{R}$.

The next lemma will be used frequently

Lemma 3.1. *For all $u \in C_0^\infty(\mathbb{R}^n)$ we have*

$$(3.13) \quad \|a_j(D_j)u\|_0^2 \leq \sum_{j=1}^n \|a_j(D_j)u\|_0^2 \leq \|u\|_{1/2, a^2}^2 .$$

PROOF. For $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} \|a_j(D_j)u\|_0^2 &= \int_{\mathbb{R}^n} a_j^2(\xi_j) |\hat{u}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} \sum_{j=1}^n a_j^2(\xi_j) |\hat{u}(\xi)|^2 d\xi \\ &\leq \int_{\mathbb{R}^n} (1 + \sum_{j=1}^n a_j^2(\xi_j)) |\hat{u}(\xi)|^2 d\xi = \|u\|_{1/2, a^2}^2 . \end{aligned}$$

4. On the bilinear form associated with $L(x, D)$.

The operator $L(x, D)$ can be regarded as a pseudo differential operator with symbol

$$L(x, \xi) = \sum_{j=1}^n b_j(x) a_j^2(\xi_j) .$$

This operator is clearly defined on $C_0^\infty(\mathbb{R}^n)$ by

$$\begin{aligned}
 L(x, D)u(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} L(x, \xi) \hat{u}(\xi) d\xi \\
 (4.1) \qquad &= (2\pi)^{-n/2} \sum_{j=1}^n b_j(x) \int_{\mathbb{R}^n} e^{ix\xi} a_j^2(\xi_j) \hat{u}(\xi) d\xi .
 \end{aligned}$$

Since $a_j^2(D_j)u \in C_\infty(\mathbb{R}^n)$ for any $u \in C_0^\infty(\mathbb{R}^n)$, see [21, Proposition 1.4] and $b_j \in L^\infty(\mathbb{R}^n)$, it follows that we can define on $C_0^\infty(\mathbb{R}^n)$ the bilinear form

$$(4.2) \qquad B(u, v) = (L(x, D)u, v)_0 = \sum_{j=1}^n (b_j(\cdot) a_j^2(D_j)u, v)_0 .$$

The bilinear form associated with $L^\lambda(x, D)$ is denoted by B_λ . Our next aim is to prove that B has a continuous extension onto $H^{a^2, 1/2}(\mathbb{R}^n)$. For this we need

Proposition 4.1. *Suppose that (3.12) holds. Then for any $\eta > 0$ there exists a constant $c(\eta) \geq 0$ such that*

$$(4.3) \qquad \|[a_j(D_j), b_j(\cdot)]u\|_{1/2, a^2} \leq \eta \|u\|_{1/2, a^2} + c(\eta) \|u\|_0$$

for all $u \in H^{a^2, 1/2}(\mathbb{R}^n)$.

As usual we denote by $[a_j(D_j), b_j(\cdot)]$ the commutator of $a_j(D_j)$ and $b_j(\cdot)$, i.e. the operator

$$u \mapsto a_j(D_j)(b_j u)(\cdot) - b_j(\cdot) a_j(D_j)u(\cdot) .$$

The proof of Proposition 4.1 is analogous to that of Corollary 3.2 in [22]. It will be given in Section 6 where more general commutator estimates are discussed. Using the commutator $[a_j(D_j), b_j]$ we can write B in a more appropriate way, namely

$$\begin{aligned}
 B(u, v) &= \sum_{j=1}^n (b_j(\cdot) a_j(D_j)u, a_j(D_j)v)_0 \\
 (4.4) \qquad &+ \sum_{j=1}^n (a_j(D_j)u, [a_j(D_j), b_j(\cdot)]v)_0 .
 \end{aligned}$$

Now we claim

Theorem 4.1. *For all $u, v \in C_0^\infty(\mathbb{R}^n)$ we have*

$$(4.5) \quad |B(u, v)| \leq c \|u\|_{1/2, a^2} \|v\|_{1/2, a^2} .$$

PROOF. Let $u, v \in C_0^\infty(\mathbb{R}^n)$. Then it follows that

$$\begin{aligned} |B(u, v)| &\leq \sum_{j=1}^n |(b_j(\cdot) a_j(D_j) u, a_j(D_j) v)_0| \\ &\quad + \sum_{j=1}^n |(a_j(D_j) u, [a_j(D_j), b_j(\cdot)] v)_0| \\ &\leq c \sum_{j=1}^n \|a_j(D_j) u\|_0 \|a_j(D_j) v\|_0 \\ &\quad + \sum_{j=1}^n \|a_j(D_j) u\|_0 \|[a_j(D_j), b_j(\cdot)] v\|_0 \\ &\leq c \|u\|_{1/2, a^2} \|v\|_{1/2, a^2} , \end{aligned}$$

where we used Lemma 3.1 and Proposition 4.1 for the last step.

Obviously (4.5) holds also for B_λ , $\lambda \in \mathbb{R}$. Thus B_λ has a continuous extension onto $H^{a^2, 1/2}(\mathbb{R}^n)$ which is again denoted by B_λ . Furthermore we have

Theorem 4.2. *For all $u \in H^{a^2, 1/2}(\mathbb{R}^n)$ we have with a suitable constant d_0*

$$(4.6) \quad B(u, u) \geq \frac{\delta_1}{2} \|u\|_{1/2, a^2}^2 - d_0 \|u\|_0^2 .$$

PROOF. It is sufficient to prove (4.6) for all $u \in C_0^\infty(\mathbb{R}^n)$. For these u we find

$$\begin{aligned} B(u, u) &= \sum_{j=1}^n (b_j(\cdot) a_j^2(D_j) u, u)_0 \\ &\geq \sum_{j=1}^n (b_j(\cdot) a_j(D_j) u, a_j(D_j) u)_0 \end{aligned}$$

$$\begin{aligned}
 & - \left| \sum_{j=1}^n (a_j(D_j)u, [a_j(D_j), b_j(\cdot)]u)_0 \right| \\
 & = B_1 - |B_2|.
 \end{aligned}$$

Now we get using B.3

$$(4.7) \quad B_1 = \int_{\mathbb{R}^n} \sum_{j=1}^n b_j(x) |a_j^2(D_j)u(x)|^2 dx \geq \delta_1 \|u\|_{1/2, a^2} - \delta_1 \|u\|_0^2.$$

Using Proposition 4.1 we can estimate B_2 as follows

$$\begin{aligned}
 |B_2| & = \left| \sum_{j=1}^n (a_j(D_j)u, [a_j(D_j), b_j(\cdot)]u)_0 \right| \\
 & \leq \sum_{j=1}^n \|u\|_{1/2, a^2} (\eta \|u\|_{1/2, a^2} + c(\eta) \|u\|_0) \\
 (4.8) \quad & \leq \varepsilon \|u\|_{1/2, a^2}^2 + c(\varepsilon) \|u\|_0^2,
 \end{aligned}$$

where $\eta > 0$ and therefore $\varepsilon > 0$ are sufficiently small constants. Thus by (4.7) and (4.8) we have

$$B(u, u) \geq (\delta_1 - \varepsilon) \|u\|_{1/2, a^2}^2 - (c(\varepsilon) + \delta_1) \|u\|_0^2,$$

which implies (4.6).

It follows that $L^\lambda(x, D)$, $\lambda \in \mathbb{R}$, has a closed extension L^λ , called the Friedrichs extension, with domain $D(L^\lambda)$ defined as the set of the functions $u \in H^{a^2, 1/2}(\mathbb{R}^n)$ such that

(4.9) there exists $f \in L^2(\mathbb{R}^n)$ such that for all $v \in H^{a^2, 1/2}(\mathbb{R}^n)$:

$$B_\lambda(u, v) = (f, v)_0.$$

Note that L^λ is the only closed extension of $L^\lambda(x, D)$ with the property that $D(L^\lambda) \subset H^{a^2, 1/2}(\mathbb{R}^n)$, see [26] or [41]. Moreover $-L^\lambda$ is the generator of an analytic semigroup of contractions provided λ is sufficiently large. Our next goal is to characterize the domain $D(L^\lambda)$.

