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Calderón-type Reproducing Formula and the Tb Theorem

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Abstract. In this paper we use the Calderón-Zygmund operator theory to prove a Calderón type reproducing formula associated with a para-accretive function. Using our Calderón-type reproducing formula we introduce a new class of the Besov and Triebel-Lizorkin spaces and prove a Tb theorem for these new spaces.

Introduction.

Let ϕ be a function with the properties: $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} \widehat{\phi} \subseteq \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, and $|\widehat{\phi}(\xi)| \geq c > 0$ if $3/5 \leq |\xi| \leq 5/3$. The classical Calderón Reproducing Formula can be stated as follows:

Theorem. (The Calderón Reproducing Formula) *Suppose that the function ϕ satisfies the properties above. Then there exists a function ψ satisfying the same properties as ϕ such that*

$$f = \sum_{k \in \mathbb{Z}} \psi_k * \phi_k * f ,$$

where $\psi_k(x) = 2^{kn}\, \psi(2^k x)$ and the series converges in L^2 norm or in $\mathcal{S}'/\mathcal{P}$, the test functions modulo polynomials.

It is well known that the classical Calderón Reproducing Formula plays an important role in harmonic analysis and wavelets analysis as well. For instance, this formula can be used to study classical function spaces, namely the Besov and Triebel-Lizorkin spaces, and obtain atomic decompositions of these spaces, and prove the boundedness of Calderón-Zygmund operators, namely the $T1$ theorem for the Besov and Triebel-Lizorkin spaces. Further applications of this formula can be found in [C1], [C2], [CF], [FJ1], [FJ2], [FJW], [GM], [P] ,[R] and [U]. Since the classical Calderón Reproducing Formula is given by the action of convolution operators, the Fourier transform is the basic tool for proving such formula.

Our concern in this paper is to establish a Calderón-type reproducing formula associated to a para-accretive function introduced in [DJS], which are not convolution operators. The new idea to establish the Calderón-type reproducing formula associated to a para-accretive function is to use the Calderón-Zygmund operator theory. More precisely, we will introduce a class of "test functions" which will be said to be the strong b-smooth molecules, b is a para-accretive function, and a class of the Calderón-Zygmund operators whose kernels satisfy a strong smoothness condition. We then prove that the Calderón-Zygmund operators in the class above are bounded on "test functions", that is, these operators map the strong b-smooth molecules into the strong b-smooth molecules. Using the approximation to the identity associated to a para-accretive function introduced in [DJS] and a Coifman's idea (see [DJS]), we will construct a Calderón-Zygmund operator whose kernel satisfies the strong smoothness condition mentioned before and use this Calderón-Zygmund operator to establish our Calderón-type reproducing formula associated to a para-accretive function.

As an application of this reproducing formula we prove a Tb theorem. To be precise, suppose that T satisfies the hypotheses of the Tb theorem of [DJS], where $b_1 = b_2 = b$. Suppose also that $Tb = T^*b = 0$. The results of [L] and [HJTW] state that TM_b is bounded on $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 \leq p,q \leq \infty$, where ε is the regularity exponent of the kernel of T and M_b denotes the operator of multiplication by b. Hence T maps $b\dot{B}_p^{\alpha,q}$ into $\dot{B}_p^{\alpha,q}$ and $b\dot{F}_p^{\alpha,q}$ into $\dot{F}_p^{\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 \leq p,q \leq \infty$. Applying this to T^*, we obtain by duality that T maps $\dot{B}_p^{-\alpha,q}$ into $b^{-1}\dot{B}_p^{-\alpha,q}$ and $\dot{F}_p^{-\alpha,q}$ into $b^{-1}\dot{F}_p^{-\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 \leq p,q \leq \infty$. However, the results of [L] and [HJTW] can not be applied to the case where $\alpha = 0$. As in the case of $\mathbb{R}^n$, using our Calderón-type reproducing formula associated to a para-accretive function, we will introduce a new class of function and distribution spaces, namely the Besov and Triebel-Lizorkin spaces associated to a para-accretive function, and prove the Tb theorem for these new spaces, which includes the case where $\alpha = 0$.

The paper is organized as follows. In Section 1 we describe the notations, definitions and some known results to be used throughout and prove a boundedness result of a class of the Calderón-Zygmund operators. In Section 2 we establish the Calderón-type reproducing formulas associated to a para-accretive function. The Besov and Triebel-Lizorkin spaces will be introduced in Section 3 and a Tb theorem will be proved there. In the last section we make several remarks.

Section 1.

We begin by reviewing some basic facts about the Calderón-Zygmund operator theory.

Definition 1.1. *A singular integral operator T is a continuous linear operator from $\mathcal{D}(\mathbb{R}^n)$ into its dual that is associated to a kernel $K(x,y)$, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$, satisfying the following conditions: for some constants $c > 0$ and $0 < \varepsilon \leq 1$,*

$$|K(x,y)| \leq c\,|x-y|^{-n}, \qquad \text{for all } x \neq y, \tag{1.2.i}$$

$$|K(x,y) - K(x',y)| \leq c\,|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon}, \tag{1.2.ii}$$

for all x, x' and y in $\mathbb{R}^n$ with $|x-x'| \leq |x-y|/2$, and

$$|K(x,y) - K(x,y')| \leq c\,|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon}, \tag{1.2.iii}$$

for all x, y and y' in $\mathbb{R}^n$ with $|y-y'| \leq |x-y|/2$.

Moreover, the operator T can be represented by

$$\langle Tf, g\rangle = \iint_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y)\, f(x)\, g(x)\, dx\, dy \tag{1.3}$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$.

Definition 1.4. *An operator T is called weakly bounded if there exists a constant $c > 0$ such that for all f and $g \in \mathcal{S}$ supported in a cube $Q \subset \mathbb{R}^n$ with diameter at most $t > 0$,*

$$(1.5) \qquad |\langle Tf, g\rangle| \le c\, t^n \left(\|f\|_\infty + t\,\|\nabla f\|_\infty\right)\left(\|g\|_\infty + t\,\|\nabla g\|_\infty\right).$$

It was shown in [DJS] that if T is a weakly bounded operator associated to a kernel satisfying (1.2.i) then T has a continuous extension from C_0^η into its dual, where C_0^η denotes the space of continuous functions f with compact support such that

$$\|f\|_\eta = \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|^\eta} < +\infty,$$

and for such operator T the weak boundedness property in Definition 1.4 can be described as follows.

Definition 1.6. *Let T be a continuous operator from C_0^η into its dual for each $\eta > 0$. We say that T is weakly bounded if, for each $\eta > 0$, there is a constant $c > 0$ such that for all cubes Q with diameter at most $t > 0$ and all $f, g \in C_0^\eta$ supported in Q,*

$$(1.7) \qquad |\langle Tf, g\rangle| \le c\, t^{1+2\eta/n}\, \|f\|_\eta \|g\|_\eta\,.$$

It was shown in [DJS] that if the kernel of T satisfies the condition (1.2.i), then (1.7) holds for all $\eta > 0$ whenever it holds for some $\eta > 0$. David and Journé gave a general criterion for the L^2 boundedness of singular integral operators defined in (1.1) ([DJ]).

Theorem 1.8. (The $T1$ Theorem of David-Journé) *Suppose that T is a singular integral operator defined in* (1.1). *Then T is bounded on L^2 if and only if: a) $T1 \in BMO$, b) $T^*1 \in BMO$, and c) T has the weak boundedness property defined in* (1.4).

Suppose that ϕ is a function with the properties as in the classical Calderón Reproducing Formula. The classical Besov spaces $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ for $\alpha \in R$ and $1 \le p, q \le \infty$ are the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \Big(\sum_{k\in Z} (2^{k\alpha}\|\phi_k * f\|_p)^q\Big)^{1/q} < +\infty,$$

and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ for $\alpha \in R$ and $1 \le p < \infty$, $1 \le q \le \infty$ are the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \|\Big(\sum_{k\in Z}(2^{k\alpha}|\phi_k * f|)^q\Big)^{1/q}\|_p < +\infty .$$

The $T1$ theorems for the classical Besov spaces $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ were proved in [L] and [HJTW] , respectively.

Theorem 1.9. (The $T1$ Theorems for the Besov and Triebel-Lizorkin Spaces) *Suppose that T is a singular integral operator whose kernel satisfies the conditions* (1.2.i), (1.2.ii) *and $T1 = 0$, and T has the weak boundedness property. Then T is bounded on the Besov spaces $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ for $0 < \alpha < \varepsilon, 1 \le p, q \le \infty$, where ε is the regularity exponent of the kernel of T.*

Replacing the function 1 in the $T1$ theorem by more general bounded function David, Journé and Semmes proved the Tb theorem ([DJS]). To state their Tb theorem we need the following definitions.

Definition 1.10. *A complex-valued bounded function b defined on $\mathbb{R}^n$ is said to be a para-accretive function if there exists a constant $c > 0$ such that for every cube $Q \subset \mathbb{R}^n$, there is a subcube $I \subseteq Q$ with*

$$\left|\frac{1}{|Q|}\int_I b(x)\,dx\right| \ge c > 0 . \tag{1.11}$$

Definition 1.12. *Suppose b_1 and b_2 are complex-valued functions whose inverse are also bounded. A singular integral operator is a continuous operator T from $b_1C_0^\eta$ into $(b_2C_0^\eta)'$ for all $\eta > 0$ for which there exists a kernel $K(x, y)$ satisfying the conditions* (i), (ii) *and* (iii) *of* (1.2) *such that for all $f, g \in C_0^\eta$ with* $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$,

$$\langle Tb_1 f, b_2 g\rangle = \iint_{\mathbb{R}^n\times\mathbb{R}^n} g(x)\, b_2(x)\, K(x,y)\, b_1(y)\, f(y)\, dx\, dy .$$

David, Journé and Semmes proved the following Tb theorem.

Theorem 1.13. (The Tb Theorem of David-Journé-Semmes) *Suppose that b_1 and b_2 are para-accretive functions and T is a singular integral operator from $b_1C_0^\eta$ into $(b_2C_0^\eta)'$ defined in* (1.12). *Then T is bounded on L^2 if and only if: a) $Tb_1 \in BMO$, b) $T^*b_2 \in BMO$, and c) $M_{b_2}TM_{b_1}$ has the weak boundedness property defined in* (1.6).

We now introduce our "test functions".

