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Rough isometries and p-harmonic functions with finite Dirichlet integral

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1. Introduction.

Let G be an open subset of a Riemannian n-manifold M^n. A function $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$, with $1 < p < \infty$, is called *p-harmonic in G* if it is a weak solution of

$$-\mathrm{div}\left(|\nabla u|^{p-2}\, \nabla u\right) = 0\,, \tag{1.1}$$

that is,

$$\int_G \left\langle |\nabla u|^{p-2}\, \nabla u, \nabla \varphi \right\rangle dm = 0 \tag{1.2}$$

for every $\varphi \in C_0^\infty(G)$. Equation (1.1) is the Euler-Lagrange equation of the variational integral

$$\int_G |\nabla u|^p \, dm\,.$$

We say that a Riemannian n-manifold M^n has the *Liouville D_p-property* if every p-harmonic function u on M^n with

$$\int_{M^n} |\nabla u|^p \, dm < +\infty$$

is constant. In this paper we study the invariance of the Liouville D_p-property under rough isometries between Riemannian manifolds; see Section 3 for the definition of a rough isometry. We prove that if M^n and N^ν are roughly isometric, and if both M^n and N^ν have bounded geometry, then M^n has the Liouville D_p-property if and only if so does N^ν (Theorem 5.13). Note that the dimension of M^n may differ from that of N^ν.

Our result is new also for harmonic functions ($p = 2$) even in 2-dimensional case. Indeed, in all previous results, excepting P. Pansu's result which will be discussed later, manifolds M^n and N^ν must be homeomorphic. It is known that the Liouville D_2-property is preserved under quasiconformal mappings between 2-dimensional Riemannian manifolds and under bilipschitz (sometimes also called quasi-isometric) maps in all dimensions $n \geq 2$. See, for instance, [SN, p. 405-411] where also slightly more general classes of maps are studied in this context. Note that rough isometries need not be continuous. Thus they form a very large class of maps which, however, have nice invariance properties. It is worth noting that a similar stability result is not true for positive (or bounded) harmonic functions even under bilipschitz maps. Indeed, Lyons [L] has constructed a manifold M and two metrics g and g', with $c^{-1}g' \leq g \leq c\,g'$ such that (M, g) has no non-constant positive harmonic functions but (M, g') carries a non-constant bounded harmonic function. It is an interesting open problem whether a similar unstability result holds for p-harmonic functions if $p \neq 2$. We remark that Pansu [P] has also studied the invariance of the Liouville D_p-property under rough isometries but under more restrictive assumptions on manifolds M^n and N^ν. He assumes that a global Sobolev inequality $\|u\|_q \leq c\,\|\nabla u\|_p$, with $q \geq p \geq 1$, holds for C_0^∞-functions of M^n and N^ν, and that cohomology groups $H^1(X, \mathbb{R})$, $X = M^n, N^\nu$, are trivial. With these additional requirements on M^n and N^ν, the same conclusion as in Theorem 5.13 can be made. He has informed the author that it is possible to obtain our result also by refining his arguments. However, our methods are different.

The proof of the result in this paper is based on ideas of A. A. Grigor'yan and M. Kanai. In [K2] Kanai showed that the positivity of 2-capacity at infinity, and so the existence of Green's function for the Laplace equation, is preserved under rough isometries between Riemannian manifolds of bounded geometry. On the other hand, Grigor'yan [G] has presented a criterion, which involves 2-capacities, for the existence of a non-constant harmonic function with L^2-integrable gradient on a Riemannian manifold. For the proof of our result, we first generalize Grigor'yan's criterion to the non-linear case at hand. Here we present somewhat shorter proofs than Grigor'yan's original ones which are not fully available in our setting. Then we show, by modifying Kanai's arguments, that the p-capacities in the criterion for the Liouville D_p-property remain essentially unchanged in rough isometries between manifolds of bounded geometry. The lack of injectivity of a rough isometry causes here some troubles which, however, can be solved by using a (semi)local Harnack inequality (Theorem 3.3). We want to emphasize that it is easy to obtain local Harnack's inequalities in the following form from known results in $\mathbb{R}^n$ by using suitable chart maps. Suppose $D \subset M^n$ is an open set and $C \subset D$ is compact. Then there exists a positive constant c such that

$$\sup_C u \le c \inf_C u \tag{1.3}$$

whenever u is a positive p-harmonic function in D. The main disadvantage of (1.3) is that, with no assumptions on the geometry of M^n, the constant c depends not only on metric parameters of C and D but also on the location of D on M^n. Such an inequality is useless in the proof of the main result. In Section 3 we prove inequality (1.3) with $D = B(x,r)$, $C = \bar{B}(x,r/2)$, and with c independent of x if M^n has bounded geometry. Here $r \le r_0 \le 2(\text{inj } M^n)/3$ and c depends on r_0 but not on r. We think that this inequality may also have independent interest.

The main result is formulated for so called *$\mathcal{A}$-harmonic functions* which are continuous solutions of

$$-\text{div } \mathcal{A}(\nabla u) = 0\,,$$

where $\langle \mathcal{A}(\nabla u), \nabla u \rangle \approx |\nabla u|^p$, with $1 < p < \infty$. The precise assumptions on $\mathcal{A}$ are given in 2.16. In [H1-2] and [HR] we studied a classification of Riemannian manifolds based on the existence of non-constant $\mathcal{A}$-harmonic functions with various properties. By [H1, Section 5], there exists a non-constant bounded p-harmonic function v in M^n, with $\int_{M^n} |\nabla v|^p \, dm < +\infty$, if and only if M^n admits a non-constant $\mathcal{A}$-harmonic function u, with $\int_{M^n} |\nabla u|^p \, dm < +\infty$, for some, or, in fact, for every $\mathcal{A} \in \mathcal{A}_p(M^n)$. Thus it suffices to consider only bounded p-harmonic functions if we want to study whether a given manifold carries a non-constant $\mathcal{A}$-harmonic function with L^p-integrable gradient and $\mathcal{A}$ of type p.

Harmonic functions and rough isometries on graphs are studied in Markvorsen, S., McGuinness, S., Thomassen, C., "Transient random walks on graphs and metric spaces with applications to hyperbolic surfaces", *Proc. London Math. Soc.* (3) **64** (1992), 1-20.

2. Preliminaries.

2.1. Terminology.

Throughout the paper we assume that M^n is a non-compact, connected, and oriented Riemannian n-manifold, where $n \geq 2$, of class C^∞ equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$. The Riemannian distance and the volume form will be denoted by d and dm, respectively, and $|A| = \int_A dm$ stands for the volume of a measurable set $A \subset M^n$. Furthermore, if $|A| > 0$, we write

$$u_A = \fint_A u\, dm = \frac{1}{|A|} \int_A u\, dm$$

for the integral average of a measurable function u of A.

A vector field $X \in \operatorname{loc} L^1(G)$ is a (distributional) gradient of a function $u \in \operatorname{loc} L^1(G)$ if

$$\int_G u \operatorname{div} Y\, dm = -\int_G \langle X, Y \rangle\, dm$$

for all vector fields $Y \in C_0^1(G)$. The space of all functions $u \in L^1_{\mathrm{loc}}(G)$ whose distributional gradient ∇u belongs to $L^p(G)$, where $1 \leq p < \infty$, will be denoted by $L^1_p(G)$. The Sobolev space $W^1_p(G)$ consists of all functions $u \in L^1_p(G)$ which belong to $L^p(G)$, too. We equip $L^1_p(G)$ and $W^1_p(G)$ with the seminorm $\|\nabla u\|_p$ and with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p ,$$

respectively. The closures of $C_0^\infty(G)$ in $L^1_p(G)$ and in $W^1_p(G)$ are denoted by $L^1_{p,0}(G)$ and $W^1_{p,0}(G)$, respectively.

Throughout the paper $c, c_0, c_1, \ldots$ will be positive constants, and $c(a, b, \ldots)$ denotes a constant depending on $a, b, \ldots$ The actual value of c may vary even within a line.

Most of the time we assume that M^n is complete and has *bounded geometry* which, in this paper, means that the Ricci curvature of M^n

is uniformly bounded from below by $-(n-1)K^2$, with $K > 0$, and the injectivity radius of M^n, denoted by $\operatorname{inj} M^n$, is positive. The well-known comparison theorems [BC, p. 253-257] and [CGT, Section 4] then give estimates

$$|B(x,r)| \leq V_K(r) \qquad \text{and} \qquad \frac{|B(x,R)|}{|B(x,r)|} \leq \frac{V_K(R)}{V_K(r)} \tag{2.2}$$

for the volumes of geodesic balls for all $x \in M^n$ and $R \geq r > 0$. Here $V_K(r)$ is the volume of a geodesic ball of radius r in the simply connected complete Riemannian n-manifold of constant sectional curvature $-K^2$. This estimate holds without the assumption on the injectivity radius. By applying (2.2) to volumes of n-balls in $\mathbb{R}^n$, we obtain

$$\frac{V_K(r)}{r^n} \leq \frac{V_K(R)}{R^n} \tag{2.3}$$

for $R \geq r > 0$. Volumes of small geodesic balls in M^n have a lower bound

$$|B(x,r)| \geq v_0 r^n \tag{2.4}$$

for all $x \in M^n$ and for all $r \leq \operatorname{inj} M^n/2$, where v_0 is a positive constant depending only on n. This estimate is proved by C. B. Croke [Cr]. Another result of Croke which will be used in this paper is the following isoperimetric inequality

$$|D|^{(n-1)/n} \leq c \operatorname{area}(\partial D),$$

where $D \subset B(x,r)$ is a domain with smooth boundary, $r \leq \operatorname{inj} M^n/2$, and c depends only on n; see [Cr, Theorem 11] and [CGT, p. 16-17]. Hence

$$|D|^{(m-1)/m} \leq c\, |B(x,r)|^{1/n-1/m} \operatorname{area}(\partial D) \tag{2.5}$$

if $m \geq n$. It is well-known that the isoperimetric inequality (2.5) implies that

$$\begin{aligned} &\left(\int_{B(x,r)} |u|^{m/(m-1)}\, dm\right)^{(m-1)/m} \\ &\qquad \leq c\, |B(x,r)|^{1/n-1/m} \int_{B(x,r)} |\nabla u|\, dm\,, \end{aligned} \tag{2.6}$$

where c is the same constant as in (2.5) and $u \in C_0^\infty(B(x,r))$; see, for example [C]. We obtain a Sobolev estimate by applying (2.6) and Hölder's inequality to functions $v = |u|^\gamma$, where $u \in C_0^\infty(B(x,r))$ and γ is suitable, and approximating.

