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Weak-type estimates for the Riesz transforms associated with the gaussian measure

Eugene B. Fabes, Cristian E. Gutiérrez and Roberto Scotto

1. Introduction.

In this paper, we will study the behavior of the Riesz transforms associated with the Gaussian measure $\gamma(x)\,dx = e^{-|x|^2}\,dx$ in the space $L^1_\gamma(\mathbb{R}^n)$. These transformations are defined by

$$\mathcal{R}_j f(y) = \lim_{\varepsilon \to 0} \int_{|y-z|>\varepsilon} k_j(y,z)\, f(z)\, dz\,,$$

where

$$k_j(y,z) = \int_0^1 \Big(\frac{1-r^2}{-\log r}\Big)^{1/2} \frac{z_j - r\, y_j}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr\,,$$

$j = 1, \dots, n$.

The study of the boundedness properties of $\mathcal{R}_j$ in the spaces $L^p_\gamma(\mathbb{R}^n)$ began with the work of B. Muckenhoupt [Mu], when the dimension is $n = 1$. He proved the boundedness of this transformation (in this case it is only one operator) when $p > 1$ and the weak-type (1,1). In higher dimensions the L^p-boundedness, $p > 1$, was first proved by

P. A. Meyer [Me], by using probabilistic methods. The same result was also proved by several authors, [Gn], [Gt], [Pi], and [U]. The proof in [Gn] is probabilistic, the others are analytic. Also, all proofs except the one in [U] give strong-type constants bounded independently of the dimension n.

The purpose of this paper is to show that $\mathcal{R}_j$ are of weak-type (1,1) in any dimension. The proof uses analytic methods, and it is carried out by decomposing the kernel in several pieces and by studying each piece in appropriate regions. Some of the ideas we use here have been developed by P. Sjögren in [Sj].

We begin by explaining the notion of Riesz's transforms for the Gaussian measure. Let L be the differential operator defined by

$$L = \frac{1}{2}\Delta - x \cdot \mathrm{grad}\,,$$

and consider the set of eigenvalues λ of the problem

$$Lu = \lambda\, u\,,$$

with boundary conditions

$$u(x) = O(|x|^k)\,, \qquad \text{for some } k \geq 0 \text{ as } |x| \to +\infty\,.$$

This set is discrete, the eigenvalues are of the form $-m$, m non-negative integer, and the corresponding eigenfunctions are the multidimensional Hermite polynomials $H_\alpha(x)$, defined below, $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = m$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The one-dimensional Hermite polynomials are defined by

$$H_0(x) = 1\,, \qquad H_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\,, \qquad n \geq 1\,.$$

They have the following basic properties

$$\int_{-\infty}^{+\infty} H_n(x)^2 e^{-x^2}\, dx = 2^n\, n!\, \sqrt{\pi}\,, \qquad n = 0, 1, \ldots,$$

$$\int_{-\infty}^{+\infty} H_0(x)\, e^{-x^2}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}\,,$$

and

$$\int_{-\infty}^{+\infty} H_n(x)\, e^{-x^2}\, dx = 0\,, \qquad \text{for } n \geq 1\,.$$

Also

$$H'_{n+1}(x) = -2(n+1)\,H_n(x)\,,$$
$$H_{n+1}(x) + 2\,xH_n(x) + 2\,nH_{n-1}(x) = 0\,, \qquad n \geq 0\,,$$
$$H_{-1}(x) = 0\,,$$

and

$$H''_n(x) - 2\,xH'_n(x) + 2\,nH_n(x) = 0\,.$$

The multidimensional Hermite polynomials are defined by taking products of one-dimensional Hermite polynomials. Indeed, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, with α_j non-negative integers, and $x = (x_1, \ldots, x_n)$, then we define

$$H_\alpha(x) = H_{\alpha_1}(x_1) \ldots H_{\alpha_n}(x_n)\,,$$

where $H_{\alpha_j}(x_j)$ are one-dimensional Hermite polynomials in the variable x_j.

The differential operator L is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup T_t defined by

$$T_t f(x) = \int_{\mathbb{R}^n} k(t,x,y)\, f(y)\, dy\,,$$

where

$$k(t,x,y) = \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} \exp\left(-\frac{|e^{-t}x - y|^2}{1-e^{-2t}}\right),$$

$t > 0, x \in \mathbb{R}^n$. This means that if we set $u(x,t) = T_t f(x)$ then u is a solution of the equation

$$u_t = \frac{1}{2}\,\Delta_x u - x \cdot \operatorname{grad}_x u\,.$$

By using the properties of the Hermite polynomials mentioned above it is easy to see that

$$L\,H_\alpha(x) = -|\alpha|\,H_\alpha(x)\,,$$

and

$$T_t\,H_\alpha(x) = e^{-|\alpha| t}\,H_\alpha(x)\,.$$

The measure $\gamma(x)\,dx$ makes the operator L self-adjoint; therefore, it is the natural measure to study the boundedness properties of the operators associated with L.

In this frame the Riesz transforms are defined as follows. Given j, $1 \le j \le n$, and $H_\alpha(x)$ a multidimensional Hermite polynomial, the j-th Riesz transform of H_α is defined by

$$\mathcal{R}_j(H_\alpha)(x) = -\frac{1}{\sqrt{|\alpha|}} \frac{\partial}{\partial x_j} H_\alpha(x) = \frac{2\alpha_j}{\sqrt{|\alpha|}} H_{\alpha - e_j}(x),$$

where e_j are the coordinate vectors in $\mathbb{R}^n$. By linearity the definition of $\mathcal{R}_j$ extends to any polynomial in $\mathbb{R}^n$.

