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The Hilbert transform and maximal function along nonconvex curves in the plane

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1. Introduction.

In this paper we study the Hilbert transform and maximal function related to a curve in $\mathbb{R}^2$. For $\Gamma(t) = (t, \gamma(t))$ with $\gamma(0) = 0$, we define the Hilbert transform associated to $\Gamma(t)$ by

$$\mathcal{H}_\Gamma f(x) = \int_{-1}^{1} f(x - \Gamma(t)) \, \frac{dt}{t} \, . \tag{1}$$

Similarly we define the maximal function by the formula

$$\mathcal{M}_\Gamma f(x) = \sup_{0<h\leq 1} \frac{1}{h} \int_0^h |f(x - \Gamma(t))| \, dt \, . \tag{2}$$

We are interested in obtaining L^p estimates of the form

$$\|\mathcal{H}_\Gamma f\|_p \leq A_p \, \|f\|_p \, , \tag{3}$$

and

$$\|\mathcal{M}_\Gamma f\|_p \leq A_p \, \|f\|_p \, . \tag{4}$$

A first stage in this study was completed in the 1970's due to the efforts of Nagel, Rivière, Stein and Wainger. Their work led to the following theorem (see [SW]).

Theorem A. *Suppose* $\Gamma(t)$ *is* C^∞ *and the curvature of* $\Gamma(t)$ *does not vanish to infinite order at the origin. Then*

$$\|\mathcal{H}_\Gamma f\|_p \leq A_p \|f\|_p , \qquad 1 < p < \infty, \tag{5}$$

and

$$\|\mathcal{M}_\Gamma f\|_p \leq A_p \|f\|_p , \qquad 1 < p \leq \infty. \tag{6}$$

Much effort has been devoted to the study of $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ without the assumption that the curvature of Γ does not vanish to infinite order. See for example, [CCVWW], [C1], [C2], [CCC], [DR], [NVWW1] and [NVWW2]. In particular, we have the following theorems (see [CCC]).

Theorem B. *Assume* $\gamma(t)$ *is convex for* $t > 0$. *If for some* $C > 1$, $\gamma'(Ct) \geq 2\gamma'(t)$ *for* $t > 0$, *then*

$$\|\mathcal{M}_\Gamma f\|_p \leq A_p \|f\|_p , \qquad 1 < p \leq \infty.$$

If in addition, $\Gamma(t)$ *is even or odd, then*

$$\|\mathcal{H}_\Gamma f\|_p \leq A_p \|f\|_p , \qquad 1 < p < \infty.$$

The hypothesis of the next theorem is expressed in terms of the functional $h(t) = t\gamma'(t) - \gamma(t)$ (see [NVWW1] and [NVWW2]).

Theorem C. *Assume* $\gamma(t)$ *is convex for* $t > 0$. *If for some* $C > 1$, $h(Ct) \geq 2h(t)$ *for* $t > 0$, *then*

$$\|\mathcal{M}_\Gamma f\|_2 \leq A \|f\|_2 .$$

If in addition $\Gamma(t)$ *is odd, then*

$$\|\mathcal{H}_\Gamma f\|_2 \leq A \|f\|_2 .$$

In this paper we wish to remove the convexity assumption. The fact that there should be some positive results is suggested by examples worked out by Wright, [W]. In [W], positive results are obtained if for example $\gamma(t)$ is $t^\alpha \sin(\log\log(1/t))$, $t^\alpha \sin(1/t^\beta)$ and $e^{-1/t^2}\sin(e^{1/t})$. These examples and Theorem B suggest the following provisional hypothesis. Let

$$u(t) = \sup_{0\le s\le t} |\gamma'(s)|$$

and assume for $t > 0$, $u(Ct) \ge 2\,u(t)$ for some $C > 1$. The fact that this hypothesis must be modified can be seen by considering certain "staircase" examples. That is, we define $\gamma(t) = 2^{-2^k}$ on $E_k = [2^{-k}, 2^{-k}(1+\delta_k)]$ and make γ linear on the complementary intervals, $F_k = [2^{-k}(1+\delta_k), 2^{-k+1}]$. Here $0 \le \delta_k \le 1$. These examples are a slight variant of examples considered in [SW].

For these examples we calculate $\mathcal{M}_\Gamma$ on the characteristic function of a rectangle with corners at $(-1,0)$, $(0,0)$, $(-1,-\varepsilon)$ and $(0,-\varepsilon)$. This calculation which is similar to that in [SW] shows that $\mathcal{M}_\Gamma$ is not bounded in L^p if $\sum \delta_k^p = +\infty$. It is easy to see that in these examples $u(2t) \ge 2\,u(t)$, and so the provisional hypothesis must be modified. Furthermore, it is not difficult to see that if $\sum \delta_k^p < +\infty$, $\mathcal{M}_\Gamma$ is bounded in L^p. In fact

$$\begin{aligned}\mathcal{M}_\Gamma f(x) &\le \sup_{0<h\le 1} \int_{\substack{0\\ t\in\cup E_k}}^h |f(x-\Gamma(t))|\,dt + \sup_{0<h\le 1} \frac{1}{h}\int_{\substack{0\\ t\in\cup F_k}}^h |f(x-\Gamma(t))|\,dt \\ &= M_1 f(x) + M_2 f(x)\,.\end{aligned}$$

M_1 can be shown to be bounded in L^p by using a square function argument as in [C2] while M_2 is bounded in L^p by arguments in [CCC]. In fact the argument shows that $\mathcal{M}_\Gamma$ is bounded in L^p no matter how Γ is defined on the intervals $\{E_k\}$. Thus these staircase examples suggest that we must add some hypothesis but that we need not require any hypothesis on a suitably small set E. The staircase examples further suggest that if $I_k = \{t :\ 2^{-k} \le t \le 2^{-k+1}\}$, then the correct assumption on the size of E should be that $\sum (2^k\,|I_k \cap E|)^p < +\infty$.

It is interesting to note that although $\sum (2^k\,|I_k \cap E|)^p < +\infty$ is the correct assumption on the size of E for the maximal function, it is not the correct size for the Hilbert transform. To see this let us consider the following variant of the above staircase example. That is, define $\gamma(t) = 9^{-2k}$ on $E_k = [2^{-k}, 2^{-k}(1+\delta_k)]$ and make γ linear on the complementary intervals $F_k = [2^{-k}(1+\delta_k), 2^{-k+1}]$. Extend γ as an odd function on $[-1,1]$ and write

$$\begin{aligned}\mathcal{H}_\Gamma f(x) &= \int\limits_{\cup E_k^1} f(x-\Gamma(t))\,\frac{dt}{t} + \int\limits_{\cup F_k^1} f(x-\Gamma(t))\,\frac{dt}{t} \\ &= H_1 f(x) + H_2 f(x)\,,\end{aligned}$$

where $E_k^1 = E_k \cup E_{-k}$ and $F_k^1 = F_k \cup F_{-k}$. As before H_2 is bounded in all L^p $(p>1)$ by arguments in [CCC]. On the other hand, H_1 (and thus $\mathcal{H}_\Gamma$) is bounded in all L^p $(p>1)$ if and only if $\sum \delta_k < +\infty$. It is easy to see that if $\sum \delta_k < +\infty$, then H_1 is bounded in all L^p $(p>1)$ by Minkowsky's inequality for integrals. However suppose that $\sum \delta_k = +\infty$ and consider the multiplier for H_1,

$$m(\xi,\eta) = \sum \int\limits_{E_k} \sin(\xi t + \eta\,\gamma(t))\,\frac{dt}{t}\,.$$

Set $\xi = 0$ and $\eta = (\pi/2)\,9^{2N}$ for some large N and note that

$$m(0, 9^{2N}\,\frac{\pi}{2}) = \sum_{k<N} \int\limits_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t} + \sum_{k\ge N} \int\limits_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t}\,.$$

Since

$$\sum_{k\ge N} \int\limits_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t} \le C\,9^{2N} \sum_{k\ge N} \delta_k\, 9^{-2k} \le C\,9^{2N} \sum_{k\ge N} 9^{-2k} \le C$$

and

$$\sum_{k<N} \int\limits_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t} = \sum_{k<N} \int\limits_{E_k} \frac{dt}{t} \ge \frac{1}{2} \sum_{k<N} \delta_k\,,$$

we see that $m(\xi,\eta)$ is an unbounded function and so H_1 is not bounded in L^2 and hence unbounded in all L^p. This example therefore suggests that the correct assumption on the size of E for the Hilbert transform is $\sum 2^k\,|I_k \cap E| < +\infty$.