Lemma 2.7. *Suppose that M^n is a complete Riemannian n-manifold, with* $\operatorname{inj} M^n > 0$, *and that* $1 \le p < m$, *where* $m \ge n$. *Then there exists a constant* $c = c(n,m,p)$ *such that*

$$\left(\int_{B(x,r)} |u|^{pm/(m-p)}\, dm\right)^{(m-p)/m} \le c\,|B(x,r)|^{p/n-p/m} \int_{B(x,r)} |\nabla u|^p\, dm \tag{2.8}$$

for every $u \in W_{p,0}^1(B(x,r))$ *and* $r \le \operatorname{inj} M^n/2$.

The above estimate will be used in the proof of Harnack's inequality together with a Poincaré inequality. We recall Buser's isoperimetric inequality [B, Section 5]

$$\frac{\operatorname{area}(\partial\Omega \cap B)}{|\Omega|} \ge \frac{c^{1+Kr}}{r}, \tag{2.9}$$

where $B = B(x,r)$, Ω is an open subset of B with smooth boundary such that $|\Omega| \le |B|/2$, and $c < 1$ depends only on n. Note that r can be arbitrary large in this inequality. Buser normalized the metric so that the lower bound for the Ricci curvature is $-(n-1)$. By rescaling the metric back to our setting, we obtain (2.9). We rewrite the right hand side of (2.9) as $r^{-1}e^{-c_n(1+Kr)}$ where $c_n > 0$ depends only on n. The analytic counterpart of (2.9) is the following local Poincaré inequality

$$\int_B |u - u_B|\, dm \le r\, e^{c_n(1+Kr)} \int_B |\nabla u|\, dm, \tag{2.10}$$

where $c_n > 0$ and $u \in W_1^1(B)$; see [C], [K2] for deducing (2.10) from (2.9).

2.11. Rough isometries and nets on manifolds.

Following Kanai [K1-3], we say that a mapping $\varphi: X \to Y$ between metric spaces X and Y is a *rough isometry* if, for some $c > 0$, the c-neighborhood of φX coincides with Y, and if there exist constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}\, d(x,y) - b \leq d\big(\varphi(x), \varphi(y)\big) \leq a\, d(x,y) + b \tag{2.12}$$

for all $x, y \in X$. Note that the mapping φ need not be continuous. Two metric spaces are said to be *roughly isometric* if there is a rough isometry between them. If $\varphi: X \to Y$ is a rough isometry satisfying (2.12) with the constants a and b, it is possible to construct a rough isometry $\psi: Y \to X$. Indeed, for any $y \in Y$, there exists at least one $x \in X$ such that $d(\varphi(x), y) < c$, where c is the constant in the definition. If we set $\psi(y) = x$, then ψ satisfies (2.12) with constants a and $a(b + 2c)$, and the $a(b + c)$-neighborhood of ψY coincides with X. Thus ψ is a rough isometry. It is called a *rough inverse* of φ. Furthermore, a composition $\psi \circ \varphi: X \to Z$ of rough isometries $\varphi: X \to Y$ and $\psi: Y \to Z$ is a rough isometry. Thus being roughly isometric is an equivalence relation.

A *net* is a countable set P with a family $\{N_p\}_{p\in P}$ of finite subsets N_p of P such that, for all $p, q \in P$, $p \in N_q$ if and only if $q \in N_p$. A sequence of points $p_0, p_1, \ldots, p_\ell$ in P is said to be a *path from p_0 to p_ℓ of length ℓ* if $p_k \in N_{p_{k-1}}$ for $k = 1, \ldots, \ell$. A net is *connected* if any two points of P can be joined by a path. For any two points p and q in a connected net P, we denote by $\delta(p, q)$ the minimum of the lengths of paths from p to q. Then δ satisfies the axioms of metric, and it is called the *combinatorial metric of P*. The *boundary* of a subset $S \subset P$ is the set $\{p \in P : \ \delta(p, S) = 1\}$ and it will be denoted by ∂S.

Suppose then that M^n is a Riemannian manifold. Let P be a maximal collection of κ-separated points, where $\kappa > 0$ is a fixed constant. Then P together with a net structure $\{N_p\}_{p\in P}$ of sets $N_p = \{q \in P : 0 < d(p,q) \leq 3\kappa\}$ is called a *κ-net on M^n*, or simply a net. Since M^n is assumed to be connected, it is easy to see that P is also connected. Next we show that a κ-net with the combinatorial metric δ is roughly isometric to M^n with no curvature assumptions on M^n; see [K1].

Lemma 2.13. *Let M^n be a Riemannian manifold and let P be a κ-net on M^n. Then (M, d) and (P, δ) are roughly isometric, and furthermore*

$$\frac{1}{3\kappa}\, d(p,q) \leq \delta(p,q) \leq \frac{1}{\kappa}\, d(p,q) + 1\,, \tag{2.14}$$

for all $p, q \in P$.

PROOF. We prove that the inclusion map $i : P \to M^n$, $i(p) = p$, is a rough isometry. Since P is a maximal κ-separated set, the κ-neighborhood of $P(= iP)$ coincides with M^n. To prove the left hand inequality, let p and q be two distinct points in P, and let $\delta(p,q) = \ell$. Then there exists a path $p_0 = p, p_1, \ldots, p_\ell = q$ of length ℓ. For each $i = 1, \ldots, \ell$, and $\varepsilon > 0$, there is a smooth curve from p_{i-1} to p_i of length at most $3\kappa + \varepsilon$. Thus there exists a piecewise smooth curve from p to q whose length is at most $3\kappa\ell + \ell\varepsilon$. Letting $\varepsilon \to 0$ we conclude that $d(p,q) \le 3\kappa\delta(p,q)$. For the right hand inequality, let p and q, with $p \ne q$, be any points in P. Again there exists a curve γ from p to q of length $l(\gamma) \le d(p,q) + \varepsilon$. Let ℓ be a positive integer such that $\kappa(\ell - 1) < l(\gamma) \le \kappa\ell$. Now there are points $x_0 = p, x_1, \ldots, x_{\ell-1}, x_\ell = q$ on γ such that $d(x_{i-1}, x_i) \le \kappa$ for all $i = 1, \ldots, \ell$. For each x_i, there exists a point $p_i \in P$ such that $d(x_i, p_i) < \kappa$ since P is a maximal κ-separated set. By the triangle inequality, $d(p_{i-1}, p_i) \le 3\kappa$, and so $\delta(p_{i-1}, p_i) \le 1$. Hence

$$\delta(p,q) \le \ell \le \frac{1}{\kappa}\, l(\gamma) + 1 \le \frac{1}{\kappa}\,(d(p,q) + \varepsilon) + 1\,,$$

and the right hand inequality follows by letting $\varepsilon \to 0$. We see that $\kappa\delta(p,q) - \kappa \le d(p,q) \le 3\kappa\delta(p,q)$, and therefore the inclusion map satisfies (2.12) with $a = \max\{3\kappa, 1/\kappa\}$ and $b = \kappa$.

A net P is said to be *uniform* if $\sup\{\#N_p : p \in P\} < +\infty$. If P is a κ-net on a complete Riemannian n-manifold M^n whose Ricci curvature is bounded from below by $-(n-1)K^2$, then

$$\#\{p \in P :\ p \in B(x,r)\} \le \mu(r)\,, \tag{2.15}$$

for every $x \in M^n$ and $r > 0$, where $\mu(r)$ depends only on r, n, K, and κ; see [K1]. In particular, such a net P is uniform.

2.16. $\mathcal{A}$-harmonic functions.

As we mentioned in the introduction, our result applies not only to p-harmonic functions but also to solutions of a wide class of equations modeled by the p-Laplace equation (1.1). Let $\mathcal{A}$ be a mapping

$\mathcal{A} : TM^n \to TM^n$ which satisfies the following assumptions for some numbers $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$:

$$\begin{array}{l} \text{the mapping } \mathcal{A}_x = \mathcal{A} \mid T_xM^n : T_xM^n \to T_xM^n \text{ is} \\ \text{continuous for a.e. } x \in M^n, \text{ and} \\ \text{the mapping } x \mapsto \mathcal{A}_x(X) \text{ is measurable} \\ \text{for all measurable vector fields } X \end{array} \tag{2.17}$$

for a.e. $x \in M^n$ and for all $h \in T_xM^n$,

$$\langle \mathcal{A}_x(h), h \rangle \ge \alpha \, |h|^p \,, \tag{2.18}$$

$$|\mathcal{A}_x(h)| \le \beta \, |h|^{p-1} \,, \tag{2.19}$$

$$\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h - k \rangle > 0, \tag{2.20}$$

whenever $h \neq k$, and

$$\mathcal{A}_x(\lambda h) = |\lambda|^{p-2} \, \lambda \, \mathcal{A}_x(h) \tag{2.21}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$.

A mapping $\mathcal{A}$ which satisfies conditions (2.17)-(2.21) with the constant p is said to be of *type* p. The class of all $\mathcal{A}$ of type p will be denoted by $\mathcal{A}_p(M^n)$.

A function $u \in W^1_{p,\mathrm{loc}}(G)$ is a (weak) solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0 \tag{2.22}$$

in G if

$$\int_G \langle \mathcal{A}(\nabla u), \nabla \varphi \rangle \, dm = 0$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (2.22) are called $\mathcal{A}$-harmonic.

Perhaps the most important feature of $\mathcal{A}$-harmonic functions is the following *comparison principle.* If u and v are $\mathcal{A}$-harmonic functions in $G \Subset M^n$ with $u \ge v$ on ∂G, then $u \ge v$ in G. The comparison principle has made it possible to develop a non-linear potential theory for solutions of (2.22). For the basic results in the non-linear potential theory in the Euclidean n-space we refer to [GLM], [HK], and to a forthcoming book [HKM]. Finally, we remark that it follows directly from the properties of $\mathcal{A}$ that $\lambda u + \mu$ is $\mathcal{A}$-harmonic if u is $\mathcal{A}$-harmonic and λ and μ are constants.

3. Local Harnack's inequality.

In this section we prove a local Harnack inequality for positive $\mathcal{A}$-harmonic functions on a complete Riemannian n-manifold with bounded geometry. We need the result later in the paper. Harnack's inequalities are usually proved using the Moser iteration method where Sobolev and Poincaré inequalities are involved. We start by recalling the following Caccioppoli-type inequality from [H2]. We assume that $\mathcal{A} \in \mathcal{A}_p(M^n)$ satisfies conditions (2.17)-(2.21) with constants α and β, and that $G \subset M^n$ is open.

Lemma 3.1. *Let u be a positive $\mathcal{A}$-harmonic function in G, and let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$ and $\mathcal{A}$ is of type p. Then*

$$\int_G \eta^p \, |\nabla v|^p \; dm \le \left(\frac{\beta \, |q|}{\alpha \, |q-p+1|} \right)^p \int_G v^p \, |\nabla \eta|^p \; dm \tag{3.2}$$

holds for every non-negative $\eta \in C_0^\infty(G)$.

The most important point in the following theorem is that the constant c_0 in (3.4) does not depend on x at all.

