REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 10, N.° 1, 1994

A Paley-Wiener theorem for step two nilpotent Lie groups

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1. Introduction.

The classical Paley-Wiener Theorem for the Fourier transform on $\mathbb{R}^n$ which characterises compactly supported functions in terms of their Fourier transforms plays an important role in many problems of Fourier analysis. It is therefore desirable to have analogues of the Paley-Wiener Theorem for compactly supported functions on general Lie groups whenever there is a group Fourier transform available. For the case of the spherical Fourier transform on semi simple Lie groups an analogue of the Paley-Wiener Theorem is known. In 1976, Ando [1] proved a Paley-Wiener type theorem for the Heisenberg group which is the simplest example of a nilpotent Lie group which is nonabelian. Recently, we have proved another Paley-Wiener theorem for the Heisenberg group, *cf.* [4]. In both papers the explicit form of the representations on the Heisenberg group has played an important role in formulating and proving Paley-Wiener theorems.

It is an interesting open problem to establish Paley-Wiener theorems for general nilpotent Lie groups. The aim of this paper is prove one such theorem for step two nilpotent Lie groups which is analogous to the Paley-Wiener theorem for the Heisenberg group proved in [4].

2. Some basic facts about nilpotent Lie groups.

In this section we briefly recall some basic results from the representation theory of nilpotent Lie groups. A general reference is the book [2] by Corwin and Greenleaf.

Let G be a nilpotent Lie group of dimension n and let $\mathfrak{g}$ be its Lie algebra. Denote the dual of the Lie algebra by $\mathfrak{g}^*$ and the centre of the enveloping algebra $u(\mathfrak{g})$ by $\zeta(\mathfrak{g})$. Every coadjoint orbit in $\mathfrak{g}^*$ is even dimensional. Let $2k$ be the maximal dimension which occurs and let $q = n - 2k$. Then there exists a nonempty Zariski-open subset Γ of $\mathbb{R}^q$ and for each $\lambda \in \Gamma$ there is a unitary irreducible representation π_λ of G realised on $L^2(\mathbb{R}^k)$. Moreover, there exists a rational function $R(\lambda)$ regular on Γ and unique up to multiplication by -1 such that the Plancherel Formula holds with $d\mu(\lambda) = |R(\lambda)| d\lambda_1 \cdots d\lambda_q$,

$$\int_G |f(g)|^2 \, dg = \int_\Gamma \|\pi_\lambda(f)\|_{HS}^2 \, d\mu(\lambda) \tag{2.1}$$

for $f \in L^1(G) \cap L^2(G)$. Here dg is the Haar measure on G and HS stands for the Hilbert-Schmidt norm.

We let K stand for the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^k)$ with the inner product $\langle T, S \rangle = \mathrm{Tr}\,(TS^*)$ and let $L^2(\Gamma, K)$ be the space of L^2 functions on Γ with values in K taken with respect to the measure $d\mu(\lambda)$. Then there is a unique bijective isometry, $\Phi : L^2(G) \longrightarrow L^2(\Gamma, K)$ such that for every f in $L^1(G) \cap L^2(G)$, $\Phi(f)$ is the function $\lambda \mapsto \pi_\lambda(f)$. Here and in (2.1) $\pi_\lambda(f)$ is the operator defined by

$$\pi_\lambda(f) = \int_G f(g)\, \pi_\lambda(g)\, dg\,. \tag{2.2}$$

The function Φ is called the group Fourier transform.

Now each representation π_λ of G defines a skew adjoint representation, also denoted by π_λ, of the Lie algebra $\mathfrak{g}$ by the formula

$$\pi_\lambda(X)\, \phi = \frac{d}{dt}\Big|_{t=0} \pi_\lambda(\exp tX)\phi\,, \tag{2.3}$$

where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map and ϕ is a C^∞ vector for π_λ. The skew adjointness of π_λ means that $\pi_\lambda(X)^* = -\pi_\lambda(X)$. Let

$X_1, X_2, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$. It then follows that if for each $\xi \in \mathbb{R}^n$ we define $U_\lambda(\xi)$ by

$$U_\lambda(\xi) = \exp\Big(-\sum_{j=1}^{n} \xi_j \, \pi_\lambda(X_j)\Big), \tag{2.4}$$

then $U_\lambda(\xi)$ becomes a unitary operator. This operator valued function is crucial for formulating a Paley-Wiener theorem.