Definition 1.14. *Fix two exponents $0 < \beta \le 1$ and $\gamma > 0$. Suppose that b is a para-accretive function. A function f defined on $\mathbb{R}^n$ is said to be a strong b-smooth molecule of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ if f satisfies the following conditions:*

$$|f(x)| \le c\frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}}, \tag{1.15.i}$$

$$|f(x)-f(x')| \le c\Big(\frac{|x-x'|}{d+|x-x_0|}\Big)^\beta \frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}}, \tag{1.15.ii}$$

for $|x-x'| \le (d+|x-x_0|)/2$, and

$$\int_{\mathbb{R}^n} f(x)\, b(x)\, dx = 0. \tag{1.15.iii}$$

This definition was first introduced in [M1] by considering the conditions (i) and (iii) of (1.15), and (ii) of (1.15) replaced by

$$\begin{aligned}|f(x)-f(x')| \le c\Big(\frac{|x-x'|}{d}\Big)^\beta \\ \cdot\Big(\frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}} + \frac{d^\gamma}{(d+|x'-x_0|)^{n+\gamma}}\Big).\end{aligned} \tag{1.16}$$

We call such f a b-smooth molecule of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. The collection of all strong b-smooth molecules of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ will be denoted by $M^{(\beta,\gamma)}(x_0, d)$. If $f \in M^{(\beta,\gamma)}(x_0, d)$, the norm of f in $M^{(\beta,\gamma)}(x_0, d)$ is defined by

$$\|f\|_{M^{(\beta,\gamma)}(x_0,d)} = \inf\{c \ge 0 : \text{ (1.9) (i), (ii) and (iii) hold }\}. \tag{1.17}$$

We denote $M^{(\beta,\gamma)}$ the class of all $f \in M^{(\beta,\gamma)}(0,1)$. It is easy to see that $M^{(\beta,\gamma)}$ is a Banach space under the norm $\|f\|_{M^{(\beta,\gamma)}} < +\infty$. We then introduce the dual space $(M^{(\beta,\gamma)})'$ consisting of all linear functionals $\mathcal{L}$ from $M^{(\beta,\gamma)}$ to $\mathbb{C}$ with the property that there exists a finite constant c such that for all $f \in M^{(\beta,\gamma)}$,

$$|\mathcal{L}(f)| \leq c \, \|f\|_{M^{(\beta,\gamma)}} . \tag{1.18}$$

We denote $\langle h, f\rangle$ the natural pairing of elements $h \in (M^{(\beta,\gamma)})'$ and $f \in M^{(\beta,\gamma)}$. It is easy to check that for $x_0 \in \mathbb{R}^n$ and $d > 0$, $M^{(\beta,\gamma)}(x_0, d) = M^{(\beta,\gamma)}$ with equivalent norms. Thus, for all $h \in (M^{(\beta,\gamma)})'$, $\langle h, f\rangle$ is well defined for all $f \in M^{(\beta,\gamma)}(x_0, d)$ with $x_0 \in \mathbb{R}^n$ and $d > 0$.

We now state and prove the main result in this section.

Theorem 1.19. *Suppose that b is a para-accretive function and T is a singular integral operator from $C_0^\eta(\mathbb{R}^n)$ into its dual for all $\eta > 0$ such that T and $b^{-1}(T^*)M_b$ satisfy the hypotheses of Theorem* 1.9 *and further, $K(x,y)$, the kernel of T, satisfies the following strong smoothness condition*

$$\begin{aligned} &|(K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)) \\ &\quad - (K(x,y')b^{-1}(y') - K(x',y')b^{-1}(y'))| \\ &\qquad\qquad \leq c\,|x-x'|^{\varepsilon}|y-y'|^{\varepsilon}|x-y|^{-n-2\varepsilon} , \end{aligned} \tag{1.20}$$

for all x, x', y and y' in $\mathbb{R}^n$ with $|x - x'| \leq |x-y|/3$ and $|y-y'| \leq |x-y|/3$.

Then T maps the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ for $0 < \beta,\gamma < \varepsilon$ where ε is the regularity exponent of the kernel of T. Moreover, denote $\|T\|$ the smallest constant in the estimates of the kernel of T, then there exists a constant $c > 0$ such that

$$\|Tf\|_{M^{(\beta,\gamma)}(x_0,d)} \leq c\,\|T\|\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} . \tag{1.21}$$

In [M2] it was shown that if T satisfies the hypotheses of Theorem 1.19 except (1.20), then T maps b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to b-smooth molecules of type (β',γ') centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, which is not available for our purposes.

To prove Theorem 1.19, we follow Meyer's idea, [M2]. Fix a function $\theta \in \mathcal{D}$ with $\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$ and $\theta = 1$ on

$\{x \in \mathbb{R}^n : |x| \le 1\}$. Suppose that f is a strong b-smooth molecule of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. We first prove that $T(f)(x)$ satisfies the size condition (i) of (1.15). To do this, consider first the case where $|x - x_0| \le 5\,d$. Set $1 = \xi(y) + \eta(y)$ where $\xi(y) = \theta(y - x_0/(10\,d))$. Then, as in [M2],

$$\begin{aligned} Tf(x) &= \int K(x,y)\,(f(y) - f(x))\,\xi(y)\,dy \\ &\quad + \int K(x,y)\,f(y)\,\eta(y)\,dy \\ &\quad + f(x)\int K(x,y)\xi(y)\,dy = \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned}$$

Using lemmas 2 and 3 in [M2], we have

$$\begin{aligned} |\mathrm{I}| &\le c \int\limits_{|x-y|\le 25\,d} |K(x,y)|\;|f(y) - f(x)|\,dy \\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|x-y|\le 25\,d} |x-y|^{-n}\,\frac{|x-y|^\beta}{d^{\beta+n}}\,dy \\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\;d^{-n}\,, \end{aligned}$$

and

$$|\mathrm{III}| \le c\,|f(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\;d^{-n}\,.$$

For the term II we have

$$\begin{aligned} |\mathrm{II}| &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|y-x_0|>10d} |x-y|^{-n}\,\frac{d^\gamma}{|y-x_0|^{n+\gamma}}\,dy \\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\;d^{-n}\,, \end{aligned}$$

since $|x - x_0| \le 5\,d$. This shows that $Tf(x)$ satisfies (i) of (1.15) for the case $|x - x_0| \le 5\,d$. Now consider the case where $|x - x_0| = R > 5\,d$. Set $1 = I(y) + J(y) + L(y)$ where $I(y) = \theta(8|x-y|/R)$, $J(y) = \theta(8|y-x_0|/R)$, and $f_1(y) = f(y)\,I(y)$, $f_2(y) = f(y)\,J(y)$, and $f_3(y) = f(y)\,L(y)$. Then it is easy to check the following estimates

$$|f_1(y)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^{n+\gamma}}\,, \tag{1.22.a}$$

$$|f_1(y) - f_1(y')| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{|y-y'|^\beta}{R^\beta}\,\frac{d^\gamma}{R^{n+\gamma}}\,, \tag{1.22.b}$$

for all y and $y' \in \mathbb{R}^n$,

$$|f_3(y)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \cdot \frac{d^\gamma}{|y-x_0|^{n+\gamma}}\,\chi_{\{|y-x_0|>R/8\}}\,, \tag{1.22.c}$$

$$\int |f_3(y)|\,dy \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^\gamma}\,, \tag{1.22.d}$$

$$\begin{aligned}\Big|\int f_2(y)\,b(y)\,dy\Big| &\le c\Big(\int |f_1(y)|\,dy + \int |f_3(y)|\,dy\Big)\\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^\gamma}\,,\end{aligned} \tag{1.22.e}$$

since$\int f(y)b(y)dy = 0$. We now have

$$\begin{aligned}Tf_1(x) &= \int K(x,y)\,(f_1(y)-f_1(x))\,u(y)\,dy + f_1(x)\int K(x,y)\,u(y)\,dy\\ &= r_1(x)+r_2(x)\,,\end{aligned}$$

where $u(y)=\theta(4|x-y|/R)$. Applying the estimates in (1.22), we obtain

$$\begin{aligned}|r_1(x)| &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|x-y|\le R/2} |x-y|^{-n}\,\frac{|x-y|^\beta}{R^\beta}\,\frac{d^\gamma}{R^{n+\gamma}}\,dy\\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^{n+\gamma}}\,,\end{aligned}$$

$$|r_2(x)| \le c\,|f_1(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^{n+\gamma}}\,.$$

For f_2 we have

$$\begin{aligned}Tf_2(x) &= \int \big(b^{-1}(y)K(x,y)-b^{-1}(x_0)K(x,x_0)\big)f_2(y)\,b(y)\,dy\\ &\quad + b^{-1}(x_0)\,K(x,x_0)\int f_2(y)\,b(y)\,dy\\ &= \sigma_1(x)+\sigma_2(x)\,.\end{aligned}$$

Using the estimates of the kernel of $b^{-1}T^*M_b$ and f_2 in (1.22),

$$|\sigma_1(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|y-x_0|\le R/4} \frac{|y-x_0|^\varepsilon}{R^{n+\varepsilon}} \frac{d^\gamma}{|y-x_0|^{n+\gamma}}\,dy$$
$$\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{d^\gamma}{R^{n+\gamma}}\,,$$

since $\gamma < \varepsilon$, and

$$|\sigma_2(x)| \le c\,R^{-n} \left| \int f_2(y)\,b(y)\,dy \right| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{d^\gamma}{R^{n+\gamma}}\,.$$

Finally,

$$|Tf_3(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{\substack{|x-y|>R/8\\|y-x_0|>R/8}} |x-y|^{-n} \frac{d^\gamma}{|y-x_0|^{n+\gamma}}\,dy$$
$$\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{d^\gamma}{R^{n+\gamma}}\,.$$

This proves that $Tf(x)$ satisfies (i) of (1.15) for $|x-x_0| \ge 5\,d$ and hence the estimate (i) of (1.15). It remains to prove that $Tf(x)$ satisfies the smoothness condition (ii) of (1.15). Set $|x-x_0| = R$ and $|x-x'| = \delta$. We consider only the case where $R > 5\,d$ and $\delta \le (d+R)/20$ (see the proof in [M2] for the case where $R \le 5\,d$). As in the above,

$$Tf_1(x) = \int K(x,y)\,(f_1(y) - f_1(x))\,\zeta(y)\,dy$$
$$+ \int K(x,y)\,f_1(y)\,\mu(y)\,dy$$
$$+ f_1(x) \int K(x,y)\,\zeta(y)\,dy\,,$$

where $1 = \zeta(y) + \mu(y)$ and $\zeta(y) = \theta(|x-y|/(2\,\delta))$. Denote the first term of right hand side above by $p(x)$ and the sum of the last two terms by $q(x)$. Then the size condition of K and the smoothness of f_1 in (1.22) yield

$$|p(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|x-y|\le 4\delta} |x-y|^{-n} \frac{|x-y|^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,dy$$
$$\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,.$$

This estimate still holds with x replaced by x' for $|x-x'|=\delta$. Thus,

$$|p(x)-p(x')| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta}\frac{d^\gamma}{R^{n+\gamma}}.$$

For $q(x)$, using the condition that $T1=0$, we have

$$\begin{aligned} q(x)-q(x') &= \int \big(K(x,y)-K(x',y)\big)\big(f_1(y)-f_1(x)\big)\,\mu(y)\,dy \\ &\quad + \big(f_1(x)-f_1(x')\big)\int K(x',y)\,\zeta(y)\,dy \\ &= \mathrm{I}+\mathrm{II}. \end{aligned}$$

Again, using lemmas 2 and 3 in [M2], and the smoothness of f_1 in (1.22), we obtain

$$|\mathrm{II}| \le c\,|f(x)-f(x')| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta}\frac{d^\gamma}{R^{n+\gamma}}.$$

Notice that

$$|f_1(y)-f_1(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{|x-y|^\beta}{R^\beta}\frac{d^\gamma}{R^{n+\gamma}},$$

for all $y\in\mathbb{R}^n$, term I is dominated by

$$\begin{aligned} &\int_{2\delta\le|x-y|} |K(x,y)-K(x',y)|\,|f_1(y)-f_1(x)|\,dy \\ &\qquad \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int_{|x-y|\ge 2\delta} \frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}}\frac{|x-y|^\beta}{R^\beta}\frac{d^\gamma}{R^{n+\gamma}}\,dy \\ &\qquad \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta}\frac{d^\gamma}{R^{n+\gamma}}, \end{aligned}$$

since $\beta<\varepsilon$. This shows that $Tf_1(x)$ satisfies the condition (ii) of (1.15) for the case $R>5\,d$. Note that x and x' are not in the supports of f_2 and f_3 and $\delta\le(d+R)/20<R/16$. Using the strong smoothness condition of the kernel of T in (1.20) and the estimates of f_2 and f_3 in