We now show that this definition formally coincides, except for a multiplicative constant, with the one given at the beginning of the section. In fact, let H_α be a multidimensional Hermite polynomial, we have

$$\begin{aligned}\int k_j(y,z)\, H_\alpha(z)\, dz &= \int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \frac{1}{2r} \frac{1}{(1-r^2)^{(n+1)/2}} \\ &\qquad \cdot \frac{\partial}{\partial y_j}\Big(\int H_\alpha(z)\, e^{-|ry-z|^2/(1-r^2)}\, dz\Big)\, dr \\ &= \int_0^1 \frac{1}{2\,r\,(-\log r)^{1/2}} \frac{\partial}{\partial y_j} (T_{-\log r} H_\alpha(y))\, dr \\ &= \int_0^1 \frac{1}{2\,r\,(-\log r)^{1/2}} \frac{\partial}{\partial y_j} \Big(e^{|\alpha|\log r} H_\alpha(y)\Big)\, dr \\ &= \frac{\partial}{\partial y_j} H_\alpha(y) \int_0^1 \frac{1}{2\,r\,(-\log r)^{1/2}}\, e^{|\alpha|\log r}\, dr\,.\end{aligned}$$

By making the change of variables $r = e^{-t^2/|\alpha|}$, the last integral equals to $\sqrt{\pi}/(2\sqrt{|\alpha|})$, and the desired conclusion follows.

Instead of studying the operators $\mathcal{R}_j$ it is enough to consider the operator

$$K^* f(y) = \sup_{\varepsilon > 0} \left| \int_{|y-z|>\varepsilon} K(y,z)\, f(z)\, dz \right|,$$

with kernel

$$K(y,z) = \int_0^1 \frac{z_j - r y_j}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr\,. \tag{1.2}$$

The kernels k_j and K have basically the same behavior. In fact, we shall show that the absolute value of its difference gives an integral operator of weak-type (1,1) with respect to $\gamma\, dx$ (see Remark at the end of Section 2). Also, by symmetry it is enough to study the case when $j = 1$.

Given $R > 0$ let

$$N_R = \{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n : |y| \le R \text{ and } |z| \le R, \\ \text{or } |z| \ge R/2 \text{ and } |y-z| \le R/|z|\},$$

and $N_R^y = \{z : (y,z) \in N_R\}$. We define the operators

$$K^* f(y) = \sup_{\varepsilon>0} \Big| \int_{|y-z|>\varepsilon} K(y,z)\, f(z)\, dz \Big| ,$$

$$K_1^* f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Big| ,$$

$$K_2^* f(y) = \int_{\mathbb{R}^n \setminus N_R^y} |K(y,z)|\, |f(z)|\, dz .$$

We clearly have

$$K^* f(y) \le K_1^* f(y) + K_2^* f(y) .$$

We shall show that K_i^*, i=1, 2 are of weak-type (1,1) with respect to γ.

The organization of the paper is as follows. In Section 2 we prove estimates of K_1^* and that it is of weak-type (1,1). This done by showing that K_1^* can be pointwise controlled in terms of certain maximal and singular integral operators appropriatly truncated. In Section 3 estimates of K_2^* are shown as well as the weak-type (1,1). The proofs requiere precise estimations of the size of various integrals in different regions. In order to make the paper comprehensible we give most of the details.

2. The estimate of $\mathbf{K_1^*}$.

We begin by introducing the following operators.

Let $b \ge a > 0$, we define the maximal operator

$$M_{a,b} f(y) = \sup_{0<r<(a\wedge b/|y|)} \frac{1}{\gamma(B_r(y))} \int_{B_r(y)} |f(z)|\, \gamma(z)\, dz ,$$

where $B_r(y)$ denotes the Euclidean ball with radius r and centered at y. Also, given a function $f \in L^1_\gamma(\mathbb{R}^n)$ and a Calderón-Zygmund convolution kernel k we define

$$K_\gamma f(y) = \sup_{\varepsilon>0} \left| \int_{\varepsilon<|y-z|\leq(a\wedge b/|y|)} k(y-z)\, f(z)\, dz \right| . \tag{2.0}$$

By $\chi_E(y)$ we denote the characteristic function of the set E.

We have the following

Lemma 1. *The operator K_γ is of weak-type (1,1) with respect to the measure $\gamma\, dz$, i.e. there exists a constant $C = C(n,a,b)$ such that*

$$\int_{E_\lambda} \gamma(y)\, dy \leq \frac{C}{\lambda} \|f\|_{L^1_\gamma} ,$$

where $E_\lambda = \{y : K_\gamma f(y) > \lambda\}$, for every $\lambda > 0$.

PROOF. We first construct a countable family of balls $\mathcal{F}$ with bounded overlapping, whose union is $\mathbb{R}^n$ and on each ball $B \in \mathcal{F}$ all values of $\gamma(x)$ are equivalent. Given $\alpha \geq 1$ we define the following sequence

$$x_1 = \alpha , \qquad x_{k+1} = x_k + \frac{1}{x_k} , \qquad k \geq 1 .$$

The sequence $\{x_k\}$ is strictly increasing and $x_k \to +\infty$ as $k \to \infty$. Set $l_0 = x_1$ and $l_k = x_{k+1} - x_k$, $k \geq 1$, then $l_{k+1} < l_k < 2\, l_{k+1}$. Let

$$R_j = \{x \in \mathbb{R}^n : x_j \leq |x| < x_{j+1}\} , \qquad j \geq 1 ,$$

the width of R_j is l_j. Let $B_1^j, \ldots, B_N^j$ be a maximal disjoint family of balls contained in R_j and such that the diameter of B_k^j is l_j for all k, $1 \leq k \leq N$. If y_k^j is the center of B_k^j then we have $|y_k^j| = (x_{j+1} + x_j)/2$. It is easy to see that $\bigcup_{k=1}^N 2B_k^j \supset R_j$, where $2B$ denotes the ball with the same center as B but twice the radius. Let us define $\tilde{B}_k^j = 2B_k^j$.