We now specialise to the case of step two nilpotent Lie groups. In this case the group admits a dilation structure. By this we mean the existence of a family $\{\delta_r : r > 0\}$ of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = \exp(A \log r)$ where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues. In the case of step two nilpotent Lie groups the eigenvalues of A can be assumed to be 1 and 2. The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphisms of G and are called dilations of G. For facts about groups admitting dilations we refer to the monograph [3] of Folland and Stein. A function f defined on G is said to be homogeneous of degree α if $f(\delta_r g) = r^\alpha f(g)$ for all $r > 0$.

If G is a simply connected step two nilpotent Lie group then the exponential map is a global diffeomorphism. Using exponential coordinates we can identify G with $\mathbb{R}^{n-m} \times \mathbb{R}^m$ and the group law can be written in the form

$$(x,t) \cdot (y,s) = (x + y, t + s + F(x,y)), \tag{2.5}$$

where F is a bilinear form from $\mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m$. Let Y_j be the right invariant vector fields agreeing with X_j at the origin. Recall that X_j and Y_j are defined by

$$X_j f(g) = \frac{d}{dt}\Big|_{t=0} f(g \cdot \exp tX_j), \tag{2.6}$$

$$Y_j f(g) = \frac{d}{dt}\Big|_{t=0} f(\exp tX_j \cdot g). \tag{2.7}$$

We then have the following lemma.

Lemma 2.1. *Let G be a step two nilpotent Lie group admitting dilations as above. Then there exists polynomials $P_{jk}(x)$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$ on $\mathbb{R}^{n-m}$ homogeneous of degree one such that*

$$(Y_j f - X_j f) = \sum_{k=1}^{m} P_{jk}(x) \frac{\partial f}{\partial t_k}, \qquad j = 1, 2, \ldots, n. \tag{2.8}$$

PROOF. This lemma can be easily proved using the definitions (2.6) and (2.7). Expressions for $X_j f$ and $Y_j f$ have been obtained in [3] (see Proposition 1.26). That P_{jk} are independent of t follows from the fact that they are homogeneous of degree one. (For details see [3]). We also remark that for some j it may happen that $X_j = Y_j$.

The above lemma is very important for our purpose and its importance will become apparent soon. First we make some observations and a definition. Direct calculation shows that

$$\pi_\lambda(-X_j f) = \pi_\lambda(f)\,\pi_\lambda(X_j)\,, \tag{2.9}$$

$$\pi_\lambda(Y_j f) = -\pi_\lambda(X_j)\,\pi_\lambda(f)\,. \tag{2.10}$$

Together these two equations imply that

$$\pi_\lambda(Y_j f - X_j f) = [\pi_\lambda(f), \pi_\lambda(X_j)]\,, \tag{2.11}$$

where $[T, S]$ stands for the commutator $TS - ST$. On the space of bounded operators on $L^2(\mathbb{R}^k)$ we can define n derivations $\delta_j(\lambda)$ by

$$\delta_j(\lambda)T = [T, \pi_\lambda(X_j)]\,. \tag{2.12}$$

Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ we define

$$\delta(\lambda)^\alpha = \delta_1(\lambda)^{\alpha_1} \cdots \delta_n(\lambda)^{\alpha_n}\,. \tag{2.13}$$

We say that an operator T is of class C^k if $\delta(\lambda)^\alpha T$ is bounded for all α with $|\alpha| \leq k$.

Now from Lemma 2.1 and equation (2.11) we have the interesting formula

$$\pi_\lambda\Big(\sum_{k=1}^{n} P_{jk}(x)\,\frac{\partial f}{\partial t_k}\Big) = \delta_j(\lambda)\,\pi_\lambda(f)\,. \tag{2.14}$$

The operations $\delta_j(\lambda)$ are derivations in the sense that

$$\delta_j(\lambda)(TS) = T\delta_j(\lambda)S + \delta_j(\lambda)TS\,.$$

The above formula connects multiplication by polynomials on the function side and derivations on the Fourier transform side and may be considered as the analogue of the formula

$$(-2\pi i x_j f)\hat{}\,(\xi) = \frac{\partial}{\partial \xi_j}\hat{f}(\xi) \tag{2.15}$$

for the euclidean Fourier transform. In the next section we formulate and prove a Paley-Wiener theorem using formula (2.14).