Since we want to impose no condition on $\gamma(t)$ in E, we modify $u(t)$ to

$$v(t) = \sup_{\substack{s<t \\ s\notin E}} |\gamma'(s)|\,.$$

If $\gamma(t)$ is convex, $\gamma'(t)$ is monotone and $\gamma(t) \le t\,\gamma'(t)$. So if we set

$$\phi(t) = \sup_{\substack{s<t \\ s\notin E}} |\gamma(s)|\,,$$

we add two more provisional hypotheses, namely that outside of E,

$$\gamma'(t) \ge \varepsilon\, v(t) \qquad \text{and} \qquad \phi(t) \le C\,t\,v(t)\,.$$

Thus one has modified the provisional hypothesis to the following:

A) There is an exceptional set E such that for the maximal function, $\sum (2^k\,|I_k \cap E|)^p < +\infty$ and for the Hilbert transform,

$$\sum 2^k\,|I_k \cap E| < +\infty\,.$$

B) Outside of E, $\phi(t) \le C\,t\,v(t)$.

C) Outside of E, $v(\lambda t) \ge 2\,v(t)$ for some $\lambda > 1$.

D) Outside of E, $|\gamma'(t)| \ge \varepsilon\, v(t)$.

Unfortunately, for the example $\gamma(t) = t^k \sin(1/t)$, the hypothesis D) is not satisfied. So we replace D) by

D′) Outside of E, $|\gamma(t)| + t\,|\gamma''(t)| \ge \varepsilon\, v(t)$.

It turns out, as we shall see by an example later on, that A), B), C) and D′) do not suffice for the L^2 boundedness of the Hilbert transform (if Γ is extended to be an odd curve).

If one attemps to prove a positive result, one naturally divides I_k into various subintervals; subintervals which belong to E, subintervals which do not belong to E and $|\gamma'(t)| \ge \varepsilon\, v(t)$, and subintervals which do not belong to E and $|\gamma'(t)| < \varepsilon\, v(t)$ but γ'' is large. Thus I_k is partitioned into a possibly large number of subintervals. Our examples show that at least in the case of the Hilbert transform, our hypothesis must depend qualitatively on the number of such subintervals. If the number of subintervals into which we have divided I_k is N_k, we might then expect to modify B) to

B') On $I_k \setminus E$, $t\,v(t) \ge \varepsilon_0\, N_k\, \phi(t)$.

This latter assumption however is not satisfied for certain examples like $\gamma(t) = e^{-1/t^2} \sin e^{1/t}$. Examples such as this can be incorporated by modifying B') to $t\,v(t/2) \ge \varepsilon_0\, N_k\, \phi(t/2)$. It turns out that the proof requires one additional hypothesis, namely that the sequence $\{2^k N_k\}$ is sufficiently spread out. In fact we will assume that for all n,

$$\sum_{k\geq n} \frac{1}{2^k N_k} \leq C \frac{1}{2^n N_n} \tag{7}$$

where C is independent of n. (7) holds whenever $\{2^k N_k\}$ forms an increasing lacunary sequence. However condition (7) allows for situations where the N_k's might not be monotone. With the above remarks in mind, the following theorem seems reasonable.

Main Theorem. *Let* $\Gamma(t) = (t, \gamma(t)) \in C^2(0,1]$ *with* $\gamma(0) = \gamma'(0) = 0$. *Suppose*

$$I_k = E_k \cup F_k \cup G_k$$

is a disjoint union with F_k *and* G_k *each a union of at most* N_k *open intervals.*

Assume that for some $\varepsilon_0 > 0$,

$$v(\lambda t) \geq 2\, v(t) \qquad \text{for some } \lambda > 1 \text{ on } I_k \setminus E_k \tag{8}$$

and

$$t\, v\left(\frac{t}{2}\right) \geq \varepsilon_0\, N_k\, \phi\left(\frac{t}{2}\right) \qquad \text{on } I_k \setminus E_k\,. \tag{9}$$

Suppose also that for some ε_1 *and* $\varepsilon_2 > 0$,

$$|\gamma'(t)| > \varepsilon_1\, v(t) \quad \text{and} \quad |t\, \gamma''(t)| < \varepsilon_2\, N_k\, v\left(\frac{t}{2}\right) \quad \text{on } F_k \tag{10}$$

and

$$|t\, \gamma''(t)| > \varepsilon_2\, N_k\, v\left(\frac{t}{2}\right) \quad \text{on } G_k\,. \tag{11}$$

Finally assume that (7) *holds. Then if* $\sum_k (2^k\, |E_k|)^p < +\infty$,

$$\|\mathcal{M}_\Gamma f\|_p \leq A_p\, \|f\|_p\,. \tag{12}$$

Also if $\Gamma(t)$ *is extended to the interval* $[-1,1]$ *as an even or odd curve and* $\sum_k 2^k\, |E_k| < +\infty$,

$$\|\mathcal{H}_\Gamma f\|_p \leq A_p\, \|f\|_p\,, \qquad 1 < p < \infty\,. \tag{13}$$

We add six further remarks.

REMARK 1. In view of Theorem C we might be tempted to replace the hypothesis $v(Ct) \geq 2\,v(t)$ by $w(Ct) \geq 2\,w(t)$ (at least in the case $p = 2$) where

$$w(t) = \sup_{\substack{s \leq t \\ s \notin E}} |s\,\gamma'(s) - \gamma(s)|\,.$$

We shall show by an example that this can not be done.

REMARK 2. Examples show that the complement of the set $\{t \in I_k : |\gamma'(t)| < \varepsilon\, u(t)\}$ is too large to be contained in E_k.

REMARK 3. The staircase curves can be modified to show that for any p_0, $1 < p_0 < \infty$, there is a smooth curve $\Gamma(t)$ so that $\mathcal{M}_\Gamma$ is bounded in L^p for $p \geq p_0$ and unbounded for $p < p_0$. Other examples have been pointed out by M. Wierdl.

REMARK 4. We are not sure if the conclusion of the main theorem holds if $v(t)$ is replaced by $u(t)$ and the N_k's are omitted. However, the hypothesis of such a theorem would not be satisfied by staircase examples with very steep slopes in E, for which we know the conclusion is true.

REMARK 5. We do not know whether the quantitative hypothesis on N_k is necessary for the conclusions concerning the maximal function.

REMARK 6. For convex curves, the hypotheses of the main theorem are satisfied whenever γ' is infinitesimal doubling, *i.e.* $\gamma'(t) \leq C\,t\,\gamma''(t)$.

2. Proof of the main theorem.

We consider first the maximal function. Let us first reduce the problem to obtaining the L^p estimate for

$$Mf = \sup_k |M_k f|\,,$$

where

$$M_k f(x,y) = 2^k \int_{I_k \setminus E_k} f(x - t, y - \gamma(t))\,dt\,,$$

by using the square function argument alluded to above. In fact if μ_k denotes the positive measure such that

$$\mu_k(f) = \frac{1}{|E_k|}\int_{E_k} f(t,\gamma(t))\,dt\,,$$

then for $f \geq 0$, $\mathcal{M}_\Gamma f \leq C\left(Mf + \sup_k (2^k\,|E_k|\,f * \mu_k)\right)$. But

$$\begin{aligned}\|\sup_k (2^k\,|E_k|\,f * \mu_k)\|_p &\leq C\Big(\sum_k \|2^k\,|E_k|\,f * \mu_k\|_p^p\Big)^{1/p}\\ &\leq C\Big(\sum_k (2^k\,|E_k|)^p\Big)^{1/p}\|f\|_p \leq C\,\|f\|_p\,.\end{aligned}$$

Therefore it suffices to prove that M is bounded in L^p. In fact we will show that M is founded in L^p for all $p > 1$. This will be done by following ideas from Christ [C3] and Wright [W].