The family $\mathcal{F}$ is the collection of all balls $\tilde{B}_k^j$ and the ball $B(0, x_1)$. It is obvious that the union is $\mathbb{R}^n$. We show that $\mathcal{F}$ has bounded overlaps. If $x \in \bigcap_{k=1}^l \tilde{B}_{i_k}^j$ then $l \leq 4^n$. This is because

$$B(x, 2\, l_j) \supset \bigcup_{k=1}^{l} \tilde{B}_{i_k}^j .$$

Let

$$\tilde{R}_j = \left\{ x \in \mathbb{R}^n : \ x_j - \frac{1}{2\,x_j} \le |x| < x_{j+1} + \frac{1}{2\,x_j} \right\}, \qquad j \ge 1\,.$$

Then $\bigcup_{k=1}^{N} \tilde{B}_k^j \subset \tilde{R}_j$. We have that

$$\tilde{R}_k \cap \tilde{R}_j = \varnothing\,, \qquad \text{for} \ \ k > j+2\,,$$

which follows from the fact that

$$x_{j+1} + \frac{1}{2\,x_j} < x_k - \frac{1}{2\,x_k}$$

by the construction of x_j. It remains to show that on each $B \in \mathcal{F}$ all the values of γ are equivalent. In fact, for $B(0, x_1)$ this is obvious. If $B \in \mathcal{F}$ then $B = B(y_k, 1/x_j)$ for some j and $|y_k| = (x_{j+1} + x_j)/2$. Consequently $B \subset B(y_k, 2/|y_k|)$ and in the last ball all values of γ are equivalent.

Take $\alpha = b/a \ge 1$ in the construction above and define

$$Tf(y) = \sup_{\varepsilon' > \varepsilon > 0} \Big| \int\limits_{\varepsilon < |y-z| < \varepsilon'} k(y-z)\, f(z)\, dz \Big|\,.$$

We write

$$E_\lambda = \bigcup_{B \in \mathcal{F}} E_\lambda \cap B\,.$$

Suppose $B = B(0,\alpha)$, if $y \in E_\lambda \cap B(0,\alpha)$ then the integration in $K_\gamma f$ is over the set $|y-z| < a$ and consequently

$$K_\gamma f(y) \le T(\chi_{B_{a+b/a}(0)} f)(y)\,.$$

If $|y| > \alpha$ then the integration in K_γ is over $|y-z| < b/|y|$. If we assume that $B = B(y_k, 1/x_j)$ with $|y_k| = (x_{j+1} + x_j)/2$ and $y \in B(y_k, 1/x_j)$ then

$$|z - y_k| \le |z-y| + |y - y_k| \le \frac{b}{|y|} + \frac{1}{x_j} \le \frac{c(a,b)}{|y_k|}\,.$$

This follows because since

$$x_j > \alpha \quad \text{and} \quad \frac{1}{x_j} < \left(1 - \left(\frac{a}{b}\right)^2\right) |y_k|$$

we have

$$|y| \geq |y_k| - \frac{1}{x_j} \geq \left(1 - \left(\tfrac{a}{b}\right)^2\right) |y_k| \, .$$

Analogously, $(1 - (a/b)^2)\,|y_k| < x_j$. Therefore

$$K_\gamma f(y) \leq T(\chi_{B(y_k, c/|y_k|)} f)(y) \, ,$$

for $y \in B(y_k, 1/x_j)$, and $|y| > \alpha$.

Consequently, by the weak-type (1,1) of T with respect to Lebesgue measure (see [St]), we have

$$\begin{aligned} \gamma(E_\lambda \cap B(0,\alpha)) &\leq |E_\lambda \cap B(0,\alpha)| \\ &\leq |\{y : \ T(\chi_{B_{a+b/a}(0)} f)(y) > \lambda\}| \\ &\leq \frac{c}{\lambda} \int_{B_{a+b/a}(0)} |f(z)| \, dz \\ &\leq \frac{c_a}{\lambda} \int_{B_{a+b/a}(0)} |f(z)| \, \gamma(z) \, dz \, . \end{aligned}$$

Also,

$$\begin{aligned} &\gamma(E_\lambda \cap \{|y| > \alpha\} \cap B(y_k, 1/x_j)) \\ &\qquad \leq c\,\gamma(y_k)\, |E_\lambda \cap \{|y| > \alpha\} \cap B(y_k, 1/x_j)| \\ &\qquad \leq c\,\gamma(y_k)\, |\{y : \ T(\chi_{B(y_k, c/|y_k|)} f)(y) > \lambda\}| \\ &\qquad \leq c\,\gamma(y_k) \frac{1}{\lambda} \int_{B(y_k, c/|y_k|)} |f(z)| \, dz \\ &\qquad \leq \frac{c}{\lambda} \int_{B(y_k, c/|y_k|)} |f(z)| \, \gamma(z) \, dz \, . \end{aligned}$$

By adding up and using the fact that the family of balls $B(0,\alpha)$, $B(y_k, c/|y_k|)$ has bounded overlaps the lemma follows.

REMARK. Since $M_{a,b}$ is pointwise dominated by the Hardy-Littlewood maximal function defined with the measure $\gamma(x)\,dx$, it follows from the Besicovitch covering lemma that $M_{a,b}$ is of weak-type (1,1) with respect to that measure.