5. A characterization of $D(L^\lambda)$.

First of all let us prove

Proposition 5.1. *The operator $L^\lambda(x, D)$, $\lambda \in \mathbb{R}$, has a continuous extension onto $H^{a^2, 1}(\mathbb{R}^n)$, i.e. $L^\lambda(x, D) : H^{a^2, 1}(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a continuous operator.*

PROOF. For $u \in C_0^\infty(\mathbb{R}^n)$ we find using B.1

$$\|L^\lambda(x, D)u\|_0 \leq \left\| \sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u \right\|_0 + |\lambda| \|u\|_0 \leq c' \|u\|_{1, a^2}.$$

The next estimate will give an important regularity result for solutions of the equation $L^\lambda(x, D)u = f$.

Theorem 5.1. *Under the assumptions B.1-B.4 and (3.12) there exists a constant c_λ such that*

$$(5.1) \quad \|u\|_{1, a^2} \leq c_\lambda (\|L^\lambda(x, D)u\|_0 + \|u\|_0)$$

holds for all $u \in L^2(\mathbb{R}^n)$ with $L^\lambda(x, D)u \in L^2(\mathbb{R}^n)$.

PROOF. Using Proposition 2.1 we find

$$\begin{aligned} & \|u\|_{1, a^2} \left\| \sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u + \lambda u \right\|_0 \\ &= \|u\|_{1, a^2} \left\| \left(1 + \sum_{l=1}^n a_l^2(D_l)\right) \left(\sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u + \lambda u\right) \right\|_{-1, a^2} \\ &\geq (u, \left(1 + \sum_{l=1}^n a_l^2(D_l)\right) \left(\sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u + \lambda u\right))_0 \\ &= \left(\left(1 + \sum_{l=1}^n a_l^2(D_l)\right)u, \sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u + \lambda u \right)_0 \\ &= \sum_{j, l=1}^n (a_l^2(D_l)u, b_j(\cdot) a_j^2(D_j)u)_0 + \lambda \|u\|_0^2 \end{aligned}$$

$$+ \lambda \left(\sum_{l=1}^n a_l^2(D_l)u, u \right)_0 + \sum_{j=1}^n (u, b_j(\cdot) a_j^2(D_j)u)_0 .$$

For $\lambda \geq 0$ we find

$$(5.2) \quad \lambda \left(\sum_{l=1}^n a_l^2(D_l)u, u \right)_0 = \lambda \sum_{l=1}^n \|a_l(D_l)u\|_0^2 \geq 0 .$$

Further we have with x_0 as in B.4

$$\begin{aligned} \sum_{j,l=1}^n (a_l^2(D_l)u, b_j(\cdot) a_j^2(D_j)u)_0 &= \sum_{j,l=1}^n (a_l^2(D_l)u, b_j(x_0) a_j^2(D_j)u)_0 \\ &\quad + \sum_{j,l=1}^n ((b_j(\cdot) - b_j(x_0)) a_j^2(D_j)u, a_l^2(D_l)u)_0 \\ &= A_1 + A_2 . \end{aligned}$$

Now, by B.3 and Lemma 2.1 we get for $\eta_1 > 0$

$$\begin{aligned} A_1 &= \sum_{j,l=1}^n (b_j(x_0) a_j^2(D_j)u, a_l^2(D_l)u)_0 \\ &\geq \delta_1 \sum_{j,l=1}^n \int_{\mathbb{R}^n} a_j^2(\xi_j) a_l^2(\xi_l) |\hat{u}(\xi)|^2 d\xi \\ &= \delta_1 \int_{\mathbb{R}^n} \left(1 + \sum_{j=1}^n a_j^2(\xi_j) \right)^2 |\hat{u}(\xi)|^2 d\xi - \delta_1 \|u\|_0^2 \\ &\quad - 2 \delta_1 \int_{\mathbb{R}^n} \sum_{j=1}^n a_j^2(\xi_j) |\hat{u}(\xi)|^2 d\xi \\ (5.3) \quad &\geq (\delta_1 - \eta_1) \|u\|_{1,a^2}^2 - c(\delta_1, \eta_1) \|u\|_0^2 . \end{aligned}$$

Now we estimate A_2 by taking into account B.4:

$$\begin{aligned} |A_2| &\leq \sum_{j,l=1}^n |(b_j(\cdot) - b_j(x_0)) a_j^2(D_j)u, a_l^2(D_l)u)_0| \\ &\leq \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \sum_{j,l=1}^n \|a_j^2(D_j)u\|_0 \|a_l^2(D_l)u\|_0 \end{aligned}$$

$$\begin{aligned}
&\leq \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \left(\sum_{j=1}^n \|a_j^2(D_j)u\|_0 \right) \left(\sum_{l=1}^n \|a_l^2(D_l)u\|_0 \right) \\
&\leq n \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \|u\|_{1,a^2}^2 \\
&\leq \frac{\delta_1}{2} \|u\|_{1,a^2}^2.
\end{aligned}$$

Thus

$$(5.4) \quad |A_2| \leq \frac{\delta_1}{2} \|u\|_{1,a^2}^2.$$

Moreover using Proposition 4.1 and Lemma 2.1 we find for any $\eta_2 > 0$

$$\begin{aligned}
&\left| \sum_{j=1}^n (u, b_j(\cdot) a_j^2(D_j)u)_0 \right| \\
&\leq c \sum_{j=1}^n |(a_j(D_j)u, a_j(D_j)u)_0| + \sum_{j=1}^n |([b_j(\cdot), a_j(D_j)]u, a_j(D_j)u)_0| \\
&\leq c \|u\|_{1/2,a^2}^2 \leq \eta_2 \|u\|_{1,a^2}^2 + c(\eta_2) \|u\|_0^2.
\end{aligned}$$

Thus we find for $\lambda \geq 0$

$$\begin{aligned}
&\|u\|_{1,a^2} \left\| \sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u + \lambda u \right\|_0 \\
&\geq (\delta_1 - \delta_1/2 - \eta_1 - \eta_2) \|u\|_{1,a^2}^2 \\
&\quad + \lambda \sum_{l=1}^n \|a_l(D_l)u\|_0^2 + (\lambda - c(\delta_1, \eta_1, \eta_2)) \|u\|_0^2.
\end{aligned}$$

For $\eta_1 = \eta_2 = \delta_1/8$ and $\lambda \geq \lambda_0 = c(\delta_1)$ we find using (3.12)

$$(5.5) \quad \|L^\lambda(x, D)u\|_0 \geq \frac{\delta_1}{4} \|u\|_{1,a^2}.$$

Now let $\lambda \in \mathbb{R}$ be arbitrary. Then it follows that

$$\begin{aligned}
\|u\|_{1,a^2} &\leq c \|L^{\lambda_0}(x, D)u\|_0 \\
&= c \|L^\lambda(x, D)u + (\lambda_0 - \lambda)u\|_0 \\
&\leq c (\|L^\lambda(x, D)u\|_0 + |\lambda_0 - \lambda| \|u\|_0)
\end{aligned}$$

$$\leq c' (\|L^\lambda(x, D)u\|_0 + \|u\|_0),$$

which proves the theorem.

Note that Theorem 5.1 also follows from Theorem 7.1 below, even the structure of the proof is the same. However our proof of Theorem 5.1 does not use Theorem 6.1, that is why we have given it separately.

Corollary 5.1. *Let us consider $L^\lambda(x, D)$ as an operator on $L^2(\mathbb{R}^n)$ with domain $H^{a^2,1}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Then $L^\lambda(x, D)$ is a closed operator.*

PROOF. We prove that $H^{a^2,1}(\mathbb{R}^n)$ equipped with the graph norm $\|u\|_0 + \|L^\lambda(x, D)u\|_0$ is a Hilbert space. By Proposition 5.1 we have

$$\|u\|_0 + \|L^\lambda(x, D)u\|_0 \leq c \|u\|_{1,a^2}$$

for all $u \in H^{a^2,1}(\mathbb{R}^n)$. Conversely, by Theorem 5.1 we find

$$c' \|u\|_{1,a^2} \leq \|u\|_0 + \|L^\lambda(x, D)u\|_0$$

for all $u \in H^{a^2,1}(\mathbb{R}^n)$, since for these u we have by Proposition 5.1 that $L^\lambda(x, D)u \in L^2(\mathbb{R}^n)$. Thus the graph norm is equivalent to the norm $\|\cdot\|_{1,a^2}$ which implies that $H^{a^2,1}(\mathbb{R}^n)$ is a Hilbert space with respect to the graph norm.