(1.22), we then have

$$
\begin{aligned}
&|Tf_2(x) - Tf_2(x')| \\
&\quad = \Big| \int \big(K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)\big)\, f_2(y)\, b(y)\, dy \Big| \\
&\quad \le \Big| \int \big(\big(K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)\big) \\
&\qquad\qquad - \big(K(x,x_0)b^{-1}(x_0) - K(x',x_0)b^{-1}(x_0)\big)\big)\, f_2(y)\, b(y)\, dy \Big| \\
&\qquad + \big|\big(K(x,x_0) - K(x',x_0)\big)b^{-1}(x_0)\big| \, \Big| \int f_2(y)\, b(y)\, dy \Big| \\
&\quad \le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \\
&\qquad\quad \int\limits_{|y-x_0| \le R/4} \frac{|x-x'|^{\varepsilon} |y-x_0|^{\varepsilon}}{|x-x_0|^{n+2\varepsilon}} \frac{d^{\gamma}}{(d+|y-x_0|)^{n+\gamma}}\, dy \\
&\qquad + c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{|x-x'|^{\varepsilon}}{|x-x_0|^{n+\varepsilon}} \frac{d^{\gamma}}{R^{\gamma}} \\
&\quad \le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^{\beta}}{R^{\beta}} \frac{d^{\gamma}}{R^{n+\gamma}},
\end{aligned}
$$

since $\beta, \gamma < \varepsilon$, and

$$
\begin{aligned}
|Tf_3(x) - Tf_3(x')| &= \Big| \int\limits_{|x-y| \ge R/8 > 2\delta} \big(K(x,y) - K(x',y)\big)\, f_3(y)\, dy \Big| \\
&\le c \int\limits_{|x-y| \ge R/8} \frac{|x-x'|^{\varepsilon}}{|x-y|^{n+\varepsilon}} |f_3(y)|\, dy \\
&\le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^{\varepsilon}}{R^{\varepsilon}} \frac{d^{\gamma}}{R^{n+\gamma}}.
\end{aligned}
$$

These estimates show that $T(f)(x)$ satisfies the condition (ii) in (1.15) for the case where $R > 5\,d$ and $\delta \le (d+R)/20$. The fact that $\int T(f)(x)\, b(x)\, dx = 0$ follows from the condition $T^*(b)(x) = 0$. This completes the proof of Theorem 1.19.

Section 2.

In this section we construct a Calderón-Zygmund operator whose kernel satisfies the strong smoothness condition (1.20) and use this operator to establish the Calderón-type reproducing formula associated to a para-accretive function. We first introduce the following definition (see [DJS]).

Definition 2.1. *A sequence $(S_k)_{k\in\mathbb{Z}}$ of operators is called to be an approximation to the identity associated to a para-accretive function b if $S_k(x,y)$, the kernel of S_k, are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that for all $k \in \mathbb{Z}$ and all x, x', y and y' in $\mathbb{R}^n$, and some $0 < \varepsilon \leq 1$ and $c > 0$,*

(2.2.i)
$$S_k(x,y) = 0\,, \ \text{if } |x-y| \geq c\,2^{-k} \qquad \text{and} \qquad \|S_k\|_\infty \leq c\,2^{kn}\,,$$

(2.2.ii)
$$|S_k(x,y) - S_k(x,y')| \leq c\,2^{k(n+\varepsilon)}\,|y-y'|^\varepsilon\,,$$

(2.2.iii)
$$|S_k(x,y) - S_k(x',y)| \leq c\,2^{k(n+\varepsilon)}\,|x-x'|^\varepsilon\,,$$

(2.2.iv)
$$\begin{aligned} &|(S_k(x,y) - S_k(x',y)) - (S_k(x,y') - S_k(x',y'))| \\ &\qquad \leq c\,2^{k(n+2\varepsilon)}\,|x-x'|^\varepsilon\,|y-y'|^\varepsilon\,, \end{aligned}$$

(2.2.v)
$$\int_{\mathbb{R}^n} S_k(x,y)\,b(y)\,dy = 1\,, \qquad \text{for all } k \in \mathbb{Z} \text{ and } x \text{ in } \mathbb{R}^n\,,$$

(2.2.vi)
$$\int_{\mathbb{R}^n} S_k(x,y)\,b(x)\,dx = 1\,, \qquad \text{for all } k \in \mathbb{Z} \text{ and } y \text{ in } \mathbb{R}^n\,.$$

In [DJS] such operators were constructed and all conditions except for (iv) in (2.2) were checked. Note that in [DJS] S_k were given by $P_k^*\{P_k b\}^{-1} P_k$ where P_k satisfy the conditions (i), (ii), and (v) with $b(x) = 1$ in (2.2). We have

$$\begin{aligned} &\big(S_k(x,y) - S_k(x',y)\big) - \big(S_k(x,y') - S_k(x',y')\big) \\ &\qquad = \int \big(P_k(z,x) - P_k(z,x')\big)\,(P_k b(z))^{-1}\,\big(P_k(z,y) - P_k(z,y')\big)\,dz\,. \end{aligned}$$

The condition (iv) in (2.2) then follows from simple calculation.

We can now state our Calderón-type reproducing formula.

Theorem 2.3. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ is an approximation to the identity defined in (2.1). Set $D_k = S_k - S_{k-1}$. Then there exists a family of operators $(\widetilde{D}_k)_{k\in\mathbb{Z}}$ such that for all $f \in M^{(\beta,\gamma)}$,*

$$f = \sum_{k\in\mathbb{Z}} \widetilde{D}_k\, M_b\, D_k\, M_b(f)\,, \tag{2.4}$$

where the series converges in the norm of L^p, $1 < p < \infty$, and $M^{(\beta',\gamma')}$ with $\beta' < \beta$ and $\gamma' < \gamma$. Moreover, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfy the following estimates: for $\varepsilon', 0 < \varepsilon' < \varepsilon$, where ε is the regularity exponent of S_k, there exists a constant $c > 0$ such that

$$|\widetilde{D}_k(x,y)| \le c\,\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}\,, \tag{2.5.i}$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le c\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}\,, \tag{2.5.ii}$$

for $|x-x'| \le (2^{-k}+|x-y|)/2$,

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)\, b(x)\, dx = 0\,, \tag{2.5.iii}$$

for all $k \in \mathbb{Z}$ and y in $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)\, b(y)\, dy = 0\,, \tag{2.5.iv}$$

for all $k \in \mathbb{Z}$ and x in $\mathbb{R}^n$.

The similar formula on spaces of homogeneous type for the case where $b(x) = 1$ was established in [HS2]. To prove theorem (2.3) we begin with a Coifman's idea. By non-degeneracy condition (v) and the size condition (i) in (2.2),

$$\mathrm{I} = \sum_{k\in\mathbb{Z}} D_k\, M_b \;\text{ in } L^2(\mathbb{R}^n)\,. \tag{2.6}$$

Coifman's idea is to rewrite (2.6) in the following way

$$
\begin{aligned}
\mathrm{I} &= \Big(\sum_{k\in\mathbb{Z}} D_k\, M_b\Big)\Big(\sum_{j\in\mathbb{Z}} D_j\, M_b\Big) \\
&= \sum_{|j|>N}\sum_{k\in\mathbb{Z}} D_{k+j}\, M_b\, D_k\, M_b \qquad (2.7)\\
&\quad + \sum_{k\in\mathbb{Z}}\sum_{|j|\le N} D_{k+j}\, M_b\, D_k\, M_b = R_N + T_N
\end{aligned}
$$

where

$$R_N = \sum_{|j|>N}\sum_{k\in\mathbb{Z}} D_{k+j}\, M_b\, D_k\, M_b$$

and

$$T_N = \sum_{k\in\mathbb{Z}} D_k^N\, M_b\, D_k\, M_b$$

with

$$D_k^N = \sum_{|j|\le N} D_{k+j}\,,$$

and N is a fix positive integer. It was shown in [DJS] that $\lim_{N\to\infty} T_N = I$ in L^2 and hence T_N^{-1} is bounded on L^2 for large N. Our goal here is to show that T_N^{-1} maps the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. To be precise, we prove the following theorem.

Theorem 2.8. *Suppose that $(D_k)_{k\in\mathbb{Z}}$ is as in Theorem 2.3, and $T_N = \sum_{k\in\mathbb{Z}} D_k^N\, M_b\, D_k\, M_b$ where $D_k^N = \sum_{|j|\le N} D_{k+j}$ and N is a large positive integer. Then T_N^{-1} maps the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. More precisely, for $0 < \beta,\gamma < \varepsilon$ there exists a constant $c > 0$ such that if N is sufficiently large,*

$$\|T_N^{-1}(f)\|_{M^{(\beta,\gamma)}(x_0,d)} \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,. \qquad (2.9)$$

The proof of theorem (2.8) is based on the following technical lemma.

Lemma 2.10. *Suppose that the hypotheses of Theorem* 2.8 *are satisfied, and* T_N *and* R_N *are as in* (2.7). *Then for* $0 < \varepsilon' < \varepsilon$ *there exist a constant* $c > 0$ *and* $\delta > 0$ *such that*

$$|R_N(x,y)| \leq c\,2^{-N\delta}\,|x-y|^{-n}, \tag{2.11}$$

$$|R_N(x,y) - R_N(x',y)| \leq c\,2^{-N\delta}\,|x-x'|^{\varepsilon'}\,|x-y|^{-n-\varepsilon'}, \tag{2.12}$$

for $|x-x'| \leq |x-y|/2$,

$$\begin{aligned}&\big|\big(R_N(x,y)b^{-1}(y) - R_N(x',y)b^{-1}(y)\big)\\ &\quad - \big(R_N(x,y')b^{-1}(y') - R_N(x',y')b^{-1}(y')\big)\big|\\ &\qquad\leq c\,2^{-N\delta}\,|x-x'|^{\varepsilon'}\,|y-y'|^{\varepsilon}\,|x-y|^{-n-2\varepsilon'},\end{aligned} \tag{2.13}$$

for $|x-x'| \leq |x-y|/3$ *and* $|y-y'| \leq |x-y|/3$,

$$|\langle R_N f, g\rangle| \leq c\,2^{-N\delta}\,t^{1+2\eta/n}\,\|f\|_\eta\,\|g\|_\eta, \tag{2.14}$$

for all $f, g \in C_0^\eta(\mathbb{R}^n)$, $\eta > 0$, *supported in* Q *with diameter at most* $t > 0$.

Assuming Lemma 2.10 for the moment and applying the same proof of (2.10) to $b^{-1}(R_N)^*M_b$, and using the facts that $R_N(1) = 0$ and $(R_N)^*(b) = 0$, by Theorem 1.19,

$$\|R_N(f)\|_{M^{(\beta,\gamma)}(x_0,d)} \leq c\,2^{-N\delta}\|f\|\|_{M^{(\beta,\gamma)}(x_0,d)}, \tag{2.15}$$

for all $f \in M^{(\beta,\gamma)}(x_0,d)$. Using the fact that $T_N^{-1} = \sum_m (R_N)^m$, we obtain

$$\begin{aligned}\|T^{-1}{}_N(f)\|_{M^{(\beta,\gamma)}(x_0,d)} &\leq \sum_{m=0} (c\,2^{-N\delta})^m \|f\|_{M^{(\beta,\gamma)}(x_0,d)}\\ &\leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)},\end{aligned} \tag{2.16}$$

for a fixed sufficiently large integer N, which shows (2.9) and hence Theorem 2.8.