Theorem 3.3. *Suppose that M^n is a complete Riemannian n-manifold with bounded geometry, and let $\mathcal{A} \in \mathcal{A}_p(M^n)$. Then there exists, for each $0 < r_0 \le 2 \operatorname{inj} M^n/3$, a constant $c_0 = c_0(n, p, K, r_0, \beta/\alpha)$ such that*

$$\sup_{B(x,r/2)} u \le c_0 \inf_{B(x,r/2)} u \,, \tag{3.4}$$

for every positive $\mathcal{A}$-harmonic function u in a geodesic ball $B(x,r) \subset M^n$, where $r \le r_0$.

PROOF. The proof is similar to that in [H2] but we want to give it in detail to work out how c_0 depends on various parameters. Fix $r_0 \le 2 \operatorname{inj} M^n/3$, and let $r \le r_0$. Suppose that u is a positive $\mathcal{A}$-harmonic function in $B(x,r) \subset M^n$. Let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$, let

$m = \max\{n, p+1\}$, and write $\lambda = m/(m-p)$. The Sobolev estimate (2.8) and the Caccioppoli inequality (3.2) imply that

$$\begin{aligned}
&\left(\int_{B(x,3r/4)} |\eta v|^{p\lambda}\, dm\right)^{1/\lambda} \\
(3.5)\quad &\le c\,|B(x,3r/4)|^{p/n-p/m} \int_{B(x,3r/4)} \left(\eta^p\,|\nabla v|^p + v^p\,|\nabla \eta|^p\right) dm \\
&\le A\left(\left(\frac{|q|}{|q-p+1|}\right)^p + 1\right) \int_{B(x,3r/4)} v^p\,|\nabla\eta|^p\, dm
\end{aligned}$$

for every non-negative $\eta \in C_0^\infty\big(B(x,3r/4)\big)$, where

$$A = c_1\,|B(x,3r/4)|^{p/n-p/m} \qquad \text{and} \qquad c_1 = c_1(n,p,\beta/\alpha)\,.$$

Let $r/2 \le t < t' \le 3r/4$, and write $t_i = t + (t'-t)2^{-i}$ and $B_i = B(x,t_i)$ for every $i = 0, 1, \ldots$ Then $(t_i - t_{i+1})^{-p} = 2^{(i+1)p}(t'-t)^{-p}$, $B_0 = B(x,t')$, and $B(x,t) \subset B_i$ for every i. For each i, we choose a non-negative $\eta_i \in C_0^\infty\big(B(x,3r/4)\big)$ such that $\eta_i = 1$ in B_{i+1}, $\eta_i = 0$ outside B_i, and $|\nabla\eta_i| \le 2(t_i - t_{i+1})^{-1}$. Next we choose $q_0 \in \mathbb{R}\setminus\{0\}$ such that

$$|q_0\lambda^i - p + 1| \ge \frac{p\,(p-1)}{2m-p} \tag{3.6}$$

for every i. Applying (3.5) to η_i and to $q = q_0\lambda^i$ yields

$$\begin{aligned}
&\left(\int_{B_{i+1}} u^{q_0\lambda^{i+1}}\, dm\right)^{1/\lambda} \\
&\qquad\le A\left(\left(\frac{|q_0\lambda^i|}{|q_0\lambda^i - p + 1|}\right)^p + 1\right) \frac{2^{(i+1)p}}{(t'-t)^p} \int_{B_i} u^{q_0\lambda^i}\, dm\,,
\end{aligned}$$

and so

$$\begin{aligned}
&\left(\int_{B_j} \left(u^{q_0}\right)^{\lambda^j} dm\right)^{1/\lambda^j} \\
&\qquad\le A^{S_j} \prod_{i=0}^{j-1}\left(\frac{|q_0\lambda^i|^p}{|q_0\lambda^i - p + 1|^p} + 1\right)^{1/\lambda^i} \frac{2^{pS_j'}}{(t'-t)^{pS_j}} \int_{B_0} u^{q_0}\, dm\,,
\end{aligned}$$

where $S_j = \sum_{i=0}^{j} \lambda^{-i}$ and $S_j' = \sum_{i=0}^{j}(i+1)\lambda^{-i}$. The condition (3.6) implies that the product above has an upper bound which depends only on n and p (note that $m = \max\{n, p+1\}$). Letting $j \to \infty$ we get $S_j \to m/p$ and

$$\sup_{B(x,t)} u^{q_0} \le \frac{c_2 A^{m/p} |B(x,t')|}{(t'-t)^m} \fint_{B(x,t')} u^{q_0}\, dm\,, \tag{3.7}$$

with $c_2 = c_2(n,p)$ provided that (3.6) holds. The condition (3.6) holds for every $q_0 < 0$. Moreover, for every $q > 0$, there can be at most one i such that

$$\left|q\lambda^i - p + 1\right| < \frac{p(p-1)}{2m-p}\,.$$

Thus every interval $[q/\lambda, q]$ contains a number q_0 which satisfies (3.6) for all i. To get rid of (3.6), suppose that $q \ne 0$. If $q < 0$, we set $q_0 = q$, otherwise, we choose $q_0 \in [q/\lambda, q]$ such that (3.6) holds for every i. Next we choose $c_3 = \max\{c_2, (2c_1^{1/p} v_0^{1/n})^{-m}\}$. Then

$$\frac{c_3 A^{m/p} |B(x,t')|}{(t'-t)^m} \ge \frac{c_3\, c_1^{m/p} v_0^{m/n} (r/2)^m}{(r/4)^m} \ge 1$$

by (2.4). It follows from (3.7) that

$$\begin{aligned} \sup_{B(x,t)} u^q &= \Big(\sup_{B(x,t)} u^{q_0}\Big)^{q/q_0} \\ &\le \left(\frac{c_3 A^{m/p} |B(x,t')|}{(t'-t)^m}\right)^{q/q_0} \left(\fint_{B(x,t')} u^{q_0}\, dm\right)^{q/q_0} \qquad (3.8)\\ &\le \frac{c_3^\lambda A^{m\lambda/p} |B(x,t')|^\lambda}{(t'-t)^{m\lambda}} \fint_{B(x,t')} u^q\, dm\,. \end{aligned}$$

This holds for every $q \ne 0$ and $r/2 \le t < t' \le 3r/4$. Next we write $\mathcal{B}(s) = B(x, r/2 + sr/4)$ for $0 \le s \le 1$. Since $A = c_1 |B(x, 3r/4)|^{p/n - p/m}$, we can write (3.8) as

$$\begin{aligned} \sup_{\mathcal{B}(s)} u^q &\le c \left(\frac{|B(x,3r/4)|}{r^n}\right)^{m\lambda/n} (s'-s)^{-m\lambda} \fint_{\mathcal{B}(s')} u^q\, dm \\ &\le c \left(\frac{V_K(3r_0/4)}{r_0^n}\right)^{m\lambda/n} (s'-s)^{-m\lambda} \fint_{\mathcal{B}(s')} u^q\, dm\,. \end{aligned}$$

Here we used volume estimates (2.2) and (2.3) to obtain first $|B(x,3r/4)| \leq V_K(3r/4)$ and then $V_K(3r/4)r^{-n} \leq V_K(3r_0/4)r_0^{-n}$. We have proved that

$$\sup_{B(s)} u \leq \left(c\,(s'-s)^{m\lambda}\right)^{-1/q} \left(⨍_{B(s')} u^q\, dm\right)^{1/q},$$

and

$$\inf_{B(s)} u \geq \left(c\,(s'-s)^{m\lambda}\right)^{1/q} \left(⨍_{B(s')} u^{-q}\, dm\right)^{-1/q}$$

for all $q > 0$ and $0 \leq s < s' \leq 1$, where $c = c(n, p, \beta/\alpha, K, r_0)$. By the refined version of the John-Nirenberg Theorem [BG],

$$\sup_{B(x,r/2)} u \leq \exp\bigl(c\, g(u)\bigr) \inf_{B(x,r/2)} u\,,$$

where

$$g(u) = \sup_{0\leq s\leq 1} \inf_{a\in\mathbb{R}} ⨍_{B(s)} |\log u - a|\; dm$$

and $c = c(n, p, \beta/\alpha, K, r_0)$. To estimate $g(u)$, we first use the local Poincaré inequality (2.10) and Hölder's inequality

$$\begin{aligned} g(u) &\leq \frac{1}{|B(x,r/2)|} \inf_{a\in\mathbb{R}} \int_{B(x,3r/4)} |\log u - a|\; dm \\ &\leq \frac{r\exp\bigl(c_n(1+Kr)\bigr)}{|B(x,r/2)|} \int_{B(x,3r/4)} |\nabla \log u|\; dm \\ &\leq \frac{r\exp\bigl(c_n(1+Kr)\bigr)\, |B(x,3r/4)|^{1-1/p}}{|B(x,r/2)|} \\ &\quad\cdot \left(\int_{B(x,3r/4)} |\nabla \log u|^p\; dm\right)^{1/p}. \end{aligned}$$

Furthermore, [HK, 2.24] implies that

$$\int_{B(x,3r/4)} |\nabla \log u|^p\; dm \leq c\,(p, \beta/\alpha) \int_{B(x,r)} |\nabla\eta|^p\; dm \tag{3.9}$$

for every $\eta \in C_0^\infty(B(x,r))$ such that $\eta = 1$ in $B(x,3r/4)$. We obtain an upper bound $c\,r^{-p}\,|B(x,r)|$ for the right hand side of (3.9) by choosing η such that $|\nabla\eta| \le 8/r$. Putting together these estimates yields

$$g(u) \le c\exp\big(c_n(1+Kr)\big)\frac{|B(x,3r/4)|}{|B(x,r/2)|}\left(\frac{|B(x,r)|}{|B(x,3r/4)|}\right)^{1/p}$$

$$\le c\exp\big(c_n(1+Kr_0)\big)\frac{V_K(3r/4)}{V_K(r/2)}\left(\frac{V_K(r)}{V_K(3r/4)}\right)^{1/p}$$

Finally, we apply (2.2) and (2.3) to volumes of n-balls in $\mathbb{R}^n$ to deduce first that $c\,r^n \le V_K(r/2)$ $\big(\le V_K(3r/4)\big)$, with $c = c(n)$, and then that

$$\frac{V_K(3r/4)}{V_K(r/2)} \le \frac{V_K(3r/4)}{c\,r^n} \le \frac{V_K(3r_0/4)}{c\,r_0^n}.$$

Similarly,

$$\frac{V_K(r)}{V_K(3r/4)} \le \frac{V_K(r_0)}{c\,r_0^n}.$$

Hence $g(u)$ has an upper bound which depends only on $n, p, \beta/\alpha, K$, and r_0. The theorem is proved.

As a consequence of the local Harnack inequality we obtain the following result.