3. A Paley-Wiener theorem for smooth functions.

For the sake of simplicity we consider functions in the Schwartz class $\mathcal{S}(G)$. Let f be a function in $\mathcal{S}(G)$ and let $\tilde{f}$ stand for the partial Fourier transform in the central variable,

$$\tilde{f}(x,t) = \int_{\mathbb{R}^m} e^{-2\pi i t\cdot s} f(x,s)\, ds\,. \tag{3.1}$$

The formula (2.14) applied to $\tilde{f}$ takes the form

$$\delta_j(\lambda)\,\pi_\lambda(\tilde{f}) = -2\pi i\,\pi_\lambda((P_j f)^{\sim})\,, \tag{3.2}$$

where the function $P_j(x,t)$ is defined by

$$P_j(x,t) = \sum_{k=1}^{m} P_{jk}(x)\, t_k\,. \tag{3.3}$$

We now define the modified Fourier transform of f in the following way. For each $\xi \in \mathbb{R}^n$, $\hat{f}(\xi)$ takes values in $L^2(\Gamma, K)$ and is given by

$$\hat{f}(\xi)(\lambda) = U_\lambda(\xi)\,\pi_\lambda(\tilde{f})\,U_\lambda(-\xi)\,. \tag{3.4}$$

Recalling the definition of $U_\lambda(\xi)$, taking derivative with respect to ξ_j and using (3.2) we obtain the interesting relation

$$\frac{\partial}{\partial \xi_j}\,\hat{f}(\xi) = -2\pi i\,(P_j f)^{\wedge}(\xi)\,, \tag{3.5}$$

which is the analogue of (2.15) for our modified Fourier transform on the group G.

The classical Paley-Wiener Theorem for the euclidean Fourier transform follows immediately from (2.15). If f is supported in $|x_j| \le B$, $j = 1, 2, \ldots, n$, then it follows that $|\partial_\xi^\alpha \hat{f}(\xi)| \le C\,(2\pi B)^{|\alpha|}$ and this leads directly to the extendability of $\hat{f}(\xi)$ as an entire function on $\mathbb{C}^n$ satisfying the estimate

$$|\hat{f}(\zeta)| \le e^{2\pi B|\operatorname{Im}\zeta|}\,. \tag{3.6}$$

In the same spirit we would like to set up an isomorphism between functions f supported in a set defined by the inequalities $|P_j(x,t)| \leq B$, $j = 1, 2, \ldots, n$ and a class of entire functions on $\mathbb{C}^n$.

To this end let $H_B(\mathbb{C}^n)$ stand for the space of entire functions $F(\zeta)$ taking values in the Hilbert space $E = L^2(\Gamma, K)$ which agrees with $\hat{f}(\xi)$ on $\mathbb{R}^n$ for some $f \in \mathcal{S}(G)$ and satisfies the estimate

$$\|F(\zeta)\|_E \leq C\, e^{2\pi B|\mathrm{Im}\,\zeta|}\,. \tag{3.7}$$

Let G_B stand for the set defined by

$$G_B = \{(x,t) \in G : \ |P_j(x,t)| \leq B,\ j = 1, 2, \ldots, n\}\,. \tag{3.8}$$

Then we have the following theorem. Let $C^\infty(G_B)$ stand for the set of all smooth f supported in G_B.

Theorem 3.1. *The modified Fourier transform sets up an isomorphism between $\mathcal{S}(G) \cap C^\infty(G_B)$ and $H_B(\mathbb{C}^n)$.*

