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An inverse Sobolev lemma

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Abstract. We establish an inverse Sobolev lemma for quasiconformal mappings and extend a weaker version of the Sobolev lemma for quasiconformal mappings of the unit ball of $\mathbb{R}^n$ to the full range $0 < p < n$. As an application we obtain sharp integrability theorems for the derivative of a quasiconformal mapping of the unit ball of $\mathbb{R}^n$ in terms of the growth of the mapping.

1. Introduction.

Suppose that u belongs to the Sobolev space $W^{1,p}(B^n(1))$, $p > n$, where $B^n(1)$ is the unit ball of $\mathbb{R}^n$. Then the Sobolev imbedding theorem states that u is uniformly Hölder continuous in $B^n(1)$ with exponent $1 - n/p$, see [GT, 7.26]. Recently, in [AK1, 4.7] we established a partial converse to this imbedding.

Theorem A. *Let f be a K-quasiconformal mapping of $B^n(1)$ into $\mathbb{R}^n$. If f is uniformly Hölder continuous in $B^n(1)$ with exponent $0 < \alpha \leq 1$, then $f \in W^{1,p}(B^n(1))$ for some $p > n$, which depends only on K, n, α.*

Thus, for quasiconformal mappings the Sobolev imbedding admits a converse. Recall that a homeomorphism $f : D \to D'$ is K-quasiconformal if $f \in W^{1,n}_{\text{loc}}(D)$ and

$$|f'(x)|^n \leq K\, J_f(x)$$

holds a.e. in D. Here $|f'(x)|$ is the operator norm of the formal derivative $f'(x)$ of f.

In this note, we study the invertibility of the Sobolev lemma, which states that if u belongs to the Sobolev space $W^{1,p}(B^n(1))$, $1 \le p < n$, then $|u|^{pn/(n-p)}$ is integrable over $B^n(1)$, *cf.* [GT, 7.26]. We prove the following inverse Sobolev lemma; see Corollary 4.6.

Theorem B. *Let $0 < p \le n$, and suppose that f is a K-quasiconformal mapping of $B^n(1)$ into $\mathbb{R}^n$. Then*

$$\int_{B^n(1)} |f'|^q \, dm < +\infty , \qquad \textit{for all } \ 0 < q < p$$

if and only if

$$\int_{B^n(1)} |f|^s \, dm < +\infty , \qquad \textit{for all } \ 0 < s < pn/(n-p) .$$

It should be observed that Theorem B extends a weaker version of the Sobolev lemma to the full range $0 < p < n$. We also point out that the inverse Sobolev lemma does not, in general, hold for Sobolev functions. It seems to be a special property of quasiconformal mappings. In fact, the inverse Sobolev lemma may even fail for non-injective mappings satisfying the above inequality and in particular for analytic functions. Indeed, there exist bounded analytic functions of the unit disc whose derivatives fail to be integrable [R].

We link the integrability of the derivative of a quasiconformal mapping to the integrability of the mapping itself by means of growth estimates for the mapping. As a handy tool we employ the notion of the *average derivative* of a quasiconformal mapping introduced by K.Astala and F.W.Gehring [AG1]. This substitute for the derivative has turned out to have a number of applications in questions related to boundary distortion of quasiconformal mappings, see [AG2], [AK2], [AK1], and [H]. Following Astala and Gehring we write

$$a_f(x) = \exp\Big(\int_{B_x} \log J_f(y) \, \frac{dm}{n|B_x|} \Big) ,$$

where $|B_x|$ is the n-measure of B_x and B_x stands for $B(x, d(x, \partial D)/2)$.

In order to establish sharp integrability results (6.1), (6.2) in terms of the dilatation K we prove a quasiconformal analogue of Koebe type growth estimates for univalent functions. In the course of our study we provide new evidence to ensure that a_f plays the role of the derivative by generalizing some classical growth estimates on the derivative of a univalent function, *cf.* [Hy, 1.3, 1.9, 3.3], [P, 1.6].

2. Preliminaries.

Our notation and terminology conform with that of [V1]. In particular, $B^n(x,r)$ and $S^{n-1}(x,r) = \partial B^n(x,r)$ are the open ball and sphere of radius r centered at x. We abbreviate $B^n(0,1)$ to $B^n(1)$ and $S^{n-1}(0,r)$ to $S^{n-1}(r)$, and we write ω_{n-1} for the $(n-1)$-measure of $S^{n-1}(1)$. D and D' will always denote proper subdomains of the n-dimensional Euclidean space $\mathbb{R}^n$, and we apply the convention $B_x = B^n(x, d(x,\partial D)/2)$ for points x in D. We write $C = C(a,\dots)$ to indicate that C depends only on the parameters $a,\dots$ Finally, for any pair E,F of disjoint, closed sets in $\overline{D}$, $M(E,F;D)$ is the modulus of the family of curves joining E,F in D, and we abbreviate $M(E,F;\mathbb{R}^n)$ to $M(E,F)$ and $M(E,\partial D;D)$ to $M(E;D)$.