Theorem 3.10. *Suppose that M^n is a complete Riemannian n-manifold with bounded geometry and that $\mathcal{A} \in \mathcal{A}_p(M^n)$. Let*

$$r_0 = \min\{1, \frac{2}{3}\operatorname{inj} M^n\}\ .$$

Then there exists a positive constant $c_4 = c_4(n,p,\beta/\alpha,K,r_0)$ such that

$$\text{(3.11)}\qquad d(x,y) > c_4\,r_0\,\max\left\{\left|\log\frac{u(x)}{u(y)}\right|, \left|\log\frac{1-u(x)}{1-u(y)}\right|\right\} - r_0\ ,$$

whenever u is $\mathcal{A}$-harmonic in M^n, with $\inf_{M^n} u = 0$ and $\sup_{M^n} u = 1$.

PROOF. Let x and y be two points in M^n. We may assume that $u(x) > u(y)$. Suppose first that $d(x,y) \ge r_0$. Let γ be a minimal geodesic from x to y, and let $\ell \ge 2$ be an integer such that $(\ell -$

$1)\,r_0/2 < d(x,y) \le \ell\, r_0/2$. Then there are points $x_0 = x, x_1, \ldots, x_\ell = y$ on γ such that $d(x_i, x_{i+1}) \le r_0/2$ for all $i = 0, 1, \ldots, \ell - 1$. Hence $B(x_i, r_0/2) \cap B(x_{i+1}, r_0/2) \ne \varnothing$ for all $i = 0, 1, \ldots, \ell - 1$. The local Harnack inequality (3.4) implies that

$$\begin{aligned} u(x) &\le \sup_{B(x_0,r_0/2)} u \le c_0 \inf_{B(x_0,r_0/2)} u \\ &\le c_0 \sup_{B(x_1,r_0/2)} u \le c_0^2 \inf_{B(x_1,r_0/2)} u \le \cdots \\ &\le c_0^\ell \sup_{B(x_\ell,r_0/2)} u \le c_0^{\ell+1} \inf_{B(x_\ell,r_0/2)} u \le c_0^{\ell+1} u(y)\,. \end{aligned}$$

Hence $\ell + 1 \ge (\log c_0)^{-1} \log\bigl(u(x)/u(y)\bigr)$, and so

$$d(x,y) > c_4\, r_0 \log \frac{u(x)}{u(y)} - r_0\,,$$

with $c_4 = (2 \log c_0)^{-1}$. If $d(x,y) < r_0$, there exists a point $z \in M^n$ such that $x, y \in B(z, r_0/2)$. Then $u(x) \le c_0\, u(y)$ by (3.4), and so $c_4\, r_0 \log\bigl(u(x)/u(y)\bigr) - r_0 \le -r_0/2$. The theorem follows by applying the same reasoning to the function $1 - u$.

4. A criterion for the Liouville D_p-property.

Manifolds which admit non-constant harmonic functions with bounded Dirichlet integral can be characterized by means of 2-capacities; see [G]. The purpose of this section is to generalize this criterion to the non-linear case (Theorem 4.6). It should be noted that M^n need not be of bounded geometry in this section.

A *condenser* is a triple $(F_1, F_2; G)$, where F_1 and F_2 are disjoint, non-empty, and closed sets in $\bar{G}$. Its *p-capacity* is the number

$$\operatorname{cap}_p(F_1, F_2; G) = \inf_u \int_G |\nabla u|^p \, dm\,,$$

where the infimum is taken over all functions $u \in L^1_p(G)$ which are continuous in $G \cup F_1 \cup F_2$ with $u = 0$ in F_1 and $u = 1$ in F_2. Such a function is called *admissible* for $(F_1, F_2; G)$. If the class of admissible functions is empty, we set $\operatorname{cap}_p(F_1, F_2; G) = +\infty$.

Let $\{B_i\}_{i=1}^{\infty}$ be an exhaustion of M^n such that $B_i \Subset B_{i+1}$ for every i. We say that a set $A \subset M^n$ is unbounded if A has common points with $M^n \setminus B_i$ for every i. For an open set $\Omega \subset M^n$ and a compact set $F \subset \bar{\Omega}$, we define

$$\operatorname{cap}_p(F, \infty; \Omega) = \lim_{i \to \infty} \operatorname{cap}_p(F, \bar{\Omega} \setminus B_i; \Omega).$$

Note that the limit exists and is independent of the exhaustion since the assumption $B_i \Subset B_{i+1}$ implies that

$$\operatorname{cap}_p(F, \bar{\Omega} \setminus B_i; \Omega) \geq \operatorname{cap}_p(F, \bar{\Omega} \setminus B_{i+1}; \Omega).$$

Definition 4.1. *An unbounded open set $\Omega \subset M^n$ is called p-hyperbolic if there exists a compact set $F \subset \bar{\Omega}$ such that* $\operatorname{cap}_p(F, \infty; \Omega) > 0$.

We remark that any open set Ω' is p-hyperbolic if there exists a p-hyperbolic subset $\Omega \subset \Omega'$. We also observe that $\operatorname{cap}_p(F, \bar{\Omega} \setminus D; \Omega) \geq \operatorname{cap}_p(F, \infty; \Omega) > 0$ for each open $D \Subset M^n$ if Ω is p-hyperbolic and F is as in the definition.

Definition 4.2. *An unbounded open set $\Omega \subset M^n$, with $\partial\Omega \neq \varnothing$, is called D_p-massive if there exists a p-harmonic function u in Ω which is continuous in $\bar{\Omega}$, with $u = 0$ in $\partial\Omega$, $\sup_\Omega u = 1$, and*

$$\int_\Omega |\nabla u|^p \, dm < +\infty .$$

It is clear from the definition that the sets $\{x : u(x) < a\}$ and $\{x : u(x) > b\}$, and even all components of these sets, are D_p-massive if u is a non-constant bounded p-harmonic function in M^n, with $|\nabla u| \in L^p(M^n)$, and $\inf u < a < b < \sup u$.

Next we explain the connection between D_p-massive and p-hyperbolic sets.

Lemma 4.3. *Every D_p-massive set is also p-hyperbolic.*

PROOF. Let Ω be D_p-massive, and let u be as in Definition 4.2. Suppose that $\{B_i\}_{i=1}^{\infty}$ is an exhaustion of M^n such that $B_i \Subset B_{i+1}$, and that

$\operatorname{cap}_p(F, \bar{\Omega} \setminus B_2; \Omega) > 0$, where $F = \bar{B}_1 \cap \partial\Omega \neq \varnothing$. Next we choose admissible functions $w_i \in W_p^1(\Omega \cap B_i)$, $i \geq 2$, for condensers $(F, \bar{\Omega} \setminus B_i; \Omega)$ such that $0 \leq w_i \leq 1$,

$$(4.4) \qquad \int_{\Omega \cap B_i} |\nabla w_i|^p \, dm \leq \operatorname{cap}_p(F, \bar{\Omega} \setminus B_i; \Omega) + \frac{1}{i},$$

and that $w_i \equiv 1$ in all those components of $\Omega \cap B_i$ whose closures do not intersect F. We choose these functions in the following way. Suppose that w_2 is chosen. Let v_2 be the unique p-harmonic function in $\Omega \cap B_2$ such that $v_2 - w_2 \in W_{p,0}^1(\Omega \cap B_2)$. We set $v_2 = 1$ in $\Omega \setminus B_2$. Then

$$\int_{\Omega \cap B_2} |\nabla v_2|^p \, dm \leq \int_{\Omega \cap B_2} |\nabla w_2|^p \, dm$$

and $v_2 \geq u$ in Ω. Next we choose w_3. Then the set $A = \{x \in \Omega : w_3(x) > v_2(x)\}$ is a subset of $\Omega \cap B_2$. If $A \neq \varnothing$,

$$\int_A |\nabla v_2|^p \, dm \leq \int_A |\nabla w_3|^p \, dm,$$

since v_2 is p-harmonic in A. We redefine w_3 by setting $w_3 = v_2$ in A. Clearly (4.4) still holds. By continuing similarly, we get a decreasing sequence of functions $\{v_i\}$ such that v_i is p-harmonic in $\Omega \cap B_i$, $v_i \geq u$, and that

$$\int_{\Omega \cap B_i} |\nabla v_i|^p \, dm \leq \int_{\Omega \cap B_i} |\nabla w_i|^p \, dm.$$

To finish the proof, suppose that Ω is not p-hyperbolic. Then $\operatorname{cap}_p(F, \bar{\Omega} \setminus B_i; \Omega) \to 0$, and so $\int_{\Omega \cap B_i} |\nabla v_i|^p \, dm \to 0$. Since $v_i \geq u$ and $\sup_\Omega u = 1$, the only possibility is that $v_i \to 1$. This is a contradiction since $\{v_i\}$ is decreasing. Hence Ω is p-hyperbolic.

Note that the assumption $\int_\Omega |\nabla u|^p \, dm < +\infty$ was not needed in the proof. The converse of Lemma 4.3 is not true, that is, there are p-hyperbolic sets which are not D_p-massive. Indeed, let $p < n$ and let $\Omega \subset \mathbb{R}^n$ be the upper half space $\{x : x_n > 0\}$. By symmetry, $\operatorname{cap}_p(\bar{B}^n(r) \cap \Omega, \infty; \Omega) = \operatorname{cap}_p(\bar{B}^n(r), \infty; \mathbb{R}^n)/2$. It is well-known that $\operatorname{cap}_p(\bar{B}^n(r), \infty; \mathbb{R}^n) = c r^{n-p} > 0$. Hence Ω is p-hyperbolic. On the other hand, Ω can not be D_p-massive. Otherwise, the lower half space would be D_p-massive by symmetry. But this implies that $\mathbb{R}^n$ does not have the Liouville D_p-property (see the end of the proof of Theorem 4.6) which leads to a contradiction with [H1, 5.9, 5.11]. The exact relation between D_p-massive and p-hyperbolic sets is given by Theorem 4.5. It says that D_p-massive sets are, in general, "broader" than p-hyperbolic sets. Indeed, a D_p-massive set Ω must contain a p-hyperbolic set Ω_1 such that $\operatorname{cap}_p(\partial\Omega, \partial\Omega_1; \Omega) < +\infty$. This is the meaning of Theorem 4.5, although we have formulated it in a slightly different way to avoid difficulties with boundary regularity.

Theorem 4.5. *An unbounded open set $\Omega \subset M^n$, with $\partial\Omega \neq \varnothing$, is D_p-massive if and only if there exists a p-hyperbolic $\Omega_1 \subset \Omega$ and a continuous function v in $\bar{\Omega}$ which is p-harmonic in $\Omega \setminus \bar{\Omega}_1$, with $v = 0$ in $\partial\Omega$, $v = 1$ in Ω_1, and $\int_\Omega |\nabla v|^p \, dm < +\infty$.*

PROOF. The idea of the proof comes from [G]. Suppose first that Ω is D_p-massive. Let u be as in Definition 4.2, and let $0 < \varepsilon < 1$. Then the set $\{x \in \Omega : \ u(x) > \varepsilon\}$ is D_p-massive, and hence p-hyperbolic. Furthermore, the function $v = \min\{u, \varepsilon\}/\varepsilon$ satisfies the assumptions of the claim.