Following [26, p. 325-326], we get

Theorem 5.2. *Let $L^\lambda(x, D)$ be as above and let L^λ be its Friedrichs extension. Then we have $D(L^\lambda) = H^{a^2,1}(\mathbb{R}^n)$ and $L^\lambda = L^\lambda(x, D)$ as operators defined on $H^{a^2,1}(\mathbb{R}^n)$.*

Note that Theorem 5.2 is a regularity result for solutions of the representation problem

$$(5.6) \quad B_\lambda(u, \varphi) = (f, \varphi)_0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n),$$

where $f \in L^2(\mathbb{R}^n)$ is a given function. For $\lambda \geq d_0$ the non-symmetric version of the Lax-Milgram theorem implies that (5.6) always has a solution in $H^{a^2,1/2}(\mathbb{R}^n)$, while the definition of $D(L^\lambda)$ together with Theorem 5.2 gives that this solution belongs already to $H^{a^2,1}(\mathbb{R}^n)$.

6. Some commutator estimates.

In order to get further regularity results for solutions of the equation $L^\lambda(x, D)u = f$ we have to prove some commutator estimates. First we note the trivial identity

$$(6.1) \quad x^m - y^m = (x - y) \sum_{l=0}^{m-1} x^l y^{m-1-l}$$

which holds for all $x, y \in \mathbb{R}$ and $m \in \mathbb{N}$.

Lemma 6.1. *Let $a^2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function and $m \in \mathbb{N}$. Then we have for all $\xi, \eta \in \mathbb{R}^n$*

$$(6.2) \quad \begin{aligned} & |(1 + a^2(\xi))^m - (1 + a^2(\eta))^m| \\ & \leq 4(a(\xi - \eta) + a^2(\xi - \eta)) \sum_{l=0}^{m-1} (1 + a^2(\xi))^{l+1/2} (1 + a^2(\eta))^{m-1-l}. \end{aligned}$$

PROOF. By (6.1) we have

$$\begin{aligned} & |(1 + a^2(\xi))^m - (1 + a^2(\eta))^m| \\ & = |(1 + a^2(\xi)) - (1 + a^2(\eta))| \sum_{l=0}^{m-1} (1 + a^2(\xi))^l (1 + a^2(\eta))^{m-1-l} \end{aligned}$$

Since a^2 is a continuous negative definite function we find using Lemma 1.2

$$\begin{aligned} & |(1 + a^2(\xi))^m - (1 + a^2(\eta))^m| \\ & \leq (4a(\xi)a(\xi - \eta) + a^2(\xi - \eta)) \sum_{l=0}^{m-1} (1 + a^2(\xi))^l (1 + a^2(\eta))^{m-1-l} \\ & \leq (4a(\xi - \eta) + a^2(\xi - \eta)) \sum_{l=0}^{m-1} (1 + a^2(\xi))^{l+1/2} (1 + a^2(\eta))^{m-1-l}, \end{aligned}$$

which proves the lemma.

The proof of Theorem 6.1 requires the following two lemmas

Lemma 6.2. ([35], Lemma 2.2.1) *For any $q \in \mathbb{R}$ and all $\xi, \eta \in \mathbb{R}^n$ the inequality*

$$(6.3) \quad (1 + |\xi|^2)^q (1 + |\eta|^2)^{-q} \leq 2^{|q|} (1 + |\xi - \eta|^2)^{|q|}$$

holds.

Lemma 6.3. ([35], Lemma 2.2.4) *Let $k \in L^1(\mathbb{R}^n)$. Then we have for all $u, v \in L^2(\mathbb{R}^n)$*

$$(6.4) \quad \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} k(\xi - \eta) u(\eta) v(\xi) d\eta \right) d\xi \right| \leq \|k\|_{L^1} \|u\|_0 \|v\|_0 .$$

Now we prove

Theorem 6.1. *Let b_j be as in B.2, in particular suppose (3.9) with $q = n + r(s + 1/2) + t m_0 + 1$. Then for all $u \in H^{a^2, m_0}(\mathbb{R}^n)$ we have*

$$(6.5) \quad \|[(1 + a^2(D))^{m_0}, b_j(\cdot)]u\|_{s_0, a^2} \leq c \|u\|_{m_0 - \delta, a^2} ,$$

where $s_0 = s - \frac{r-t}{t} m_0$ with m_0, s and δ as in Section 3.

PROOF. First we note that for $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} [(1 + a^2(D))^{m_0}, b_j(\cdot)]u &= [(1 + a^2(D))^{m_0}, d_j(\cdot) + c_j]u \\ &= [(1 + a^2(D))^{m_0}, d_j(\cdot)]u , \end{aligned}$$

thus we only have to prove (6.3) with d_j instead of b_j . By a straightforward calculation we find for $u \in C_0^\infty(\mathbb{R}^n)$ that

$$\begin{aligned} &([(1 + a^2(D))^{m_0}, d_j(\cdot)]u)^\wedge(\xi) \\ &= \int_{\mathbb{R}^n} \hat{d}_j(\xi - \eta) ((1 + a^2(\xi))^{m_0} - (1 + a^2(\eta))^{m_0}) \hat{u}(\eta) d\eta . \end{aligned}$$

Furthermore, for $v \in L^2(\mathbb{R}^n)$ we have

$$\begin{aligned} &|([(1 + a^2(D))^{m_0}, d_j(\cdot)]u, v)_0| \\ &= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \overline{\hat{d}_j(\xi - \eta)} ((1 + a^2(\xi))^{m_0} - (1 + a^2(\eta))^{m_0}) \overline{\hat{u}(\eta)} \hat{v}(\xi) d\eta \right) d\xi \right| \end{aligned}$$

$$\begin{aligned}
& \stackrel{(6.2)}{\leq} c \sum_{l=0}^{m_0-1} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)| (a(\xi - \eta) + a^2(\xi - \eta)) \right. \\
& \quad \left. \cdot (1 + a^2(\xi))^{l+1/2} (1 + a^2(\eta))^{m_0-1-l} |\hat{u}(\eta)| |\hat{v}(\xi)| d\eta \right) d\xi \\
& \leq c \sum_{l=0}^{m_0-1} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)| (1 + |\xi - \eta|^2) \right. \\
& \quad \cdot \frac{(1 + a^2(\xi))^l (1 + a^2(\xi))^{1/2}}{(1 + a^2(\eta))^l (1 + a^2(\eta))^{-m_0+1}} \\
& \quad \cdot (1 + a^2(\xi))^s (1 + a^2(\eta))^{-m_0+\delta} (1 + a^2(\xi))^{-s} |\hat{v}(\xi)| \\
& \quad \left. \cdot (1 + a^2(\eta))^{m_0-\delta} |\hat{u}(\eta)| d\eta \right) d\xi \\
& \stackrel{(3.8)}{\leq} c \sum_{l=0}^{m_0-1} c_l \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)| (1 + |\xi - \eta|^2) \right. \\
& \quad \cdot \frac{(1 + |\xi|^2)^{r/2} (1 + |\xi|^2)^{r(s+1/2)/2}}{(1 + |\eta|^2)^{t/2} (1 + |\eta|^2)^{t(1-\delta)/2}} \\
& \quad \left. \cdot (1 + a^2(\xi))^{-s} |\hat{v}(\xi)| (1 + a^2(\eta))^{m_0-\delta} |\hat{u}(\eta)| d\eta \right) d\xi \\
& \stackrel{(3.6)}{=} c \sum_{l=0}^{m_0-1} c_l \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)| (1 + |\xi - \eta|^2) (1 + |\xi|^2)^{l(r-t)/2} \right. \\
& \quad \cdot \left(\frac{1 + |\xi|^2}{1 + |\eta|^2} \right)^{(r(s+1/2)+tl)/2} (1 + a^2(\xi))^{-s} |\hat{v}(\xi)| \\
& \quad \left. \cdot (1 + a^2(\eta))^{m_0-\delta} |\hat{u}(\eta)| d\eta \right) d\xi \\
& \stackrel{(3.8),(6.3)}{\leq} c \sum_{l=0}^{m_0-1} \tilde{c}_l \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)| (1 + |\xi - \eta|^2) \right. \\
& \quad \cdot (1 + |\xi - \eta|^2)^{(r(s+1/2)+tl)/2} \\
& \quad \cdot (1 + a^2(\xi))^{-s+(r-t)(l/t)} \\
& \quad \left. \cdot |\hat{v}(\xi)| (1 + a^2(\eta))^{m_0-\delta} |\hat{u}(\eta)| d\eta \right) d\xi \\
& \stackrel{(3.9)}{\leq} c \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-q} (1 + |\xi - \eta|^2)^{(r(s+1/2)+tm_0+2)/2} \right. \\
& \quad \left. \cdot (1 + a^2(\xi))^{-s+(r-t)(m_0/t)} |\hat{v}(\xi)| \right)
\end{aligned}$$

$$\begin{aligned} & \cdot (1 + a^2(\eta))^{m_0 - \delta} |\hat{u}(\eta)| d\eta \Big) d\xi \\ (6.4) \quad & \leq c \|v\|_{-s_0, a^2} \|u\|_{m_0 - \delta, a^2} . \end{aligned}$$

Thus we find

$$\frac{|([(1 + a^2(D))^{m_0}, d_j(\cdot)]u, v)_0|}{\|v\|_{-s_0, a^2}} \leq c \|u\|_{m_0 - \delta, a^2} ,$$

which implies by Proposition 2.1

$$\|[(1 + a^2(D))^{m_0}, d_j(\cdot)]u\|_{s - 2(r-t)(m_0/t)} \leq c \|u\|_{m_0 - \delta, a^2} ,$$

thus the theorem is proved.