We will decompose M_k into a sum of four operators. To do this, let

$$R_k = \{\zeta = (\xi,\eta) \in \mathbb{R}^2 :\ v(2^{-k-N}) \leq |\xi/\eta| \leq v(2^{-k+N})\}\,,$$

where N is some large number to be determined later and define

$$S_k f = (\chi_{R_k}\cdot \hat{f})^\vee\,.$$

Also let

$$T_k = \begin{pmatrix}\alpha_k & 0\\ 0 & \beta_k\end{pmatrix},$$

where

$$\alpha_k = \sum_{k\leq j}\frac{1}{2^j N_j} \qquad \text{and} \qquad \beta_k = \sum_{k\leq j}\frac{v(2^{-j-1})}{2^j N_j}\,.$$

Next choose $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi(0) = 1$ and define $\Phi_k = (\varphi\circ T_k)^\vee$. Write

$$\begin{aligned}M_k &= \Phi_k * M_k + (\delta - \Phi_k) * (I - S_k)M_k + (\delta - \Phi_k) * S_k M_k\\ &\stackrel{\text{def}}{=} \Phi_k * M_k + M_k^1 + M_k^2\,,\end{aligned}$$

where δ denotes the Dirac mass at the origin. Since $\Phi_k * M_k$ will not in general be dominated by the usual maximal functions, we will also apply a g-functions argument to it by further writing $\Phi_k * M_k$ as a sum of two operators. To do this, let $\omega \in C_c^\infty(\mathbb{R})$ with $\omega(0) = 1$ and define K_k by

$$\hat{K}_k(\xi, \eta) = 2^k \omega(\beta_k \eta) \int_{I_k \setminus E_k} e^{i\xi t} \, dt \, .$$

Write $\sigma_k = \Phi_k * M_k - K_k$ so that $\Phi_k * M_k = K_k + \sigma_k$. Finally we have the desired decomposition,

$$M_k f = K_k * f + \sigma_k * f + M_k^1 f + M_k^2 f \, .$$

Note that $|K_k * f| \leq C \mathcal{M}_s f$, where $\mathcal{M}_s$ denotes the strong maximal function, since K_k is dominated pointwise by

$$\frac{1}{2^{-k}\beta_k} \frac{1}{1 + |2^k x|^2} \frac{1}{1 + |y/\beta_k|^2} \, .$$

Thus we have

$$\begin{aligned} Mf \leq \; & C \Big(\mathcal{M}_s f + \big(\sum |\sigma_k * f|^2 \big)^{1/2} \\ & + \big(\sum |M_k^1 f|^2 \big)^{1/2} + \big(\sum |M_k^2 f|^2 \big)^{1/2} \Big) \, . \end{aligned} \tag{14}$$

By an argument used in [NSW], we will prove the L^p estimates for M by repeated applications of the following three lemmas.

Lemma 1. *M is bounded in L^2.*

Lemma 2. *If*

$$\|(\sum |M_k f_k|^2)^{1/2}\|_{p_0} \leq C_{p_0} \|(\sum |f_k|^2)^{1/2}\|_{p_0}$$

for some $p_0 < 2$, then M is bounded in L^p for $p_0 < p \leq 2$.

Lemma 3. *If M is bounded in L^{p_0} for some $p_0 \leq 2$, then*

$$\|(\sum |M_k f_k|^2)^{1/2}\|_p \leq C_p \|(\sum |f_k|^2)^{1/2}\|_p \, ,$$

for

$$\frac{1}{p} \leq \frac{1}{2}\Big(\frac{1}{p_0} + 1\Big) \, .$$

Lemma 3 follows from interpolation since the operators $\{M_k\}$ are positive and uniformly bounded in L^p, $1 \le p \le \infty$, as in [NSW]. The main estimates needed in the proofs of lemmas 1 and 2 are contained in the following two lemmas.

Lemma 4.

a) $|\hat{\sigma}_k(\zeta)| \le C\,|T_k\zeta|^{-1}$.

b) $|\hat{\sigma}_k(\zeta)| \le C\,|T_{k-3}\zeta|$.

PROOF. Recall that $\hat{\sigma}_k(\zeta) = \varphi(T_k\zeta)\hat{M}_k(\zeta) - \hat{K}_k(\zeta)$ where

$$\hat{K}_k(\xi,\eta) = 2^k\,\omega(\beta_k\eta)\int_{I_k\setminus E_k} e^{i\xi t}\,dt\,.$$

Note that

$$\left|2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\right| \le C\,\frac{2^k N_k}{|\xi|} \le C\,\frac{1}{\alpha_k|\xi|}\,.$$

The last inequality follows from (7) since

$$\alpha_k = \sum_{k\le j}\frac{1}{2^j N_j} \le C\,\frac{1}{2^k N_k}\,. \tag{15}$$

Also $|\omega(\beta_k\eta)| \le C\,|\beta_k\eta|^{-1}$ and so $|\hat{K}_k(\zeta)| \le C\,|T_k\zeta|^{-1}$. Furthermore

$$|(\Phi_k * M_k)^\wedge(\zeta)| = |\varphi(T_k\zeta)\hat{M}_k(\zeta)| \le C\,|T_k\zeta|^{-1}\,,$$

which gives us part *a*). For *b*) note that

$$\begin{aligned}
\hat{\sigma}_k(\zeta) &= \varphi(T_k\zeta)\Big(2^k\int_{I_k\setminus E_k} e^{i(\xi t+\eta\,\gamma(t))}\,dt - 2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\Big)\\
&\quad + 2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\,\big(\varphi(T_k\zeta) - \omega(\beta_k\eta)\big)\\
&= \varphi(T_k\zeta)\Big(2^k\int_{I_k\setminus E_k}\big(e^{i(\xi t+\eta\,\gamma(t))} - e^{i\xi t}\big)\,dt\Big)\\
&\quad + 2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\,\big((\varphi(T_k\zeta)-1) - (\omega(\beta_k\eta)-1)\big)\,,
\end{aligned}$$

and so

$$|\hat{\sigma}_k(\zeta)| \le C\left(2^k|\eta|\int_{I_k\setminus E_k}|\gamma(t)|\,dt + |T_k\zeta|\right)$$

$$
\begin{aligned}
&\le C\left(|\eta|\phi(t_*) + |T_k\zeta|\right)\\
&\le C\left(|\eta|\frac{1}{N_{k-2}\,2^{k-2}}\,v(t_*) + |T_k\zeta|\right)\\
&\le C\left(|\eta|\frac{1}{N_{k-2}\,2^{k-2}}\,v(2^{-k+2}) + |T_k\zeta|\right)\\
&\le C\left(|\eta|\,\beta_{k-3} + |T_{k-3}\zeta|\right) \le C\,|T_{k-3}\zeta|\,,
\end{aligned}
$$

where $t_* \in I_{k-1}$ such that $2t_* \in I_{k-2} \setminus E_{k-2}$. The third inequality follows from applying (9) on $I_{k-2} \setminus E_{k-2}$. The second to last inequality uses the fact that the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are monotone and the estimate $2^j N_j \le C\, 2^k N_k$ for $j \le k$ which follows from (7). This gives us b) and thus completes the proof of the lemma.

The next estimate is based on a lemma of Van der Corput whose proof can be found in [Z].

Van der Corput's lemma. *Let $f \in C^2[a,b]$ be a real-valued function such that $|f''(t)| \le \lambda$ on $[a,b]$. Then*

$$
\left|\int_a^b e^{i\,f(t)}\,dt\right| \le C\,\frac{1}{\sqrt{\lambda}}\,,
$$

where C is independent of f, a and b.

Lemma 5.

a) $|\hat{M}_k^1(\zeta)| \le C\,|T_k\zeta|\,.$

b) $|\hat{M}_k^1(\zeta)| \le C\,|T_k\zeta|^{-1/2}\,.$

PROOF. Note that

$$
\hat{M}_k^1(\zeta) = \big(1 - \varphi(T_k\zeta)\big)\big(1 - \chi_{R_k}(\zeta)\big)\hat{M}_k(\zeta)\,, \tag{16}
$$

where

$$
\hat{M}_k = 2^k \int_{I_k \setminus E_k} e^{i(\xi t + \eta\,\gamma(t))}\,dt = 2^k \int_{F_k} e^{i\,f(t)}\,dt + 2^k \int_{G_k} e^{i\,f(t)}\,dt
$$

and $f(t) = \xi t + \eta\,\gamma(t)$. The estimate in a) is clear from (16) since $\varphi(0) = 1$. We turn now to the proof of b). We may assume $\zeta \notin R_k$. We will consider two cases.

Case 1. $v(2^{-k+N}) \leq |\xi/\eta|$. On $I_k \setminus E_k$,

$$\begin{aligned} |f'(t)| &\geq |\xi| - |\eta|\,\gamma'(t)| \\ &\geq |\xi| - |\eta|\,v(2^{-k+1}) \geq |\xi|\Big(1 - \frac{v(2^{-k+1})}{v(2^{-k+N})}\Big)\,. \end{aligned} \tag{17}$$

From assumption (8) and the fact that E is "thin", we see that

$$\frac{v(2^{-k+1})}{v(2^{-k+N})} < \frac{1}{2} \tag{18}$$

for N sufficiently large. So from (17) and (18) we see that

$$|f'(t)| \geq \frac{|\xi|}{2} \tag{19}$$

for N sufficiently large.