We define the following operators

$$T_1 f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \Big| \, \chi_{B_R^c(0)}(y),$$

$$T_2 f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\, f(z)\, dz \Big| \, \chi_{B_R(0)}(y),$$

$$T_3 f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\geq R \\ \varepsilon<|y-z|\leq 1}} k(y-z)\, f(z)\, dz \Big| \, \chi_{B_R(0)}(y).$$

These operators are of the form (2.0).

We have the following

Theorem 1. *Let $n \geq 2$, $R \geq 4$ and $k(z) = z_1/|z|^{n+1}$. There exist a constant $C = C(n, R)$ and kernels $k_1(y,z), k_2(y,z)$ satisfying*

$$|k_1(y,z)| \leq C \frac{|y|^{1/2}}{|y-z|^{n-1/2}}, \qquad \text{for } |y| > R \text{ and } |y-z| \leq 2R/|y|,$$

$$|k_2(y,z)| \leq C \frac{1}{|y-z|^{n-1}}, \qquad \text{for } |y| \leq R \text{ and } |y-z| \leq 2R,$$

and such that

$$K_1^* f(y) \leq C \Big(\sum_{i=1}^{6} T_i f(y) + M_{2,2R} f(y) + M_{1,R} f(y) \Big),$$

where T_i, $i = 4, 5, 6$ are defined by

$$T_4 f(y) = \Big(\int_{\substack{|z|\geq R/2 \\ |y-z|\leq R/|z|}} |k_1(y,z)|\, |f(z)|\, dz \Big) \chi_{B_R^c(0)}(y),$$

$$T_5 f(y) = \Big(\int_{\substack{|z|\leq R \\ |y-z|\leq 2R}} |k_2(y,z)|\, |f(z)|\, dz \Big) \chi_{B_R(0)}(y),$$

and

$$T_6 f(y) = \Bigg(\int\limits_{\substack{|z|>R \\ |y-z|\leq R/|z|}} |k_2(y,z)|\,|f(z)|\,dz \Bigg) \chi_{B_R(0)}(y)\,.$$

Corollary. *Let $n \geq 2$ and $R \geq 4$. The operator K_1^* is of weak-type (1,1) with respect to the measure $\gamma\,dz$.*

PROOF OF THEOREM 1. Assume $|y| > R$ and $\varepsilon < |y-z| \leq \dfrac{2R}{|y|}$. We write

$$\begin{aligned} K(y,z) &= (z_1 - y_1) \int_0^1 \frac{1}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\ &\quad + y_1 \int_0^1 (1-r) \frac{1}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \qquad (2.1) \\ &= K_1(y,z) + K_2(y,z)\,. \end{aligned}$$

Note that

$$\frac{|y-z|}{|y|} \leq \frac{2R}{|y|^2} < \frac{2}{R} < 1\,, \qquad \text{for } R > 2\,.$$

Then we have

$$\begin{aligned} |K_2(y,z)| &\leq |y_1| \int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|ry-z|^2/(2(1-r))}\, dr \\ &= |y_1| \int_0^1 \frac{1}{r^{(n+1)/2}}\, e^{-|(1-r)y-z|^2/(2r)}\, dr \\ &\leq |y| \Bigg(\int_0^{|y-z|/|y|} + \int_{|y-z|/|y|}^1 \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(2r)} \\ &\qquad\qquad \cdot e^{-r|y|^2/2}\, dr \Bigg) e^{y\cdot(y-z)} \\ &\leq e^{2R} |y| \Bigg(\int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(2r)}\, dr \\ &\qquad + \int_{|y-z|/|y|}^{\infty} \frac{1}{r^{(n+1)/2}}\, dr \Bigg) \\ &\leq e^{2R} |y| \Bigg(\frac{2^{(n+1)/2}}{|y-z|^{n-1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{\infty} u^{n-2}\, e^{-u^2}\, du \end{aligned}$$

$$+\frac{2}{n-1}\Big(\frac{|y|}{|y-z|}\Big)^{(n-1)/2}\Big)$$

$$\leq C_n(R)\left(\frac{|y|}{|y-z|^{n-1}}+\frac{|y|^{(n+1)/2}}{|y-z|^{(n-1)/2}}\right).$$

We also write

$$\begin{aligned}K_1(y,z) &= (z_1-y_1)\int_0^{1-|y-z|/|y|}\frac{1}{(1-r^2)^{(n+3)/2}}\,e^{-|ry-z|^2/(1-r^2)}\,dr\\ &\quad+(z_1-y_1)\int_{1-|y-z|/|y|}^{1}\frac{1}{(1-r^2)^{(n+3)/2}}\,e^{-|ry-z|^2/(1-r^2)}\,dr\\ &= K_3(y,z)+K_4(y,z).\end{aligned}$$

As in the estimate of K_2 we get

$$\begin{aligned}|K_3(y,z)| &\leq e^{2R}\,|z_1-y_1|\int_{|y-z|/|y|}^{1}\frac{1}{r^{(n+3)/2}}\,e^{-|y-z|^2/(2r)}\,dr\\ &\leq C_n(R)\,\frac{|z_1-y_1|}{|y-z|^{n+1}}\int_{|y-z|/\sqrt{2}}^{(|y|\,|y-z|/2)^{1/2}}u^n e^{-u^2}\,du\\ &\leq C_n(R)\,\frac{|y|^{1/2}|y-z|^{1/2}}{|y-z|^n}\\ &= C_n(R)\,\frac{|y|^{1/2}}{|y-z|^{n-1/2}}.\end{aligned}$$

Let

$$\psi(t)=\frac{1}{(2-t)^{(n+3)/2}}\,e^{-|(1-r)y-z|^2/((2-t)r)}, \qquad 0\leq t<2. \tag{2.2}$$