Next, we collect a number of results used in our proofs. First we state the following well known modulus estimate, see [G1].

Suppose that $E,F \subset \mathbb{R}^n$ are disjoint, non-degenerate, closed, connected sets with E bounded and F unbounded. Then

$$M(E,F) \geq \omega_{n-1}\Big(\log C\,\big(1+\frac{d(E,F)}{\operatorname{diam}(E)}\big)\Big)^{1-n}, \tag{2.1}$$

where $C = C(n)$.

We will frequently employ the following basic property of quasiconformal mappings; see [V1, 18.1], [V2, 2.4].

Lemma 2.2. *Let $f : D \to D'$ be K-quasiconformal. Then for any $0 < \lambda < 1$ there exist positive constants C_1, C_2 depending only on n, K, λ such that*

$$B^n(f(x), C_1 d') \subset f(B^n(x, C_2 d)) \subset B^n(f(x), \lambda d'),$$

where $d = d(x,\partial D)$ and $d' = d(f(x),\partial D')$. Moreover, there is a constant C_3 depending only on n, K such that

$$B^n(f(x), d'/C_3) \subset f(B_x) \subset B^n(f(x), C_3 d')$$

and

$$d(f(B_x), \partial D') \geq d'/C_3 .$$

Next, from Lemma 2.2, [G2, Lemma 4] and [IN, Theorem 2] we deduce

Lemma 2.3. *There exists a constant $C_1 = C_1(n,K) \geq 1$ such that if f is K-quasiconformal in D and $B = B^n(x,r) \subset D$ satisfies $r \leq d(x,\partial D)/C_1$, then for any $0 < p < n$*

$$\int_B |f'|^n dm \leq C_2\, r^{n(1-n/p)} \Big(\int_{C_1 B} |f'|^p \, dm \Big)^{n/p} ,$$

where $C_2 = C_2(n,K,p)$ and $C_1 B = B^n(x, C_1 r)$.

Furthermore, from [IN, Proposition 3] we have

Lemma 2.4. *For each $0 < p < \infty$ and any $K \geq 1$ there is a constant $C_2 = C_2(p,n,K)$ such that for all K-quasiconformal mappings f of D*

$$|f(x)| \leq C_2\, r^{-n/p} \Big(\int_{B^n(x,r)} |f|^p \, dm \Big)^{1/p}$$

whenever $B^n(x, C_1 r) \subset D$, where $C_1 = C_1(K,n)$.

We continue with a quasiconformal analogue [AG2, 1.8] of the Koebe distortion theorem.

Lemma 2.5. *Let $f : D \to D'$ be K-quasiconformal. There is a constant C, which depends only on n, K, so that for each $x \in D$*

$$\frac{1}{C}\, d(f(x), \partial D') \leq a_f(x)\, d(x, \partial D) \leq C\, d(f(x), \partial D')$$

and

$$\frac{1}{C} \Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|} \Big)^{1/n} \leq a_f(x) \leq C \Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|} \Big)^{1/n}$$

PROOF. First of our claims is [AG2, 1.8]. Moreover,

$$\Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|}\Big)^{1/n} \leq \Big(K\frac{|fB_x|}{|B_x|}\Big)^{1/n}$$

and, by [V1, 34.5-6],

$$\Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|}\Big)^{1/n} \geq \Big(K^{1-n}\frac{|fB_x|}{|B_x|}\Big)^{1/n}.$$

Therefore, owing to Lemma 2.2, our second chain of inequalities is a consequence of the first.

We conclude with a lemma which will prove useful.

Lemma 2.6. *Let $f : D \to D'$ be K-quasiconformal. If $\gamma \subset D$ is a rectifiable curve with $l(\gamma) \geq d(\gamma, \partial D)$, then*

$$\operatorname{diam}(f\gamma) \leq C\int_\gamma a_f(x)\,ds\,.$$

Here C depends only on n, K.

PROOF. Pick a cover $B_1, \dots, B_k$ of γ where each ball B_i is of the form $B_i = B_{x_i}$ with x_i on γ so that no point in D lies in more than $C = C(n)$ of these balls; this is possible by the Besicovitch covering theorem. Now Lemma 2.2 yields

$$\operatorname{diam}(f\gamma) \leq \sum \operatorname{diam}(f(B_i)) \leq C_1 \sum d(f(x_i), \partial D')\,,$$

where $C_1 = C_1(n, K)$. On the other hand, from the assumption on the length of γ, we deduce that for each i the one-dimensional measure of $\gamma \cap B_i$ cannot be less than $d(x_i, \partial D)/2$. So, appealing to Lemmas 2.2 and 2.5, we obtain

$$\int_\gamma a_f(x)\,ds \geq C_2 \sum d(x_i, \partial D)\, a_f(x_i) \geq C_3 \sum d(f(x_i), \partial D')\,,$$

where the constants C_2, C_3 depend only on n, K. The claim follows.