Since F_k and G_k are open sets, we will write

$$F_k = \bigcup_{\ell=1}^{L_k}(a_\ell, b_\ell) \qquad \text{and} \qquad G_k = \bigcup_{\ell=1}^{M_k}(c_\ell, d_\ell)$$

where L_k and M_k are at most N_k. Integration by parts gives

$$\begin{aligned} \Big|2^k \int_{F_k} e^{i\,f(t)}\,dt\Big| &= \Big|2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} e^{i\,f(t)}\,dt\Big| \\ &\leq C\Big(\frac{2^k N_k}{|\xi|} + 2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\Big) \\ &\leq C\Big(\frac{2^k N_k}{|\xi|} + \frac{2^k N_k\,|\eta|}{|\xi|^2} \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{v(t/2)}{t}\,dt\Big) \\ &\leq C\Big(\frac{2^k N_k}{|\xi|} + \frac{2^k N_k}{|\xi|}\,\frac{v(2^{-k})}{v(2^{-k+N})} \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{dt}{t}\Big) \\ &\leq C\,\frac{2^k N_k}{|\xi|} \leq C\,\frac{1}{\alpha_k\,|\xi|} \leq C\,|T_k\zeta|^{-1}\,. \end{aligned}$$

The first inequality uses (19) while the second inequality uses both (10) and (19). The second to last inequality follows from (15). To prove the last inequality, note that it suffices to prove that $\beta_k/\alpha_k \le C\,|\xi/\eta|$ in this case. But since $v(2^{-k+N}) \le |\xi/\eta|$,

$$\beta_k = \sum_{k\le j} \frac{v(2^{-j-1})}{2^j N_j} \le v(2^{-k+N}) \sum_{k\le j} \frac{1}{2^j N_j} \le \alpha_k \left|\frac{\xi}{\eta}\right|.$$

Next note that

$$\begin{aligned}\left|2^k \int_{G_k} e^{i\,f(t)}\,dt\right| &= \left|2^k \sum_{\ell=1}^{M_k} \int_{c_\ell}^{d_\ell} e^{i\,f(t)}\,dt\right| \\ &\le C\left(\frac{2^k N_k}{|\xi|} + 2^k \sum_{\ell=1}^{M_k} \int_{c_\ell}^{d_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\right) \\ &= C\left(\frac{2^k N_k}{|\xi|} + 2^k \sum_{\ell=1}^{M_k} \left|\int_{c_\ell}^{d_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\right|\right) \\ &\le C\,\frac{2^k N_k}{|\xi|} \le C\,\frac{1}{\alpha_k\,|\xi|} \le C\,|T_k\zeta|^{-1}.\end{aligned}$$

The second equality holds since (11) implies that $f''(t)$ is single-signed on each (c_ℓ, d_ℓ). The first and third to last inequalities follow from (19). The final inequality was already used in the treatment of the F_k. Thus $|\hat{M}_k^1(\zeta)| \le C\,|T_k\zeta|^{-1}$.

Case 2. $|\xi/\eta| \le v(2^{-k-N})$. On F_k,

$$\begin{aligned}|f'(t)| &\ge |\eta\,\gamma'(t)| - |\xi| \ge \varepsilon_1\,|\eta|\,v(t) - |\xi| \\ &= \varepsilon_1\,|\eta|\,v(t)\Big(1 - \left|\frac{\xi}{\eta}\right| \frac{1}{\varepsilon_1\,v(t)}\Big) \ge \frac{\varepsilon_1}{2}\,|\eta|\,v(t)\end{aligned} \tag{20}$$

for N large enough. Integration by parts shows

$$\begin{aligned}\left|2^k \int_{F_k} e^{i\,f(t)}\,dt\right| &= \left|2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} e^{i\,f(t)}\,dt\right| \\ &\le C\left(\frac{2^k N_k}{|\eta|\,v(2^{-k})} + 2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\right) \\ &\le C\left(\frac{2^k N_k}{|\eta|\,v(2^{-k})} + \frac{2^k N_k\,|\eta|}{v^2(2^{-k})\,|\eta|^2} \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{v(t/2)}{t}\,dt\right)\end{aligned}$$

$$\leq C \frac{2^k N_k}{|\eta|\, v(2^{-k})} \leq C\, |T_k \zeta|^{-1}.$$

The first inequality uses (20) while the second inequality uses both (10) and (20). To prove the last inequality, note that from (15) and $|\xi/\eta| \leq v(2^{-k-N})$, $\alpha_k\,|\xi| \leq C\,(2^k N_k)^{-1}|\xi| \leq C\,(2^k N_k)^{-1} v(2^{-k})\,|\eta|$. Also $\beta_k\,|\eta| \leq v(2^{-k})\,(2^k N_k)^{-1}|\eta|$. Hence the last inequality follows.

On G_k, $|f''(t)| \geq \varepsilon_2\,|\eta|\,N_k\, v(t/2)/t$ and so by Van der Corput's lemma,

$$\begin{aligned}
\Big|2^k \int_{G_k} e^{i\,f(t)}\,dt\Big| &= \Big|2^k \sum_{\ell=1}^{M_k} \int_{c_\ell}^{d_\ell} e^{i\,f(t)}\,dt\Big| \\
&\leq C\, 2^k N_k \frac{1}{\sqrt{2^k |\eta| N_k\, v(2^{-k-1})}} \\
&\leq C \sqrt{\frac{2^k N_k}{v(2^{-k-1})\,|\eta|}} \leq C\,|T_k \zeta|^{-1/2}.
\end{aligned}$$

The proof of the last inequality is the same as in the treatment of F_k. Thus $|\hat{M}_k^1(\zeta)| \leq C\,|T_k\zeta|^{-1/2}$ in this case as well which finishes part *b*) and thus the lemma.

We turn now to the proofs of lemmas 1 and 2. First observe that our family $\{T_k\}$ satisfies the norm estimate

$$\|T_k^{-1} T_{k+1}\| \leq \alpha < 1. \tag{21}$$

In fact,

$$T_k^{-1} T_{k+1} = \begin{pmatrix} \alpha_{k+1}/\alpha_k & 0 \\ 0 & \beta_{k+1}/\beta_k \end{pmatrix}$$

and so to prove (21) it suffices to show that there is an $\alpha < 1$ such that for all k,

$$\frac{\alpha_{k+1}}{\alpha_k} \leq \alpha \qquad \text{and} \qquad \frac{\beta_{k+1}}{\beta_k} \leq \alpha.$$

This however follows easily from (7).

To show that Mf is bounded in L^2 we see from (14) that it suffices to prove that $(\sum |\sigma_k * f|^2)^{1/2}$, $(\sum |M_k^1 f|^2)^{1/2}$, and $(\sum |M_k^2 f|^2)^{1/2}$ are bounded in L^2. With the aid of Plancherel's Theorem, the L^2 estimates of the first two square functions reduce to showing that $\sum |\hat{\sigma}_k(\zeta)|^2$ and

$\sum |\hat{M}_k^1(\zeta)|^2$ are bounded functions of ζ. This easily follows from lemmas 4 and 5 together with (21). For the third square function, note that

$$M_k^2 f \le C\left(M_k S_k f + \mathcal{M}_s M_k S_k f\right) \le C\,\mathcal{M}_s M_k S_k f\,. \tag{22}$$

Then since $\sum \chi_{R_k}(\zeta) \le 2N$ (where N appears in the definition of the R_k's),

$$\begin{aligned}\|(\sum |M_k^2 f|^2)^{1/2}\|_2^2 &\le C \sum \int (\mathcal{M}_s M_k S_k f)^2 \le C \sum \int |M_k S_k f|^2 \\ &\le C \sum \int |S_k f|^2 = C \int \sum \chi_{R_k}(\zeta)\,|\hat{f}(\zeta)|^2\,d\zeta \\ &\le 2\,N\,C \int |\hat{f}(\zeta)|^2\,d\zeta = 2\,N\,C\,\|f\|_2^2\,.\end{aligned}$$

We have used the fact that the strong maximal function is bounded in L^2 and that the M_k's are uniformly bounded in L^2. Thus $(\sum |M_k^2 f|^2)^{1/2}$ and hence Mf is bounded in L^2. This completes the proof of Lemma 1.