It remains to prove Lemma 2.10. In fact, we prove the following estimates: for $0 < \varepsilon'' < \varepsilon$ there exists a constant c such that

$$\begin{aligned}&|D_{k+j}\,M_b\,D_k\,M_b(x,y)|\\ &\qquad\leq c\,2^{-|j|\varepsilon}\,\frac{2^{-[(k+j)\wedge k]\varepsilon}}{(2^{-[(k+j)\wedge k]} + |x-y|)^{n+\varepsilon}},\end{aligned} \tag{2.17}$$

$$\begin{aligned}&|D_{k+j}\,M_b\,D_k\,M_b(x,y) - D_{k+j}\,M_b\,D_k\,M_b(x',y)|\\ &\qquad\leq c\left(\frac{|x-x'|}{2^{-[(k+j)\wedge k]}}\right)^{\varepsilon''}\frac{2^{-[(k+j)\wedge k]\varepsilon''}}{(2^{-[(k+j)\wedge k]} + |x-y|)^{n+\varepsilon''}},\end{aligned} \tag{2.18}$$

for $|x-x'| \le |x-y|/2$,

$$\begin{aligned} &\big|\big(D_{k+j}\, M_b\, D_k(x,y) - D_{k+j}\, M_b\, D_k(x',y)\big) \\ &\quad - \big(D_{k+j}\, M_b\, D_k(x,y') - D_{k+j}\, M_b\, D_k(x',y')\big)\big| \\ &\qquad \le c\left(\frac{|x-x'|}{2^{-[(k+j)\wedge k]}}\right)^{\varepsilon''}\left(\frac{|y-y'|}{2^{-[(k+j)\wedge k]}}\right)^{\varepsilon''} \\ &\qquad\quad \cdot \frac{2^{-[(k+j)\wedge k]\varepsilon''}}{(2^{-[(k+j)\wedge k]}+|x-y|)^{n+\varepsilon''}}, \end{aligned} \tag{2.19}$$

for $|x-x'| \le |x-y|/3$ and $|y-y'| \le |x-y|/3$, where D_k are as in Lemma 2.10 and $a \wedge b$ denotes the minimum of a and b.

It is easy to see (2.17). For instance, suppose $j \ge 0$, then

$$\begin{aligned} &|D_{k+j}\, M_b\, D_k\, M_b(x,y)| \\ &\qquad = \Big|\int D_{k+j}(x,z)\, b(z)\, D_k(z,y)\, b(y)\, dz\Big| \\ &\qquad = \Big|\int D_{k+j}(x,z)\, b(z)\, \big(D_k(z,y) - D_k(x,y)\big)\, b(y)\, dz\Big| \\ &\qquad \le c \int_{|z-x|\le 2^{-(k+j)}} 2^{(k+j)n}\, |z-x|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz \\ &\qquad \le c\, 2^{-j\varepsilon}\, 2^{kn}\, \chi_{\{|x-y|\le c\,2^{-k}\}} \end{aligned}$$

which shows (2.17) for the case $j \ge 0$. To see (2.18), suppose $j \ge 0$. Then there exists a constant c such that for $|x-x'| \le |x-y|/2$ and all α , $0 < \alpha < \varepsilon$,

$$\begin{aligned} &|D_{k+j}\, M_b\, D_k\, M_b(x,y) - D_{k+j}\, M_b\, D_k\, M_b(x',y)| \\ &\quad = \Big|\int \big(D_{k+j}(x,z) - D_{k+j}(x',z)\big)\, b(z)\, D_k(z,y)\, b(y)\, dz\Big| \\ &\quad = \Big|\int \big(D_{k+j}(x,z) - D_{k+j}(x',z)\big)\, b(z) \\ &\qquad\qquad \cdot \big(D_k(z,y) - D_k(x,y)\big)\, b(y)\, dz\Big| \\ &\quad \le c \int_{|x-z|\le c\,2^{-(k+j)}} 2^{(k+j)(n+\varepsilon)}\, |x-x'|^{\varepsilon}\, |z-x|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz \\ &\qquad + c \int_{|x'-z|\le c 2^{-(k+j)}} |D_{k+j}(x,z) - D_{k+j}(x',z)| \\ &\qquad\qquad \cdot |z-x|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz \end{aligned}$$

$$
\begin{aligned}
&\le c\,|x-x'|^{\varepsilon}\, 2^{k(n+\varepsilon)} \\
(2.20)\qquad &+ c \int_{|x'-z|\le c2^{-(k+j)}} |D_{k+j}(x,z) - D_{k+j}(x',z)| \\
&\qquad\qquad \cdot \left(|z-x'|^{\varepsilon} + |x-x'|^{\varepsilon}\right) 2^{k(n+\varepsilon)}\, dz \\
&\le c\,|x-x'|^{\varepsilon}\, 2^{k(n+\varepsilon)} \\
&+ c \int_{|x'-z|\le c2^{-(k+j)}} 2^{(k+j)(n+\alpha)} |x-x'|^{\alpha} \\
&\qquad\qquad \cdot |x-x'|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz \\
&\le c\,|x-x'|^{\varepsilon}\, 2^{k(n+\varepsilon)} + c\,2^{\alpha}|x-x'|^{(\alpha+\varepsilon)} 2^{k(n+\alpha+\varepsilon)}\,.
\end{aligned}
$$

Note that for $|x-x'| \le |x-y|/2$ the estimate of (2.17) implies

$$
(2.21)\quad |D_{k+j}\, M_b\, D_k\, M_b(x,y) - D_{k+j}\, M_b\, D_k\, M_b(x',y)| \le c\,2^{-j\varepsilon}\, 2^{kn}\,.
$$

If choose α small enough, the geometric mean of (2.20) and (2.21) and the fact that the support of $D_{k+j}\, M_b\, D_k\, M_b(x,y) - D_{k+j}\, M_b\, D_k\, M_b(x',y)$ is contained in the set $\{|x-y| \le c\,2^{-k}\} \cup \{|x'-y| \le c\,2^{-k}\}$ yield (2.18) for the case $j \ge 0$. The proof of (2.18) for the case $j < 0$ is similar but easier.

The proof of (2.19) is similar. Suppose $j \ge 0$. Then there exists a constant c such that for $|x-x'| \le |x-y|/3$, $|y-y'| \le |x-y|/3$ and all α , $0 < \alpha < \varepsilon$,

$$
\begin{aligned}
&\Big|\big(D_{k+j}\, M_b\, D_k\, M_b(x,y) - D_{k+j}\, M_b\, D_k\, M_b(x',y)\big) \\
&\quad - \big(D_{k+j}\, M_b\, D_k\, M_b(x,y') - D_{k+j}\, M_b\, D_k\, M_b(x',y')\big)\Big| \\
&\qquad = \Big| \int \big(D_{k+j}(x,z) - D_{k+j}(x',z)\big) \\
&\qquad\qquad \cdot b(z)\,\big(D_k(z,y) - D_k(z,y')\big)\, b(y)\, dz \Big| \\
&\qquad = \Big| \int \big(D_{k+j}(x,z) - D_{k+j}(x',z)\big)\, b(z) \\
&\qquad\qquad \cdot \big(\big(D_k(z,y) - D_k(z,y')\big) \\
&\qquad\qquad\quad - \big(D_k(x,y) - D_k(x,y')\big)\big)\; b(y)\, dz \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le c \int_{|x-z|\le c\,2^{-(k+j)}} 2^{(k+j)(n+\varepsilon)}\,|x-x'|^{\varepsilon} \\
&\qquad\qquad \cdot |z-x|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)}\,dz \\
&\quad + c \int_{|x'-z|\le c\,2^{-(k+j)}} |D_{k+j}(x,z)-D_{k+j}(x',z)| \qquad (2.22)\\
&\qquad\qquad \cdot |z-x|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)}\,dz \\
&\le c\,|x-x'|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)} \\
&\quad + c \int_{|x'-z|\le c\,2^{-(k+j)}} 2^{(k+j)(n+\alpha)}\,|x-x'|^{\alpha} \\
&\qquad\qquad \cdot |x-x'|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)}\,dz \\
&\le c\,|x-x'|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)} \\
&\quad + c\,2^{j\alpha}\,|x-x'|^{(\alpha+\varepsilon)}\,|y-y'|^{\varepsilon}\,2^{k(n+\alpha+2\varepsilon)}\,.
\end{aligned}
$$

Note that for $|x-x'|\le |x-y|/3$ and $|y-y'|\le |x-y|/3$ the estimates of (2.18) and (2.21) imply

(2.23)

$$
\begin{aligned}
&\big|\big(D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)\big)\\
&\quad-\big(D_{k+j}\,M_b\,D_k\,M_b(x,y')-D_{k+j}\,M_b\,D_k\,M_b(x',y')\big)\big|\\
&\qquad\qquad\le c\big(|x-x'|^{\varepsilon'}\,2^{k(n+\varepsilon')}\wedge 2^{-j\varepsilon}2^{kn}\big)\,.
\end{aligned}
$$

If choose α small enough, the geometric mean of (2.22) and (2.23) and the fact that the support of

$$
\begin{aligned}
&\big(D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)\big)\\
&\quad-\big(D_{k+j}\,M_b\,D_k\,M_b(x,y')-D_{k+j}\,M_b\,D_k\,M_b(x',y')\big)
\end{aligned}
$$

is contained in the set

$$
\{|x-y|\le c\,2^{-k}\}\cup\{|x'-y|\le c\,2^{-k}\}\cup\{|x-y'|\le c\,2^{-k}\}\cup\{|x'-y'|\le c\,2^{-k}\}
$$

yield (2.19) for the case $j\ge 0$. The proof of (2.19) for the case $j<0$ is similar but easier.

Considering the cases $j\ge 0$ and $j<0$ separately, and summing over k, and then, (2.17) implies (2.11), (2.18) and (2.19) imply (2.12) and (2.13) with constant $c\,2^{-N\delta}$ replaced by a constant c, respectively. By taking the geometric means with (2.11) we obtain (2.12) and (2.13). We leave these details to the reader. Finally, the estimate of (2.14) follows the following simple calculations.

$$\langle R_N f, g\rangle = \sum_{|j|>N}\sum_{k\in\mathbb{Z}} \iint D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, g(x)\, dy\, dx\,.$$

Since

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, dy\Big| \\
&\quad= \Big|\int D_{k+j}\, M_b\, D_k\, M_b(x,y)\, \big(f(y)-f(x)\big)\, dy\Big| \\
&\quad\le c\int_{\mathbb{R}^n} 2^{-|j|\varepsilon}\, 2^{[(k+j)\wedge k]n}\, \chi_{\{|x-y|\le c\,2^{-[(k+j)\wedge k]}\}}\, |f(y)-f(x)|\, dy \\
&\quad\le c\,2^{-|j|\varepsilon}\, 2^{-[(k+j)\wedge k]\eta}\, \|f\|_\eta\,,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, dy\Big| \\
&\qquad\le c\int_{\mathbb{R}^n} 2^{-|j|\varepsilon}\, 2^{[(k+j)\wedge k]n}\, \chi_{\{|x-y|\le c\,2^{-[(k+j)\wedge k]}\}}\, |f(y)|\, dy \\
&\qquad\le c\,2^{-|j|\varepsilon}\, 2^{[(k+j)\wedge k]n}\, \|f\|_\infty\, |Q|\,,
\end{aligned}$$

and denote $|Q| = 2^{-k_0 n}$, we then, by the estimates above, have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, dy\Big| \\
&\qquad\le \begin{cases} c\,2^{-|j|\varepsilon}\, 2^{-([(k+j)\wedge k]-k_0)\eta}\, \|f\|_\eta\, |Q|^{\eta/n}\,; \\ c\,2^{-|j|\varepsilon}\, 2^{([(k+j)\wedge k]-k_0)n}\, \|f\|_\eta\, |Q|^{\eta/n}\,. \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\iint D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, g(x)\, dy\, dx\Big| \\
&\qquad\le c\,2^{-|j|\varepsilon}\, 2^{-([(k+j)\wedge k]-k_0)\eta}\, \|f\|_\eta\, |Q|^{1+2\eta/n}\, \|g\|_\eta\,,
\end{aligned}$$

and

$$\Big|\iint D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, g(x)\, dy\, dx\Big| \le c\, 2^{-|j|\varepsilon}\, 2^{([(k+j)\wedge k]-k_0)n}\, \|f\|_\eta\, |Q|^{1+2\eta/n}\, \|g\|_\eta\,.$$

which yields (2.14). This shows Lemma 2.10.