To prove the converse, let $\{B_i\}_{i=1}^\infty$ be an exhaustion of M^n, with $B_i \Subset B_{i+1}$. For $i \geq 2$, we write

$$\Omega_i = \Omega_1 \setminus \bar{B}_i, \quad G_1 = \Omega \setminus \bar{\Omega}_1, \quad G_i = \Omega \setminus \bar{\Omega}_i, \quad \text{and} \quad G_i^k = G_i \cap B_k.$$

Let u_i^k be the unique p-harmonic function in G_i^k with boundary values $u_i^k - v \in W_{p,0}^1(G_i^k)$. We set $u_i^k = v$ in $\Omega \setminus G_i^k$. Now $0 \leq u_i^k \leq v$ and $u_{i+1}^k \leq u_i^k$ in Ω. Since the sequence $\{u_i^k\}_{k=1}^\infty$ is uniformly bounded, it is equicontinuous in G_i by the Hölder-continuity estimate [T, Theorem 2.2]. By Ascoli's theorem, there exists a subsequence, still denoted by $\{u_i^k\}_{k=1}^\infty$, which converges locally uniformly in G_i to a function u_i. We set $u_i = v$ in $\Omega \setminus G_i$. Then u_i is p-harmonic in G_i and the sequence $\{u_i\}_{i=1}^\infty$ is decreasing. By Harnack's principle [HK, 3.3], the limit function $u = \lim_{i\to\infty} u_i$ is p-harmonic in Ω. If we set $u = 0$ in $\partial\Omega$, then u is continuous in $\bar{\Omega}$ since $0 \leq u \leq v$ and $v \in C(\bar{\Omega})$, with $v = 0$ in $\partial\Omega$.

Next we shall show that u (multiplied by a suitable constant) satisfies the conditions in the definition of D_p-massiveness. First we observe that

$$\begin{aligned}\int_\Omega |\nabla u_i^k|^p \, dm &= \int_{G_i^k} |\nabla u_i^k|^p \, dm + \int_{\Omega\setminus G_i^k} |\nabla v|^p \, dm \\ &\leq \int_{G_i^k} |\nabla v|^p \, dm + \int_{\Omega\setminus G_i^k} |\nabla v|^p \, dm\end{aligned}$$

$$= \int_\Omega |\nabla v|^p \, dm < +\infty \, .$$

Passing to a subsequence we conclude that there exists a vector field $X \in L^p(\Omega)$ such that $\nabla u_i^k \to X$ weakly in $L^p(\Omega)$ as $k \to \infty$. But the convergence of u_i^k implies that $X = \nabla u_i$. Now $u_i - v \in L^1_{p,0}(\Omega)$ since $u_i^k - v \in L^1_{p,0}(\Omega)$. This in turn implies that

$$\begin{aligned} \int_\Omega |\nabla u_i|^p \, dm &= \int_{G_i} |\nabla u_i|^p \, dm + \int_{\Omega_i} |\nabla v|^p \, dm \\ &\le \int_{G_i} |\nabla v|^p \, dm + \int_{\Omega_i} |\nabla v|^p \, dm \\ &= \int_\Omega |\nabla v|^p \, dm < +\infty \, . \end{aligned}$$

By repeating the above reasoning, we get that $\int_\Omega |\nabla u|^p \, dm < +\infty$ and $u - v \in L^1_{p,0}(\Omega)$. It follows from Maz'ya's lemma [M, Lemma 2], which obviously holds in our situation, that

$$|\nabla u_i|^{p-2} \nabla u_i \to |\nabla u|^{p-2} \nabla u$$

weakly in $L^{p/(p-1)}(\Omega)$. It remains to show that $u \not\equiv 0$. Since Ω_1 is p-hyperbolic, there exists a compact set $F \subset \bar{\Omega}_1$ such that $\operatorname{cap}_p(F, \infty; \Omega_1) > 0$. Let $U \Subset M^n$ be a sufficiently large connected neighborhood of F so that $U \setminus \bar{\Omega}$ is non-empty. We write $\Omega_1' = \Omega_1 \cup U$ and $F_1 = \bar{U} \setminus \Omega$. Now Ω_1' is also p-hyperbolic, and $\operatorname{cap}_p(F_1, \infty; \Omega_1') > 0$ since F_1 and F lie in a same component of Ω_1'. For each i, u_i is admissible for the condenser $(\partial\Omega, \partial\Omega_i; G_i)$. Using this fact and well-known properties of capacities we get that

$$\begin{aligned} \int_\Omega |\nabla u_i|^p \, dm &\ge \operatorname{cap}_p(\partial\Omega, \partial\Omega_i; G_i) \\ &= \operatorname{cap}_p(M^n \setminus \Omega, \bar{\Omega}_i; M^n) \\ &\ge \operatorname{cap}_p(F_1, \bar{\Omega}_1' \setminus B_i; \Omega_1') \\ &\ge \operatorname{cap}_p(F_1, \infty; \Omega_1') > 0 \end{aligned}$$

if i is large enough. Furthermore,

$$\begin{aligned} \int_\Omega |\nabla u_i|^p \, dm &= \int_\Omega \langle |\nabla u_i|^{p-2} \nabla u_i, \nabla v \rangle \, dm \\ &\to \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle \, dm \, , \end{aligned}$$

and so ∇u can not vanish identically in Ω. We conclude that u is non-constant. Multiplying u by a suitable constant, if necessary, we get a function which satisfies all the conditions in the definition of D_p-massiveness. The theorem is proved.

An open set $G \Subset M^n$ is called *regular* if, for all functions $h \in C(\bar{G}) \cap W_p^1(G)$,

$$\lim_{x \to y} u(x) = h(y)$$

holds at every boundary point $y \in \partial G$ whenever u is the unique p-harmonic function in G with $u - h \in W_{p,0}^1(G)$. We refer to [M], [KM], and [LM] for the results concerning the boundary regularity. For example, all domains $\Omega \Subset M^n$ with C^1-boundaries are regular for all p.

Theorem 4.6. *A Riemannian n-manifold M^n admits a non-constant p-harmonic function u, with $\int_{M^n} |\nabla u|^p \, dm < +\infty$, if and only if there exist two p-hyperbolic sets Ω_1, $\Omega_2 \subset M^n$ such that* $\operatorname{cap}_p(\bar{\Omega}_1, \bar{\Omega}_2; M^n) < +\infty$.

PROOF. If M^n does not have the Liouville D_p-property, there exists a non-constant bounded p-harmonic function u in M^n, with $\int_{M^n} |\nabla u|^p dm < +\infty$. Let $\inf u < a < b < \sup u$. Then the sets $\Omega_1 = \{x : u(x) < a\}$ and $\Omega_2 = \{x : u(x) > b\}$ are D_p-massive, hence p-hyperbolic. Moreover,

$$\operatorname{cap}_p(\bar{\Omega}_1, \bar{\Omega}_2; M^n) \le \frac{1}{(b-a)^p} \int_{M^n} |\nabla u|^p \, dm < +\infty \,,$$

since the function

$$v = \max\left\{0, \min\left\{\frac{u-a}{b-a}, 1\right\}\right\}$$

is admissible for the condenser $(\bar{\Omega}_1, \bar{\Omega}_2; M^n)$.

Suppose then that Ω_1 and Ω_2 are p-hyperbolic, with $\operatorname{cap}_p(\bar{\Omega}_1, \bar{\Omega}_2; M^n) < +\infty$. Then there exists an admissible function w for the condenser $(\bar{\Omega}_1, \bar{\Omega}_2; M^n)$. By taking slightly larger open sets Ω_1' and Ω_2' with smooth boundaries and containing Ω_1 and Ω_2, respectively, such that $\Omega_1' \subset \{x : w(x) < 1/4\}$ and $\Omega_2' \subset \{x : w(x) > 3/4\}$, we obtain p-hyperbolic sets Ω_1' and Ω_2', with $\operatorname{cap}_p(\bar{\Omega}_1', \bar{\Omega}_2'; M^n) < +\infty$. Now there exists a continuous function u in M^n which is p-harmonic in

$M^n\backslash(\bar{\Omega}_1'\cup\bar{\Omega}_2')$ with $u=0$ in $\bar{\Omega}_1'$, $u=1$ in $\bar{\Omega}_2'$, and $\int_{M^n}|\nabla u|^p\,dm<+\infty$. By Theorem 4.5, the sets $\{x:\ u(x)>b\}$ and $\{x:\ u(x)<a\}$ are disjoint D_p-massive sets for $0<a<b<1$. Call them G_1 and G_2. Let $\{B_i\}$ be an exhaustion of M^n such that B_i is regular for every i. Let u_j, $j=1,2$, be a p-harmonic function in G_j satisfying the conditions in Definition 4.2. We extend u_j to M^n by setting $u_j=0$ in $M^n\setminus G_j$. Let $v_i\in C(\bar{B}_i)$ be p-harmonic in B_i such that $v_i=u_1$ in ∂B_i. Then

$$u_1\le v_i\le 1-u_2$$

in B_i. Furthermore,

$$\int_{B_i}|\nabla v_i|^p\,dm\le\int_{B_i}|\nabla u_1|^p\,dm\le\int_{M^n}|\nabla u_1|^p\,dm<+\infty\,.$$

Thus there exists a subsequence, denoted again by v_i, which converges locally uniformly in M^n to a p-harmonic function v. Now $u_1\le v\le 1-u_2$ in M^n and $\int_{M^n}|\nabla v|^p\,dm<+\infty$. Since $\sup u_1=\sup u_2=1$, v can not be constant. The theorem is proved.