We turn now to Lemma 2. Note that

$$\|(\sum |M_k f_k|^2)^{1/2}\|_p \le C_p\,\|(\sum |f_k|^2)^{1/2}\|_p\,, \qquad p_0 \le p \le 2\,,$$

by interpolation and so by (22),

$$\begin{aligned}\|(\sum |M_k^2 f|^2)^{1/2}\|_p &\le C_p\,\|(\sum |\mathcal{M}_s M_k S_k f|^2)^{1/2}\|_p \\ &\le C_p\,\|(\sum |M_k S_k f|^2)^{1/2}\|_p \\ &\le C_p\,\|(\sum |S_k f|^2)^{1/2}\|_p \\ &\le C_p\,\|f\|_p\,, \qquad p_0 \le p \le 2\,.\end{aligned}$$

We have used the fact that the strong maximal function satisfies vector-valued estimates, see [FS]. The last inequality follows from [NSW] and [CF] since the sequence $\{v(2^{-k})\}$ satisfies (18). Hence by (14), it suffices to prove that $(\sum |\sigma_k * f|^2)^{1/2}$ and $(\sum |M_k^1 f|^2)^{1/2}$ are bounded in L^p, $p_0 < p \le 2$. This is proved by an argument used in [CCVWW]. We will only sketch the argument here (the interested reader should consult [CCVWW] for more details). The argument is based on a general Littlewood-Paley decomposition developed in [CCVWW] and [CVWW].

Suppose that we have a family of invertible linear transformations on $\mathbb{R}^n$, $\{A_j\}_{j\in\mathbb{Z}}$, which satisfy the Rivière condition

$$\text{(R)} \qquad \|A_j^{-1}A_{j+1}\| \le \alpha < 1\,.$$

See [R]. Choose a smooth function $\phi(x)$ such that $\hat{\phi}(\xi) = 1$ for $|\xi| \le 1$ and $\hat{\phi}(\xi) = 0$ for $|\xi| \ge 2$. Now define the multipliers

$$m_j(\xi) = \hat{\phi}(A_{j+1}^*\xi) - \hat{\phi}(A_j^*\xi)$$

and the corresponding linear operators

$$P_j f(x) = \left(m_j \hat{f}\right)^{\vee}(x)\,.$$

The following theorem can be found in [CVWW].

Theorem D. *Under the conditions stated above, we have*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |P_j f|^2\Big)^{1/2}\Big\|_p \le C_p\, \|f\|_p\,, \qquad 1 < p < \infty\,,$$

and

$$\Big\|\sum_{j\in\mathbb{Z}} P_j f_j\Big\|_p \le C_p\, \Big\|\Big(\sum_{j\in\mathbb{Z}} |P_j f|^2\Big)^{1/2}\Big\|_p\,, \qquad 1 < p < \infty\,.$$

Also for each $\xi \neq 0$,

$$\sum_{j\in\mathbb{Z}} m_j(\xi) = 1\,.$$

We will use our family of invertible linear transformations on $\mathbb{R}^2$, $\{T_k\}$ and note that (R) is simply (21) in this case. To prove the L^p estimates, for say $(\sum |M_k^1 f|^2)^{1/2}$ (the same reasoning applies to $(\sum |\sigma_k * f|^2)^{1/2}$), we will decompose f with respect to the operators $\{P_j\}$. Write

$$M_k^1 f = M_k^1 \sum_{j\in\mathbb{Z}} P_{j+k} f = \sum_{j\in\mathbb{Z}} P_{j+k} M_k^1 f\,,$$

which implies

$$\Big(\sum_k |M_k^1 f|^2\Big)^{1/2} = \Big(\sum_k \Big|\sum_{j\in\mathbb{Z}} P_{j+k} M_k^1 f\Big|^2\Big)^{1/2}$$

$$\leq \sum_{j\in\mathbb{Z}} \Big(\sum_k |P_{j+k} M_k^1 f|^2 \Big)^{1/2} = \sum_{j\in\mathbb{Z}} G_j f ,$$

where $G_j f = (\sum_k |P_{j+k} M_k^1 f|^2)^{1/2}$. We will prove that

a) $\|G_j f\|_{p_0} \leq C\, \|f\|_{p_0}$, and

b) $\|G_j f\|_2 \leq C\, 2^{-\varepsilon |j|}\, \|f\|_2$,

for some $\varepsilon > 0$ and C independent of j. For *a)* note that

$$|M_k^1 f_k| \leq C\, (\mathcal{M}_s M_k f_k + \mathcal{M}_s S_k M_k f_k) .$$

Therefore using the hypothesis of Lemma 2 that the family of operators $\{M_k\}$ satisfy an ℓ^2-valued L^{p_0} estimate, we obtain the same conclusion for the family $\{M_k^1\}$,

$$\begin{aligned} \|(\sum |M_k^1 f_k|^2)^{1/2}\|_{p_0} &\leq C\, \|(\sum |\mathcal{M}_s S_k M_k f_k|^2)^{1/2}\|_{p_0} \\ &\leq C\, \|(\sum |S_k M_k f_k|^2)^{1/2}\|_{p_0} \\ &\leq C\, \|(\sum |M_k f_k|^2)^{1/2}\|_{p_0} \\ &\leq C\, \|(\sum |f_k|^2)^{1/2}\|_{p_0} . \end{aligned}$$

Again we used the angular Littlewood-Paley theory developed in [NSW] and the fact that the strong maximal function satisfies vector-valued L^p estimates. Using Theorem D, we see that

$$\begin{aligned} \|G_j f\|_{p_0} &= \|(\sum_k |M_k^1 (P_{j+k} f)|^2)^{1/2}\|_{p_0} \\ &\leq \|(\sum_k |P_{j+k} f|^2)^{1/2}\|_{p_0} \leq \|f\|_{p_0} . \end{aligned}$$

which gives *a)*. *b)* is proved by using Lemma 5 and (21). See [CCVWW] for details.

By interpolating the estimates in *a)* and *b)*, we see that $\|G_j f\|_p \leq C\, 2^{-\varepsilon_p |j|}\, \|f\|_p$, $p_0 < p \leq 2$ for some $\varepsilon_p > 0$. Summing these L^p estimates for G_j gives us the desired L^p estimates for $(\sum |M_k^1 f|^2)^{1/2}$. This finishes the proof of Lemma 2. The treatment of the maximal function is now complete.

We turn to the Hilbert transform. Note that

$$\mathcal{H}_\Gamma f(x) = \int\limits_{\cup E_k^1} f(x-\Gamma(t))\,\frac{dt}{t} + \int\limits_{\cup E_k^2} f(x-\Gamma(t))\,\frac{dt}{t}$$

$$\stackrel{\text{def}}{=} H_E f(x) + H_G f(x)\,,$$

where $E_k^1 = E_k \cup (-E_k)$ and $E_k^2 = (I_k \setminus E_k) \cup (-(I_k \setminus E_k))$. The L^p estimates for $H_E f$ follow from Minkowski's inequality for integrals and the assumption that $\sum 2^k |E_k| < +\infty$. For $H_G f$, write

$$H_k f(z) = \int\limits_{E_k^2} f(z-\Gamma(t))\,\frac{dt}{t}$$

and note that $\sum_{k\geq 1} H_k f(z) = H_G f(z)$. We will decompose $H_k f$ into a sum of several operators. First let us consider a variant of the kernel K_k. Denote by B_k the function such that

$$b_k(\xi,\eta) = \hat{B}_k(\xi,\eta) = \omega(\beta_k \eta) \int\limits_{E_k^2} e^{i\xi t}\,\frac{dt}{t}$$

where β_k and ω are the same as in K_k. Note that $B_k = C_k - D_k$ where

$$\hat{C}_k(\xi,\eta) = \omega(\beta_k\eta) \int\limits_{I_k \cup (-I_k)} e^{i\xi t}\,\frac{dt}{t}$$

and

$$\hat{D}_k(\xi,\eta) = \omega(\beta_k\eta) \int\limits_{E_k^1} e^{i\xi t}\,\frac{dt}{t}\,.$$

Since the L^1 norm of D_k is no larger that $2^k|E_k|$, we see that the operator $\sum_k D_k * f$ is bounded in all L^p. Also

a) $\big|\,|C_k| * f\,\big| \leq C\,\mathcal{M}_s f$, and

b) $|\hat{C}_k(\xi,\eta)| \leq C \min\{2^{-k}|\xi|, (2^{-k}|\xi|)^{-1}\}$.

We may apply Theorem D′ in [DR] to the operator $\sum_k C_k * f$ and find that it is bounded in all L^p, $p > 1$. Therefore it suffices to estimate the operator $\sum_k (H_k - B_k)$. Write

$$(H_k - B_k) f(x) = S_k(H_k - B_k) f(x) + (I - S_k)(H_k - B_k) f(x)$$

$$\stackrel{\text{def}}{=} H_k^1 f(x) + H_k^2 f(x)\,.$$

The L^p estimates of $\sum_k (H_k - B_k)$ will be based on the following lemma.