Now we turn to the proof of Theorem 2.3. Let $\widetilde{D}_k = T_N^{-1} D_k^N$, where D_k^N is defined in (2.7) and N is a fixed large integer such that T_N^{-1} maps the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ by Theorem 2.8. It is easy to see that $D_k^N(x,y)$, the kernel of D_k^N, is a strong b-smooth molecule of type $(\varepsilon,\varepsilon)$ centered at y with width $2^{-k} > 0$. Thus, $\widetilde{D}_k(x,y) = T_N^{-1}[D_k^N(\cdot,y)](x)$, the kernel of $\widetilde{D}_k$, is a strong b-smooth molecule of type $(\varepsilon',\varepsilon')$ centered at y with width $2^{-k} > 0$ for $0 < \varepsilon' < \varepsilon$ by Theorem 2.8. This shows that $\widetilde{D}_k(x,y)$ satisfies the conditions (i), (ii) and (iii) of (2.5). The condition (iv) of (2.5) follows from the fact that $(D_k^N)(b) = 0$. All we need to do now is to prove that the series in (2.4) converges in the norm of L^p and $M^{(\beta',\gamma')}$. Suppose first that $f \in M^{(\beta,\gamma)}$. Then the convergence of the series in (2.4) in $M^{(\beta',\gamma')}$ is equivalent to

$$\lim_{M\to\infty} \|\sum_{|k|\le M} \widetilde{D}_k\, M_b\, D_k\, M_b(f) - f\|_{M^{(\beta',\gamma')}} = 0\,, \tag{2.24}$$

for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$. Since

$$\begin{aligned}\sum_{|k|\le M} \widetilde{D}_k\, M_b\, D_k\, M_b(f) &= T_N^{-1}\Big(\sum_{|k|\le M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\\ &= T_N^{-1}\Big(T_N - \sum_{|k|>M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\\ &= f - \lim_{m\to\infty} R_N^m(f)\\ &\quad - T_N^{-1}\Big(\sum_{|k|>M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\,,\end{aligned}$$

to show (2.24), it suffices to prove

$$\lim_{m\to\infty} \|R_N^m(f)\|_{M^{(\beta',\gamma')}} = 0\,, \tag{2.25}$$

$$\lim_{M\to\infty} \Big\|T_N^{-1}\Big(\sum_{|k|>M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\Big\|_{M^{(\beta,'\gamma')}} = 0\,. \tag{2.26}$$

By (2.15),

$$\|R_N^m(f)\|_{M(\beta',\gamma')} \le (c\,2^{-N\delta})^m \,\|f\|_{M(\beta',\gamma')} \le (c\,2^{-N\delta})^m \,\|f\|_{M(\beta,\gamma)} ,$$

since $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, which gives (2.25). The proof of (2.26) is based on the following estimate

$$\Big\| \sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_{M(\beta',\gamma')} \le c\,2^{-M\sigma}\,\|f\|_{M(\beta,\gamma)} , \tag{2.27}$$

for all $0 < \beta' < \beta, 0 < \gamma' < \gamma$ and some $\sigma > 0$, and a constant c which is independent of f and M.

Assuming (2.27) for the moment, by Theorem 2.8, for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$,

$$\begin{aligned} \Big\|T_N^{-1} \sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_{M(\beta',\gamma')} & \\ &\le c \Big\| \sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_{M(\beta',\gamma')} \\ &\le c\,2^{-M\sigma}\,\|f\|_{M(\beta,\gamma)} , \end{aligned}$$

which gives (2.26).

To prove (2.27), it suffices to show that for $0 < \beta'' < \beta$ and $0 < \gamma' < \gamma$ there exist a constant c which is independent of f and M and some $\sigma > 0$ such that

$$\begin{aligned} \Big| \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\Big| & \\ &\le c\,2^{-M\sigma}(1+|x|)^{-(n+\gamma')}\,\|f\|_{M(\beta,\gamma)} , \end{aligned} \tag{2.28}$$

$$\begin{aligned} \Big| \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) - \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| & \\ &\le c\Big(\frac{|x-x'|}{1+|x|}\Big)^{\beta''}\frac{1}{(1+|x|)^{n+\gamma'}}\,\|f\|_{M(\beta,\gamma)} , \end{aligned} \tag{2.29}$$

for $|x-x'| \le (1+|x|)/2$.

To see this, by taking the geometric average between (2.29) and the following estimate

$$\begin{aligned}&\Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) - \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| \\ &\qquad \le \Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\Big| + \Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| \\ &\qquad \le c\, 2^{-M\sigma}(1+|x|)^{-(n+\gamma')}\|f\|_{M^{(\beta,\gamma)}},\end{aligned}$$

for $|x-x'| \le (1+|x|)/2$, we have

$$\begin{aligned}&\Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) - \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| \\ &\qquad \le c\, 2^{-M\sigma'}|x-x'|^{\beta'}(1+|x|)^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}},\end{aligned} \tag{2.30}$$

for $|x-x'| \le (1+|x|)/2$.

Now (2.28) and (2.30) together with the fact that

$$\begin{aligned}&\int_{\mathbb{R}^n} \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\, b(x)\, dx \\ &\qquad = \int_{\mathbb{R}^n} \sum_{|k|>M} M_b D_k M_b(f)(x)\,(D_k^N)^*(b)(x)\, dx = 0,\end{aligned}$$

show that

$$\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) \in M^{(\beta',\gamma')}$$

and

$$\Big\|\sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_{M^{(\beta',\gamma')}} \le c\, 2^{-M\sigma}\|f\|_{M^{(\beta,\gamma)}},$$

which gives (2.27).

Now we prove (2.28). Denote $E_k = D_k^N M_b D_k$. It is easy to check that $E_k(x,y)$, the kernel of E_k, satisfies the conditions (2.2.i), (2.2.ii), and (2.2.iii) with ε replaced by ε', $0<\varepsilon'<\varepsilon$, and $E_k(b)=0$. Consider first the case where $|x| \leq 2$, then

$$
\begin{aligned}
&\Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\Big| \\
&\quad = \Big|\sum_{|k|>M} E_k M_b(f)(x)\Big| \\
&\quad \leq \Big|\sum_{k>M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, (f(y)-f(x))\, dy\Big| \\
&\qquad + \Big|\sum_{k<-M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, f(y)\, dy\Big| \quad (\text{ since } E_k(b)=0) \\
&\quad \leq c \sum_{k>M} 2^{-k\beta}\, \|f\|_{M(\beta,\gamma)} + c \sum_{k<-M} 2^{kn}\, \|f\|_{M(\beta,\gamma)} \\
&\quad \leq c\,(2^{-M\beta} + 2^{-Mn})\, \|f\|_{M(\beta,\gamma)} \\
&\quad \leq c\, 2^{-M\sigma}\, (1+|x|)^{-(n+\gamma')}\, \|f\|_{M(\beta,\gamma)}\,, \qquad (\text{ since } |x| \leq 2)
\end{aligned} \tag{2.31}
$$

and, where $\sigma > 0$ is a constant and $0 < \gamma' < \gamma$.

This proves (2.28) for$|x| \leq 2$. If $|x| > 2$, then

$$
\begin{aligned}
\Big|\sum_{|k|>M} E_k M_b(f)(x)\Big| &\leq \Big|\sum_{k>M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)(f(y)-f(x))\, dy\Big| \\
&\quad + \Big|\sum_{k<-M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, f(y)\, dy\Big| = \mathrm{I} + \mathrm{II}\,.
\end{aligned}
$$

Since $|x-y| \leq c\,2^{-k} < c\,2^{-M}$ for $k > M$ and hence $|x-y| < 1$ if M is larger than $\log_2 c$. This gives that $|y| \geq |x| - |x-y| \geq |x|/2$ for $M > \log_2 c$ and term I is now bounded by a constant times

$$
\begin{aligned}
&\sum_{k>M} \int_{\mathbb{R}^n} |E_k(x,y)|\, |x-y|^{\beta} \\
&\qquad \cdot \big((1+|y|)^{-(n+\gamma)} + (1+|x|)^{-(n+\gamma)}\big)\, dy\, \|f\|_{M(\beta,\gamma)} \\
&\quad \leq c\Big(\sum_{k>M} 2^{-k\beta}\Big)(1+|x|)^{-(n+\gamma)}\, \|f\|_{M(\beta,\gamma)} \\
&\quad \leq c\, 2^{-M\beta}\, (1+|x|)^{-(n+\gamma')}\, \|f\|_{M(\beta,\gamma)}\,.
\end{aligned} \tag{2.32}
$$

To estimate the term II, by use of the fact that $\int_{\mathbb{R}^n} f(y)b(y)dy = 0$, we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, f(y)\, dy\Big| \\
&\quad = \Big|\int_{\mathbb{R}^n} \big(E_k(x,y) - E_k(x,0)\big)\, b(y)\, f(y)\, dy\Big| \\
&\quad \le c \int\limits_{|y|\le|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\qquad + c \int\limits_{|x|/2<|y|<3|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\qquad + c \int\limits_{|y|\ge 3|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy\,.
\end{aligned}
\tag{2.33}
$$

Since

$$
|E_k(x,y) - E_k(x,0)| \le c\, 2^{k(n+\varepsilon)} \Big(\frac{|y|}{|x|}\Big)^{\varepsilon},
$$

the size condition of f yields

$$
\begin{aligned}
&\int\limits_{|y|\le|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\quad \le c \int\limits_{|y|\le|x|/2} 2^{k(n+\varepsilon)} \Big(\frac{|y|}{|x|}\Big)^{\varepsilon} \frac{1}{|y|^{n+\gamma}}\, dy \\
&\qquad\quad \cdot \chi_{\{k:\ 2^k \le c|x|^{-1}\}}\, \|f\|_{M^{(\beta,\gamma)}} \\
&\quad \le c\, 2^{k\varepsilon}\, |x|^{-(n+\gamma)}\, \|f\|_{M^{(\beta,\gamma)}}\,.
\end{aligned}
\tag{2.34}
$$

Similarly,

$$
\begin{aligned}
&\int\limits_{|y|\ge 3|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\quad \le c\, 2^{kn} \chi_{\{k:\ 2^k \le c|x|^{-1}\}}\, |x|^{-\gamma}\, \|f\|_{M^{(\beta,\gamma)}} \\
&\quad \le c\, 2^{k\sigma}\, |x|^{-(n+\gamma')}\, \|f\|_{M^{(\beta,\gamma)}}
\end{aligned}
\tag{2.35}
$$

and

$$\int_{|x|/2<|y|<3|x|/2} |E_k(x,y) - E_k(x,0)|\,|f(y)|\,dy$$

$$\text{(2.36)} \qquad \leq c \int_{|x|/2<|y|<3|x|/2} \left(|E_k(x,y)| + |E_k(x,0)|\right)|f(y)|\,dy$$

$$\leq c\,2^{k\sigma}\,|x|^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}},$$

where $\sigma = \gamma - \gamma' > 0$.