Since $|y-z|/|y|<1$, for $R>2$ we write

$$\begin{aligned}K_4(y,z) &= (z_1-y_1)\int_0^{|y-z|/|y|}\frac{1}{r^{(n+3)/2}}\,\psi(0)\,dr\\ &\quad+(z_1-y_1)\int_0^{|y-z|/|y|}\frac{1}{r^{(n+3)/2}}\,(\psi(r)-\psi(0))\,dr\\ &= K_5(y,z)+K_6(y,z).\end{aligned}$$

Observe that for $0 \leq t \leq r \leq 1$

$$|\psi'(t)| \leq c_n \left(1 + \frac{|(1-r)y - z|^2}{r}\right) e^{-|(1-r)y-z|^2/((2-t)r)}$$
$$\leq c_n\, e^{-|(1-r)y-z|^2/(4r)} .$$

This follows since there exists $c > 0$ such that

$$(1+s)\, e^{-s/(2-t)} \leq c\, e^{-s/4} , \qquad \text{for } \ 0 \leq t \leq 1 .$$

Hence

$$|K_6(y,z)| \leq c_n\, |z_1 - y_1| \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|(1-r)y-z|^2/(4r)}\, dr$$
$$\leq C_n(R) \int_0^1 \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(4r)}\, dr$$
$$\leq C_n(R) \frac{1}{|y-z|^{n-1}} .$$

To estimate K_5 we write

$$K_5(y,z) = c_n\, (z_1 - y_1)\, e^{y\cdot(y-z)} \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-r|y|^2/2} \cdot \frac{1}{r^2}\, e^{-|y-z|^2/(2r)}\, dr$$
$$= 2\, c_n \frac{z_1 - y_1}{|y-z|^2}\, e^{y\cdot(y-z)} \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-r|y|^2/2} \cdot \frac{d}{dr}\left(e^{-|y-z|^2/(2r)}\right) dr .$$

By integrating by parts we get

$$K_5(y,z) = 2\, c_n \frac{z_1 - y_1}{|y-z|^2}\, e^{y\cdot(y-z)}\, e^{-|y|\,|y-z|} \left(\frac{|y|}{|y-z|}\right)^{(n-1)/2}$$
$$+ c_n \frac{z_1 - y_1}{|y-z|^2}\, |y|^2\, e^{y\cdot(y-z)} \cdot \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)}\, e^{-r|y|^2/2}\, dr$$

$$+ (n-1)\, c_n \frac{z_1 - y_1}{|y-z|^2}$$
$$\cdot \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|(1-r)y-z|^2/(2r)}\, dr$$
$$= K_7(y,z) + K_8(y,z) + K_9(y,z)\,.$$

We have

$$|K_7(y,z)| \le c_n \frac{|y|^{(n-1)/2}}{|y-z|^{(n+1)/2}}\,,$$

and

$$|K_8(y,z)| \le c_n\, e^{2R} \frac{|z_1 - y_1|}{|y-z|^2}\, |y|^2 \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)}\, dr$$
$$\le c_n \frac{|y|^{1/2}}{|y-z|^{n-1/2}}\,.$$

In the last estimate we have considered the cases $n = 2$ and $n > 2$ separately and used the fact that

$$\int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-4}\, e^{-u^2}\, du \le \frac{c}{(|y|\,|y-z|)^{1/2}}\,,$$

for $n > 2$.

To estimate K_9 we define

$$\phi(t) = e^{-|(1-t)y-z|^2/(2r)}\,, \qquad 0 \le t < 1\,, \tag{2.3}$$

and write

$$K_9(y,z) = (n-1)\, c_n \frac{z_1 - y_1}{|y-z|^2} \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, \phi(0)\, dr$$
$$+ (n-1)\, c_n \frac{z_1 - y_1}{|y-z|^2} \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, (\phi(r) - \phi(0))\, dr$$
$$= K_{10}(y,z) + K_{11}(y,z)\,.$$

Observe that

$$\phi'(t) = -\frac{1}{r}\,((1-t)y - z)\cdot y\, \phi(t)\,,$$

and consequently for $0 < t \le r < 1$ and $|y-z| \le 2R/|y|$ we have

$$|\phi'(t)| \le |y| \, \frac{|(1-t)y-z|}{r} \, \phi(t) \le C(R) \, \frac{|y|}{\sqrt{r}} \, e^{-|y-z|^2/(4r)} .$$

Hence

$$\begin{aligned} |K_{11}(y,z)| &\le C_n(R) \, \frac{|y|}{|y-z|} \int_0^{|y-z|/|y|} \frac{1}{r^{n/2}} \, e^{-|y-z|^2/(4r)} \, dr \\ &\le C_n(R) \, \frac{|y|}{|y-z|^{n-1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-3} \, e^{-u^2} \, du \\ &\le C_n(R) \, \frac{|y|^{1/2}}{|y-z|^{n-1/2}} . \end{aligned}$$

Now we write

$$\begin{aligned} K_{10}(y,z) &= c_n' \, \frac{z_1-y_1}{|y-z|^{n+1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-2} \, e^{-u^2} \, du \\ &= c_n'' \, \frac{z_1-y_1}{|y-z|^{n+1}} \\ &\quad - c_n' \, \frac{z_1-y_1}{|y-z|^{n+1}} \int_0^{(|y-z|\,|y|/2)^{1/2}} u^{n-2} \, e^{-u^2} \, du \\ &= c_n' \, \frac{z_1-y_1}{|y-z|^{n+1}} + K_{12}(y,z) , \end{aligned}$$

and we obtain

$$|K_{12}(y,z)| \le c_n \, \frac{1}{|y-z|^n} \, (|y-z|\,|y|)^{1/2} = c_n \, \frac{|y|^{1/2}}{|y-z|^{n-1/2}} .$$