Lemma 6.

$$\Big\| \Big(\sum_{k\geq 1} |H_k f_k|^2 \Big)^{1/2} \Big\|_p \leq C_p \Big\| \Big(\sum_{k\geq 1} |f_k|^2 \Big)^{1/2} \Big\|_p, \qquad 1 < p < \infty,$$

and

$$\Big\| \Big(\sum_{k\geq 1} |B_k f_k|^2 \Big)^{1/2} \Big\|_p \leq C_p \Big\| \Big(\sum_{k\geq 1} |f_k|^2 \Big)^{1/2} \Big\|_p, \qquad 1 < p < \infty.$$

PROOF. We shall first derive the estimate for H_k. This will be proved by an interpolation argument. In fact we will prove that

$$(23) \qquad \Big\| \Big(\sum_{k\geq 1} |H_k f_k|^q \Big)^{1/q} \Big\|_p \leq C_{p,q} \Big\| \Big(\sum_{k\geq 1} |f_k|^q \Big)^{1/q} \Big\|_p$$

for certain $1 \leq p, q \leq \infty$. Since the operators H_k are uniformly bounded in L^p, $1 \leq p \leq \infty$, we see that (23) holds for $p = q \geq 1$ and so in particular we have the lemma for $p = 2$. If γ is even, we have the pointwise estimate

$$(24) \quad \sup_{k\geq 1} |H_k f_k|(x,y) \leq C\Big(M\big(\sup_{k\geq 1} |f_k|\big)(x,y) + M\big(\sup_{k\geq 1} |f_k^1|\big)(-x,y)\Big)$$

where $f_k^1(x,y) = f_k(-x,y)$. Also if γ is odd, we have

$$(25) \quad \sup_{k\geq 1} |H_k f_k|(x,y) \leq C\Big(M\big(\sup_{k\geq 1} |f_k|\big)(x,y) + M\big(\sup_{k\geq 1} |f_k^2|\big)(-x,-y)\Big)$$

where $f_k^2(x,y) = f_k(-x,-y)$. Therefore from the L^p estimates for M, we see that (23) holds for $q = \infty$ and $p > 1$. Interpolating between $L^1(\ell^1)$ and $L^p(\ell^\infty)$, $p > 1$, establishes the lemma for $1 < p < 2$ and then duality gives us the full range. The argument for the operators $\{B_k\}$ is similar. We must only replace M by $\mathcal{M}_s$ in (24) and (25).

Lemma 6 is sufficient to give the L^p estimates for H_k^1. In fact for $1 < p < \infty$,

$$\begin{aligned} \Big\| \sum_{k\geq 1} H_k^1 f \Big\|_p &= \Big\| \sum_{k\geq 1} S_k(H_k - B_k) f \Big\|_p = \Big\| \sum_{k\geq 1} S_k^2(H_k - B_k) f \Big\|_p \\ &\leq C_p \Big\| \Big(\sum_{k\geq 1} |(H_k - B_k) S_k f|^2 \Big)^{1/2} \Big\|_p \qquad (26) \\ &\leq C_p \Big\| \Big(\sum_{k\geq 1} |S_k f|^2 \Big)^{1/2} \Big\|_p \leq C_p \, \|f\|_p \,, \end{aligned}$$

where we have used again the results of [NSW] and Lemma 6. For H_k^2, we will need the analogous estimates to Lemma 5 for the multiplier h_k^2 of the operator H_k^2. Note that $h_k^2(\zeta) = (1 - \chi_{R_k}(\zeta))(h_k(\zeta) - b_k(\zeta))$. Here h_k is the multiplier for H_k, *i.e.*,

$$h_k(\zeta) = \int_{E_k^2} e^{i(\xi t + \eta\,\gamma(t))} \, \frac{dt}{t} = h_k^+(\zeta) + h_k^-(\zeta)$$

where

$$h_k^+(\zeta) \stackrel{\text{def}}{=} \int_{I_k \setminus E_k} e^{i\zeta\,\Gamma(t)} \, \frac{dt}{t} \,, \qquad h_k^-(\zeta) \stackrel{\text{def}}{=} \int_{-(I_k \setminus E_k)} e^{i\zeta\,\Gamma(t)} \, \frac{dt}{t}$$

and b_k is the multiplier for B_k defined above. A direct consequence of Lemmas 4 and 5 is the following lemma.

Lemma 7.

a) $|h_k^2(\zeta)| \leq C \, |T_{k-3}\zeta|$,

b) $|h_k^2(\zeta)| \leq C \, |T_k \zeta|^{-1/2}$.

PROOF. Note that

$$h_k(\zeta) - b_k(\zeta) = \int_{E_k^2} \left(e^{i(\xi t + \eta\,\gamma(t))} - e^{i\xi t} \right) \frac{dt}{t} + (1 - \omega(\beta_k \eta)) \int_{E_k^2} e^{i\xi t} \, \frac{dt}{t}$$

and so as in Lemma 4,

$$|h_k(\zeta) - b_k(\zeta)| \leq C \left(2^k |\eta| \int_{I_k} \phi(t) \, dt + |\eta| \, \beta_k \right) \leq C \, |T_{k-3}\zeta| \,.$$

This gives us *a*). For *b*), integration by parts shows that Lemma 5 implies that *b*) holds for h_k^+. If γ is even, we see that $h_k^-(\xi,\eta) = h_k^+(-\xi,\eta)$ and if γ is odd, $h_k^-(\xi,\eta) = h_k^+(-\xi,-\eta)$. Therefore *b*) also holds for h_k^- and thus for h_k. Similarly for b_k and this completes the lemma.

Now if we follow the same arguments for the maximal function, using Lemma 7, we see that $\sum_k H_k^2$ is bounded in L^p, $p > 1$. This completes the proof of the main theorem.

3. Examples.

In this section, we will construct the two curves mentioned in the introduction. We will first construct an odd monotone curve $\Gamma(t) = (t,\gamma(t))$ on $[-1,1]$ which satisfies A), B), C) and D$'$) in the introduction but whose Hilbert transform is unbounded in L^2. We begin with a continuous piecewise linear curve. For $r \geq 0$, we will construct γ on $[2^{-2^{r+1}}, 2^{-2^r}]$. For $2^r+1 \leq k \leq 2^{r+1}$, write

$$[2^{-k}, 2^{-k+1}] = \bigcup_{\ell=0}^{N_k-1} [a_\ell^k, b_\ell^k] \cup \bigcup_{\ell=0}^{N_k-1} [b_\ell^k, a_{\ell+1}^k] \stackrel{\text{def}}{=} F_k \cup E_k\ ,$$

where $a_0^k = 2^{-k}$, $a_{N_k}^k = 2^{-k+1}$ and $N_k = 2^{2^{r+2}-2k}$ so that

$$\Delta_k = b_\ell^k - a_\ell^k = \frac{1}{2^k N_k}\Big(1 - \frac{1}{k^2}\Big)\,, \qquad \text{for } 0 \leq \ell \leq N_k - 1\,,$$

and

$$\delta_k = a_{\ell+1}^k - b_\ell^k = \frac{1}{2^k N_k\, k^2}\,, \qquad \text{for } 0 \leq \ell \leq N_k - 1\,.$$

On $[a_\ell^k, b_\ell^k]$, define

$$\gamma' = \frac{\pi}{2^{2^{r+2}}\Delta_k} \stackrel{\text{def}}{=} m_k \sim 2^{-k}\,.$$

On $[b_\ell^k, a_{\ell+1}^k]$, define

$$\gamma' = \frac{\pi}{2^{2^{r+2}}\delta_k} \stackrel{\text{def}}{=} M_k \sim k^2\, 2^{-k}\,.$$

Then

$$\gamma(b_\ell^k) - \gamma(a_\ell^k) = m_k \Delta_k = \frac{\pi}{2^{2^r+2}} \tag{27}$$

and

$$\gamma(a_{\ell+1}^k) - \gamma(b_\ell^k) = M_k \delta_k = \frac{\pi}{2^{2^r+2}}\,. \tag{28}$$

This will now define γ uniquely on $[0, 1/2]$ once we choose $\gamma(1/2)$. We will do so to make $\gamma(0) = 0$. Note that by (27) and (28),

$$\begin{aligned}\gamma(2^{-k+1}) - \gamma(2^{-k}) &= \sum_{\ell=0}^{N_k-1} \big(\gamma(a_{\ell+1}^k) - \gamma(b_\ell^k)\big) \\ &\quad + \sum_{\ell=0}^{N_k-1} \big(\gamma(b_\ell^k) - \gamma(a_\ell^k)\big) \\ &= \frac{\pi N_k}{2^{2^r+2}} + \frac{\pi N_k}{2^{2^r+2}} = \frac{2\pi}{2^{2k}}\,.\end{aligned} \tag{29}$$

Thus

$$\gamma\Big(\frac{1}{2}\Big) - \gamma\Big(\frac{1}{2^N}\Big) = \sum_{\ell=2}^{N} \big(\gamma(2^{-\ell+1}) - \gamma(2^{-\ell})\big) = \sum_{\ell=2}^{N} \frac{2\pi}{2^{2\ell}}$$

and so if $\gamma(1/2) = \pi/6$, $\gamma(0) = 0$. We will show that $\mathcal{H}_\Gamma$ is unbounded in L^2.