Combining (2.33), (2.34), (2.35) and (2.36) shows

$$\begin{aligned} \text{II} &\leq c \sum_{k<-M} 2^{k\sigma}\,|x|^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}} \\ &\leq c\,2^{-M\sigma}\,(1+|x|^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}}, \end{aligned}$$

which together with (2.31) and (2.32) implies (2.28).

It remains to prove (2.29). We need only to check that

$$\sum_{|k|>M} D_k^N\,M_b\,D_k\,M_b \qquad \text{and} \qquad b^{-1}\Big(\sum_{|k|>M} D_k^N\,M_b\,D_k\,M_b\Big)^*M_b,$$

as operators, satisfy the hypotheses of Theorem 1.19 and the estimates of the kernels are independent of M. Since

$$b^{-1}\Big(\sum_{|k|>M} D_k^N\,M_b\,D_k\,M_b\Big)^*M_b = \sum_{|k|>M} D_k^*\,M_b\,(D_k^N)^*\,M_b,$$

and D_k and D_k^* satisfy the same conditions, so it suffices to check that $\sum_{|k|>M} D_k^N M_b D_k M_b$ satisfies the hypotheses of Theorem 1.19 with the constants independent of M. This follows from the simple computation. We leave these details to the reader.

Finally, to see that the series in (2.4) converges in L^p for $1 < p < \infty$, by the proof above, we only need to show that (2.25) and (2.26) still hold with the norm of $M^{(\beta',\gamma')}$ replaced by the norm of L^p for $1 < p < \infty$. The estimates in Lemma 2.10 show that $R_N b^{-1}$ is a Calderón-Zygmund operator with the operator norm at most $c\,2^{-N\delta}$ and hence R_N is bounded on L^p for $1 < p < \infty$ with the operator norm at most $c\,2^{-N\delta}$. This yields (2.25) and also implies that T_N^{-1} is bounded on L^p for $1 < p < \infty$. To see that (2.26) still holds with the norm of $M^{(\beta',\gamma')}$ replaced by the norm of L^p for $1 < p < \infty$, it suffices to show $\lim_{M\to\infty} \| \sum_{|k|>M} D_k^N M_b D_k M_b(f)\|_p = 0$ for $f \in L^p, 1 < p < \infty$. This can be proved by a result in [DJS]. More precisely,

$$\Big\| \sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_p$$

$$= \sup_{\|g\|_{p'}\leq 1} \langle \sum_{|k|>M} D_k^N M_b D_k M_b(f), g \rangle$$

$$\leq \sup_{\|g\|_{p'}\leq 1} \Big\|\Big(\sum_{|k|>M} |D_k M_b(f)|^2\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_{|k|>M} |M_b (D_k^N)^*(g)|^2\Big)^{1/2}\Big\|_{p'}$$

(by a result of [DJS])

$$\leq c \sup_{\|g\|_{p'}\leq 1} \Big\|\Big(\sum_{|k|>M} |D_k M_b(f)|^2\Big)^{1/2}\Big\|_p \|g\|_{p'}$$

$$\leq c \Big\|\Big(\sum_{|k|>M} |D_k M_b(f)|^2\Big)^{1/2}\|_p ,$$

where again by a result of [DJS] the last term tends to zero as M tends to infinity . This ends the proof of Theorem 2.3.

By an argument of duality we obtain the following Calderón-type reproducing formula on $(M^{(\beta,\gamma)})'$:

Theorem 2.37. *Suppose that $(D_k)_{k\in\mathbb{Z}}$ is as in Theorem 2.3. Then there exists a family of operators $(\widetilde{D}_k)_{k\in\mathbb{Z}}$ whose kernels satisfy the same properties as in Theorem 2.3 such that for all $f \in (M^{(\beta,\gamma)})'$,*

$$f = \sum_{k\in\mathbb{Z}} M_b \widetilde{D}_k M_b D_k(f) , \tag{2.38}$$

where the series converges in the sense that for all $g \in M^{(\beta',\gamma')}$ with $\beta' > \beta$ and $\gamma' > \gamma$,

$$\lim_{M\to\infty} \langle \sum_{|k|\leq M} M_b \widetilde{D}_k M_b, D_k(f), g \rangle = \langle f, g\rangle . \tag{2.39}$$

We leave the details to the reader.

Section 3.

In this section we introduce a new class of the Besov and Triebel-Lizorkin spaces associated to a para-accretive function, which generalizes the classical Besove and Triebel-Lizorkin spaces on $\mathbb{R}^n$ and prove a Tb theorem on these spaces. We begin with the following proposition.

Proposition 3.1. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are approximations to the identity defined in* (2.1). *Set $D_k = S_k - S_{k-1}$ and $E_k = Q_k - Q_{k-1}$. Then for all $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$, where ε is the regularity exponent of the approximations to the identity, there exist two constants c_1 and $c_2 > 0$ such that*

$$\begin{aligned} c_1 \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|E_k(f)\|_p)^q\Big)^{1/q} &\le \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|D_k(f)\|_p)^q\Big)^{1/q} \\ &\le c_2 \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|E_k(f)\|_p)^q\Big)^{1/q}, \end{aligned} \tag{3.2}$$

for $-\varepsilon < \alpha < \varepsilon$, $1 \le p, q \le \infty$,

$$\begin{aligned} c_1 \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |E_k(f)|)^q\Big)^{1/q} \Big\|_p &\le \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |D_k(f)|_p)^q\Big)^{1/q} \Big\|_p \\ &\le c_2 \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |E_k(f)|^q\Big)^{1/q} \Big\|_p, \end{aligned} \tag{3.3}$$

for $-\varepsilon < \alpha < \varepsilon$, $1 < p, q < \infty$.

PROOF. We first prove (3.2). Without loss of generality we may assume that

$$\Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|E_k(f)\|_p)^q\Big)^{1/q} < +\infty$$

and

$$\Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|D_k(f)\|_p)^q\Big)^{1/q} < +\infty\,.$$

Since $D_k(\cdot, y) \in M^{(\varepsilon,\varepsilon)}$, by the Calderón-type reproducing formula in (2.37), there exists a family of operators $(\widetilde{E}_j)_{j\in Z}$ such that

$$\begin{aligned} D_k(f)(x) = \langle D_k(x,\cdot), f\rangle &= \langle D_k(x,\cdot), \sum_{j\in\mathbb{Z}} M_b\, \widetilde{E}_j\, M_b\, E_j(f)\rangle \\ &= \sum_{j\in\mathbb{Z}} D_k\, M_b\, \widetilde{E}_j\, M_b\, E_j(f)(x)\,. \end{aligned}$$

Thus,

$$\begin{aligned} \|D_k(f)\|_p &\le \sum_{j\in\mathbb{Z}} \|D_k\, M_b\, \widetilde{E}_j\, M_b\, E_j(f)\|_p \\ &\le \sum_{j\in\mathbb{Z}} \|D_k\, M_b\, \widetilde{E}_j\|_{p,p}\, \|M_b\, E_j(f)\|_p\,. \end{aligned} \tag{3.4}$$

The estimate in (2.17) still holds with D_k replaced by $\widetilde{E}_j$ and hence implies that $\|D_k\, M_b\, \widetilde{E}_j\|_{p,p} \le c\, 2^{-|k-j|\varepsilon''}$. Thus,

$$\begin{aligned} \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha}\, \|D_k(f)\|_p)^q\Big)^{1/q} &\le \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''}\, \|E_j(f)\|_p)^q\Big)^{1/q} \\ &\le c\Big(\sum_{k\in\mathbb{Z}} \big(\sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha}\big)^{q/q'} \\ &\qquad \cdot \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha}\, (2^{j\alpha}\, \|E_j(f)\|_p)^q\Big)^{1/q} \\ &\le c\Big(\sum_{j\in\mathbb{Z}} (2^{j\alpha}\, \|E_j(f)\|_p)^q\Big)^{1/q} < +\infty\,, \end{aligned}$$

since we may choose $-\varepsilon'' < \alpha < \varepsilon''$ and hence

$$\sum_{k\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha} + \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha} < +\infty\,.$$

The same proof can be applied to prove the other inequality in (3.2).

To prove (3.3), we will use the Fefferman-Stein vector-valued maximal function inequality. As in the proof above, we have

$$\begin{aligned} |D_k(f)(x)| &\le \sum_{j\in\mathbb{Z}} |D_k\, M_b\, \widetilde{E}_j\, M_b\, E_j(f)(x)| \\ &\le c \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''}\, M(M_b E_j(f))(x)\,, \end{aligned}$$

where M is the Hardy-Littlewood maximal function. This gives

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{k\alpha}|D_k(f)|)^q\Big)^{1/q}\Big\|_p$$

$$\le c\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\sum_{j\in\mathbb{Z}}2^{-|k-j|\varepsilon''}M(M_bE_j(f))^q\Big)^{1/q}\Big\|_p$$

$$\le c\Big\|\Big(\sum_{k\in\mathbb{Z}}\big(\sum_{j\in\mathbb{Z}}2^{-|k-j|\varepsilon''+(k-j)\alpha}\big)^{q/q'}$$

$$\cdot\sum_{j\in\mathbb{Z}}2^{-|k-j|\varepsilon''+(k-j)\alpha}\big(2^{k\alpha}M(M_bE_j(f))^q\Big)^{1/q}\Big\|_p$$

$$\le c\Big\|\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}M(M_bE_j(f))^q\Big)^{1/q}\Big\|_p$$

(since $\sum_{k\in\mathbb{Z}}2^{-|k-j|\varepsilon''+(k-j)\alpha}+\sum_{j\in Z}2^{-|k-j|\varepsilon''+(k-j)\alpha}<+\infty$)

$$\le c\Big\|\Big(\sum_{j\in\mathbb{Z}}2^{j\alpha}|M_bE_j(f)|^q\Big)^{1/q}\Big\|_p$$

(by the Fefferman-Stein vector-valued maximal function inequality for $1<p,q<\infty$)

$$\le c\Big\|\Big(\sum_{j\in Z}2^{j\alpha}|E_j(f)|^q\Big)^{1/q}\Big\|_p$$

(since $b\in L^\infty(\mathbb{R}^n)$), which shows one inequality in (3.3). The other inequality in (3.3) can be proved by same manner.

We remark that if the kernels of E_k satisfy the conditions (i), (ii), (iii), and (iv) of (2.5), the first inequality in (3.2) and (3.3) still hold.

The proposition above allows us to introduce the following Besov and Triebel-Lizorkin spaces associated to a para-accretive function.