It is easy to see that in the region $|y| > R$ and $|y-z| \le \dfrac{2R}{|y|}$ the kernels

$$\frac{|y|^{(n-1)/2}}{|y-z|^{(n+1)/2}} , \quad \frac{1}{|y-z|^{n-1}} , \quad \frac{|y|}{|y-z|^{n-1}} , \quad \frac{|y|^{(n+1)/2}}{|y-z|^{(n-1)/2}} ,$$

are all dominated by

$$\frac{|y|^{1/2}}{|y-z|^{n-1/2}} .$$

Consequently, in case $|y| > R$ and $\varepsilon < |y-z| \le 2R/|y|$ we obtain that

$$K(y,z) = c_n\, k(y-z) + k_1(y,z) \tag{2.4}$$

where

$$k_1 = K_1 + \cdots + K_{12}\ ,$$

and

$$|k_1(y,z)| \le C_n(R)\,\frac{|y|^{1/2}}{|y-z|^{n-1/2}}\ .$$

We now assume $|y| \le R$, and $\varepsilon < |y-z| \le 2R$. By (2.1)

$$K(y,z) = K_1(y,z) + K_2(y,z)\ ,$$

and

$$\begin{aligned}|K_2(y,z)| &\le R\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|ry-z|^2/(2(1-r))}\, dr\\ &\le R\, e^{2R^2}\int_0^1 \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(2r)}\, dr\\ &\le C_n(R)\,\frac{1}{|y-z|^{n-1}}\ .\end{aligned}$$

We write

$$\begin{aligned}K_1(y,z) &= (z_1-y_1)\int_0^1 \frac{1}{r^{(n+3)/2}}\,\psi(0)\, dr\\ &\quad + (z_1-y_1)\int_0^1 \frac{1}{r^{(n+3)/2}}\,(\psi(r)-\psi(0))\, dr\\ &= \overline{K}_3(y,z) + \overline{K}_4(y,z)\ ,\end{aligned}$$

where $\psi(t)$ is defined by (2.2).

As in the estimate of $K_6(y,z)$ we obtain

$$|\overline{K}_4(y,z)| \le C_n(R)\,\frac{1}{|y-z|^{n-1}}\ .$$

If ϕ is defined by (2.3) then

$$\begin{aligned}\overline{K}_3(y,z) &= c_n\,(z_1-y_1)\int_0^1 \frac{1}{r^{(n+3)/2}}\,\phi(0)\,dr \\ &\quad + c_n\,(z_1-y_1)\int_0^1 \frac{1}{r^{(n+3)/2}}\,(\phi(r)-\phi(0))\,dr \\ &= c_n\,\frac{z_1-y_1}{|y-z|^{n+1}}\int_{|y-z|/\sqrt{2}}^{+\infty} u^n\,e^{-u^2}\,du + \overline{K}_6(y,z) \\ &= \tilde{c}_n\,\frac{z_1-y_1}{|y-z|^{n+1}} \\ &\quad - c_n\,\frac{z_1-y_1}{|y-z|^{n+1}}\int_0^{|y-z|/\sqrt{2}} u^n\,e^{-u^2}\,du + \overline{K}_6(y,z) \\ &= \tilde{c}_n\,\frac{z_1-y_1}{|y-z|^{n+1}} + \overline{K}_5(y,z) + \overline{K}_6(y,z)\,.\end{aligned}$$

We have

$$|\overline{K}_5(y,z)| \le \frac{C_n}{|y-z|^{n-1}}\,,$$

and

$$\begin{aligned}|\overline{K}_6(y,z)| &\le C_n(R)\,|z_1-y_1|\,|y|\int_0^1 \frac{1}{r^{(n+2)/2}}\,e^{-|y-z|^2/(4r)}\,dr \\ &\le C_n(R)\,\frac{|z_1-y_1|}{|y-z|^n}\,|y|\int_{|y-z|/2}^{+\infty} u^{n-1}\,e^{-u^2}\,du \\ &\le \frac{C_n(R)}{|y-z|^{n-1}}\,.\end{aligned}$$

Therefore, in case $|y|\le R$, and $|y-z|\le 2R$, we obtain

$$K(y,z) = k(y-z) + k_2(y,z)\,, \tag{2.5}$$

with

$$k_2 = \overline{K}_1(y,z) + \cdots + \overline{K}_6(y,z)\,,$$

and

$$|k_2(y,z)| \le \frac{C_n}{|y-z|^{n-1}}\,.$$

If $|y| > R$ then $(y, z) \in N_R$ if and only if $|z| \geq R/2$ and $|y - z| \leq R/|z|$. Hence $|y| \approx |z|$, *i.e.*

$$\left(1 + \frac{4}{R}\right)^{-1} |z| \leq |y| \leq \left(1 + \frac{4}{R}\right) |z| \, ,$$

in particular $|y| \leq 2R$ for $R \geq 4$. Then by (2.4)

$$\int\limits_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz = c_n \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz + \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k_1(y,z)\, f(z)\, dz \, .$$

Therefore

$$\sup_{\varepsilon>0} \Bigg| \int\limits_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Bigg| \leq c_n \sup_{\varepsilon>0} \Bigg| \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz \Bigg| + \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} |k_1(y,z)|\, |f(z)|\, dz \, .$$

Since $R \geq 4$ then $|z|/2 \leq |y| \leq 2|z|$ and so

$$\frac{R}{2|y|} \leq \frac{R}{|z|} \leq \frac{2R}{|y|} \, .$$

We have

$$\int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz = \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \quad \int\limits_{\substack{|z|\geq R/2 \\ R/|z|<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \, .$$