Corollary 6.1. *Let b_j be as in Theorem 6.1. Then for any $\eta > 0$ there exists $c(\eta) \geq 0$ such that*

$$(6.6) \quad \|[(1 + a^2(D))^{m_0}, b_j(\cdot)]u\|_0 \leq \eta \|u\|_{m_0, a^2} + c(\eta) \|u\|_0$$

holds for all $u \in H^{a^2, m_0}(\mathbb{R}^n)$.

PROOF. For $u \in H^{a^2, m_0}(\mathbb{R}^n)$ we have using (6.5) and Lemma 2.1

$$\begin{aligned} \|[(1 + a^2(D))^{m_0}, b_j(\cdot)]u\|_0 & \leq \|[(1 + a^2(D))^{m_0}, b_j(\cdot)]u\|_{s - (r-t)(m_0/t), a^2} \\ & \leq c \|u\|_{m_0 - \delta, a^2} \\ & \leq \eta \|u\|_{m_0, a^2} + c(\eta) \|u\|_0 . \end{aligned}$$

Finally we have to give the

PROOF OF PROPOSITION 4.1. Let $\alpha_l \in (0, 1 - r_l/2t_l)$ and set $s_l = (1 - \delta)t_l/r_l - 1/2$. Then we may proceed as in the proof of Theorem 6.1 (or as in the proof of Theorem 3.1 in [22]) to get for $u \in C_0^\infty(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^n)$

$$\begin{aligned} & |([a_l(D_l), b_j(\cdot)]u, v)_0| = |([a_l(D_l), d_j(\cdot)]u, v)_0| \\ & \leq c \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)|(1 + |\xi - \eta|^2) \frac{(1 + a_l^2(\xi_l))^{s_l + 1/2}}{(1 + a_l^2(\eta_l))^{1 - \alpha_l}} \right. \\ & \quad \cdot (1 + a_l^2(\xi_l))^{-s_l} |\hat{v}(\xi)|(1 + a_l^2(\eta_l))^{1 - \alpha_l} |\hat{u}(\eta)| d\eta \Big) d\xi \\ (3.12) \quad & \leq c \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{d}_j(\xi - \eta)|(1 + |\xi - \eta|^2) \left(\frac{1 + |\xi|^2}{1 + |\eta|^2} \right)^{r_l(s_l + 1/2)/2} \right. \\ & \quad \cdot (1 + a_l^2(\xi_l))^{-s_l} |\hat{v}(\xi)|(1 + a_l^2(\eta_l))^{1 - \alpha_l} |\hat{u}(\eta)| d\eta \Big) d\xi , \end{aligned}$$

which implies

$$\| [a_l(D_l), b_j(\cdot)]u \|_{s_l, a^2} \leq c \|u\|_{1-\alpha_l, a^2},$$

from which the proposition follows as Corollary 6.1 follows from Theorem 6.1.

7. A regularity result.

We will need a stronger regularity result for solutions of the equation

$$(7.1) \quad L^\lambda(x, D)u = f$$

For this we give

Theorem 7.1. *Let $L^\lambda(x, D)$ be as before, i.e. assume B.1-B.4. Suppose further that $L^\lambda(x, D)u \in H^{a^2, m_0}(\mathbb{R}^n)$ for some $u \in L^2(\mathbb{R}^n)$. Then $u \in H^{a^2, m_0+1}(\mathbb{R}^n)$ and*

$$(7.2) \quad \|u\|_{m_0+1, a^2} \leq c (\|L^\lambda(x, D)u\|_{m_0, a^2} + \|u\|_0)$$

holds.

PROOF. Let u be as stated in the theorem. Then we have using Proposition 2.1

$$\begin{aligned} & \|u\|_{m_0+1, a^2} \|L^\lambda(x, D)u\|_{m_0, a^2} \\ &= \|u\|_{m_0+1, a^2} \|(1+a^2(D))^{1+2m_0} L^\lambda(x, D)u\|_{-m_0-1, a^2} \\ &\geq (u, (1+a^2(D))^{1+2m_0} L^\lambda(x, D)u)_0 \\ &= ((1+a^2(D))^{1+m_0} u, (1+a^2(D))^{m_0} L^\lambda(x, D)u)_0 \\ &= ((1+a^2(D))^{1+m_0} u, (1+a^2(D))^{m_0} \left(\sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u + \lambda u \right))_0 \\ &= \lambda ((1+a^2(D))^{1+m_0} u, (1+a^2(D))^{m_0} u)_0 \\ &\quad + ((1+a^2(D))^{1+m_0} u, (1+a^2(D))^{m_0} \sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u)_0 \\ &= \lambda \|u\|_{m_0+1/2, a^2}^2 + D_1 + D_2, \end{aligned}$$

where

$$D_1 = ((1 + a^2(D))^{1+m_0}u, (1 + a^2(D))^{m_0} \sum_{j=1}^n b_j(x_0)a_j^2(D_j)u)_0$$

and

$$D_2 = ((1 + a^2(D))^{1+m_0}u, (1 + a^2(D))^{m_0} \sum_{j=1}^n (b_j(\cdot) - b_j(x_0))a_j^2(D_j)u)_0 .$$

Here x_0 is again the fixed point given by (3.11). First we estimate D_1 using (3.10):

$$\begin{aligned} D_1 &= \int_{\mathbb{R}^n} (1 + a^2(\xi))^{1+m_0}(1 + a^2(\xi))^{m_0} \sum_{j=1}^n b_j(x_0)a_j^2(\xi_j)|\hat{u}(\xi)|^2 d\xi \\ &\geq \delta_1 \int_{\mathbb{R}^n} (1 + a^2(\xi))^{1+2m_0} \sum_{j=1}^n a_j^2(\xi_j)|\hat{u}(\xi)|^2 d\xi \\ (7.3) \quad &\geq \delta_1 \|u\|_{m_0+1, a^2}^2 - \delta_1 \|u\|_{m_0+1/2, a^2}^2 . \end{aligned}$$

Now let us turn to D_2 :

$$\begin{aligned} D_2 &= ((1 + a^2(D))^{1+m_0}u, (1 + a^2(D))^{m_0} \sum_{j=1}^n (b_j(\cdot) - b_j(x_0))a_j^2(D_j)u)_0 \\ &= \sum_{j=1}^n ((1 + a^2(D))^{1+m_0}u, (b_j(\cdot) - b_j(x_0))(1 + a^2(D))^{m_0}a_j^2(D_j)u)_0 \\ &\quad + \sum_{j=1}^n ((1 + a^2(D))^{1+m_0}u, [(1 + a^2(D))^{m_0}, b_j(\cdot)]a_j^2(D_j)u)_0 \\ &= D_{12} + D_{22} . \end{aligned}$$

By (3.11) we get

$$\begin{aligned} |D_{12}| &\leq \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \\ &\quad \cdot \sum_{j=1}^n \|(1 + a^2(D))^{1+m_0}u\|_0 \|(1 + a^2(D))^{m_0}a_j^2(D_j)u\|_0 \\ &\leq \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \end{aligned}$$

$$\begin{aligned}
& \cdot \|(1 + a^2(D))^{1+m_0}u\|_0 \sum_{j=1}^n \|(1 + a^2(D))^{m_0}a_j^2(D_j)u\|_0 \\
& \leq n \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |b_j(x) - b_j(x_0)| \|u\|_{1+m_0, a^2}^2 \\
(7.4) \quad & \leq \frac{\delta_1}{2} \|u\|_{1+m_0, a^2}^2.
\end{aligned}$$