PROOF. The direct part of this theorem is easy. If $f \in \mathcal{S}(G) \cap C^\infty(G_B)$ then iteration of (3.5) gives us

$$\partial_\xi^\alpha \hat{f}(\xi) = (-2\pi i)^{|\alpha|}(P^\alpha f)^\wedge(\xi)\,, \tag{3.9}$$

where $P^\alpha(x,t) = P_1(x,t)^{\alpha_1} \cdots P_n(x,t)^{\alpha_n}$. It then follows that

$$\|\partial_\xi^\alpha \hat{f}(\xi)\|_E^2 = (2\pi)^{2|\alpha|} \int_\Gamma \|\pi_\lambda(P^\alpha f)^\wedge\|_{HS}^2\, d\mu\,, \tag{3.10}$$

which by Plancherel Theorem gives the estimate

$$\begin{aligned} \|\partial_\xi^\alpha \hat{f}(\xi)\|_E^2 &= (2\pi)^{2|\alpha|} \int_G |P^\alpha(g)\, f(g)\|^2\, dg \\ &\leq (2\pi B)^{2|\alpha|} \int_G |f(g)|^2\, dg\,. \end{aligned} \tag{3.11}$$

From these estimates it follows that the series

$$F(\zeta) = \sum \frac{\partial_\xi^\alpha \hat{f}(0)}{\alpha!}\, \zeta^\alpha \tag{3.12}$$

converges in the norm of E and represents an entire function. Moreover, the expansion

$$F(\zeta) = \sum \frac{\partial_\xi^\alpha \hat{f}(\xi)}{\alpha!} (i\eta)^\alpha , \qquad \zeta = \xi + i\eta , \tag{3.13}$$

gives the estimate

$$\|F(\zeta)\|_E \leq C\, e^{2\pi B|\mathrm{Im}\,\zeta|} , \tag{3.14}$$

This proves that $F(\zeta) \in H_B(\mathbb{C}^n)$.

We now turn to the converse. Let $F \in H_B(\mathbb{C}^n)$ and let f be the Schwartz class function such that $\hat{f}(\xi) = F(\xi)$ for $\xi \in \mathbb{R}^n$. We need to show that f is supported in G_B. This will follow immediately if we can show that

$$\int_G |P_j(x,t)|^{2k} |f(x,t)|^2 \, dg \leq B^{2k} \|f\|_2^2 \tag{3.15}$$

for all k. Again in view of the equation (3.11) it is enough to show that

$$\left\| \left(\frac{\partial}{\partial \xi_j} \right)^k \hat{f}(0) \right\|_E^2 \leq (2\pi B)^{2k} \|f\|_2^2 . \tag{3.16}$$

In order to establish this we proceed as follows.

Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be a real valued function supported in $|x| \leq 1$ and $\int |\theta(x)|^2 \, dx = 1$. For $\varepsilon > 0$ let $\theta_\varepsilon(x) = \varepsilon^{-n/2}\theta(x/\varepsilon)$ so that $\hat{\theta}_\varepsilon(\xi) = \varepsilon^{n/2}\hat{\theta}(\varepsilon\xi)$. (Here $\hat{\theta}$ stands for the euclidean Fourier transform on $\mathbb{R}^n$). By the classical Paley-Wiener Theorem we know that $\hat{\theta}_\varepsilon$ extends to an entire function which verifies the estimate

$$|\hat{\theta}_\varepsilon(\zeta)| \leq C_\varepsilon\, e^{2\pi\varepsilon|\mathrm{Im}\,\zeta|} . \tag{3.17}$$

As $\theta \in C_0^\infty$ we also know that $\hat{\phi}_\varepsilon \in L^2(\mathbb{R}^n)$.

We now consider the function $M_\varepsilon(\zeta) = \hat{\theta}_\varepsilon(\zeta)\, F(\zeta)$. This is an entire function taking values in E and satisfies

$$\|M_\varepsilon(\zeta)\|_E \leq C_\varepsilon\, e^{2\pi(B+\varepsilon)|\mathrm{Im}\,\zeta|} . \tag{3.18}$$

Moreover the calculation

$$\int_{\mathbb{R}^n} \|M_\varepsilon(\xi)\|_E^2 \, d\xi = \|f\|_2^2 \int_{\mathbb{R}^n} |\hat{\theta}_\varepsilon(\xi)|^2 \, d\xi = \|f\|_2^2 \tag{3.19}$$

shows that $M_\varepsilon \in L^2(\mathbb{R}^n, E)$. Now we can appeal to the classical Paley-Wiener Theorem to conclude that there is a function $T_\varepsilon \in L^2(\mathbb{R}^n, E)$ supported in $|x| \leq (B+\varepsilon)$ such that

$$M_\varepsilon(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} T_\varepsilon(x)\, dx\,. \tag{3.20}$$