Observe that the behavior of the quasiconformal mapping $f(x) = x|x|^{-1/2}$ of $B^n(1)$ at the origin shows that some assumption on the diameter of the curve γ in Lemma 2.6 is necessary.

3. A local inverse Sobolev lemma.

Throughout this section f will be a K-quasiconformal mapping of $B^n(1)$, and we assume that $f(0) = 0$, $d(0, \partial f(B^n(1))) = 1$. We prove that integrability conditions over hyperbolic balls are characterized by the growth of f.

Theorem 3.1. *The following two conditions are equivalent for* $0<p<\infty$.

a) $\displaystyle\int_{B_x} |f|^p dm \leq C_2$.

b) $|f(x)| \leq C_1(1-|x|)^{-n/p}$.

The constants C_1, C_2 *depend only on* p, n, K *and on each other.*

PROOF. First note that, by Lemma 2.4, *a)* implies *b)*. For the converse, observe that $d(f(x), \partial f(B^n(1))) \leq 1 + |f(x)|$; hence Lemma 2.2 gives

$$|f(y)| \leq (1 + C_3)(1 + |f(x)|)$$

for all $y \in B_x$, where $C_3 = C_3(n, K)$. The desired implication follows.

Theorem 3.2. *Let* $0 < p \leq n$, *fix* $x \in B^n(1)$, *and suppose that* $|f(x)| \leq C_1(1-|x|)^{1-n/p}$. *Then*

$$\int_{B_x} |f'|^p \, dm \leq C_2 ,$$

where $C_2 = C_2(p, n, K, C_1)$.

PROOF. Observe from Lemma 2.2 that for all $y \in B_x$,

$$|f(y)| \leq C_3 \, (1-|x|)^{1-n/p} ,$$

where $C_3 = C_3(n, K, C_1)$. Now

$$\int_{B_x} |f'|^n \, dm \leq K \, |f(B_x)| \leq C_4 \, (1-|x|)^{(1-n/p)n} ,$$

where $C_4 = C_4(p, n, K, C_1)$. Next, Hölder's inequality yields

$$\int_{B_x} |f'|^p \, dm \leq |B_x|^{1-p/n} \Big(\int_{B_x} |f'|^n \, dm \Big)^{p/n} ,$$

and the claim follows.

Theorems 3.1 and 3.2 are local in nature. It is not too difficult to produce examples where the converse to Theorem 3.2 fails. Nevertheless, we have a global version of the converse statement which completes the chain of implications.

Theorem 3.3. *Suppose that* $\int_{B_x} |f'|^p \, dm \le C_1$ *for all* $x \in B^n(1)$ *where* $0 < p < n$. *Then*

$$|f(x)| \le C_2 \, C_1^{1/p} (1 - |x|)^{1-n/p},$$

for all $x \in B^n(1)$, *where* $C_2 = C_2(p, n, K)$.

PROOF. We conclude from Lemma 2.3 that

$$\begin{aligned}\int_B |f'|^n \, dm &\le C_4 (1 - |x|)^{n(1-n/p)} \Big(\int_{B_x} |f'|^p \, dm \Big)^{n/p} \\ &\le C_5 (1 - |x|)^{n(1-n/p)},\end{aligned}$$

where $B = B^n(x, (1-|x|)/C_3), C_3 = C_3(n, K)$, and $C_5 = C_4(p, n, K) C_1^{n/p}$. Since the inverse mapping of f is K^{n-1}-quasiconformal, Lemma 2.2 shows that

$$\int_B |f'|^n \, dm \ge \frac{1}{C_6} \, d(f(x), \partial f(B^n(1)))^n$$

with $C_6 = C_6(n, K)$, which permits us to deduce

$$\frac{d(f(x), \partial f(B^n(1)))}{1 - |x|} \le C_7 \, C_1^{1/p} (1 - |x|)^{-n/p},$$

for all $x \in B^n(1)$, where $C_7 = C_7(p, n, K)$. Thus, by Lemma 2.5, we have

$$a_f(x) \le C_8 \, C_1^{1/p} (1 - |x|)^{-n/p}$$

in $B^n(1)$, where $C_8 = C_8(p, n, K)$. Since $f(0) = 0$, the claim follows from Lemma 2.6 by integrating $a_f(x)$ along the line segment joining 0 to x.

