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A characterization of 2-knots groups

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An n-knot group is the fundamental group of the complement of an n-sphere smoothly embedded in S^{n+2}.

Artin gave in 1925 ([A]) an algebraic characterization of 1-knot groups:

Theorem ([A]). *A group is a 1-knot group if and only if it has a presentation* $(x_1, \dots, x_n : x_j^{-1}\beta_j,\ \ 1 \leq j \leq n)$ *such that*

1) *For* $j = 1, \dots, n$ β_j *is conjugate to* $x_{\mu(j)}$ *in the free group* F *generated by* $x_1, \dots, x_n$,

2) $\prod_{j=1}^{n} \beta_j = \prod_{j=1}^{n} x_j$ *in* F, *and*

3) μ *is the permutation* $(1\ 2 \cdots n)$.

M. Kervaire gave in 1965 ([K]) an algebraic characterization of n-knot groups for $n \geq 3$.

Theorem ([K]). *Let* $n \geq 3$. *A group* G *is an* n*-knot group if and only if*

i) G *is finitely presented,*

ii) G *is normally generated by one element,*

iii) $H_1(G) = \mathbb{Z}$, *and*

iv) $H_2(G) = 0$.

We remark that if we drop the smoothness asumption in the definition of a knot, then their groups do not satisfy ii) in general. There are examples of wild 1-knots whose groups are free products $\mathbb{Z} * H$ that cannot be normally generated by one element (see [D] and [AF, example 2.1]). It is a conjecture, known to be true for $n = 1$, that (smooth) n-knot groups cannot be free products. Condition i) does not hold in general for groups of wild knots. In fact the group of a wild knot is a (smooth) 1-knot group if and only if it is finitely generated ([GHM]).

The problem of characterizing algebraically 2-knot groups has been posed several times (see for example [Su, Problem 4.7]). Ribbon 2-knot groups have been characterized algebraically by Yajima [Y].

We will give here a characterization of 2-knot groups in terms of presentations. It has the flavor of Artin's characterization of 1-knot groups. S. Kamada has independently, obtained another characterization of 2-knot groups ([Ka]).

A presentation $\mathfrak{G}$ of non positive deficiency $-h$ is *saddled* if it is of the form

$$(*)\quad \mathfrak{G} = \left\{x_1, \ldots, x_n :\ x_{2i-1}^{-1} x_{2i}\,,\ \ x_j^{-1}\beta_j\,,\ \ 1 \leq i \leq h,\ \ 1 \leq j \leq n\right\},$$

where

1) For $j = 1, \ldots, n$ β_j is conjugate to $x_{\mu(j)}$ in the free group F generated by $x_1, \ldots, x_n$,

2) $\prod_{j=1}^{n} \beta_j = \prod_{j=1}^{n} x_j$ in F.

If, in addition, the permutations μ and $\nu = \prod_{i=1}^{h}(2i-1, 2i)$ generate a transitive group of permutations of $\{1, 2, \ldots, n\}$ then we call $\mathfrak{G}$ *connected.*

The *genus* of the connected saddled presentation $\mathfrak{G}$ is $1 - (|\mu| - h + |\mu\nu|)/2$ where $|\pi|$ is the number of cycles of the permutation π.

The saddled presentation (*) is *unlinked* if

$$(x_1, \ldots, x_n : x_j^{-1}\beta_j \ \ 1 \leq j \leq n)$$

and

$$(x_1, \ldots, x_n : x_j^{-1}\beta'_j \ \ 1 \leq j \leq n)$$

present free groups, where

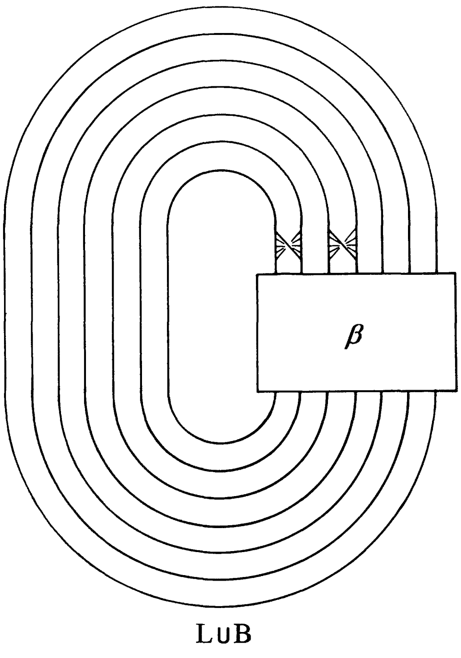
$$\beta'_j = \begin{cases} \beta_j\beta_{j+1}\beta_j^{-1}\,, & \text{if } j \text{ is odd and } j < 2h\,, \\ \beta_{j-1}\,, & \text{if } j \text{ is even and } j \leq 2h\,, \\ \beta_j\,, & \text{if } j > 2h \end{cases}$$

(Notice these two presentations are saddled of deficiency 0. Also $\beta_j\beta_{j+1}$ β_j^{-1} can be replaced by $x_{j+1}\beta_{j+1}x_{j+1}^{-1}$).