Differentiating this k times with respect to ξ_j we get the relation

$$\Big(\frac{\partial}{\partial\xi_j}\Big)^k M_\varepsilon(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}(-2\pi i x_j)^k T_\varepsilon(x)\, dx\,. \tag{3.21}$$

By the euclidean Plancherel Theorem this gives the estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \Big\|\Big(\frac{\partial}{\partial\xi_j}\Big)^k M_\varepsilon(\xi)\Big\|_E^2\, d\xi &= \int_{\mathbb{R}^n} \|(2\pi x_j)^k T_\varepsilon(x)\|_E^2\, dx \\ &\leq (2\pi(B+\varepsilon))^{2k} \int_{\mathbb{R}^n} \|T_\varepsilon(x)\|_E^2\, dx \\ &= (2\pi(B+\varepsilon))^{2k} \int_{\mathbb{R}^n} \|M_\varepsilon(\xi)\|_E^2\, d\xi \\ &= (2\pi(B+\varepsilon))^{2k}\, \|f\|_2^2\,.\end{aligned} \tag{3.22}$$

Finally, as $M_\varepsilon(\xi)$ is the product of $\hat{\theta}_\varepsilon(\xi)$ and $F(\xi)$ we have by Leibnitz Formula the relation

$$\partial_j^k M_\varepsilon(\xi) = \sum_{\ell=0}^{k} \binom{k}{\ell} \partial_j^\ell \hat{\theta}_\varepsilon(\xi)\, \partial_j^{k-\ell} \hat{f}(\xi)\,, \tag{3.23}$$

where ∂_j stands for $\partial/\partial\xi_j$. From the above relation we calculate

$$\begin{aligned}\|\partial_j^k M_\varepsilon(\xi)\|_{HS}^2 = \sum_{\ell=0}^{k}\sum_{i=0}^{k} \binom{k}{\ell}\binom{k}{i} \partial_j^\ell \hat{\theta}_\varepsilon(\xi) \partial_j^i \hat{\theta}_\varepsilon(\xi) \\ \cdot \operatorname{Tr}\big((\partial_j^{k-\ell}\hat{f}(\xi))(\partial_j^{k-i}\hat{f}(\xi))^*\big)\,.\end{aligned} \tag{3.24}$$

But now

$$\operatorname{Tr}\big((\partial_j^{k-\ell}\hat{f}(\xi))(\partial_j^{k-i}\hat{f}(\xi))^*\big) = \operatorname{Tr}\big((\partial_j^{k-\ell}\hat{f}(0))(\partial_j^{k-i}\hat{f}(0))^*\big)$$

and we have the inequality

$$\sum_{\ell=0}^{k}\sum_{i=0}^{k}\binom{k}{\ell}\binom{k}{i}\Big(\int_{\mathbb{R}^n}\partial_j^{\ell}\hat{\theta}_\varepsilon(\xi)\partial_j^{i}\hat{\theta}_\varepsilon(\xi)\,d\xi \cdot \int_\Gamma \mathrm{Tr}\left((\partial_j^{k-\ell}\hat{f}(0))(\partial_j^{k-i}\hat{f}(0))^*\right)d\mu(\lambda) \leq (2\pi(B+\varepsilon))^{2k}\|f\|_2^2\,. \tag{3.25}$$

As $\hat{\theta}_\varepsilon(\xi) = \varepsilon^{n/2}\hat{\theta}(\varepsilon\xi)$ it follows that the integral

$$\int_{\mathbb{R}^n}\partial_j^{\ell}\hat{\theta}_\varepsilon(\xi)\,\partial_j^{i}\hat{\theta}_\varepsilon(\xi)\,d\xi = \varepsilon^{\ell+i}\int_{\mathbb{R}^n}(\partial_j^{\ell}\hat{\theta})(\xi)\,(\partial_j^{i}\hat{\theta})(\xi)\,d\xi$$

tends to 0 as $\varepsilon \to 0$ unless $\ell = i = 0$. Therefore, if we let $\varepsilon \to 0$ in (3.25) the only surviving term is the one with $\ell = i = 0$ and we get

$$\int_\Gamma \mathrm{Tr}\left((\partial_j^{k}\hat{f}(0))\,(\partial_j^{k}\hat{f}(0))^*\right)d\mu(\lambda) \leq (2\pi B)^{2k}\|f\|_2^2\,, \tag{3.26}$$

which proves (3.16).