Combining Theorems 3.1-3.3 we obtain

Theorem 3.4. *Let $0 < p < n$. Then the following conditions are equivalent.*

a) $|f(x)| \le C_1(1-|x|)^{1-n/p}$ *in* $B^n(1)$.

b) $d(f(x), \partial f(B^n(1))) \le C_2(1-|x|)^{1-n/p}$ *in* $B^n(1)$.

c) $a_f(x) \le C_3\,(1-|x|)^{-n/p}$ *in* $B^n(1)$.

d) $\int_{B_x} |f'|^p\, dm \le C_4$ *for all* $x \in B^n(1)$.

e) $\int_{B_x} |f|^{pn/(n-p)}\, dm \le C_5$ *for all* $x \in B^n(1)$.

Here all constants depend only on p, n, K and each other.

PROOF. Conditions *a*), *d*), and *e*) are equivalent by Theorems 3.1-3.3. Furthermore, *b*) follows from *a*) by the triangle inequality, and *c*) from *b*) by Lemma 2.5. Finally, Lemma 2.6 enables us to deduce *a*) from *c*).

We point out that Theorem 3.4 gives the following somewhat surprising corollary.

Corollary 3.5. *Let $s > 0$. Then*

$$|f(x)| \le C_1\,(1-|x|)^{-s} \quad in\ B^n(1)$$

if and only if

$$d(f(x), \partial f(B^n(1))) \le C_2\,(1-|x|)^{-s} \quad in\ B^n(1).$$

For completeness, let us comment on the Sobolev lemma for quasiconformal mappings in the case $p \ge n$. From [N, 1.4] and Lemma 2.5 we have

REMARK 3.6. The following conditions are equivalent.

a) $\int_{B_x} |f'|^n\, dm \le C_1$ for all $x \in B^n(1)$.

b) $f \in \mathrm{BMO}(B^n(1))$.

c) $d(f(x), \partial f(B^n(1))) \le C_2$ for all $x \in B^n(1)$.

Contrary to the case $0 < p < n$, one cannot characterize the integrability condition a) of Remark 3.6 by means of the growth of f. In fact, the argument of the proof of Theorem 3.4 gives the estimate $|f(x)| \leq C \log(1/(1-|x|))$, whereas there exist univalent functions of $B^2(1)$ of slower growth but not belonging to $\mathrm{BMO}(B^2(1))$. Examples of this type can easily be constructed with the help of the equivalence on a) and c) in Remark 3.6 and the methods employed in Section 5.

Finally, here is the case $p > n$.

REMARK 3.7. If f is uniformly Hölder continuous in $B^n(1)$ with some exponent $\alpha > 0$, then $\int_{B_x} |f'|^p \, dm \leq C$ for all $x \in B^n(1)$, for an exponent $p = p(n, K, \alpha) > n$. Moreover, if f is conformal, one may take $p = n/(1-\alpha)$. Conversely, if $\int_{B_x} |f'|^p \, dm \leq C$ for some $p > n$ for all $x \in B^n(1)$, then f is uniformly Hölder continuous in $B^n(1)$ with exponent $1 - n/p$. Indeed, the assertion is a consequence of [AK1, 4.7] and [GM, 2.24].

4. The global case.

In this section we present global versions of the results of the preceding section. As earlier, we assume throughout this section that f is a K-quasiconformal mapping of $B^n(1)$ with $f(0) = 0$ and $d(0, \partial f(B^n(1))) = 1$. We begin with an extension of the Sobolev lemma to the full range $0 < p < n$ for which record the following lemma due to K. Astala [AK2].

Lemma 4.1. *For each $p > 0$ and for all $1/2 < r < 1$*

$$\int_{S^{n-1}(r)} |f|^p \, d\sigma \leq C \int_0^r M(t, f)^p \, (1-t)^{n-2} \, dt \,,$$

where $C = C(p, n, K)$ and $M(t, f) = \max_{|x|=t} |f(x)|$.

Theorem 4.2. *Suppose that $\int_{B_x} |f'|^p \, dm \leq C$, $0 < p < n$, for all $x \in B^n(1)$. Then for all $0 < q < pn/(n-p)$*

$$\int_{B^n(1)} |f|^q \, dm \leq C_1 \, C^{q/p} \,,$$

where $C_1 = C_1(p, q, n, K)$.

PROOF. Notice first that by Lemmas 2.2 and 2.3 it suffices to establish the assertion with $B^n(1)$ replaced by $B = B^n(1) \setminus B^n(0,1/2)$. Now Theorem 3.3 asserts that $M(t,f) \le C_0\, C^{1/p}\,(1-t)^{1-n/p}$ for $0 < t < 1$, where $C_0 = C_0(p,n,K)$. Hence the claim follows by integrating the inequality in Lemma 4.1.