Furthermore we find

$$\begin{aligned}
|D_{22}| &= \left| \sum_{j=1}^n ((1 + a^2(D))^{1+m_0}u, [(1 + a^2(D))^{m_0}, b_j(\cdot)]a_j^2(D_j)u)_0 \right| \\
&\leq \sum_{j=1}^n \|u\|_{1+m_0, a^2} \|[(1 + a^2(D))^{m_0}, b_j(\cdot)]a_j^2(D_j)u\|_0 \\
(7.5) \quad &\leq \varepsilon \|u\|_{1+m_0, a^2}^2 + c(\varepsilon) \|u\|_0^2,
\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary number. Combining (7.3)-(7.5) we find

$$\begin{aligned}
\|u\|_{1+m_0, a^2} \|L^\lambda(x, D)u\|_{m_0, a^2} &\geq \left(\delta_1 - \frac{\delta_1}{2} - \varepsilon\right) \|u\|_{1+m_0, a^2}^2 \\
&\quad + (\lambda - \delta_1) \|u\|_{m_0+1/2, a^2}^2 - c(\varepsilon) \|u\|_0^2.
\end{aligned}$$

Since $\|u\|_0 \leq \|u\|_{m_0+1/2, a^2}$ we get for $\lambda \geq \delta_1$

$$\begin{aligned}
& \|u\|_{1+m_0, a^2} \|L^\lambda(x, D)u\|_{m_0, a^2} \\
& \geq \left(\frac{\delta_1}{2} - \varepsilon\right) \|u\|_{1+m_0, a^2}^2 + (\lambda - \delta_1 - c(\varepsilon)) \|u\|_0^2.
\end{aligned}$$

Thus taking $\varepsilon = \delta_1/4$ and $\lambda \geq \delta_1 + c(\delta_1/4)$ we find

$$\|L^\lambda(x, D)u\|_{m_0, a^2} \geq \frac{\delta_1}{4} \|u\|_{m_0+1, a^2}.$$

Now let $\lambda \in \mathbb{R}$ be arbitrary and set $\lambda_0 = \delta_1 + c(\delta_1/4)$. Then it follows using Lemma 2.1 for any $\eta > 0$ that

$$\|u\|_{m_0+1, a^2} \leq \frac{4}{\delta_1} \|L^{\lambda_0}(x, D)u\|_{m_0, a^2}$$

$$\begin{aligned} &\leq \frac{4}{\delta_1} (\|L^\lambda(x, D)u\|_{m_0, a^2} + |\lambda - \lambda_0| \|u\|_{m_0, a^2}) \\ &\leq \frac{4}{\delta_1} (\|L^\lambda(x, D)u\|_{m_0, a^2} \\ &\quad + |\lambda - \lambda_0| \eta \|u\|_{m_0+1, a^2} + c(\eta) |\lambda - \lambda_0| \|u\|_0). \end{aligned}$$

For $\eta = \delta_1 |\lambda - \lambda_0|/8$ we finally get

$$\begin{aligned} \|u\|_{m_0+1, a^2} &\leq \frac{8}{\delta_1} \|L^\lambda(x, D)u\|_{m_0, a^2} + \tilde{c} \|u\|_0 \\ &\leq c (\|L^\lambda(x, D)u\|_{m_0, a^2} + \|u\|_0), \end{aligned}$$

which proves the theorem.

From Theorem 7.1 it follows that any solution $u \in L^2(\mathbb{R}^n)$ of the equation

$$(7.6) \quad L^\lambda(x, D)u = f, \quad f \in C_0^\infty(\mathbb{R}^n),$$

belongs to $H^{a^2, m_0+1}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$. Furthermore, by Theorem 4.2 and Theorem 5.2 we know that for $\lambda \geq d_0$ there exists for any $f \in L^2(\mathbb{R}^n)$ a unique solution $u \in H^{a^2, 1}(\mathbb{R}^n)$ of (7.6).

We close this section with

Theorem 7.2. *For $\lambda \in \mathbb{R}$ the operator $L^\lambda(x, D)$ maps $H^{a^2, m_0+1}(\mathbb{R}^n)$ continuously into the space $H^{a^2, m_0}(\mathbb{R}^n)$.*

PROOF. Let $u \in C_0^\infty(\mathbb{R}^n)$. Then we have using Corollary 6.1 and Lemma 2.1

$$\begin{aligned} &\|L^\lambda(x, D)u\|_{m_0, a^2} \\ &\leq \|(1 + a^2(D))^{m_0} \sum_{j=1}^n b_j(\cdot) a_j^2(D_j)u\|_0 + |\lambda| \|u\|_{m_0, a^2} \\ &\leq \sum_{j=1}^n \|b_j(\cdot)(1 + a^2(D))^{m_0} a_j^2(D_j)u\|_0 + |\lambda| \|u\|_{m_0, a^2} \\ &\quad + \sum_{j=1}^n \|[(1 + a^2(D))^{m_0}, b_j(\cdot)] a_j^2(D_j)u\|_0 \end{aligned}$$

$$\begin{aligned} &\leq c \|u\|_{m_0+1, a^2} + |\lambda| \|u\|_{m_0, a^2} + c' \|u\|_{m_0+1, a^2} + \tilde{c} \|u\|_0 \\ &\leq \bar{c} \|u\|_{m_0+1, a^2} . \end{aligned}$$

REMARK 7.1. The proofs of Theorem 7.1 and Theorem 7.2 together with assumptions B.1-B.4 show that both theorems hold for any $k \in \mathbb{N}$, $k \leq m_0$, instead of m_0 .

8. On the operator $[L^\lambda]^{m_0}$.

Let $L^\lambda(x, D)$ be as in the previous section. In order to apply results of [17] and [25], see also [16], we need a characterization of $D([L^\lambda]^{m_0})$, where L^λ is the Friedrichs extension of $L^\lambda(x, D)$, see Section 4. Since L^μ is self-adjoint we can define the operator $[L^\mu]^k$ using the functional calculus or by iteration. It is well known, see [12, Corollary XII.2.8., p. 1200], that these two definitions coincide and that $[L^\mu]^k$ is a closed operator on its domain $D([L^\mu]^k)$. Furthermore we have (see [12, Definition XII, 1.1, p. 1186])

$$(8.1) \quad D([L^\mu]^k) = \{u \in D([L^\mu]^{k-1}) : [L^\mu]^{k-1}u \in D(L^\mu)\}$$

and

$$(8.2) \quad D([L^\mu]^k) = \{u \in D([L^\mu]) : L^\mu u \in D([L^\mu]^{k-1})\} .$$

Now we claim

Theorem 8.1. *Let L^μ be the Friedrichs extension of the operator $L^\mu(x, D)$, where $L(x, D)$ satisfies the assumptions B.1-B.4. Then we have for any $k \leq m_0$*

$$(8.3) \quad D([L^\mu]^k) = H^{a^2, k}(\mathbb{R}^n) .$$

PROOF. We prove (8.3) by induction. For $k = 1$ (8.3) was proved in Theorem 5.2. Next we prove that $D([L^\mu]^k) \subset H^{a^2, k}(\mathbb{R}^n)$ provided we know that $D([L^\mu]^{k-1}) = H^{a^2, k-1}(\mathbb{R}^n)$. Let $u \in D([L^\mu]^k)$. Then we have $[L^\mu]^k u = [L^\mu]^{k-1} L^\mu u$ and $L^\mu u \in D([L^\mu]^{k-1}) = H^{a^2, k-1}(\mathbb{R}^n)$. But by Theorem 7.1 and Remark 7.1 it follows now that $u \in H^{a^2, k}(\mathbb{R}^n)$. Finally let us prove that $H^{a^2, k}(\mathbb{R}^n) \subset D([L^\mu]^k)$ assuming that

$D([L^\mu]^{l-1}) = H^{a^2, l-1}(\mathbb{R}^n)$ for $l \leq k$. Let $u \in H^{a^2, k}(\mathbb{R}^n)$, then by Theorem 7.2 and Remark 7.1 we find that $[L^\mu]^{k-1}u \in H^{a^2, 1}(\mathbb{R}^n)$ but $H^{a^2, 1}(\mathbb{R}^n) = D(L^\mu)$, which by (8.1) proves the theorem.