To estimate the second integral on the right hand side we write

$$\begin{aligned}
I_1 &= \Big| \int_{\substack{|z|\geq R/2 \\ R/|z|<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \Big| \\
&\leq \int_{R/2|y|<|y-z|\leq 2R/|y|} |k(y-z)|\, |f(z)|\, dz \\
&\leq c \int_{R/2|y|<|y-z|\leq 2R/|y|} \frac{1}{|y-z|^n}\, |f(z)|\, dz \\
&\leq \frac{c}{\left(\frac{R}{2|y|}\right)^n} \int_{|y-z|\leq 2R/|y|} |f(z)|\, dz\,.
\end{aligned}$$

It easy to see that $e^{-|z|^2} \approx e^{-|y|^2}$ for $|y-z| \leq 2R/|y|$ and $|z| \leq 2|y|$. Therefore

$$\begin{aligned}
I_1 &\leq \frac{c}{\gamma(B_{2R/|y|}(y))} \int_{B_{2R/|y|}(y)} |f(z)|\, \gamma(z)\, dz \\
&\leq c\, M_{2,2R} f(y)\,,
\end{aligned}$$

since $|y| > R$.

Consequently if $|y| > R$ then

$$\begin{aligned}
\sup_{\varepsilon>0} \Big| \int_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Big| \leq c_n \sup_{\varepsilon>0} \Big| & \int_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \Big| \\
&+ M_{2,2R} f(y) \\
&+ \int_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} |k_1(y,z)|\, |f(z)|\, dz\,.
\end{aligned}$$

Let us consider the case $|y| \leq R$. We write

$$\int_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz = \int_{\substack{|z|\leq R \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz$$

$$+ \int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} K(y,z)\, f(z)\, dz$$

and $|z| \leq R$ implies $|y-z| \leq 2R$. Consequently we may apply (2.5) and get

$$\int\limits_{\substack{|z|\leq R \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz = c_n \int\limits_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\, f(z)\, dz + \int\limits_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k_2(y,z)\, f(z)\, dz\,.$$

Thus,

$$\sup_{\varepsilon>0} \Big| \int\limits_{\substack{|z|\leq R \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Big| \leq c_n \sup_{\varepsilon>0} \Big| \int\limits_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\, f(z)\, dz \Big| + \int\limits_{\substack{|z|\leq R \\ |y-z|\leq 2R}} |k_2(y,z)|\, |f(z)|\, dz\,.$$

If $|z| > R$ and $|y-z| \leq R/|z|$ then $|y-z| < 1$ and since $|y| \leq R$ we may apply (2.5) to write

$$\int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} K(y,z)\, f(z)\, dz = \int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz + \int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} k_2(y,z)\, f(z)\, dz$$

$$= J_1 + J_2\,.$$

We have

$$|J_2| \leq \int\limits_{\substack{|z|>R \\ |y-z|\leq R/|z|}} |k_2(y,z)|\, |f(z)|\, dz\,.$$

Since $R/|z| < 1$ we write

$$J_1 = \int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|<1}} k(y-z)\,f(z)\,dz - \int\limits_{\substack{|z|>R \\ R/|z|<|y-z|<1}} k(y-z)\,f(z)\,dz$$

$$= J_3 - J_4\,.$$

In the region $|z| > R$ and $R/|z| \leq |y-z| \leq 1$ we have that $|z| \leq 1+R$. Since $|y| \leq R$ we have $e^{-|z|^2} \approx e^{-|y|^2}$, for $z \in B_1(y)$. Consequently

$$|J_4| \leq C\left(\frac{1+R}{R}\right)^n \frac{1}{\gamma(B_1(y))}\int_{B_1(y)} |f(z)|\,\gamma(z)\,dz$$
$$\leq C_R\, M_{1,R}f(y)\,.$$

Therefore

$$\sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} K(y,z)\,f(z)\,dz\Bigg|$$
$$\leq c_n \sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{|z|>R \\ \varepsilon<|y-z|\leq 1}} k(y-z)\,f(z)\,dz\Bigg|$$
$$+ \int\limits_{\substack{|z|>R \\ |y-z|\leq R/|z|}} |k_2(y,z)|\,|f(z)|\,dz$$
$$+ M_{1,R}f(y)\,.$$

Then for $|y| \leq R$ we obtain

$$\sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\,f(z)\,dz\Bigg| \leq c_n\Bigg(\sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\,f(z)\,dz\Bigg|$$
$$+ \sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{|z|<R \\ \varepsilon<|y-z|\leq 1}} k(y-z)\,f(z)\,dz\Bigg|$$

$$+ \int\limits_{\substack{|z|\le R\\ |y-z|\le 2R}} |k_2(y,z)|\,|f(z)|\,dz$$

$$+ \int\limits_{\substack{|z|>R\\ |y-z|\le R/|z|}} |k_2(y,z)|\,|f(z)|\,dz$$

$$\left. + M_{1,R}f(y)\right) .$$

This completes the proof of Theorem 1.

PROOF OF THE COROLLARY. By Lemma 1, the operators T_1, T_2 and T_3 are of weak-type (1,1) with respect to the Gaussian measure.