We point out that one cannot take $q = pn/(n-p)$ in Theorem 4.2; see Remarks 4.8 below. Next we establish an inverse Sobolev lemma for quasiconformal mappings of $B^n(1)$.

Theorem 4.3. *Let $0 < p < n$. Then for any $q > p$ there is a constant $C = C(p,q,n,K)$ such that*

$$\int_{B^n(1)} |f'|^p\, dm \le C \Big(\int_{B^n(1)} |f|^{qn/(n-p)}\, dm \Big)^{(n-p)/n}$$

PROOF. Let $q > p$. It suffices to establish the integrability condition with $B^n(1)$ replaced by $B = B^n(1)\setminus B^n(0,1/2)$. Now Hölder's inequality gives

$$\begin{aligned}\int_B |f'|^p\, dm &= \int_B |f'|^p |f|^{-q} |f|^q\, dm \\ &\le \Big(\int_B |f'|^n |f|^{-nq/p}\, dm \Big)^{p/n} \Big(\int_B |f|^{qn/(n-p)}\, dm \Big)^{(n-p)/n}\end{aligned}$$

Next, the quasiconformality of f yields

$$\int_B |f'|^n |f|^{-nq/p}\, dm \le K \int_{f(B)} |x|^{-nq/p}\, dm\,.$$

With the help of Lemma 2.2 we conclude that $f(B) \subset \mathbb{R}^n \setminus B^n(0, C_2)$. Hence

$$\int_{f(B)} |x|^{-nq/p}\, dm \le C_3 \int_{C_2}^{\infty} t^{n-1-nq/p}\, dt\,.$$

The assertion follows from this string of inequalities because

$$n - 1 - \frac{nq}{p} < -1\,.$$

From the proof of Theorem 4.3 we further deduce

Theorem 4.4. *Let $0 < p < n$, and suppose that*

$$\int_{f(B^n(1))\setminus B^n(1)} |x|^{-n}\,dm = C < +\infty\,.$$

Then

$$\int_{B^n(1)} |f'|^p\,dm \le C_1\,C^{p/n}\Big(\int_{B^n(1)} |f|^{pn/(n-p)}\,dm\Big)^{(n-p)/n}$$

Combining Theorems 3.4, 4.2 and 4.3 we have

Corollary 4.5. *Let $0 < p < n$ and suppose that $\int_{B_x} |f'|^p\,dm \le M$ in $B^n(1)$ or that $\int_{B_x} |f|^{pn/(n-p)}\,dm \le M$ in $B^n(1)$. Then*

$$\int_{B^n(1)} |f'|^q\,dm < \infty\,, \qquad \text{for any } \ 0 < q < p\,.$$

Next, combining Theorems 3.4, 4.2, 4.3 and Corollary 4.5, we deduce

Corollary 4.6. *Let $0 < p \le n$. Then the following conditions are equivalent.*

a) $\displaystyle\int_{B^n(1)} |f'|^q\,dm < +\infty,$ *for all* $0 < q < p\,.$

b) $\displaystyle\int_{B^n(1)} |f|^s\,dm < +\infty,$ *for all* $0 < s < pn/(n-p)\,.$

c) $|f(x)| \le C_1\,(1-|x|)^{1-n/q}$ *in* $B^n(1)$, *for each* $0 < q < p$, *for some* C_1 .

d) $\displaystyle\int_{B_x} |f'|^q\,dm \le C_2$, *for all* $x \in B^n(1)$, *for each* $0 < q < p$, *for some* C_2 .

e) $\displaystyle\int_{B_x} |f|^s\,dm \le C_3$, *for all* $x \in B^n(1)$, *for each* $0 < s < pn/(n-p)$, *for some* C_3 .

4.7. OPEN QUESTIONS.

(a) Suppose that $\int_{B^n(1)} |f'|^p \, dm < +\infty$ for some $0 < p < n$. Does it follow that $\int_{B^n(1)} |f|^q \, dm < +\infty$ with $q = pn/(n-p)$? By the Sobolev lemma this is the case for $1 \le p < n$, and from Theorem 4.2 we know that this integral converges for $0 < p < 1$ provided $0 < q < pn/(n-p)$.

(b) Suppose that $\int_{B_x} |f'|^p \, dm \le M$ for all $x \in B^n(1)$. If $0 < p \le n$, then Corollary 4.5 ensures that $\int_{B^n(1)} |f'|^q \, dm < +\infty$ for all $0 < q < p$. On the other hand, one can apply the example in [K] to show that this is not, in general, true for $p > n$. Is the conclusion nevertheless valid for analytic univalent functions of the unit disc for all $0 < p < \infty$? If this is the case, then Remark 3.7 would show that $\int_{B^2(1)} |f'|^q \, dm < +\infty$ for all $0 < q < 2/(1-\alpha)$ provided f is uniformly Hölder continuous in $B^2(1)$ with exponent $0 < \alpha < 1$. With some work one can show that this, in turn, would yield that the Hausdorff dimension of $\partial f(B^2(1))$ is at most $2/(1+\alpha)$.