By the Sobolev embedding theorem we get

Corollary 8.1. *Let L^μ as above. Then we have by (3.5)*

$$(8.4) \quad D([L^\mu]^{m_0}) \subset C_\infty(\mathbb{R}^n).$$

9. On the Feller semigroup generated by $-L^\lambda(x, D)$.

By definition a Feller semigroup on \mathbb{R}^n is a family of linear operators $(T_t)_{t \geq 0}, T_t : C_\infty(\mathbb{R}^n) \rightarrow C_\infty(\mathbb{R}^n)$, satisfying the following conditions

- F.1. For all $t, s \geq 0$ we have $T_{s+t} = T_s T_t$ and $T_0 = I$.
- F.2. For all $u \in C_\infty(\mathbb{R}^n)$ it follows that $\lim_{t \rightarrow 0} \|T_t u - u\|_\infty = 0$.
- F.3. Let $u \in C_\infty(\mathbb{R}^n)$ and $0 \leq u \leq 1$ in \mathbb{R}^n . Then it is required that $0 \leq T_t u \leq 1$ holds for all $t \geq 0$.

The generator of a Feller semigroup is the operator

$$(9.1) \quad Au = \lim_{t \rightarrow 0} \frac{T_t u - u}{t},$$

which is defined on $D(A) \subset C_\infty(\mathbb{R}^n)$, where $D(A)$ consists of all $u \in C_\infty(\mathbb{R}^n)$ such that (9.1) exists. The following theorem, often called the Yosida-Hille-Ray Theorem, will be of greater importance to us.

Theorem 9.1. ([7, p. 3-44], or [13, p. 165]) *Let $D(A)$ be a linear subspace of $C_\infty(\mathbb{R}^n)$ and let $A : D(A) \rightarrow C_\infty(\mathbb{R}^n)$ be a linear operator. Suppose further that $D(A)$ is dense in $C_\infty(\mathbb{R}^n)$, that A satisfies the positive maximum principle on $D(A)$, i.e. if $u \in D(A)$ and $x_0 \in \mathbb{R}^n$ such that $\sup_{x \in \mathbb{R}^n} u(x) = u(x_0) \geq 0$ then it follows that $Au(x_0) \leq 0$, and suppose that for some $\lambda \geq 0$ the operator $\lambda I - A$ maps $D(A)$ onto a dense subspace of $C_\infty(\mathbb{R}^n)$. Then A has a closed extension which is the generator of a Feller semigroup.*

It was Ph.Courrège who gave a characterization of operators satisfying the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$.

Theorem 9.2. ([6, p. 2-36]) *Let $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that for each $x \in \mathbb{R}^n$ the function $\xi \mapsto a(x, \xi)$ is negative definite. Then the operator $-a(x, D)$ defined on $C_0^\infty(\mathbb{R}^n)$ by*

$$(9.2) \quad -a(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} (-a(x, \xi)) \hat{u}(\xi) d\xi$$

satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$.

Theorem 9.3. *Suppose that $L(x, D)$ satisfies B.1-B.4. Then for each $x \in \mathbb{R}^n$ the function*

$$(9.3) \quad \xi \mapsto \sum_{j=1}^n b_j(x) a_j^2(\xi_j)$$

is negative definite and $-L(x, D)$ satisfies the positive maximum principle as an operator defined on $H^{a^2, m_0+1}(\mathbb{R}^n)$.

PROOF. The fact that the function defined by (9.3) is negative definite follows from Definition 1.1 and (3.10). By (3.5) we know that $H^{a^2, m_0+1}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$. Furthermore, Theorem 7.2 combined with the Sobolev embedding theorem gives

$$(9.4) \quad \|L(x, D)u\|_\infty \leq c \|u\|_{m_0+1, a^2}.$$

Now let $u \in H^{a^2, m_0+1}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ such that $u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geq 0$. Take $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x_0) = 1$ and $\chi|_{\mathbb{R}^n \setminus \{x_0\}} < 1$. Then for any $\eta > 0$ the function $u + \eta\chi$ belongs to $H^{a^2, m_0+1}(\mathbb{R}^n)$, $\sup_{x \in \mathbb{R}^n} (u + \eta\chi)(x) = u(x_0) + \eta > 0$ and

$$(9.5) \quad (u + \eta\chi)|_{\mathbb{R}^n \setminus \{x_0\}} < u(x_0) + \eta.$$

Let $(\varphi_\nu^\eta)_{\nu \in \mathbb{N}}$, $\varphi_\nu^\eta \in C_0^\infty(\mathbb{R}^n)$, be a sequence converging to $u + \eta\chi$ in $H^{a^2, m_0+1}(\mathbb{R}^n)$ and therefore also in $C_\infty(\mathbb{R}^n)$. Denote by $x_\nu \in \mathbb{R}^n$ a point defined by $\varphi_\nu^\eta(x_\nu) = \sup_{x \in \mathbb{R}^n} \varphi_\nu^\eta(x)$. Since $\varphi_\nu^\eta \rightarrow u + \eta\chi$ in $C_\infty(\mathbb{R}^n)$ it follows that $\varphi_\nu^\eta(x_\nu) \rightarrow u(x_0) + \eta$. We claim that a subsequence of $(x_\nu)_{\nu \in \mathbb{N}}$ converges to x_0 . If no subsequence of $(x_\nu)_{\nu \in \mathbb{N}}$ converges to x_0 , then there exists an open neighbourhood $U_\delta(x_0)$ such that at most a finite number of members of that sequence lie in $U_\delta(x_0)$. By (9.5) we can find some ε , $0 < \varepsilon < \eta$, such that

$$(u + \eta\chi)|_{\mathbb{R}^n \setminus U_\delta(x_0)} < u(x_0) + \eta - \varepsilon.$$

But this is a contradiction to the fact that $\varphi_\nu^\eta(x_\nu) \rightarrow u(x_0) + \eta$. In the following we may suppose that the whole sequence $(x_\nu)_{\nu \in \mathbb{N}}$ converges to x_0 , since otherwise we have to take a subsequence. Since we have $\varphi_\nu^\eta(x_\nu) \rightarrow u(x_0) + \eta \geq \eta > 0$, we can also suppose that $\varphi_\nu^\eta(x_\nu) \geq 0$ for all $\nu \in \mathbb{N}$. By Theorem 9.2 the operator $-L(x, D)$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$. Thus we have $-L(x, D)\varphi_\nu^\eta(x_\nu) \leq 0$. But this implies $-L(x, D)(u + \eta\chi)(x_0) \leq 0$, where we used (9.4) and the convergence properties of $(\varphi_\nu^\eta)_{\nu \in \mathbb{N}}$. Thus we have for any $\eta > 0$ that

$$-L(x, D)u(x_0) \leq \eta(L(x, D)\chi)(x_0),$$

and for $\eta \rightarrow 0$ the theorem follows.

Now from Theorem 4.2, Theorem 5.2, Theorem 7.1 and Theorems 9.1-9.3 we get

Theorem 9.4. *Let $L(x, D)$ satisfy the assumptions of Theorem 9.3. Then for all $\lambda \geq 0$ the operator*

$$-L(x, D) : H^{a^2, m_0+1}(\mathbb{R}^n) \rightarrow H^{a^2, m_0}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$$

has a closed extension which is the generator of a Feller semigroup on \mathbb{R}^n .

An immediate consequence of Theorem 9.4 is

Corollary 9.1. *Suppose that $L(x, D)$ satisfies the assumptions of Theorem 9.3 and is symmetric. Then B_λ is a regular Dirichlet form with domain $H^{a^2, 1/2}(\mathbb{R}^n)$.*

10. Examples.

In this section we want to give examples of operators $L(x, D)$ we can apply to Theorem 9.4 and the results of the theorems leading to Theorem 9.4. For this we have to recall some basic properties of continuous negative definite functions. Our standard reference is the book [1]. First we want to note the following representation formula, see [9, p. 5-9] or [1, p. 184].

Theorem 10.1. (Lévy-Khinchin Formula) *Every real-valued continuous negative definite function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ has the following representation*

$$(10.1) \quad a(\xi) = c + Q(\xi) + \int_{\mathbb{R}^n} (1 - \cos(\xi, \eta)) \frac{1 + |\eta|^2}{|\eta|^2} d\sigma(\eta),$$

where $c \geq 0$, Q is a non-negative quadratic form on \mathbb{R}^n and σ is a positive measure on \mathbb{R}^n which does not charge the origin and has finite total mass. Conversely, given c, Q , and σ with the properties mentioned above, then the function a defined by (10.1) is a continuous negative definite function.