Theorem 1. *A group is a* 2*-knot group if and only if it has a connected, unlinked, saddled presentation of genus* 0 (*c.u.s.p.* 0).

PROOF. A saddled presentation (*) determines, in a constructible way, a compact orientable 2-manifold S properly embedded in $S^3 \times [-1, 1]$ as follows. First one can construct a (unique) braid β on n strings whose corresponding automorphism sends x_j to β_j. (See Birman's book [B, Corollary 1.8.3 and proof of theorem 1.9])

Let $L \subset S^3$ be the closed braid determined by β.



L∪B

Figure 1

Let B be the union of the bands joining the $(2i-1)$-string to the $2i$-string, $i = 1, \dots, h$ indicated in figure 1. Let $L' = \overline{L \cup \partial B - L \cap B}$, that is, L' is the closed braid determined by $\beta \prod_{i=1}^{h} \sigma_{2i-1}$.

Then $S = L \times [-1, 0] \cup B \times \{0\} \cup L' \times [0, 1] \subset S^3 \times [-1, 1]$. The fundamental group of $S^3 \times [-1, 1] - S$ is presented by $\mathfrak{G}$. S is connected if and only if $\mathfrak{G}$ is connected and if this is the case, the genus of S is the genus of $\mathfrak{G}$.

Every compact orientable 2-manifold properly and smoothly embedded in $S^3 \times [-1, 1]$ with no elliptic points is isotopic to a surface determined by a saddled presentation. (The proof is similar to that of Alexander's Theorem ([B, Theorem 2.1]))

The group of $S^3 - L$ is presented by $(x_1, \dots, x_n : x_j^{-1}\beta_j \;\; j = 1, \dots, n)$ and the group of $S^3 - L'$ is presented by $(x_1, \dots, x_n : x_j^{-1}\beta'_j \;\; j = 1, \dots, n)$ so L and L' are trivial if and only if $\mathfrak{G}$ is unlinked. If this is the case then L bounds D, a union of disjoint disks in S^3, and L' bounds D', a union of disjoint disks in S^3, so that, if in addition, $\mathfrak{G}$ is connected and of genus 0, then

$$\Sigma^2 = D \times \{-1\} \cup S \cup D' \times \{1\} \subset S^3 \times [-1, 1] \subset \partial(B^4 \times [-1, 1]) = S^4$$

is a 2-knot whose group is still presented by $\mathfrak{G}$.

Every smooth 2-knot in S^4 is isotopic to one constructed as above. Hence G is a 2-knot group if and only if it has a c.u.s.p. 0.

Figure 2 describes a deformation of a description of the spun trefoil by a link with two bands so that a c.u.s.p. 0 can be read in Figure 3. Group generators are numbers, $\overline{n}$ denotes the inverse of n and $x^y = y^{-1}xy$.

It is easy to decide if a given finite presentation $\mathfrak{G}$ is saddled and connected and, if so, to compute its genus. Since one can decide whether a given link is trivial ([H],[S, Satz 4.1]), one can decide whether $\mathfrak{G}$ is unlinked. Hence the set of c.u.s.p. 0's is a recursive subset of the set of finite presentations.

Thus the set of smooth 2-knots is recursively enumerable (Markov's Theorem (see [B, Theorem 2.3]) helps to do the enumeration a little less inefficient). It is then possible to construct, from a given presentation of an 2-knot group G a 2-knot with that group: recursively enumerate all finite presentations of G until one finds a c.u.s.p. 0 $\mathfrak{G}$ and then construct the 2-knot determined by $\mathfrak{G}$.