REMARKS 4.8.

(a) Theorem 4.2 does not hold for $q = pn/(n-p)$ and Corollary 4.5 does not extend to the case $q = p$. Indeed, a simple counterexample is provided by the quasiconformal mapping $f(x) = (x - w)\,|x - w|^{-1-a}$, $a > 0$, where $w \in S^{n-1}(1)$. An appropriate modification of f shows that the assumption on $f(B^n(1))$ in Theorem 4.4 is necessary and that Theorem 4.3 fails for $q = p$.

(b) Corollary 4.6 shows that for $0 < p < n$ the global integrability of a quasiconformal mapping of $B^n(1)$ and that of its derivative are more or less completely characterized by the growth of the mapping.

(c) A look at the proof of Theorem 4.3 shows that we did not need the fact that the domain in consideration is a ball. Consequently, Theorem 4.3 extends to any domain D.

5. Koebe type distortion theorems.

Motivated by Corollary 4.6 we turn our attention towards distortion estimates for quasiconformal mappings. We establish the following distortion theorem that for plane univalent functions reduces to the classical results, *e.g.* [Hy, 1.3, 1.9], [P, 1.6]. Some parts of the theorem are apparently folklore, but we have not been able to locate these results in the literature except for the upper bound in (a), which is a special case of [FMV, 4.2].

Theorem 5.1. *Let f be K-quasiconformal in $B^n(1)$, and assume that $f(0) = 0$, $d(f(0), \partial f(B^n(1))) = 1$. Set $a = K^{1/(n-1)}$ and $b = (2K)^{1/(n-1)}$. Then*

a) $|f(x)| \geq |x|^a/C$ *and* $|f(x)| \leq C(1-|x|)^{-b}$.

b) $(1-|x|)^{b-1}/C \leq a_f(x) \leq C(1-|x|)^{-b-1}$.

c) If $fB^n(1)$ is convex, then

$$|f(x)| \leq C(1-|x|)^{-a}$$

and

$$(1-|x|)^{a-1}/C \leq a_f(x) \leq C(1-|x|)^{-a-1}.$$

d) For $n \geq 3$ there is $a' = a'(n,K)$ with $a' \to 1$ as $K \to 1$ such that c) holds without the convexity assumption if a is replaced with a'.

Here $C = C(n,K)$.

We divide the proof of Theorem 5.1 into several lemmas. To simplify our statements we assume in Lemmas 5.2-5.5 that $f : B^n(1) \to D = f(B^n(1))$ is K-quasiconformal, $f(0) = 0$, and $d(f(0), \partial D) = 1$.

Lemma 5.2. *We have*

$$|f(x)| \leq C(1-|x|)^{-b} \quad \textit{and} \quad a_f(x) \leq C(1-|x|)^{-b-1},$$

where $b = (2K)^{1/(n-1)}$ and $C = (n,K)$.

PROOF. By Lemmas 2.2 and 2.5 we may assume that $|x| \geq 1/2$. For each such x set $E_x = \overline{B}_x$, and let $F = \overline{B}^n(0,1/5)$. Using a standard modulus argument, we deduce from (2.1) that

$$2\,M(E_x, F; B^n(1)) \geq \omega_{n-1}\left(\log\frac{C_1}{1-|x|}\right)^{1-n},$$

where $C_1 = C_1(n)$. On the other hand, (2.2) shows that $f(F) \subset B^n(C_2)$ and $|f(x)| \leq C_2|f(y)|$ whenever $|f(x)| \geq C_3$, where C_2, C_3 depend only on n, K. Hence

$$M(f(E_x), f(F); D) \leq \omega_{n-1}\left(\log(C_4\,|f(x)|)\right)^{1-n}$$

provided $|f(x)| \geq C_3$, where both constants C_3, C_4 depend only on n, K. By the quasiconformality of f

$$M(E_x, F; B^n(1)) \leq K\, M(f(E_x), f(F); D),$$

which permits as to infer that

$$\log(C_4\, |f(x)|) \leq b \log\big(\frac{C_1}{1-|x|}\big)$$

provided $|f(x)| \geq C_3$ and the proof for our first claim is complete.

Finally, the estimate for a_f follows from the first claim and Theorem 3.4.

Lemma 5.3. *We have*

$$|f(x)| \geq C\,|x|^a \quad \textit{and} \quad a_f(x) \geq C\,(1-|x|)^{b-1},$$

where $a = K^{1/(n-1)}$, $b = (2K)^{1/(n-1)}$, *and* $C = C(n, K)$.