Sometimes it is convenient to consider continuous negative definite functions of the form

$$(10.2) \quad a(\xi) = \int_{\mathbb{R}^n} (1 - \cos(\xi, \eta)) k(\eta) d\eta.$$

We will call k the kernel associated with a . For $0 < s < 1$ an example is $\lambda^{2s}(\xi) = |\xi|^{2s}$, where the associated kernel is given by $K_{2s}(\eta) = c(n, s)|\eta|^{-n-2s}$.

Clearly the set of all continuous negative definite functions forms a convex cone. Further, if $a_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}$, $j = 1, 2$, are two continuous negative definite functions then the function $a : \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{R}$ defined by $(\xi, \eta) \mapsto a_1(\xi) + a_2(\eta)$ is again a continuous negative definite function. This fact is verified by a direct calculation using (1.3). Since for $0 < s \leq 1$ the function λ^{2s} is a continuous negative definite one, it follows that for s_j , $0 < s_j \leq 1$, and $b_j \geq 0$ by $\xi \mapsto \sum_{j=1}^n b_j \lambda^{2s_j}(\xi_j)$ a continuous negative definite function is given. It is rather easy to construct continuous negative definite functions on \mathbb{R} . By Proposition 10.6 in [1] any continuous function $a : \mathbb{R} \rightarrow [0, \infty)$ which is even and when restricted to $[0, \infty)$ increasing and concave is negative definite.

From the previous considerations it follows that for any choice of t_j and r_j , $0 < t_j \leq r_j \leq 2$, there are a lot of continuous negative definite functions $a_j^2 : \mathbb{R} \rightarrow \mathbb{R}$, $1 \leq j \leq n$, satisfying

$$(10.3) \quad c_j (1 + |\xi_j|^2)^{t_j/4} \leq (1 + a_j^2(\xi_j))^{1/2} \leq \tilde{c}_j (1 + |\xi_j|^2)^{r_j/4}.$$

Since the square root of a_j^2 is again a continuous negative definite function, we can now start to construct an operator $L(x, D)$ satisfying the

assumptions of Theorem 9.3. It is clear that

$$(10.4) \quad a^2(\xi) = \sum_{j=1}^n a_j^2(\xi_j)$$

is a continuous negative definite function on \mathbb{R}^n and that

$$(10.5) \quad c(1 + |\xi|^2)^{t/2} \leq (1 + a^2(\xi)) \leq \tilde{c}(1 + |\xi|^2)^{r/2}$$

holds, where $t = \min_{1 \leq j \leq n} t_j$ and $r = \max_{1 \leq j \leq n} r_j$. Thus given t and n , we can find m_0 such that (3.5) holds. Then we will determine $r \geq t$ such that $0 < 1 - r/(2t) - r(r - t)m_0/t^2$ holds, i.e. $r \in [t, t(1 + \bar{m}) \wedge 2]$, where

$$\bar{m} = \frac{1}{4m_0} (((2m_0 + 1)^2 + 8m_0)^{1/2} - (2m_0 + 1)).$$

Now we take continuous negative definite functions $a_j^2 : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (10.3), where $t \leq t_j \leq r_j \leq r$. Note that $r = t$ is the elliptic case, i.e. in that case we will handle an elliptic pseudodifferential operator. Then it is possible to let m_0 tend to infinity and the regularity results are just (hypo-)elliptic regularity results. In particular, we get for coefficients $b_j = d_j + c_j, d_j \in S(\mathbb{R}^n)$, satisfying B.1-B.4, that any solution $u \in L^2(\mathbb{R}^n)$ of the equation $L^\lambda(x, D)u = f, f \in H^\infty(\mathbb{R}^n)$, lies also in $H^\infty(\mathbb{R}^n)$. However, when we want to handle non-elliptic operators m_0 must be finite and therefore in general we do not get hypoellipticity results.

Now, taking $b_j : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying B.1-B.4 it is clear that the operator

$$(10.6) \quad \sum_{j=1}^n b_j(x) a_j^2(D_j)$$

satisfies all assumptions of Theorem 9.3, in particular all results proved in sections 4-8 do hold.

11. Some probabilistic consequences.

In this section we will show that the validity of estimate (4.6) is not only helpful in constructing a Feller semigroup or a Dirichlet form, but once it is known that B_λ is a symmetric Dirichlet form on $H^{a^2, 1/2}(\mathbb{R}^n)$

and that the continuous negative definite function a^2 satisfies (3.8), then (4.6) has some probabilistic consequences for the stochastic process associated with B_λ . We will consider two of these consequences.

For the first one we recall a result due to M. Fukushima [14].

Theorem 11.1. ([14], Theorem 2) *Suppose a symmetric Dirichlet form E on $L^2(\mathbb{R}^n)$ is regular and satisfies*

$$(11.1) \quad \|u\|_{L^q}^2 \leq c(E(u, u) + c_0 \|u\|_0^2)$$

for some $q > 2$ and $c_0 \geq 0$. Then the associated standard Markov process M possesses the following properties. There exists a Borel set N of zero capacity such that $\mathbb{R}^n \setminus N$ is M -invariant and further the following assertions hold:

i) The resolvent kernel $R_\mu(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure for each $\mu > 0$ and $x \in \mathbb{R}^n \setminus N$.

ii) The transition function $p_t(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure for each $t > 0$ and $x \in \mathbb{R}^n \setminus N$.

iii) A set $A \subset \mathbb{R}^n \setminus N$ is of zero capacity if and only if A is polar, that is, almost all sample paths starting at $x \in \mathbb{R}^n \setminus N$ do not hit A at positive time.

Now we claim

Theorem 11.2. *Let M^λ be the standard Markov process associated with B_λ , where B_λ is generated by $L^\lambda(x, D)$ which is assumed to be symmetric and to fulfill the assumptions of Theorem 9.3. Then the assertions of Theorem 11.1 do hold for M^λ .*

PROOF. It remains to prove that

$$(11.2) \quad \|u\|_{L^q}^2 \leq c(B_\lambda(u, u) + c_0 \|u\|_0^2)$$

holds for some $q > 2$ and a constant $c_0 \geq 0$. But combining (3.8) with (4.6) we have with a suitable constant c

$$\|u\|_{t/2}^2 \leq c(B_\lambda(u, u) + d_0 \|u\|_0^2)$$

for all $u \in H^{a^2, 1/2}(\mathbb{R}^n) = D(B_\lambda)$. Now applying the Sobolev inequality (see [38, p. 20]) we find

$$\|u\|_{L^q} \leq c' \|u\|_{t/2}$$

with $q = 2n/(n - t)$, hence $q > 2$. Thus with this value of q we get (11.2).

Note that one can use (11.1) to get L^∞ -bounds for the resolvent of the semigroup generated by $-L^\lambda$, see [14] and also [20].

Our second application is concerned with the asymptotic behaviour of the semigroup $(T_t)_{t>0}$ generated by $-L^\lambda$ on $L^2(\mathbb{R}^n)$. In their work [5] E. Carlen, S. Kusuoka and D. Stroock proved the following result which we formulate here for our special situation again assuming that B_λ is symmetric.

Theorem 11.3. ([5], Theorem 2.16) *Let $\nu \in (2, \infty)$ and $q = 2\nu/(\nu - 2) > 2$. Suppose further that with some constants c_1 and c_2*

$$(11.3) \quad \|u\|_{L^q}^2 \leq c_1 (B_\lambda(u, u) + c_2 \|u\|_0^2)$$

holds for all $u \in H^{a^2, 1/2}(\mathbb{R}^n)$. Then there exist constants c'_1 and c'_2 such that the Nash-type inequality

$$(11.4) \quad \|u\|_0^{2+4/\nu} \leq c'_1 (B_\lambda(u, u) + c'_2 \|u\|_0^2) \|u\|_{L^1}^{4/\nu}$$

holds.

Further they showed the next theorem, which we again state only in a formulation convenient for our purposes.