Definition 3.5. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ is an approximations to the identity defined in* (2.1). *Set $D_k=S_k-S_{k-1}$. The Besov spaces $b\dot{B}_p^{\alpha,q}$, for $-\varepsilon<\alpha<\varepsilon$ and $1\le p,q\le\infty$, are the collection of $f\in(M^{(\beta,\gamma)})'$, for $0<\beta,\gamma<\varepsilon$, such that*

$$\|f\|_{b\dot{B}_p^{\alpha,q}}=\Big(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|D_k(f)\|_p)^q\Big)^{1/q}<+\infty. \tag{3.6}$$

The Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}$, *for* $-\varepsilon < \alpha < \varepsilon$ *and* $1 < p, q < \infty$, *are the collection of* $f \in (M^{(\beta,\gamma)})'$, *for* $0 < \beta, \gamma < \varepsilon$, *such that*

$$\|f\|_{b\dot{F}_p^{\alpha,q}} = \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |D_k(f)|)^q \Big)^{1/q} \Big\|_p < +\infty \,. \tag{3.7}$$

To prove the Tb theorem on the Besov spaces $b\dot{B}_p^{\alpha,q}$ and Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}$ we need the following proposition.

Proposition 3.8. *Suppose that* $f \in (M^{(\beta,\gamma)})'$ *with* $0 < \beta, \gamma < \varepsilon$ *and* $\|f\|_{b\dot{B}_p^{\alpha,q}} < \infty$ *for* $-\varepsilon < \alpha < \varepsilon$ *and* $1 \le p, q \le \infty$ *(respectively,* $\|f\|_{b\dot{F}_p^{\alpha,q}} < \infty$ *for* $-\varepsilon < \alpha < \varepsilon$ *and* $1 < p, q < \infty$*). Then there exists a sequence* $\{f_n\}_{n=1}^{\infty}, f_n \in bM^{(\varepsilon',\varepsilon')}$ *with* $0 < \varepsilon' < \varepsilon, n = 1, 2, \ldots,$ *such that*

$$\lim_{n\to\infty} \|f_n - f\|_{b\dot{B}_p^{\alpha,q}} = 0 \qquad (\textit{respectively, } \lim_{n\to\infty} \|f_n - f\|_{b\dot{F}_p^{\alpha,q}} = 0).$$

PROOF. Suppose $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ and $\|f\|_{b\dot{B}_p^{\alpha,q}} < +\infty$ (respectively, $\|f\|_{b\dot{F}_p^{\alpha,q}} < +\infty$). It suffices to show that for any $\delta > 0$, there exists a $g \in bM^{(\varepsilon',\varepsilon')}$ such that

$$\|g - f\|_{b\dot{B}_p^{\alpha,q}} < \delta \qquad (\text{respectively, } \|g - f\|_{b\dot{F}_p^{\alpha,q}} < \delta)\,.$$

To see this, it follows from the Calderón-type reproducing formula in (2.37) and the proof of (3.1) that

$$\Big\| \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) - f \Big\|_{b\dot{B}_p^{\alpha,q}} \le c \Big(\sum_{|j|>M} (2^{j\alpha} \|D_j(f)\|_p)^q \Big)^{1/q}$$

and hence

$$\lim_{M\to\infty} \Big\| \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) - f \Big\|_{b\dot{B}_p^{\alpha,q}} = 0$$

$$(\text{respectively, } \lim_{M\to\infty} \Big\| \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) - f \Big\|_{b\dot{F}_p^{\alpha,q}} = 0).$$

Now set

$$g_M(x) = \sum_{|j|\le M} \int_{|y|\le M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy\,.$$

It is easy to check that $g_M \in bM^{(\varepsilon',\varepsilon')}$. All we need to do now is to show that for any given $\delta > 0$ there exists $M_0 > 0$ such that for all $M > M_0$

$$\Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{B}_p^{\alpha,q}} < \delta$$

$$\text{(respectively, } \Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{F}_p^{\alpha,q}} < \delta).$$

To do this, we have

$$g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f)(x) = - \sum_{|j|\le M} \int_{|y|>M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy .$$

By the definition (3.5), then it follows from the proof of (3.1) that

$$\Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{B}_p^{\alpha,q}}$$
$$= \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \Big\| D_k \sum_{|j|\le M} \int_{|y|>M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy \Big\|_p)^q \Big)^{1/q}$$

(respectively,

$$\Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{F}_p^{\alpha,q}}$$
$$= \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |D_k \sum_{|j|\le M} \int_{|y|>M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy|)^q \Big)^{1/q} \Big\|_p)$$

$$\le \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \sum_{|j|\le M} \|D_k M_b \widetilde{D}_j\|_{p,p} \int_{|y|>M} |b(y) D_j(f)(y)|\, dy)^q \Big)^{1/q}$$

(respectively,

$$\le \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \sum_{|j|\le M} 2^{-|k-j|\varepsilon'} M(\chi_M D_j(f)))^q \Big)^{1/q} \Big\|_p ,$$

where $\chi_M = \chi_{\{y:\ |y|\ge M\}}$)

$$\le c \Big(\sum_{|j|\le M} (2^{j\alpha} (\int_{|y|>M} |D_j(f)(y)|^p\, dy)^{1/p})^q \Big)^{1/q}$$

(respectively,
$$\leq c\Big\| \sum_{|j|\leq M} \big(2^{k\alpha}\, |\chi_M\, D_j(f)|^q\big)^{1/q}\Big\|_p).$$

Since $\|f\|_{b\dot{B}_p^{\alpha,q}} < +\infty$ (respectively, $\|f\|_{b\dot{F}_p^{\alpha,q}} < +\infty$), so that there exists $M_1 > 0$ such that for all $M \geq M_1$,

$$\Big(\sum_{|j|\geq M} (2^{j\alpha}\, \|D_j(f)\|_p)^q \Big)^{1/q} < \frac{1}{2c}\,\delta$$

$$\text{(respectively, } \Big\|\Big(\sum_{|j|\geq M} (2^{j\alpha}\, \|D_j(f))^q \Big)^{1/q}\Big\|_p < \frac{1}{2c}\,\delta\,).$$

It is easy to see that there exists $M_0 > M_1$ such that for all $M \geq M_0$

$$\Big(\sum_{|j|\leq M_0} (2^{j\alpha}(\int_{|y|>M} |D_j(f)(y)|^p\, dy)^{1/p})^q \Big)^{1/q} < \frac{1}{2c}\,\delta$$

$$\text{(respectively, } \Big\|\Big(\sum_{|j|\leq M_0} (2^{j\alpha}\, |\chi_M\, D_j(f)|)^q \Big)^{1/q}\Big\|_p < \frac{1}{2c}\,\delta\,).$$

Thus, for $M \geq M_0$ we then have

$$\begin{aligned}
&\Big\| g_M - \sum_{|j|\leq M} M_b\, \widetilde{D}_j\, M_b\, D_j(f) \Big\|_{\dot{B}_p^{\alpha,q}(b)} \\
&\leq c\Big(\sum_{|j|\leq M} (2^{j\alpha}(\int_{|y|>M} |D_j(f)(y)|^p\, dy)^{1/p})^q \Big)^{1/q} \\
&\leq c\Big(\sum_{|j|\geq M_0} (2^{j\alpha}\, \|D_j(f)\|_p)^q \Big)^{1/q} \\
&\quad + c\Big(\sum_{|j|\leq M_0} (2^{j\alpha}(\int_{|y|>M} |D_j(f)(y)|^p\, dy)^{1/p})^q \Big)^{1/q} \\
&\leq c\,\frac{1}{2c}\,\delta + c\,\frac{1}{2c}\,\delta = \delta\,.
\end{aligned}$$

(respectively,

$$\begin{aligned}\Big\| g_M - \sum_{|j|\le M} M_b\, \widetilde{D}_j\, M_b\, D_j(f)\Big\|_{b\dot{F}_p^{\alpha,q}} &\le c\Big\|\Big(\sum_{|j|\le M} \big(2^{j\alpha}\,|\chi_M\, D_j(f)|\big)^q\Big)^{1/q}\Big\|_p \\ &\le c\Big\|\Big(\sum_{|j|\ge M_0} \big(2^{j\alpha}\,|D_j(f)|\big)^q\Big)^{1/q}\Big\|_p \\ &\quad + c\Big\|\Big(\sum_{|j|\le M_0} \big(2^{j\alpha}\,\chi_M\, D_j(f)|\big)^q\Big)^{1/q}\Big\|_p \\ &\le c\frac{1}{2c}\,\delta + c\frac{1}{2c}\,\delta = \delta\,).\end{aligned}$$

This completes the proof of Proposition 3.8.

We now state the Tb theorem for $b\dot{B}_p^{\alpha,q}$ and $b\dot{F}_p^{\alpha,q}$. First notice that if T is a singular integral operator defined in (1.25), then T can be extended to a continuous linear operator from $bM^{(\beta,\gamma)}, 0 < \beta, \gamma$, to $(bC_0^\eta)'$. To see this, let $f \in M^{(\beta,\gamma)}$ with $0 < \beta, \gamma$ and $g \in bC_0^\eta(\mathbb{R}^n)$, and choose $\theta \in C_0^1(\mathbb{R}^n)$ with $\theta(x) = 1$ for $x \in \operatorname{supp} g$, we then define

$$\langle Tbf, g\rangle = \langle T(bf\theta), g\rangle + \langle T(bf(1-\theta)), g\rangle\,. \tag{3.9}$$

Since $bf\theta \in bC_0^\eta(\mathbb{R}^n)$, so $\langle Tbf\theta, g\rangle$ is well defined. Using the facts that $bf(1-\theta)$ and g belong to $L^1(\mathbb{R}^n)$ and $\operatorname{supp} bf(1-\theta) \cap \operatorname{supp} g = \varnothing$, then

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y)\, b(y)\, f(y)\, (1-\theta(y))\, g(x)\, dy\, dx$$

converges absolutely, and hence $\langle T(bf(1-\theta)), g\rangle$ is well defined. It is also easy to check that (3.9) is independent of the choice of θ.

Theorem 3.10. *Suppose that T is a singular integral operator defined in (1.25) and $T(b) = T^*(b) = 0$, and M_bTM_b has the weak boundedness property. Then T can be extended to a bounded operator from $b\dot{B}_p^{\alpha,q}$ to $b^{-1}\dot{B}_p^{\alpha,q}$ for $-\varepsilon < \alpha < \varepsilon$, $1 \le p, q \le \infty$, and from $b\dot{F}_p^{\alpha,q}$ to $b^{-1}\dot{F}_p^{\alpha,q}$ for $-\varepsilon < \alpha < \varepsilon$, $1 < p, q < \infty$, where $f \in b^{-1}\dot{B}_p^{\alpha,q}$ if and only if*

$bf \in b\dot{B}_p^{\alpha,q}$, and $f \in b^{-1}\dot{F}_p^{\alpha,q}$ if and only if $bf \in b\dot{F}_p^{\alpha,q}$, and ε is the regularity exponent of the kernel of T.