PROOF. For each $0 < r < 1$, let $E_r = \overline{B}^n(r)$. Then $M(E_r; B^n(1)) = \omega_{n-1}\,(\log(1/r))^{1-n}$. On the other hand, if $\operatorname{diam}(f(E_r)) < 1/2$, then $M(f(E_r); D) \leq \omega_{n-1}\,(\log(1/\operatorname{diam}(f(E_r)))^{1-n}$. Since f is K-quasiconformal, we conclude that

$$\Big(\log \frac{1}{\operatorname{diam}(f(E_r))}\Big)^{n-1} \leq K\,\Big(\log \frac{1}{r}\Big)^{n-1}$$

Hence $\operatorname{diam}(f(E_r)) \geq C_1\, r^a$, where $a = K^{1/(n-1)}$ and $C_1 = C_1(n, K)$. The desired bound for $|f(x)|$ is now a consequence of Lemma 2.2.

Next we estimate $a_f(x)$. By Lemma 2.5 it suffices to show that

$$d(f(x), \partial D) \geq C_2\,(1-|x|)^b,$$

for each $x \in B^n(1)$ for some constant C_2 which only depends on K, n. Lemmas 2.2 and 2.5 permit us to assume that $|x| \geq 1/2$. Set again $E_x = \overline{B}_x$ for each such x and define $F = \overline{B}^n(0, 1/5)$. From Lemma 2.2 and the argument of the proof of Lemma 5.2 we conclude that it suffices to find constants C_3 and δ_1, which depend only on n, K, such that

$$M(f(E_x), f(F); D) \leq \omega_{n-1}\,\Big(\log \frac{C_3}{\operatorname{diam}(f(E_x))}\Big)^{1-n},$$

whenever diam $(f(E_x)) \leq \delta_1$. Now Lemma 2.2 yields that $d(f(F), \partial D) \geq \delta_2 > 0$, where $\delta_2 = \delta_2(n, K)$. Applying again Lemma 2.2 we find a constant $\delta_1 = \delta_1(n, K, \delta_2) < \delta_2/2$ such that

$$f(F) \subset D \setminus B^n(f(x), \delta_2/2)$$

whenever diam $(f(E_x)) \leq \delta_1$. Since $f(E_x) \subset B^n(f(x), \operatorname{diam}(f(E_x)))$, the desired modulus inequality follows with $C_3 = \delta_2/2$, and the proof is complete.

Lemma 5.4. *Suppose that D is convex. Then*

$$|f(x)| \leq C\,(1 - |x|)^{-a}$$

and

$$(1 - |x|)^{a-1}/C \leq a_f(x) \leq C\,(1 - |x|)^{-a-1}\,,$$

where $a = K^{1/(1-n)}$ and $C = C(n, K)$.

PROOF. Since D is convex, we have [V1, 7.7] for each $z \in \partial D$ that

$$2\,M(\overline{B}^n(z, r) \cap \overline{D}, \overline{D} \setminus B^n(z, R); D) \leq \omega_{n-1} \left(\log \frac{R}{r}\right)^{1-n} \tag{5.5}$$

whenever $0 < r < R$. So, using the notation of the proof of Lemma 5.2, we obtain

$$2\,M(f(E_x), f(F); D) \leq \omega_{n-1} \left(\log(C_5\,|f(x)|)\right)^{1-n},$$

whenever $|f(x)| \geq C_6$, where C_5, C_6 both depend only on n, K. The desired bound for $|f(x)|$ follows as in the proof of Lemma 5.2, and Theorem 3.4 yields the analogous bound for $a_f(x)$.

The lower bound for $a_f(x)$ is obtained using (5.5) in an appropriate step in the proof of Lemma 5.3.

Lemma 5.6. *For $n \geq 3$ there is $a' = a'(n, K)$ with $a' \to 1$ as $K \to 1$ and such that the estimates in Lemma 5.4 hold without the convexity assumption if a is replaced with a'.*

PROOF. As established in [AH, 1.2] and in [T], for $K \leq K_1(n)$, f has a

K'-quasiconformal extension $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$, where $K' = K'(n, K)$ satisfies $K' \to 1$ as $K \to 1$; see [V1, 13] for the definition of a quasiconformal mapping of $\overline{\mathbb{R}}^n$. Hence

$$M(E_x, F) \le K' \, M(f(E_x), f(F)),$$

where E_x, F are as in the proof of Lemma 5.2. Thus, replacing K with K', the argument of the proof of Lemma 5.2 gives the desired upper bounds for $|f(x)|$ and $a_f(x)$. The lower bound for $a_f(x)$ follows by modifying the proof of Lemma 5.3.

EXAMPLE 5.7. Theorem 5.1 is sharp for each $K \ge 1$ for $n = 2$ and for convex images for general n.