Since the Hilbert transform is a multiplier transformation, it suffices to show that the corresponding multiplier m is un unbounded function. Since γ is an odd function on $[-1, 1]$, the multiplier reduces to a sine integral,

$$m(\xi, \eta) = \int_0^1 \sin(f(t))\,\frac{dt}{t}\,,$$

where $f(t) = \xi t - \eta\,\gamma(t)$. We will take $\xi = 0$ and show that

$$\int_0^{1/2} \sin(\eta\,\gamma(t))\,\frac{dt}{t}$$

is unbounded as $\eta \to \infty$. Let us first note that

$$\begin{aligned}\Big| \int_{\cup E_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} \Big| &\le C \sum_{k=2}^{\infty} \int_{E_k} \frac{dt}{t} \\ &\le C \sum_{k=2}^{\infty} 2^k |E_k| \le C \sum_{k=2}^{\infty} \frac{1}{k^2} \le C\,.\end{aligned}$$

Now let r be a large integer and set $\eta = 2^{2^{r+2}}$.

Claim 1.

$$\sum_{k=2}^{2^r} \int_{F_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} + \sum_{k=2^{r+1}+1}^{\infty} \int_{F_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} \stackrel{\text{def}}{=} \mathrm{I}(r) + \mathrm{II}(r)$$

remains bounded as $r \to \infty$. In fact,

$$\begin{aligned} |\mathrm{II}(r)| &\le C\,\eta \sum_{k\ge 2^r+1} \int_{F_k} \frac{\gamma(t)}{t}\,dt \le C\,\eta \sum_{k\ge 2^r+1} \gamma(2^{-k+1}) \\ &\le C\,\eta \sum_{k\ge 2^r+1} 2^{-2k} \le C\,\frac{\eta}{2^{2(2^r+1)}} \le C\,. \end{aligned}$$

The third inequality follows from (29) since

$$(30)\quad \gamma(2^{-k+1}) = \sum_{\ell=k}^{\infty} \left(\gamma(2^{-\ell+1}) - \gamma(2^{-\ell})\right) = 2\pi \sum_{\ell=k}^{\infty} 2^{-2\ell} = \frac{8\pi}{3}\,\frac{1}{2^{2k}}\,.$$

Also by integrating by parts,

$$\begin{aligned} |\mathrm{I}(r)| &\le C \sum_{k=2}^{2^r} \sum_{\ell=0}^{N_k-1} \frac{1}{\eta\, m_k\, a_\ell^k} \le C \sum_{k=2}^{2^r} \frac{N_k\, 2^{2k}}{\eta} \\ &\le C\,\frac{2^{2^{r+1}}}{\eta} \sum_{k=2}^{2^r} 1 \le C\, 2^r\, \frac{2^{2^{r+1}}}{2^{2^{r+2}}} \le C\,. \end{aligned}$$

The third inequality holds since $N_k \le 2^{2^{r+1}-2k}$ for $k \le 2^r$ and this finishes the claim. For $2^r + 1 \le k \le 2^{r+1}$, write

$$\begin{aligned} \int_{F_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} &= \sum_{\ell=0}^{N_k-1} \int_{a_\ell^k}^{b_\ell^k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} \\ &= \sum_{\ell=0}^{N_k-1} \left(\frac{\cos(\eta\,\gamma(a_\ell^k))}{a_\ell^k} - \frac{\cos(\eta\,\gamma(b_\ell^k))}{b_\ell^k} \right) \frac{1}{\eta\, m_k} \\ &\quad - \sum_{\ell=0}^{N_k-1} \int_{a_\ell^k}^{b_\ell^k} \frac{\cos(\eta\,\gamma(t))}{t^2\,\eta\, m_k}\,dt \\ &\stackrel{\text{def}}{=} \mathrm{I}_k + \mathrm{II}_k\,. \end{aligned}$$

Since

$$|\mathrm{II}_k| \le C\,\frac{2^{2k}}{\eta\, m_k} \sum_{\ell=0}^{N_k-1} \int_{a_\ell^k}^{b_\ell^k} dt \le C\,\frac{2^{2k}}{\eta}\,,$$

we have

$$\sum_{k=2^r+1}^{2^{r+1}} |\mathrm{II}_k| \le C\,\frac{1}{\eta}\, 2^{2(2^{r+1})} \le C\,.$$

Therefore it suffices to show that

$$\sum_{k=2^r+1}^{2^{r+1}} \mathrm{I}_k \quad \text{is unbounded as} \quad r \to \infty\,.$$

Claim 2. $\cos(\eta\,\gamma(a_\ell^k)) = -1/2$ for $0 \le \ell \le N_k - 1$. By (27) and (28), it suffices to show that $\cos(\eta\,\gamma(2^{-k+1})) = -1/2$. By (28) and (30),

$$\eta\,\gamma(2^{-k+1}) = \frac{8\pi}{3}\, 2^{2^{r+2}-2k} = \frac{8\pi}{3}\, N_k\,.$$

Note that for $2^r + 1 \le k \le 2^{r+1}$, $2^{r+2} - 2k = 2\ell$ for some positive integer ℓ and so

$$\eta\,\gamma(2^{-k+1}) = \frac{\pi}{3}\, 2^{2\ell+3} = \frac{\pi}{3}\left(2(2^{2(\ell+1)} - 1) + 2\right).$$

Observe that if p is a positive integer, $2^{2p} - 1$ is a multiple of 3. In fact,

$$\begin{aligned} 3\,(1 + 2^2 + 2^4 + 2^6 + \cdots + 2^{2(p-1)}) &= (4-1)(1 + 4 + 4^2 + \cdots + 4^{p-1}) \\ &= 4^p - 1 = 2^{2p} - 1\,. \end{aligned}$$

Therefore,

$$\eta\,\gamma(2^{-k+1}) = \frac{\pi}{3}\,(2 \cdot 3n + 2) = 2\pi n + \frac{2\pi}{3}$$

for some positive integer n. This gives us the claim.

From (27) and (28), we see that $\cos(\eta\,\gamma(b_\ell^k)) = 1/2$ for $0 \le \ell \le N_k - 1$ and so

$$\mathrm{I}_k = -\,\frac{1}{\eta\, m_k} \sum_{\ell=0}^{N_k-1} \left(\frac{1}{a_\ell^k} + \frac{1}{b_\ell^k}\right) \le -\varepsilon\,\frac{N_k\, 2^{2k}}{\eta}$$

$$= -\varepsilon\,, \qquad \text{for some } \varepsilon > 0\,.$$

Thus

$$\Big| \sum_{k=2^r+1}^{2^{r+1}} \mathrm{I}_k \Big| \geq \varepsilon\, 2^r$$

and this finishes the proof that $\mathcal{H}_\Gamma$ is unbounded in L^2. By smoothing out γ on the exceptional set $\cup E_k$, we obtain a smooth curve whose Hilbert transform is still unbounded in L^2. Since in this case, $\phi(t) \sim t^2$ and $v(t) \sim t$, it is easy to see that A), B), C) and D$'$) are satisfied by this curve.