Theorem 11.4 ([5], Theorem 2.2) *Suppose that (11.4) holds for all $u \in H^{a^2, 1/2}(\mathbb{R}^n)$. Then there exists a constant $d > 0$ such that for the semigroup $(T_t^\lambda)_{t>0}$ generated by B_λ on $L^2(\mathbb{R}^n)$ the estimate*

$$(11.5) \quad \|T_t^\lambda\|_{L^1 \rightarrow L^\infty} \leq d \frac{e^{c_2 t}}{t^{\nu/2}}$$

holds for $t > 0$. Here $\|\cdot\|_{L^1 \rightarrow L^\infty}$ denotes the operator norm for continuous linear operators mapping $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$.

Again using (3.8) and (4.6) we get by the Sobolev embedding theorem estimate (11.3). Thus (11.5) follows with $\nu = 2n/t > 2$. In [5] further results related to these of Theorem 11.3 and Theorem 11.4 are given.

We also want to mention that Theorem 8.1 enables us to apply some results of the theory of (r, p) -capacities developed by M.

Fukushima and H. Kaneko in [17] and by H. Kaneko in [25]. For details we refer to the paper [16].

Finally let us remark that it seems to us to be possible to use some of the Feller semigroups constructed in this paper to obtain examples for balayage spaces in the sense of [3].

Acknowledgement. Parts of this work had been done while the author, supported by DFG-contract Ja 522/1-1, was visiting the University of Osaka. The author wants to thank Prof. M. Fukushima for his hospitality and many discussions about the topics of this paper. Further the author wants to thank W. Hoh, Erlangen, for discussions and support while writing this paper.

NOTE ADDED IN THE PROOFS. Meanwhile some of our results had been improved. A report on these results is given in N. Jacob: Pseudodifferential operators with negative definite functions as symbol: Applications in probability theory and mathematical physics. In *Operator theory: Advances and Applications* **57** (1992), 149-161.

References.

- [1] Berg, C. and Forst, G. *Potential theory on locally compact Abelian groups*. Springer Verlag, 1975.
- [2] Beurling, A. and Deny, J. Dirichlet spaces. *Proc. Natl. Acad. Sci. U.S.A.* **45** (1959), 208-215.
- [3] Bliedtner, J. and Hansen, W. *Potential theory - An analytic and probabilistic approach to balayage*. Springer Verlag, 1986.
- [4] Bony, J. M., Courrège, Ph., and Priouret, P. Semi-groupes de Feller sur une variété à bord compacte et problème aux limites intégrodifférentiels du second ordre donnant lieu au principe du maximum. *Ann. Inst. Fourier*, **18** (1968), 369-521.
- [5] Carlen, E.A., Kusuoka, S., and Stroock, D.W. Upper bounds for symmetric Markov transition functions. *Ann. Inst. Henri Poincaré, Probabilités et Statistique*, Sup. au n° 2, **23** (1987), 245-287.
- [6] Courrège, Ph. Sur la forme intégrodifférentielle des opérateurs de C_K^∞ dans C satisfaisant du principe du maximum. *Sém. Théorie du Potentiel* (1965/66). 38 p.

- [7] Courrège, Ph. Sur la forme intégro-différentielle du générateur infinitésimal d'un semi-groupe de Feller sur une variété. *Sém. Théorie du Potentiel* (1965/66), 48 p.
- [8] Demuth, M., Van Casteren, J. A. On spectral theory of self-adjoint Feller generators. *Rev. Math. Phys.* **1** (1989), 325-414.
- [9] Deny, J. Sur les espaces de Dirichlet. *Sém. Théorie du Potentiel* (1957), 12 p.
- [10] Deny, J. Méthodes Hilbertiennes et théorie du potentiel, in *Potential Theory, C.I.M.E., Roma* (1970), 123-201.
- [11] Doppel, K. and Jacob, N. Zur Konstruktion periodischer Lösungen von Pseudodifferentialgleichungen mit Hilfe von Operatoralgebren. *Ann. Acad. Sci. Fenn. Ser. A.I. Math.* **8** (1983), 193-217.
- [12] Dunford, N., Schwartz, J. T. *Linear Operators, II.*, John Wiley Interscience Publ., 1963.
- [13] Ethier, S. N. and Kurtz, Th. G. *Markov processes - characterization and convergence.* John Wiley & Sons, 1985.
- [14] Fukushima, M. On an L^p -estimate of resolvents of Markov processes. *Publ. R.I.M.S.* **13** (1977), 277-284.
- [15] Fukushima, M. *Dirichlet forms and Markov processes.*, North Holland Pub. Co., 1980.
- [16] Fukushima, M., Jacob, N., and Kaneko, H. On $(r, 2)$ -capacities for a class of elliptic pseudodifferential operators. *Math. Ann.* **293** (1992), 343-348.
- [17] Fukushima, M. and Kaneko, H. On (r, p) -capacities for general Markov semi-groups. *Infinite-dimensional analysis and stochastic processes.* Proc. USP-meeting at Bielefeld 1983, Pitman Research Notes in Math. **124**, (1985), 41-47.
- [18] Hörmander, L. Pseudodifferential operators. *Comm. Pure Appl. Math.* **18** (1965), 501-517.
- [19] Jacob, N. Dirichlet forms and pseudodifferential operators. *Expo. Math.* **6** (1988), 313-351.
- [20] Jacob, N. L^∞ -bounds for solutions of pseudodifferential boundary problems in Dirichlet spaces. *Expo. Math.* **6** (1988), 363-371.
- [21] Jacob, N. A Gårding inequality for certain anisotropic pseudodifferential operators with non-smooth symbols. *Osaka J. Math.* **26** (1989), 857-879.
- [22] Jacob, N. Commutator estimates for pseudodifferential operators with negative definite functions as symbols. *Forum Math.* **2** (1990), 155-162.
- [23] Jacob, N. Feller semigroups, Dirichlet forms, and pseudodifferential operators. *Forum Math.* **4** (1992), 433-446.

- [24] Jacob, N. A class of elliptic pseudodifferential operators generating symmetric Dirichlet forms. *Potential Anal.* **1** (1992), 221-232.
- [25] Kaneko, H. On (r,p) -capacities for Markov processes. *Osaka J. Math.* **23** (1986), 325-336.
- [26] Kato, T. *Perturbation theory for linear operators.*, Springer Verlag, 1966.
- [27] Kochubei, A. N. Parabolic pseudodifferential equations, hypersingular integrals, and Markov processes. *Math. U.S.S.R. Izvestiya* **33** (1989), 233-259.
- [28] Kohn, J. J. and Nirenberg, L. On the algebra of pseudodifferential operators. *Comm. Pure Appl. Math.* **18** (1965), 269-305.
- [29] Komatsu, T. Markov processes associated with certain integrodifferential operators. *Osaka J. Math.* **10** (1973), 271-303.
- [30] Komatsu, T. On the martingale problem for generators of stable processes with perturbations. *Osaka J. Math.* **21** (1984), 113-132.
- [31] Komatsu, T. Pseudodifferential operators and Markov processes. *J. Math. Soc. Japan* **36** (1984), 387-418.
- [32] Komatsu, T. Continuity estimates for solutions of parabolic equations associated with jump type Dirichlet forms. *Osaka J. Math.* **25** (1988), 697-728.
- [33] Kumano-go, H. *Pseudodifferential operators.* M.I.T. Press, 1981.
- [34] Lions, J. L. and Magenes, E. *Nonhomogeneous boundary value problems and applications I.*, Springer-Verlag, 1972.
- [35] Oleinik, O. A. and Radkevich, E. V. *Second order equations with non-negative characteristic form.* Amer. Math. Soc. Plenum Press, 1973.
- [36] Samko, S. G. *Hypersingular integrals and their applications.* Izdat. Rostov Univ., Rostov-on Don, 1984 (in Russian).
- [37] Taira, K. *Diffusion processes and partial differential equations.* Academic Press, 1988.
- [38] Taylor, M. *Pseudodifferential operators,* Princeton University Press, 1981.
- [39] von Waldenfels, W. *Eine Klasse stationärer Markowprozesse.* Berichte der Kernforschungsanlage Jülich, 1961.
- [40] von Waldenfels, W. Fast positive Operatoren. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **4** (1965), 159-174.

- [41] Weidmann, J. *Lineare Operatoren in Hilberträumen*. Mathematische Leitfäden, B.G. Teubner, 1976.

Recibido: 5 de marzo de 1.992

Niels Jacob
Mathematisches Institut
Universität Erlangen-Nürnberg
D-8520 Erlangen, GERMANY