PROOF. By Proposition 3.8, $bM^{(\varepsilon',\varepsilon')}$, $0 < \varepsilon' < \varepsilon$, is dense in $b\dot{B}_p^{\alpha,q}$ for $-\varepsilon' < \alpha < \varepsilon'$ and $1 \le p,q \le \infty$, and in $b\dot{F}_p^{\alpha,q}$ for $-\varepsilon' < \alpha < \varepsilon'$, $1 < p,q < \infty$. Thus, it suffices to show that for $f \in bM^{(\varepsilon',\varepsilon')} \cap b\dot{B}_p^{\alpha,q}$ with $-\varepsilon' < \alpha < \varepsilon'$ and $1 \le p,q \le \infty$,

$$\|Tf\|_{b^{-1}\dot{B}_p^{\alpha,q}} \le c\,\|f\|_{b\dot{B}_p^{\alpha,q}} \tag{3.11}$$

and for $f \in bM^{(\varepsilon',\varepsilon')} \cap b\dot{F}_p^{\alpha,q}$ with $-\varepsilon' < \alpha < \varepsilon'$ and $1 < p,q < \infty$,

$$\|Tf\|_{b^{-1}\dot{F}_p^{\alpha,q}} \le c\,\|f\|_{b\dot{F}_p^{\alpha,q}}\ , \tag{3.12}$$

where c is a constant which is independent of f.

To do this, since T can be extended to a continuous linear operator from $bM^{(\beta,\gamma)}, 0 < \beta,\gamma < \varepsilon$, to $(bC_0^\eta)'$, for $f \in bM^{(\varepsilon',\varepsilon')} \cap b\dot{B}_p^{\alpha,q}$ with $0 < \beta,\gamma < \varepsilon'$ we then have

$$Tf = \sum_{k\in\mathbb{Z}} T\,M_b\,D_k\,M_b\,\widetilde{D}_k(f) \qquad \text{in } \ (bC_0^\eta)',$$

since $M_b^{-1}f \in M^{(\varepsilon',\varepsilon')}$ and hence, by Theorem 2.3,

$$M_b^{-1}f = \sum_{k\in\mathbb{Z}} D_k\,M_b\,\widetilde{D}_k\,M_b\,(M_b^{-1}f) = \sum_{k\in\mathbb{Z}} D_k\,M_b\,\widetilde{D}_k(f)$$

in the norm of $M^{(\beta,\gamma)}$ with $0 < \beta,\gamma < \varepsilon'$. Therefore,

$$\begin{aligned}\|Tf\|_{b^{-1}\dot{B}_p^{\alpha,q}} &= \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\,\|D_j\,M_b(Tf)\|_p\big)^q\Big)^{1/q}\\ &= \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\,\|D_j\,M_b \sum_{k\in\mathbb{Z}} T\,M_b\,D_k\,M_b\,\widetilde{D}_k(f)\|_p\big)^q\Big)^{1/q}\end{aligned}$$

(since $D_j(\cdot,y)b(y) \in bC_0^\eta$)

$$\le c\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\sum_{k\in\mathbb{Z}} \|D_j\,M_b\,T\,M_b\,D_k\|_{p,p}\,\|\widetilde{D}_k(f)\|_p\big)^q\Big)^{1/q}$$

To estimate the last term above, we need the following lemma (see [HS1]).

Lemma 3.13. *Suppose that T satisfies the hypotheses of Theorem 3.10. Then for $0 < \varepsilon' < \varepsilon$ there exists a constant $c > 0$ such that $D_j\, M_b\, T\, M_b\, D_k(x,y)$, the kernel of $D_j\, M_b\, T\, M_b\, D_k$, satisfies the following estimate:*

$$|D_j\, M_b\, T\, M_b\, D_k(x,y)| \le c\, 2^{-|k-j|\varepsilon'} \frac{2^{-(k\wedge j)\varepsilon'}}{(2^{-(k\wedge j)} + |x-y|)^{n+\varepsilon'}}\,. \tag{3.14}$$

Assuming Lemma 3.13 for the moment, we have

$$\begin{aligned}
\|Tf\|_{b^{-1}\dot{B}_p^{\alpha,q}} &\le c\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} 2^{-|k-j|\varepsilon}\, \|\widetilde{D}_k(f)\|_p\big)^q\Big)^{1/q} \\
&\le c\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, \|\widetilde{D}_k(f)\|_p\big)^q\Big)^{1/q} \\
&\le c\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, \|D_k(f)\|_p\big)^q\Big)^{1/q} = c\,\|f\|_{b\dot{B}_p^{\alpha,q}}
\end{aligned}$$

by the remark following (3.1). Similarly,

$$\begin{aligned}
\|Tf\|_{b^{-1}\dot{F}_p^{\alpha,q}} &\le c\,\Big\|\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} D_j\, M_b\, T\, M_b\, D_k\, M_b\, \widetilde{D}_k(f)|\big)^q\Big)^{1/q}\Big\|_p \\
&\le c\,\Big\|\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} 2^{-|k-j|\varepsilon}\, M(M_b\, \widetilde{D}_k(f))\big)^q\Big)^{1/q}\Big\|_p \\
&\le c\,\Big\|\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, M(M_b\, \widetilde{D}_k(f))\big)^q\Big)^{1/q}\Big\|_p \\
&\le c\,\Big\|\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, |b\, \widetilde{D}_k(f)|\big)^q\Big)^{1/q}\Big\|_p
\end{aligned}$$

(by the Fefferman-Stein vector-valued maximal function inequality)

$$\begin{aligned}
&\le c\,\Big\|\Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, |D_k(f)|\big)^q\Big)^{1/q}\Big\|_p \\
&= c\,\|f\|_{b\dot{F}_p^{\alpha,q}}
\end{aligned}$$

by the remark following (3.1).

All we need to do now is to show Lemma 3.13. We prove the estimate (3.14) in the crucial case where $j \geq k$ and $|x-y| \leq c\,2^{-k}$. The three remaining cases: $j \geq k$ and $|x-y| > c\,2^{-k}$, $j < k$ and $|x-y| \leq c\,2^{-j}$, $j < k$ and $|x-y| > c\,2^{-j}$, are similar or easier. Let $\eta_0 \in C^\infty(\mathbb{R}^n)$ be 1 on the unit ball and 0 outside its double. Set $\eta_1 = 1 - \eta_0$. Then, following the proof of Lemma 7 in Section 6 of [DJS], we have

$$
\begin{aligned}
&D_j\, M_b\, T\, M_b\, D_k(x,y) \\
(3.15) \qquad &= \iint D_j(x,u)\, b(u)\, K(u,v)\, b(v)\, D_k(v,y)\, du\, dv \\
&= \iint D_j(x,u)\, b(u)\, K(u,v)\, b(v)\, \big(D_k(v,y) - D_k(x,y)\big)\, du\, dv
\end{aligned}
$$

since $T(b) = 0$, so

$$
\begin{aligned}
&D_j\, M_b\, T\, M_b\, D_k(x,y) \\
&\qquad = \iint D_j(x,u)\, b(u)\, K(u,v)\, b(v) \\
&\qquad\qquad \cdot \big(D_k(v,y) - D_k(x,y)\big)\, \eta_0\Big(\frac{v-x}{c\,2^{-j}}\Big)\, du\, dv \\
(3.16) \qquad &\qquad + \iint D_j(x,u)\, b(u)\, \big(K(u,v) - K(x,v)\big)\, b(v) \\
&\qquad\qquad \cdot \big(D_k(v,y) - D_k(x,y)\big)\, \eta_1\Big(\frac{v-x}{c\,2^{-j}}\Big)\, du\, dv \\
&\qquad = \mathrm{I} + \mathrm{II},
\end{aligned}
$$

since $1 = \eta_0 + \eta_1$ and $D_j(b) = 0$. Now with $\psi(u) = D_j(x,u)$ and $\phi(v) = (D_k(v,y) - D_k(x,y))\eta_0((v-x)/c\,2^{-j})$,

$$
\begin{aligned}
|\mathrm{I}| &= |\langle M_b\, T\, M_b \phi, \psi\rangle| \\
&\leq c\, 2^{-j(n+2\eta)}\, \|\phi\|_{\mathrm{Lip}\,\eta}\, \|\psi\|_{\mathrm{Lip}\,\eta}
\end{aligned}
$$

(by the weak boundedness property of $M_b T M_b$)

$$
\begin{aligned}
&\leq c\, 2^{-j(n+2\eta)} \big(2^{(k-j)\varepsilon}\, 2^{kn}\, 2^{j\eta}\big)\, \big(2^{-jn} 2^{\eta j}\big) \\
&\leq c\, 2^{(k-j)\varepsilon}\, 2^{kn}\,,
\end{aligned}
$$

which is dominated by the right side of (3.14) for the case where $j \geq k$ and $|x-y| \leq c2^{-k}$. Using the smoothness of $K(u,v)$ together with

$$|D_k(v,y) - D_k(x,y)| \leq c\,2^{k(n+\varepsilon)}\,|v-x|^{\varepsilon}\,\chi_{\{|v-y|\leq c2^{-k}\}\cup\{|x-y|\leq c2^{-k}\}}\,,$$

we have

$$\begin{aligned}|\mathrm{II}| \leq c \iint\limits_{\substack{\{|v-y|\leq c2^{-k}\}\cup\{|x-y|\leq 2^{-k}\}\\ |v-x|\geq c2^{-j}}} & |D_j(x,u)|\,\frac{|u-x|^{\varepsilon}}{|v-x|^{n+\varepsilon}} \\ & \cdot 2^{k(n+\varepsilon)}\,|v-x|^{\varepsilon}\,du\,dv\end{aligned}$$

$$\begin{aligned}&\leq c\,2^{-j\varepsilon}\,2^{kn} \int\limits_{|v-x|\geq c2^{-k}} |v-x|^{-(n+\varepsilon)}\,dv \\ &\quad + c\,2^{(k-j)\varepsilon}\,2^{kn} \int\limits_{c2^{-k}\geq|v-x|\geq c2^{-j}} |v-x|^{-n}\,dv \\ &\leq c\,2^{(j-k)\varepsilon'}\,2^{kn}\,,\end{aligned}$$

which again is dominated by the right side of (3.14) for the case where $j \geq k$ and $|x-y| \leq c2^{-k}$. This proves (3.14) for this crucial case and completes the proof of Theorem 3.13.

Section 4.

We remark that our Calderón-type reproducing formula still holds if the conditions on the approximation to the identity are replaced by the following more general conditions:

(i) $$|S_k(x,y)| \leq c\,\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}\,;$$

(ii) $$\begin{aligned}&|S_k(x,y) - S_k(x,y')| \\ &\quad\leq c\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}\,,\end{aligned}$$

for $|y-y'| \leq (2^{-k}+|x-y|)/2$;

$$\text{(iii)}\quad |S_k(x,y)-S_k(x',y)| \leq c \left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon} \frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}},$$

for $|x-x'| \leq (2^{-k}+|x-y|)/2$;

$$\text{(iv)}\quad \begin{aligned} &|(S_k(x,y)-S_k(x',y)-(S_k(x,y')-S_k(x',y'))| \\ &\leq c \left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon} \left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon} \frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}, \end{aligned}$$

for $|x-x'| \leq (2^{-k}+|x-y|)/3$ and $|x-x'| \leq (2^{-k}+|x-y|)/3$;

$$\text{(v)}\quad \int_{\mathbb{R}^n} S_k(x,y)\, b(y)\, dy = 1, \qquad \text{for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n;$$

$$\text{(vi)}\quad \int_{\mathbb{R}^n} S_k(x,y)\, b(x)\, dx = 1, \qquad \text{for all } k \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n.$$

Using our Calderón-type reproducing formula associated to a para-accretive function one can prove the atomic decomposition, duality, and interpolation for $b\dot{B}_p^{\alpha,q}$ and $b\dot{F}_p^{\alpha,q}$ as for the classical Besov and Triebel-Lizorkin spaces. Since the Fourier transform, translation and dilation are not used so all results in this paper can be generalized to spaces of homogeneous type introduced in [CW]. We will discuss these details elsewhere.

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