EXAMPLE 4.7. We close this section by an example where the Liouville D_p-property essentially depends on p. Let $M^n=S^{n-1}\times\mathbb{R}$ be equipped with a metric

$$f^2d\vartheta^2+dt^2\,,$$

where $d\vartheta^2$ is the standard metric of the sphere S^{n-1} normalized so that $m_{n-1}(S^{n-1})=1$. We assume that f is a positive C^∞-function of M^n which depends only on t-coordinate of $(\vartheta,t)\in S^{n-1}\times\mathbb{R}$ and $f(\cdot\,,-t)=f(\cdot\,,t)$. We abbreviate $f(t)=f(\cdot\,,t)$. Then

$$m_{n-1}\big(\{(\vartheta,t)\in M^n:\ t=r\}\big)=f(r)^{n-1}$$

for every $r\in\mathbb{R}$. We claim that M^n has the Liouville D_p-property if and only if the integral

$$I=\int_1^\infty f(t)^{(1-n)/(p-1)}\,dt$$

diverges. To show this, suppose that $I<+\infty$. Let $\Omega=\{(\vartheta,t)\in M^n:\ t>1\}$, $F=\{(\vartheta,t)\in M^n:\ t=1\}$ $(=\partial\Omega)$, and $B_i=\{(\vartheta,t)\in M^n:\ |t|<i\}$ for $i=2,3,\ldots$ If u is an admissible function for $(F,\bar{\Omega}\setminus B_i;\Omega)$, we have

$$\int_{\gamma_\vartheta}|\nabla u|\,ds\ge 1$$

for each curve $\gamma_\vartheta : [1, i] \to M^n$, $\gamma_\vartheta(t) = (\vartheta, t)$, where $\vartheta \in S^{n-1}$. By Hölder's inequality,

$$\begin{aligned} 1 &\le \left(\int_{\gamma_\vartheta} |\nabla u| \, ds \right)^p \\ &= \left(\int_1^i |\nabla u(\vartheta, t)| \, f(t)^{(n-1)/p} f(t)^{(1-n)/p} \, dt \right)^p \\ &\le \left(\int_1^i |\nabla u(\vartheta, t)|^p f(t)^{n-1} \, dt \right) \left(\int_1^i f(t)^{(1-n)/(p-1)} \, dt \right)^{p-1}, \end{aligned}$$

and so

$$\int_1^i |\nabla u(\vartheta, t)|^p f(t)^{n-1} \, dt \ge \left(\int_1^i f(t)^{(1-n)/(p-1)} \, dt \right)^{1-p}$$

Integrating with respect to ϑ yields

$$\begin{aligned} \int_\Omega |\nabla u|^p \, dm &\ge \int_{S^{n-1}} \left(\int_1^i |\nabla u(\vartheta, t)|^p f(t)^{n-1} \, dt \right) d\vartheta \\ &\ge \left(\int_1^i f(t)^{(1-n)/(p-1)} \, dt \right)^{1-p} \ge I^{1-p} > 0 \end{aligned}$$

since we normalized $\int_{S^{n-1}} d\vartheta = 1$. Taking the infimum over u and then letting $i \to \infty$ yields $\operatorname{cap}_p(F, \infty; \Omega) > 0$, that is, Ω is p-hyperbolic. Similarly, $\Omega' = \{(\vartheta, t) \in M^n : t < -1\}$ is p-hyperbolic. Furthermore, $\operatorname{cap}_p(\bar{\Omega}, \bar{\Omega}'; M^n) < +\infty$, and therefore M^n has a non-constant p-harmonic function v with $\int_{M^n} |\nabla v|^p \, dm < +\infty$.

Conversely, suppose that the integral I diverges. For each $r > 0$, let $D(r) = \{(\vartheta, t) \in M^n : |t| < r\}$. Fix r and R such that $R > r > 0$. For each integer $k \ge 1$ and $i = 0, 1, \ldots, k$, let $t_i = r + i(R - r)/k$. By, for instance [HKM, Section 2],

$$\begin{aligned} &\left(\operatorname{cap}_p(\bar{D}(r), M^n \setminus D(R); M^n) \right)^{1/(1-p)} \\ &\qquad \ge \sum_{i=0}^{k-1} \left(\operatorname{cap}_p(\bar{D}(t_i), M^n \setminus D(t_{i+1}); M^n) \right)^{1/(1-p)}. \end{aligned}$$

We get an estimate

$$\operatorname{cap}_p(\bar{D}(t_i), M^n \setminus D(t_{i+1}); M^n) \le \left(|D(t_{i+1})| - |D(t_i)| \right) \left(t_{i+1} - t_i \right)^{-p}$$

by choosing an admissible function u such that $u(\vartheta,t)=(t-t_i)/(t_{i+1}-t_i)$ in $D(t_{i+1})\setminus D(t_i)$. Hence

$$\begin{aligned}(4.8)\qquad &\left(\operatorname{cap}_p(\bar D(r),M^n\setminus D(R);M^n)\right)^{1/(1-p)}\\ &\qquad\ge\sum_{i=0}^{k-1}\left(\frac{|D(t_{i+1})|-|D(t_i)|}{t_{i+1}-t_i}\right)^{1/(1-p)}(t_{i+1}-t_i)\,.\end{aligned}$$

Next we observe that

$$\lim_{\varepsilon\to 0}\frac{|D(t+\varepsilon)|-|D(t)|}{\varepsilon}=m_{n-1}\big(\partial D(t)\big)=2\,f(t)^{n-1}\,.$$

Thus the right hand side of (4.8) tends to an integral

$$2^{1/(1-p)}\int_r^R f(t)^{(1-n)/(p-1)}\,dt$$

as $k\to\infty$. Hence $\lim_{R\to\infty}\operatorname{cap}_p(\bar D(r),M^n\setminus D(R);M^n)=0$ if the integral I diverges. Since r is arbitrary, this implies that $\operatorname{cap}_p(C,\infty;M^n)=0$ for every compact $C\subset M^n$. It follows from [H1, Section 5] that every p-harmonic function v on M^n with $\int_{M^n}|\nabla v|^p\,dm<+\infty$ is constant.

5. p-hyperbolic nets and the main result.

Throughout this section we assume that M^n and N^ν are complete Riemannian manifolds with bounded geometry, and that $P\subset M^n$ and $Q\subset N^\nu$ are κ-nets, with $\kappa\le\min\{\operatorname{inj}M^n,\operatorname{inj}N^\nu\}/2$.

We shall define p-hyperbolicity on nets, and therefore we need a discrete counterpart for p-capacity. For each $q\in P$ (or Q) and a real-valued function u in N_q, we set

$$|Du(q)|=\Big(\sum_{q'\in N_q}\big(u(q')-u(q)\big)^2\Big)^{1/2}$$

In many occasions we use the fact that, for a uniform net P,

$$\begin{aligned}(5.1)\qquad &c_4^{-1}\sum_{q'\in N_q}|u(q')-u(q)|^p\le|Du(q)|^p\\ &\qquad\le c_4^{p/2}\sum_{q'\in N_q}|u(q')-u(q)|^p\,,\end{aligned}$$

where $c_4 = \sup\{\#N_q : q \in P\}$. Suppose that $S \subset P$ (or Q) is a connected infinite subnet. We say that S is ***p-hyperbolic***, with $1< p <\infty$, if there exists a finite non-empty set $E \subset S$ such that

$$\operatorname{cap}_p(E, \infty; S) = \inf_u \sum_{q\in S} |Du(q)|^p > 0\,,$$

where the infimum is taken over all finitely supported functions u of $S \cup \partial S$, with $u = 1$ in E. Such functions are called admissible for $(E, \infty; S)$. Recall that $\partial S = \{q : \delta(q, S) = 1\}$.

Lemma 5.2. *Suppose that $S' \subset P$ is a connected subnet, $\Omega = \{x \in M^n : d(x, S' \cup \partial S') < 7\kappa\}$, and that $S = \{q \in P : d(q, \Omega) < \kappa\}$. Then Ω is a domain and S is a connected subnet.*

PROOF. Let x and y be any two points in Ω. Then there are points $q, q' \in S' \cup \partial S'$ such that $d(x, q) < 7\kappa$ and $d(y, q') < 7\kappa$. Since also $S' \cup \partial S'$ is connected, we can find a path in $S' \cup \partial S'$ from q to q'. Then the 7κ-neighborhood of this path is a connected subset of Ω which contains both x and y. This shows that Ω is connected and therefore a domain since clearly Ω is open. To show that S is connected, let q and q' be any two points of S. Then there are points $x, y \in \Omega$ such that $d(x, q) < \kappa$ and $d(y, q') < \kappa$. Since Ω is a domain, there exists a rectifiable curve which connects x and y in Ω. As in the proof of Lemma 2.13, we see that the κ-neighborhood of this curve contains a path in P, and hence in S, from q to q'. Thus S is connected.

Next we shall study how p-hyperbolicity of nets is related to p-hyperbolicity of open sets and vice versa. Although some parts of the following could be found in Kanai's paper [K2], we include all details for the convenience of the reader. We assume that S', Ω, and S are as in 5.2. First we attach to each continuous function $u \in W^1_{p,\mathrm{loc}}(\Omega)$ a function u^* of $S' \cup \partial S'$ by setting

$$u^*(q) = \fint_{B(q,4\kappa)} u\, dm\,. \tag{5.3}$$

Then we have the following.

Lemma 5.4. *Let u and u^* be as above. Then there exists a constant $c = c(n, \kappa, K, p)$ such that*

$$\sum_{q\in S'} |Du^*(q)|^p \le c \int_\Omega |\nabla u|^p\, dm\,.$$

PROOF. In the following c will be a positive constant which is not necessarily the same at each occurence but, however, may depend only on n, κ, K, and p. For each $q \in S' \cup \partial S'$, the volume estimate $|B(q, 4\kappa)| \leq V_K(4\kappa)$, Hölder's inequality, and the local Poincaré inequality (2.12) imply that

$$V_K(4\kappa)^{p-1} \int_{B(q,4\kappa)} |\nabla u|^p \, dm \geq \left(\int_{B(q,4\kappa)} |\nabla u| \, dm \right)^p$$
$$\geq c \left(\int_{B(q,4\kappa)} |u(x) - u^*(q)| \, dm \right)^p .$$

Let $q \in S'$ and $q' \in N_q$. Then $q' \in S' \cup \partial S'$, and by the previous estimate,

$$\begin{aligned}
\int_{B(q,7\kappa)} |\nabla u|^p \, dm &\geq \frac{1}{2} \left(\int_{B(q,4\kappa)} |\nabla u|^p \, dm + \int_{B(q',4\kappa)} |\nabla u|^p \, dm \right) \\
&\geq c \Bigg(\Big(\int_{B(q,4\kappa)} |u(x) - u^*(q)| \, dm \Big)^p \\
&\qquad + \Big(\int_{B(q',4\kappa)} |u(x) - u^*(q')| \, dm \Big)^p \Bigg) \qquad (5.5) \\
&\geq c \left(\int_{B(q,4\kappa) \cap B(q',4\kappa)} |u^*(q) - u^*(q')| \, dm \right)^p \\
&\geq c (v_0 \kappa^n)^p \, |u^*(q) - u^*(q')|^p .
\end{aligned}$$

We recall that $\# N_q \leq c_4$, with c_4 independent of q. Using this fact and (5.1), we obtain from (5.5) that

$$|Du^*(q)|^p \leq c \int_{B(q,7\kappa)} |\nabla u|^p \, dm .$$

By (2.15), every point $x \in \Omega$ belongs to at most c balls $B(q, 7\kappa)$, where $q \in P$ and c is independent of x. Thus

$$\sum_{q \in S'} \int_{B(q,7\kappa)} |\nabla u|^p \, dm \leq c \int_{\Omega} |\nabla u|^p \, dm ,$$

and so

$$\sum_{q\in S'} |Du^*(q)|^p \le c \int_\Omega |\nabla u|^p \, dm$$

as claimed.