This completes the proof of the theorem.

4. Some remarks and an example.

We have established a Paley-Wiener theorem for Schwartz class functions that are supported in sets of the form G_B and this class includes $C_0^\infty(G)$. The sets G_B are not compact and this is in sharp contrast with the classical case where one has to consider C_0^∞ functions for the holomorphic extendability of the Fourier transform. Nevertheless, we can say something more about the sets for a class of nilpotent Lie groups of step two which includes the famous Heisenberg groups.

Let G be a step two nilpotent Lie group with one dimensional centre so that $G = \mathbb{R}^{n-1} \times \mathbb{R}$. Then the polynomials $P_j(x,t)$ take the form

$$P_j(x,t) = p_j(x)\,t\,, \tag{4.1}$$

where $p_j(x)$ are homogeneous of degree one. Let

$$p_j(x) = \sum_{k=1}^{n-1} C_{jk}\,x_k \tag{4.2}$$

and further assume that the matrix (C_{kj}) is invertible. Under this assumption the conditions $|x_j| \leq B$, $j = 1, 2, \ldots, n-1$, will be equivalent to $|p_j(x)| \leq aB$, $j = 1, 2, \ldots, n-1$, for some $a > 0$. In this situation, though the set G_B is not compact its projection onto $\mathbb{R}^{n-1}$ is compact for each fixed t. Therefore, if $F \in H_B(\mathbb{C}^n)$ and $F(\xi) = \hat{f}(\xi)$ then for each fixed t, $f(x,t)$ will be compactly supported in a set of the form $|x_j| \leq aB$, $j = 1, 2, \ldots, n-1$. But regarding the support of f as a function of t we could say nothing.

The above situation is well explained by the example of the Heisenberg group $H_n = \mathbb{R}^{2n} \times \mathbb{R}$. In this case the group law is given by

$$(x,t)(x',t') = (x + x', t + t' + F(x,x')) \tag{4.3}$$

with

$$F(x,x') = \frac{1}{2} \sum_{j=1}^{n} (x'_j x_{j+n} - x_j x'_{j+n}) .$$

The left invariant vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} x_{n+j} \frac{\partial}{\partial t} , \qquad j = 1, 2, \ldots, n , \tag{4.4}$$

$$X_{j+n} = -\frac{\partial}{\partial x_{n+j}} - \frac{1}{2} x_j \frac{\partial}{\partial t} , \qquad j = 1, 2, \ldots, n . \tag{4.5}$$

A calculation shows that

$$Y_j - X_j = x_{n+j} \frac{\partial}{\partial t} , \qquad j = 1, 2, \ldots, n , \tag{4.6}$$

$$Y_{j+n} - X_{j+n} = x_j \frac{\partial}{\partial t} , \qquad j = 1, 2, \ldots, n . \tag{4.7}$$

As $X_{2n+1} = Y_{2n+1} = \partial/\partial t$ we neglect the difference $X_{2n+1} - Y_{2n+1}$. Thus $p_j(x) = x_{n+j}$, $j = 1, 2, \ldots, n$, $p_j(x) = x_{j-n}$, $j = n+1, \ldots, 2n$, and we are in the above situation.

Another interesting feature of the Heisenberg group is the fact that $\pi_\lambda(x,t) = \pi_\lambda(x)\, e^{i\lambda t}$ and each $\pi_\lambda(x)$ defines a projective representation of $\mathbb{R}^{2n}$. Therefore, one could completely discard the variable t and consider functions on $\mathbb{R}^{2n}$ and define the so called Weyl transform. For the Weyl transform we have proved a Paley-Wiener theorem in [4] and there the isomorphism is between $C_0^\infty(\mathbb{R}^{2n})$ and a class of entire functions taking values in K.

Acknowledgements. The author wishes to thank E. M. Stein and R. S. Strichartz for useful conversations regarding this paper. He also wants to thank the referee and L. Richardson for pointing out a mistake in the original version of the paper.

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Recibido: 12 de abril de 1.993

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