We will now construct a curve $\Gamma(t) = (t, \gamma(t))$ for $0 \leq t \leq 1$ where $v(t) = \sup_{s \leq t} |\gamma'(s)|$ is not doubling, $w(t) = \sup_{s \leq t} |s\,\gamma'(s) - \gamma(s)|$ is doubling and otherwise satisfies the conditions of the theorem (the set E is empty in this example). If γ is extended as an odd function on $[-1,1]$, we will see that $\mathcal{M}_\Gamma$ and $\mathcal{H}_\Gamma$ are unbounded in every L^p, $p \geq 1$. We begin by considering a saw-toothed curve $\Gamma_1(t) = (t, \gamma_1(t))$, $0 \leq t \leq 1$. We simply require that Γ_1 be continuous, piecewise linear and for each $n \geq 1$, $\gamma_1(9^{-(n+1/2)}) = 0$ and $\gamma_1(9^{-n}) = 9^{-n}/n$. To see that $\mathcal{M}_{\Gamma_1}$ is unbounded in L^p, take a large integer N and let f_N be the characteristic function of the parallelogram

$$P_N = \Big\{ (x,y) \in \mathbb{R}^2 :\ -2 \leq y \leq 0 \ \text{ and } \ \frac{2}{3} N\, y - 9^{-2N} \leq x \leq \frac{2}{3} N\, y \Big\}\,.$$

For $N \leq n \leq 2\,N$, consider smaller translated versions of P_N,

$$Q_n = \Big\{ (x,y) \in \mathbb{R}^2 : \\ -1 \leq y \leq 0, \frac{2}{3} N\, y - 9^{-(2N+1)} \leq x - 9^{-(n+1/2)} \leq \frac{2}{3} N\, y \Big\}\,.$$

Note that the Q_n's are disjoint. Also it is not hard to see that there is a positive δ and ε independent of N such that $\mathcal{M}_{\Gamma_1} f_N \geq \delta$ on each Q_n and $|Q_n| \geq \varepsilon |P_N|$, $N \leq n \leq 2\,N$. From this we see that $\|\mathcal{M}_{\Gamma_1} f_N\|_p^p \geq \varepsilon\, \delta^p\, N\, |P_N|$ whereas $\|f_N\|_p^p \leq |P_N|$ and so $\mathcal{M}_{\Gamma_1}$ is unbounded in L^p.

To see that $\mathcal{H}_{\Gamma_1}$ is unbounded in L^2, let us again consider the multiplier

$$m(\xi, \eta) = \int_0^1 \sin(f(t))\, \frac{dt}{t}\,,$$

where $f(t) = \xi t - \eta \gamma_1(t)$. Note that in $[9^{-(n+1)}, 9^{-n}]$,

$$\gamma_1(t) = -\frac{1}{2(n+1)}\, t + \frac{3}{2(n+1)}\, 9^{-(n+1)}, \tag{31.1}$$

if $9^{-(n+1)} \le t \le 9^{-(n+1/2)}$, and

$$\gamma_1(t) = \frac{3}{2n}\, t - \frac{1}{2n}\, 9^{-n}, \tag{31.2}$$

if $9^{-(n+1/2)} \le t \le 9^{-n}$.

Let n_0 be a large integer and set

$$\eta = \pi\, n_0^2\, 9^{n_0} \qquad \text{and} \qquad \xi = \frac{3}{2\, n_0}\, \eta = \frac{3}{2}\, \pi\, n_0\, 9^{n_0}.$$

Choose $n_1 > n_0$ such that $n_1\, 9^{n_1} \le n_0^2\, 9^{n_0} \le (n_1 + 1)\, 9^{n_1+1}$. Write

$$\int_0^{1/9} \sin(f(t))\, \frac{dt}{t} = \sum_{n=1}^{\infty} \int_{9^{-(n+1)}}^{9^{-(n+1/2)}} \sin(f(t))\, \frac{dt}{t} + \sum_{n=1}^{\infty} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\, \frac{dt}{t}.$$

We will show that the second sum is unbounded as $n_0 \to \infty$. The fact that the first sum is bounded as $n_0 \to \infty$ is somewhat easier.

Claim 1.

$$\sum_{n=n_1}^{\infty} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\, \frac{dt}{t} \qquad \text{and} \qquad \sum_{n=1}^{n_0-1} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\, \frac{dt}{t}$$

are bounded as $n_0 \to \infty$. To show that the first sum is bounded we need the following relationship between n_0 and n_1. Since

$$\frac{1}{9}\, \frac{n_0^2}{n_1 + 1} \le 9^{n_1 - n_0} \le \frac{n_0^2}{n_1} \le n_0,$$

we have that

$$\frac{1}{9 \log 9}\, \log \frac{n_0}{2} \le n_1 - n_0 \le \frac{\log n_0}{\log 9} \tag{32}$$

for n_0 large enough. The first inequality follows from the second inequality. Since

$$\Big|\int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\,\frac{dt}{t}\Big| \le \int_{9^{-(n+1/2)}}^{9^{-n}} \Big|\xi - \eta\,\frac{\gamma_1(t)}{t}\Big|\,dt$$
$$\le C\left(\frac{n_0\,9^{n_0}}{9^n}\,\big(1-\frac{n_0}{n}\big) + \frac{n_0^2\,9^{n_0}}{n\,9^n}\right),$$

we see that the first sum is bounded by a constant times

$$1 + \frac{n_0\,9^{n_0}}{n_1\,9^{n_1}}\,(n_1 - n_0)\,.$$

This term is bounded as $n_0 \to \infty$. One can see this from (32) and the definition of n_1.

For the second sum, let us note that

$$f'(t) = \xi - \eta\,\gamma_1'(t) = \frac{\pi\,n_0}{2}\,9^{n_0}\,\big(3 - 2\,n_0\,\gamma_1'(t)\big)$$
$$= \frac{3}{2}\,\pi\,n_0\,9^{n_0}\big(1 - \frac{n_0}{n}\big)$$

for $9^{-(n+1/2)} \le t \le 9^{-n}$. Thus

$$|f'(t)| \ge \frac{3\,\pi}{2}\,\frac{n_0 - n}{n}\,n_0\,9^{n_0}$$

and so

$$\Big|\int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\,\frac{dt}{t}\Big| \le C\,\frac{n\,9^n}{n_0\,9^{n_0}}\,\frac{1}{n_0 - n}$$

by integrating by parts. This shows that the second sum is bounded establishing the claim. Therefore it suffices to show that

$$\sum_{n=n_0}^{n_1-1} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\,\frac{dt}{t} \stackrel{\text{def}}{=} \sum_{n=n_0}^{n_1-1} \mathrm{I}_n$$

is unbounded as $n_0 \to \infty$. From (31) we see that

$$f(t) = \xi\,t - \eta\,\gamma_1(t) = \frac{3}{2}\,\pi\,n_0\,9^{n_0}\Big(1 - \frac{n_0}{n}\Big)t + \frac{\pi}{2}\,\frac{n_0^2\,9^{n_0}}{n\,9^n}$$

for $9^{-(n+1/2)} \leq t \leq 9^{-n}$. Thus

$$\sum_{n=n_0}^{n_1-1} \Big| \mathrm{I}_n - \int_{9^{-(n+1/2)}}^{9^{-n}} \sin\Big(\frac{\pi\, n_0^2\, 9^{n_0}}{2n\, 9^n}\Big)\frac{dt}{t}\Big| \leq C \sum_{n=n_0}^{n_1-1} n_0\, 9^{n_0-n}\Big(1-\frac{n_0}{n}\Big) \leq C\,.$$

And

$$\Big|\sum_{n=n_0}^{n_1-1}\Big(\sin\Big(\frac{\pi\, n_0^2\, 9^{n_0}}{2n\, 9^n}\Big) - \sin\Big(\frac{\pi\, n_0\, 9^{n_0}}{2\, 9^n}\Big)\Big)\Big| \leq C \sum_{n=n_0}^{n_1-1} n_0\, 9^{n_0-n}\Big(1-\frac{n_0}{n}\Big) \leq C\,.$$

Therefore it suffices to show that

$$\sum_{n=n_0}^{n_1-1} \sin\Big(\frac{\pi}{2}\, n_0\, 9^{n_0-n}\Big) = \sum_{k=0}^{n_1-n_0-1} \sin\Big(\frac{\pi}{2}\, n_0\, 9^{-k}\Big)$$

is unbounded as $n_0 \to \infty$. Take $n_0 = 9^N$ for some N and note that since $k \leq n_1 - n_0$ implies that $k \leq N$ by (32), we have

$$\sum_{k=0}^{n_1-n_0-1} \sin\Big(\frac{\pi}{2}\, 9^{N-k}\Big) = n_1 - n_0 \geq \varepsilon \log n_0$$

for some $\varepsilon > 0$ by (32) and this completes the proof that $\mathcal{H}_{\Gamma_1}$ is unbounded in L^2. It is easy to modify γ_1 to obtain a smooth γ whose maximal function and Hilbert transform is still unbounded in every L^p and such that $v(t) \sim -1/\log t$ does not have the doubling property, but $w(t) \sim -t/\log t$ is doubling.

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