Let $n = 2$, fix $K \ge 1$, and let k denote the Koebe function. Define $f(x) = x|x|^{K-1}$. Then f is K-quasiconformal and, consequently, $h = f \circ k$ is K-quasiconformal. Now a simple calculation shows that for $x = (0, t)$, $0 < t < 1$, we have $h(x) \ge C(1 - |x|)^{-2K}$, $a_h(x) \ge C(1 - |x|)^{-2K-1}$, and $a_h(-x) \le C\,(1 - |x|)^{2K-1}$. Furthermore, $|f(x)| = |x|^K$. Finally, for the convex case set $g(x) = (x - w)|x - w|^{-1-K}$ for some $w \in S^1(1)$ (for $n > 2$ set $g(x) = (x - w)\,|x - w|^{-1-a}$ for some $w \in S^{n-1}(1)$, where $a = K^{1/(n/1)}$).

REMARK 5.8. The proofs of Lemmas 5.2 and 5.4 show that if $f(B^n(1))$ is contained in a half space, then $|f(x)| \le C\,(1 - |x|)^{-a}$; the constant C will in this case also depend on the distance from the origin to the boundary of the half space.

6. Sharp integrability exponents.

The results of sections 4 and 5 yield sharp integrability exponents.

Theorem 6.1. *Suppose that f is a K-quasiconformal mapping of $B^n(1)$. Then*

$$\int_{B^n(1)} |f'|^p \, dm < +\infty,$$

for all $0 < p < n/(1+(2K)^{1/(n-1)})$. Moreover, if $f(B^n(1))$ is contained in a half space, then $2K$ may be replaced with K. Furthermore,

$$\int_{B^n(1)} |f|^{pn/(n-p)} \, dm < +\infty$$

for the indicated values of p.

PROOF. The claim follows from Theorems 3.4, 4.3, 5.1 and Remark 5.8.

REMARKS 6.2.

(a) A result of Jerison and Weitsman [JW] implies that $|f|^q$ is integrable for some exponent $q = q(n, K)$, but the exponent obtained from their work is not sharp. The exponents in Theorem 6.1 are sharp for each $K \geq 1$ in the plane and for each $K \geq 1$ in $\mathbb{R}^n$, $n > 2$, for mappings into a half space. This follows via a simple calculation for the functions in Example 5.7. We refer the reader to [AK2] for the H^p-theory of quasiconformal mappings.

(b) We deduce the following from Theorem 6.1. If $D \subset \mathbb{R}^2$ is any simply connected domain and $f : D \to B^2(1)$ is K-quasiconformal, then $\int_D |f'(x)|^p dx < +\infty$ for all $2 - 2/(1 + 2K) < p \leq 2$; compare with [AK1, 4.10]. We point out that the standard factorization argument combined with the analogous result for univalent functions, *cf.* [B], due to F.W.Gehring and W.K.Hayman, fails to give this sharp bound.

(c) Suppose that f is K-quasiconformal in $\mathbb{R}^n \setminus \{0\}$. Then the arguments used in Section 5 apply to verify that $|f(x)| \leq C\,|x|^{-a}$ in $B^n(1)$, where $a = K^{1/(n-1)}$. Now integrating this estimate we observe that $\int_{B^n(1)} |f|^p\, dm < +\infty$, for $0 < p < n/K^{1/(n-1)}$. Hence, by and Theorem 4.3 and Remark 4.8, the analogous integrability result for $|f'|$ holds for $0 < p < n/(1 + K^{1/(n-1)})$. As easily seen, the above upper bounds for p are sharp.

We do not know whether the claim of Theorem 6.1 holds for all K-quasiconformal mappings of $B^n(1)$ in the case $n \geq 3$ if $2K$ is replaced with K. Next we produce an estimate which is asymptotically sharp as $K \to 1$; observe that if $w \in S^{n-1}(1)$, then $|f'|^{n/2}$ is not integrable over $B^n(1)$ for the Möbius transformation $f(x) = (x - w)\,|x - w|^{-2}$.

Theorem 6.3. *Let f be K-quasiconformal in $B^n(1)$, $n \geq 3$. Then*

$$\int_{B^n(1)} |f'|^p\, dx < +\infty\,, \qquad \text{for all } 0 < p < p_0(n, K)\,,$$

where $p_0(n, K) \to n/2$ as $K \to 1$.

PROOF. First note that by Lemma 5.6

$$|f(x)| \leq C(1-|x|)^{-a}, \qquad x \in B^n(1),$$

with $a = a(n, K) \to 1$ as $K \to 1$. Then Theorem 3.4 and Corollary 4.5 yield the desired estimate.

ADDED IN PROOF. We have recently (Buckley, S. and Koskela, P., Sobolev-Poincaré inequalities for $0 < p < 1$) answered 4.7.(a) in the positive.

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