Conversely, for each function $\bar{v}$ of $S \cup \partial S$, we define a function $v \in C^\infty(\Omega)$ as follows. For each $q \in P$, we choose functions $\eta_q \in C_0^\infty(M^n)$ such that $0 \le \eta_q \le 1$, $\eta_q = 1$ in $B(q,\kappa)$, $\eta_q = 0$ outside $B(q, 3\kappa/2)$, and $|\nabla \eta_q| \le 4\kappa^{-1}$. For $x \in \Omega$, we set $P_x = P \cap B(x, 2\kappa)$. We remark that $\#P_x \le c_4 + 1$ since $P_x \subset N_q \cup \{q\}$ for some q. Then we define $v : \Omega \to \mathbb{R}$ by

$$v(x) = \frac{\sum_{q\in P_x} \bar{v}(q)\eta_q(x)}{\sum_{q\in P_x} \eta_q(x)} . \tag{5.6}$$

The function v will depend on the choice of η_q. Observe that $\sum_{q\in P_x} \eta_q(x) \ge 1$ since every x belongs to at least one $B(q,\kappa)$, with $q \in P_x$.

Lemma 5.7. *If $\bar{v}$ and v are as above, then $v \in C^\infty(\Omega)$ and there exists a constant $c = c(n,\kappa,K,p)$ such that*

$$\int_\Omega |\nabla v|^p \, dm \le c \sum_{q\in S} |D\bar{v}(q)|^p .$$

PROOF. Again c may vary even within a line but it can depend at most on n, κ, K, and p. Let $x \in \Omega$. First we note that $P_x \subset S \cup \partial S$, and thus $\bar{v}(q)$ is defined for every $q \in P_x$. This means that v is defined in Ω. To show that v is a C^∞-function, we observe that $d(y,q) \ge 3\kappa/2$ if $y \in \Omega \cap B(x,\kappa/2)$ and $q \in P_y \setminus P_x$. This implies that $\eta_q(y) = 0$, and therefore it is sufficient to consider only functions η_q, with $q \in P_x$, in the definition of $v(y)$ if $y \in \Omega \cap B(x,\kappa/2)$. Hence $v \in C^\infty(\Omega)$. To show the inequality in the claim, we abbreviate

$$\xi_q(x) = \frac{\eta_q(x)}{\sum_{q'\in P_x} \eta_{q'}(x)} .$$

Then

$$\begin{aligned} |\nabla \xi_q(x)| \le{}& |\nabla \eta_q(x)| \Big(\sum_{q'\in P_x} \eta_{q'}(x) \Big)^{-1} \\ &+ \eta_q(x) \Big(\sum_{q'\in P_x} \eta_{q'}(x) \Big)^{-2} \sum_{q'\in P_x} |\nabla \eta_{q'}(x)| \\ \le{}& 4\kappa^{-1}(1+c), \end{aligned}$$

where $c = \sup\{\#P_y : y \in M^n\}$. Suppose that $x \in B(q,\kappa) \cap \Omega$, where $q \in S$. Then

$$
\begin{aligned}
\nabla v(x) &= \sum_{q' \in P_x} \bar{v}(q')\, \nabla \xi_{q'}(x) \\
&= \sum_{q' \in N_q \cup \{q\}} \bar{v}(q')\, \nabla \xi_{q'}(x) \qquad (5.8)\\
&= \sum_{q' \in N_q} \big(\bar{v}(q') - \bar{v}(q)\big) \nabla \xi_{q'}(x)\,.
\end{aligned}
$$

On the first line in (5.8) we used the facts that $P_x \subset N_q \cup \{q\}$ if $x \in B(q,\kappa)$, and that $\xi_{q'}(x) = 0$ if $q' \notin P_x$. For the last equality in (5.8), observe that

$$
\sum_{q' \in N_q \cup \{q\}} \bar{v}(q)\, \nabla \xi_{q'}(x) = \bar{v}(q)\, \nabla \Big(\sum_{q' \in N_q \cup \{q\}} \xi_{q'}(x) \Big) = 0
$$

since $\sum_{q' \in N_q \cup \{q\}} \xi_{q'}(x) = 1$. It follows from (5.8) and from the uniformness of P that, for every $x \in B(q,\kappa)$,

$$
\begin{aligned}
|\nabla v(x)|^p &\le c \Big(\sum_{q' \in N_q} |\bar{v}(q') - \bar{v}(q)| \Big)^p \\
&\le c \Big((\#N_q)^2 \max_{q' \in N_q} |\bar{v}(q') - \bar{v}(q)|^2 \Big)^{p/2} \\
&\le c \Big(\sum_{q' \in N_q} |\bar{v}(q') - \bar{v}(q)|^2 \Big)^{p/2} = c\, |D\bar{v}(q)|^p\,.
\end{aligned}
$$

Finally, $\Omega \subset \cup_{q \in S} B(q,\kappa)$ and $|B(q,\kappa)| \le V_K(\kappa)$, and therefore

$$
\int_\Omega |\nabla v(x)|^p\, dm \le \sum_{q \in S} \int_{B(q,\kappa} |\nabla v(x)|^p\, dm \le c\, V_K(\kappa) \sum_{q \in S} |D\bar{v}(q)|^p\,.
$$

The lemma is proved.

Lemma 5.9. *Let S', Ω, and S be as in Lemma* 5.2. *Then Ω is p-hyperbolic if S' is p-hyperbolic. Conversely, if Ω is p-hyperbolic, then S is p-hyperbolic.*

PROOF. Let $\{B_i\}$ be an exhaustion of M^n. Suppose first that S' is p-hyperbolic. Then there exists a finite non-empty set $E \subset S' \cup \partial S'$ such that $\operatorname{cap}_p(E, \infty; S') > 0$. We set $C = \cup_{q \in E} \bar{B}(q, 4\kappa)$. Let $u \in L^1_p(\Omega)$ be continuous in $\Omega \cup C \cup \bar{\Omega} \setminus B_i$ such that $u = 1$ in C and $u = 0$ in $\bar{\Omega} \setminus B_i$ for some (not fixed) i. We observe that $1 - u$ is admissible for $(C, \bar{\Omega} \setminus B_i)$. We define $u^* : S' \cup \partial S' \to \mathbb{R}$ by (5.3). Then u^* is admissible for $(E, \infty; S')$, that is, $u^* = 1$ in E and it has a finite support. By Lemma 5.4,

$$\int_\Omega |\nabla u|^p \, dm \geq c \sum_{q \in S'} |Du^*(q)|^p \geq c \operatorname{cap}_p(E, \infty; S').$$

Taking the infimum over all such functions u (and i) gives

$$\operatorname{cap}_p(C, \infty; \Omega) \geq c \operatorname{cap}_p(E, \infty; S') > 0,$$

and so Ω is p-hyperbolic.

For the proof of the second claim, we choose a compact set $C \subset \Omega$ such that $\operatorname{cap}_p(C, \infty; \Omega) > 0$. Let $E = \{q \in S \cup \partial S : \ d(q, C) < 2\kappa\}$. Then E is finite and non-empty. Let $\bar{v}$ be an admissible function for $(E, \infty; S)$. We define a function $v \in C^\infty(\Omega)$ by (5.6). Since $\bar{v}$ has a finite support, $v = 0$ in $\Omega \setminus K$ for some compact set $K \subset M^n$. For each $x \in C$,

$$v(x) = \frac{\sum_{q \in P_x} \bar{v}(q)\, \eta_q(x)}{\sum_{q \in P_x} \eta_q(x)} = 1$$

since $P_x \subset E$ and $\bar{v}(q) = 1$ in E. Hence $1 - v$ is admissible for $(C, \bar{\Omega} \setminus B_i; \Omega)$ whenever $K \subset B_i$. By Lemma 5.7,

$$\sum_{q \in S} |D\bar{v}(q)|^p \geq c \int_\Omega |\nabla v|^p \, dm \geq c \operatorname{cap}_p(C, \infty; \Omega) > 0.$$

Since this holds for all admissible functions $\bar{v}$ we get

$$\operatorname{cap}_p(E, \infty; S) \geq c \operatorname{cap}_p(C, \infty; \Omega) > 0.$$

This ends the proof.

For the next two lemmas, let $\psi: M^n \to N^\nu$ be a rough isometry. Then it induces a rough isometry $\varphi: P \to Q$ with respect to the combinatorial metrics of P and Q. Let a and b be the constants of φ in (2.12).

Lemma 5.10. *Let $S \subset P$ be connected, and let $S' = \{q \in Q : \delta(q, \varphi(S \cup \partial S)) \le a + b\}$. Then S' is connected. Furthermore, let v be a function of $S' \cup \partial S'$ and $u = v \circ \varphi$. Then*

$$\sum_{x \in S} |Du(x)|^p \le c \sum_{q \in S'} |Dv(q)|^p ,$$

where c is independent of v.

PROOF. Let q and q' be are any two points in S' and let x and y be points in $S \cup \partial S$ such that $\delta(q, \varphi(x)) \le a + b$ and $\delta(q', \varphi(y)) \le a + b$. Hence there exist paths in S' from q to $\varphi(x)$ and from q' to $\varphi(y)$, respectively. Since also $S \cup \partial S$ is connected, there exists a path $q_0 = x, q_1, \ldots, q_l = y$ in $S \cup \partial S$. For every $i = 0, 1, \ldots, l-1$, $\delta(\varphi(q_i), \varphi(q_{i+1})) \le a + b$, and thus there is a path in S' from $\varphi(q_i)$ to $\varphi(q_{i+1})$ for every $i = 0, 1, \ldots, l-1$. Hence S' is connected. To prove the other part of the claim, let v be any function in $S' \cup \partial S'$. We abbreviate $c_6 = a + b$. Let $x \in S$ and $y \in S \cup \partial S$ be such that $\delta(x, y) = 1$. Then $\delta(\varphi(x), \varphi(y)) \le c_6$. Thus there is a path $q_0 = \varphi(x), q_1, \ldots, q_\ell = \varphi(y)$ in S' of length $\ell \le c_6$. Now $u(x) - u(y) = v(q_0) - v(q_1) + v(q_1) - \cdots + v(q_{\ell-1}) - v(q_\ell)$, and therefore

$$|u(x) - u(y)|^p \le c_6^p \sum_{i=0}^{\ell-1} |v(q_i) - v(q_{i+1})|^p .$$

Since Q is uniform, the number of points $q \in Q$, with $\delta(q, q_0) \le c_6$, is bounded by a constant which is independent of q_0. Hence we get, by also using (5.1), an estimate

$$|u(x) - u(y)|^p \le c \max_{\delta(q, \varphi(x)) \le c_6} |Dv(q)|^p .$$

Now the uniformness of P implies that

$$|Du(x)|^p \le c_7 \max_{\delta(q, \varphi(x)) \le c_6} |Dv(q)|^p . \tag{5.11}$$