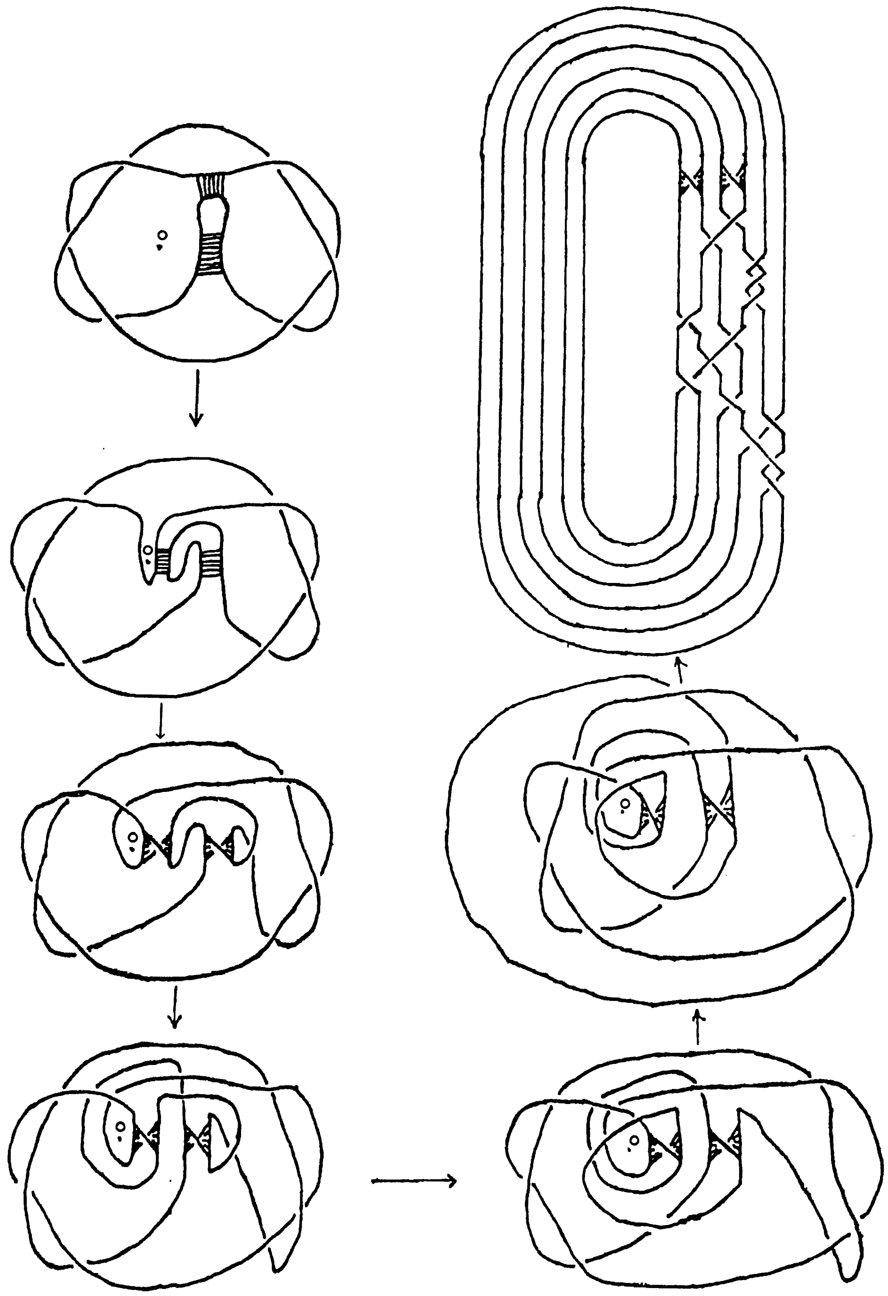


Figure 2

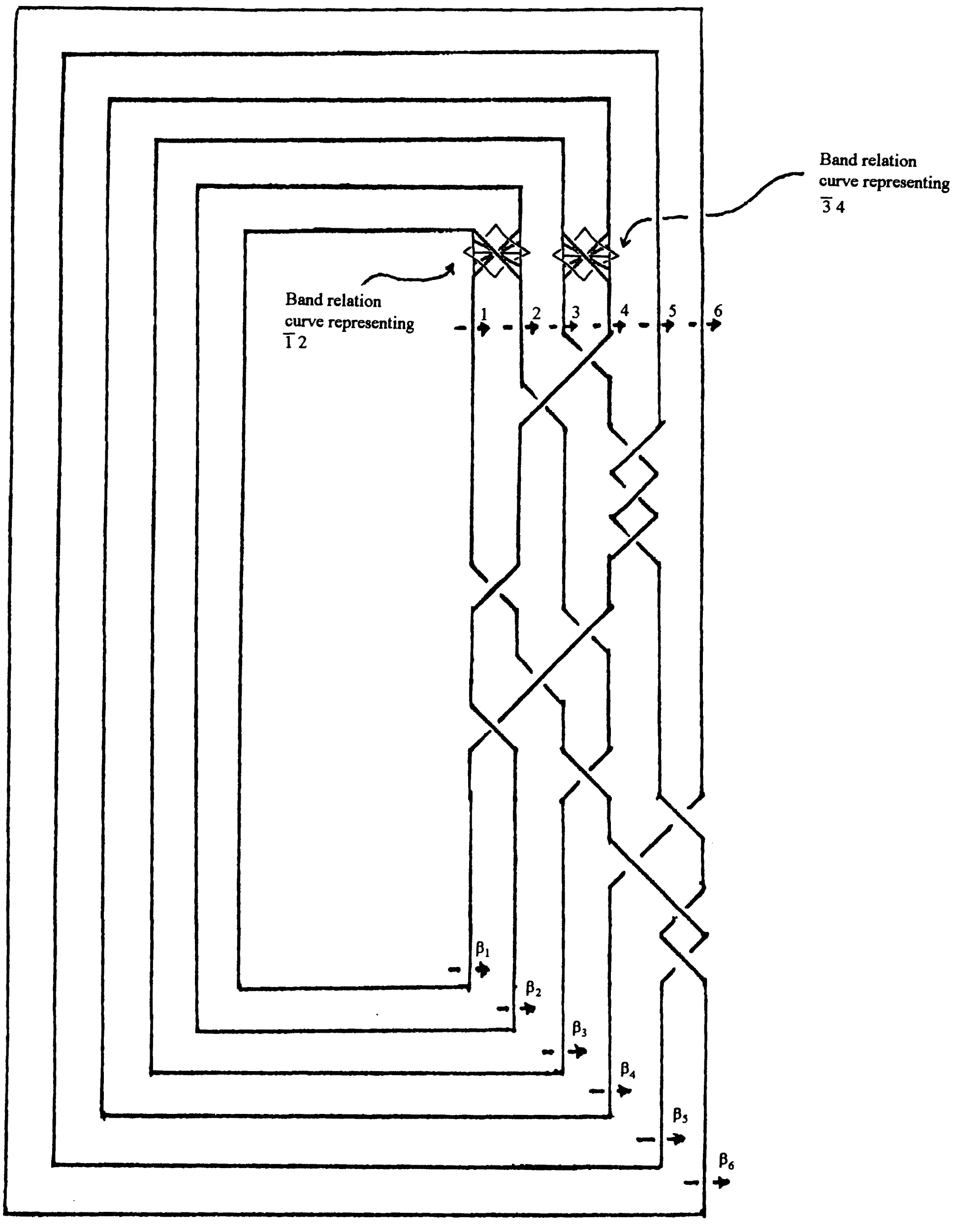

Band relation
curve representing
$\bar{3}\,4$
Band relation
curve representing
$\bar{1}\,2$
1
2
3
4
5
6
β_1
β_2
β_3
β_4
β_5
β_6

Figure 3

Theorem 1 can be generalized to treat the case of embeddings of a given compact orientable 2-manifold with empty boundary in S^4. Consider the saddled presentation (*) and the permutation

$$\nu = \prod_{i=1}^{h} (2i-1, 2i).$$

Denote by $T_1, \ldots, T_r$ the orbits of the elements of $\{1, \ldots, n\}$ under the action of the group generated by μ and ν. For $k = 1, \ldots, r$ let μ_k and ν_k be the restrictions of μ and ν to T_k and let h_k be the number of nontrivial cycles of ν_k. Write $g_k = 1-(|\mu_k|-h_k+|\mu_k\nu_k|)/2$. We call the unordered sequence $(g_1, \ldots, g_r)$ the *type* of the saddled presentation.

Theorem 2. *Let $M^2 = \coprod_{i=1}^{r} M_{g_i}$ be the 2-manifold with components $M_{g_1}, \ldots, M_{g_r}$ where M_g denotes a closed orientable surface of genus g. Then G is the group of the complement of a smooth submanifold of S^4 diffeomorphic to M^2 if and only if G has a saddled unlinked presentation of type $(g_1, \ldots, g_r)$.*

The proof is similar to that of Theorem 1.

Similar characterizations can be given to deal with groups of 2-manifolds M properly embedded in D^4. One would require the saddled presentation to be only "partially unlinked": $(x_1, \ldots, x_n : x_j^{-1}\beta_j \;\; 1 \leq j \leq n)$ should present a free group but $(x_1, \ldots, x_n : x_j^{-1}\beta_j' \;\; 1 \leq j \leq n)$ should present a free product $L * F_{|\mu\nu|-|\partial M|}$ the second factor being a free group on $|\mu\nu|-$(number of components of ∂M) generators.

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