Next we sum both sides of (5.11) over all $x \in S$. Then some terms $|Dv(q)|^p$ may appear several times on the right hand side. However, since φ is a rough isometry and P is uniform, there is a constant c_8 such that, for each $x \in S$, there can be at most c_8 points $x' \in S$ with $\delta(\varphi(x), \varphi(x')) \le 2c_6$. Hence

$$\sum_{x \in S} |Du(x)|^p \le c_7 \sum_{x \in S} \max_{\delta(q, \varphi(x)) \le c_6} |Dv(q)|^p \le c_7\, c_8 \sum_{q \in S'} |Dv(q)|^p .$$

Lemma 5.12. *Suppose that $\Omega \subset M^n$ is connected and p-hyperbolic. Let $S = \{q \in P : d(q, \Omega) < \kappa\}$. Then the set $\Omega' = \{y \in N^\nu : d(y, \varphi(S \cup \partial S)) < c_9\}$, where $c_9 = \max\{3\kappa(a+b), a+b\} + 7\kappa$, is a domain and p-hyperbolic.*

PROOF. Clearly Ω is open. To show that it is connected, let x and y be any points of Ω'. Then there are points q and q' in $S \cup \partial S$ such that $x \in B(\varphi(q), c_9)$ and $y \in B(\varphi(q'), c_9)$. Both of these balls are contained in Ω'. Futhermore, since S is a connected subnet by (the proof of) Lemma 5.2, so does $S \cup \partial S$. Thus there exists a path in $S \cup \partial S$, say $q_0 = q, q_1, \ldots, q_\ell = q'$, from q to q'. By (2.14),

$$d(\varphi(q_i), \varphi(q_{i+1})) \le 3\kappa\, \delta(\varphi(q_i), \varphi(q_{i+1})) \le 3\kappa(a+b) < c_9\,,$$

and therefore $\cup_{i=0}^{\ell} B(\varphi(q_i), c_9)$ is a connected open subset of Ω' containing x and y. This implies that Ω' is a domain. It remains to prove that Ω' is p-hyperbolic. First we observe that S is p-hyperbolic by Lemma 5.9. Thus there exists a finite set $E \subset S \cup \partial S$ such that $\operatorname{cap}_p(E, \infty; S) > 0$. Let v be an admissible function in $S' \cup \partial S'$ for $(\varphi(E), \infty; S')$, that is, v has a finite support and $v = 1$ in $\varphi(E)$. For each $q \in S \cup \partial S$, we set $u(q) = v(\varphi(q))$. Then $u = 1$ in E. Since the support of v is finite, there is a point $\tilde{q} \in S$ and $\delta_0 > 0$ such that $u(q) = v(\varphi(q)) = 0$ if $\delta(\varphi(\tilde{q}), \varphi(q)) \ge \delta_0$. Since φ is a rough isometry, there exists $\delta_1 > 0$ such that, $\delta(\varphi(\tilde{q}), \varphi(q)) \ge \delta_0$, and so $u(q) = 0$, if $\delta(\tilde{q}, q) \ge \delta_1$. The uniformness of P implies that there can be only finitely many points $q \in P$ with $\delta(\tilde{q}, q) < \delta_1$. Hence the support of u is finite and u is admissible for $(E, \infty; S)$. Lemma 5.10 then implies that

$$\sum_{q \in S'} |Dv(q)|^p \ge c \sum_{x \in S} |Du(x)|^p \ge c \operatorname{cap}_p(E, \infty; S) > 0\,.$$

This is true for every admissible v. Hence $\operatorname{cap}_p(\varphi(E), \infty; S') > 0$ and S' is p-hyperbolic. It follows from Lemma 5.9 that the 7κ-neighborhood of $S' \cup S'$ is p-hyperbolic. Hence also Ω' is p-hyperbolic as a larger set.

We are now ready to prove the main theorem.

Theorem 5.13. *Let M^n and N^ν be complete Riemannian manifolds with bounded geometry and roughly isometric to each other. Then M^n has the Liouville D_p-property if and only if so does N^ν.*

PROOF. Fix $\kappa \leq \min\{\operatorname{inj} M^n/2, \operatorname{inj} N^\nu/2\}$. Let P and Q be κ-nets in M^n and in N^ν, respectively. Since M^n and N^ν are roughly isometric, there exists an induced rough isometry $\varphi: P \to Q$. Let $\psi: Q \to P$ be a rough inverse of φ. Suppose that M^n does not have the Liouville D_p-property. By [H1, Section 5], there exists a non-constant bounded p-harmonic function u in M^n with $\int_{M^n} |\nabla u|^p \, dm < +\infty$. We normalize u such that $\inf_{M^n} u = 0$ and $\sup_{M^n} u = 1$. Since being roughly isometric is an equivalence relation it is sufficient to prove that also N^ν admits a non-constant p-harmonic function with L^p-integrable gradient. For each $a, b \in]0,1[$, we denote by Ω_a and Ω^b any component of sets $\{x \in M^n : u(x) < a\}$ and $\{x \in M^n : u(x) > b\}$, respectively. Then Ω_a and Ω^b are p-hyperbolic domains. Let $0 < s < 1/4$ and $3/4 < t < 1$. We write $S_s = \{q \in P : d(q, \Omega_s) < \kappa\}$ and $S^t = \{q \in P : d(q, \Omega^t) < \kappa\}$. Then the sets $D_s = \{x \in N^\nu : d(x, \varphi(S_s \cup \partial S_s)) < c_9\}$ and $D^t = \{x \in N^\nu : d(x, \varphi(S^t \cup \partial S^t)) < c_9\}$ are p-hyperbolic by Lemma 5.12. We claim that, for some $0 < s < 1/4$ and $3/4 < t < 1$, $\operatorname{cap}_p(\bar{D}_1, \bar{D}^2; N^\nu) < +\infty$ which then proves the theorem by Theorem 4.6. Let

$$v = \max\left\{0, \min\{2(u - 1/4), 1\}\right\}.$$

Now $v = 0$ in $\Omega_{1/4}$ and $v = 1$ in $\Omega^{3/4}$. Then we set, for each $q \in P$,

$$v^* = \fint_{B(q,4\kappa)} v \, dm.$$

Next we define $\bar{w} : Q \to P$ by $\bar{w} = v^* \circ \psi$, where ψ is a rough inverse of φ. Finally, we attach to $\bar{w}$ a function $w \in C^\infty(N^\nu)$ as in (5.6). By lemmas 5.4, 5.7 and 5.10, we have

$$\int_{N^\nu} |\nabla w|^p \, dm \leq c \int_{M^n} |\nabla v|^p \, dm \leq 2^p c \int_{M^n} |\nabla u|^p \, dm < +\infty.$$

It remains to show that w is admissible for $(\bar{D}_s, \bar{D}^t; N^\nu)$ if s and t are properly chosen. Recall that

$$w(y) = \frac{\sum_{q \in Q_y} \bar{w}(q)\, \eta_q(y)}{\sum_{q \in Q_y} \eta_q(y)},$$

where $Q_y = Q \cap B(y, 2\kappa)$ and $\eta_q \in C_0^\infty(N^\nu)$ such that $\eta_q = 1$ in $B(q, \kappa)$ and $\eta_q = 0$ outside $B(q, 3\kappa/2)$. Since ψ is a rough inverse of φ, there exists a constant c_{10} depending only on a, b, and κ such that $d(x, \psi(\varphi(x))) \le c_{10}$ for every $x \in P$. Let $q \in Q$ be such that $d(q, \bar{D}_s) \le 2\kappa$. Then there is $y \in \bar{D}_s$, with $d(q, y) \le 4\kappa$. Moreover, $d(y, \varphi(z)) \le 2c_9$ for some $z \in S_s \cup \partial S_s$, and so $d(q, \varphi(z)) \le 2c_9 + 4\kappa$. Since ψ is a rough isometry, $d(\psi(q), \psi(\varphi(z))) \le c_{11}$, where c_{11} depends only on a, b, and κ. Hence $d(\psi(q), z) \le c_{10} + c_{11}$. On the other hand, there is $z' \in S_s$ such that $d(z, z') \le 3\kappa$. Finally, $d(z, x) \le \kappa$ for some $x \in \Omega_s$. Hence, for every $y \in B(\psi(q), 4\kappa)$,

$$d(y, x) \le c_{10} + c_{11} + 8\kappa \stackrel{\text{def}}{=} c_{12}\,, \tag{5.14}$$

where c_{12} is independent of q and x. Thus we can attach to each $q \in Q$, with $d(q, \bar{D}_s) \le 2\kappa$, a point $x \in \Omega_s$ such that $d(y, x) \le c_{12}$ whenever $y \in B(\psi(q), 4\kappa)$. By Theorem 3.10,

$$d(\partial\Omega_s, \partial\Omega_{1/4}) > c_4\, r_0 \log \frac{1}{4s} - r_0\,.$$

Hence we can choose $0 < s < 1/4$ such that

$$d(\partial\Omega_s, \partial\Omega_{1/4}) \ge 2\, c_{12}\,. \tag{5.15}$$

It follows from (5.14) and (5.15) that $B(\psi(q), 4\kappa) \subset \Omega_{1/4}$ whenever $q \in Q$, with $d(q, \bar{D}_s) \le 2\kappa$. But this implies that $\bar{w}(q) = v^*(\psi(q)) = 0$ for such q, and so $w(x) = 0$ for every $x \in \bar{D}_s$. Similarly, we can choose $3/4 < t < 1$ such that $d(\partial\Omega^t, \partial\Omega^{3/4}) \ge 2c_{12}$. Then $B(\psi(q), 4\kappa) \subset \Omega^{3/4}$ if $q \in Q$ and $d(q, \bar{D}^t) \le 2\kappa$. Hence $w(x) = 1$ for every $x \in \bar{D}^t$. We have showed that w is admissible for $(\bar{D}_s, \bar{D}^t; N^\nu)$ which then proves the theorem.

FINAL REMARK 5.16. In [H1] we proved that there exists Green's function for (2.22) on M^n, that is, a certain positive solution of

$$-\operatorname{div} \mathcal{A}(\nabla g) = \delta_y\,,$$

where $y \in M^n$ and $\mathcal{A}$ is of type p, if and only if $\operatorname{cap}_p(M^n, C) > 0$ for some compact set $C \subset M^n$. In the light of the previous consideration, it is clear that Kanai's Theorem [K2, Theorem 1] is true for every $1 < p < \infty$, that is, for a fixed $p \in]1, \infty[$, the existence of Green's function for equation (2.22) is preserved under rough isometries between Riemannian manifolds of bounded geometry.

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