We shall show that the operators T_4, T_5 and T_6 are bounded in $L^1_\gamma(\mathbb{R}^n)$, and consequently of weak-type (1,1) with respect to the Gaussian measure. Let us first consider T_5 and T_6. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} T_5 f(y)\, e^{-|y|^2}\, dy &= \int\limits_{|y|\le R} e^{-|y|^2} \int\limits_{\substack{|z|\le R\\ |y-z|\le 2R}} |k_2(y,z)|\,|f(z)|\,dz\,dy \\
&\le e^{R^2} c_n(R) \int\limits_{|y|\le R} \int\limits_{|y-z|\le 2R} \frac{|f(z)|}{|y-z|^{n-1}}\, e^{-|z|^2}\, dz\,dy \\
&\le c_n(R) \int_{\mathbb{R}^n} |f(z)|\, e^{-|z|^2} \int\limits_{|y-z|\le 2R} \frac{1}{|y-z|^{n-1}}\, dy\,dz \\
&\le C_n(R)\, \|f\|_{L^1_\gamma} .
\end{aligned}$$

If $|y| \le R < |z|$, and $|y-z| \le R/|z|$ then $|z|^2 \le 1 + 2R + |y|^2$ and consequently

$$\int_{\mathbb{R}^n} T_6 f(y)\, e^{-|y|^2}\, dy \le e^{1+2R} c_n(R) \int\limits_{|y|\le R} e^{-|y|^2} \int\limits_{|y-z|\le 1} \frac{|f(z)|}{|y-z|^{n-1}}\, e^{|y|^2-|z|^2}\, dz\,dy$$

$$\leq c_n(R) \int_{\mathbb{R}^n} |f(z)|\, e^{-|z|^2} \int_{|y-z|\leq 1} \frac{1}{|y-z|^{n-1}}\, dy\, dz$$

$$\leq C_n(R) \|f\|_{L^1_\gamma} .$$

Let us now consider T_4. If $|y| > R$, $|z| \geq R/2$, and $|y-z| \leq R/|z|$, then $|y| \approx |z|$, and by (2.4) we have

$$\int_{\substack{|z|\geq R/2\\ |y-z|\leq R/|z|}} |k_1(y,z)|\, |f(z)|\, dz \leq C_n(R) \int_{\substack{|z|\geq R/2\\ |y-z|\leq R/|z|}} \frac{|y|^{1/2}}{|y-z|^{n-1/2}} |f(z)|\, dz$$

$$= \tilde{T}_4 f(y) .$$

We have

$$\int_{|y|>R} \tilde{T}_4 f(y)\, e^{-|y|^2}\, dy \leq c_n(R) \int_{|y|>R} e^{-|y|^2} \int_{|y-z|\leq R/|z|} |z|^{1/2} \frac{|f(z)|}{|y-z|^{n-1/2}}\, e^{|y|^2-|z|^2}\, dz\, dy$$

$$\leq c_n(R) \int_{\mathbb{R}^n} |f(z)|\, e^{-|z|^2} |z|^{1/2} \int_{|y-z|\leq R/|z|} \frac{1}{|y-z|^{n-1/2}}\, dy\, dz$$

$$\leq C_n(R)\, \|f\|_{L^1_\gamma} .$$

This ends the proof of the corollary.

REMARK. We show here that the kernels k_j and K have basically the same behavior when $j = 1$. The remaining values of j are treated in a similar way. We set

$$\varphi(r) = \left(\frac{1-r^2}{-\log r}\right)^{1/2} .$$

The function φ is increasing in $(0,1)$, $\varphi(1)=\sqrt{2}$, and the inequality

$$-\log r \leq \frac{1-r^2}{2\,r^2}$$

valid for $0<r<1$ implies that $\varphi(r)\geq\sqrt{2}\,r$ and consequently

$$\varphi(1)-\varphi(r)\leq\sqrt{2}\,(1-r)\,.$$

We write

$$\begin{aligned} k_1(y,z) &= k_1(y,z)-\sqrt{2}\,K(y,z)+\sqrt{2}\,K(y,z)\\ &= H(y,z)+\sqrt{2}\,K(y,z)\,. \end{aligned}$$

We have

$$\begin{aligned} -H(y,z) &= (z_1-y_1)\int_0^1 \frac{\varphi(1)-\varphi(r)}{(1-r^2)^{(n+3)/2}}\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &\quad + y_1\int_0^1 \frac{\varphi(1)-\varphi(r)}{(1-r^2)^{(n+3)/2}}\,(1-r)\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &= H_1(y,z)+H_2(y,z)\,. \end{aligned}$$

We have

$$\begin{aligned} |H_1(y,z)| &\leq c\,|z_1-y_1|\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &= \tilde{H}_1(y,z)\,, \end{aligned}$$

and

$$\begin{aligned} |H_2(y,z)| &\leq c\,|y_1|\int_0^1 \frac{1}{(1-r)^{(n-1)/2}}\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &= \tilde{H}_2(y,z)\,. \end{aligned}$$

We now proceed as in the proof of Theorem 1. We first assume $|y|>R$ and $\varepsilon<|y-z|\leq 2R/|y|$. We begin by estimating $\tilde{H}_2$. As in the estimate of K_2 in Theorem 1 we get

$$\begin{aligned}
\tilde{H}_2(y,z) &\le |y| \Big(\int_0^{|y-z|/|y|} + \int_{|y-z|/|y|}^1 \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)} \\
&\qquad \cdot e^{-r|y|^2/(2)}\, dr \Big)\, e^{y\cdot(y-z)} \\
&\le e^{2R}\, |y| \Big(\int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)}\, dr \\
&\qquad + \int_{|y-z|/|y|}^1 \frac{1}{r^{(n-1)/2}}\, dr \Big) \\
&\le e^{2R}\, |y| \Big(\frac{c_n}{|y-z|^{n-3}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-4}\, e^{-u^2}\, du \\
&\qquad + \Big(\frac{|y|}{|y-z|}\Big)^{(n-3)/2} \Big) \\
&\le C_n(R)\, |y| \Big(\frac{c}{|y-z|^{n-3}}\, \frac{1}{(|y||y-z|)^{1/2}} \\
&\qquad + \Big(\frac{|y|}{|y-z|}\Big)^{(n-3)/2} \Big).
\end{aligned}$$

We now estimate $\tilde{H}_1$. We write

$$\begin{aligned}
\tilde{H}_1(y,z) &= |z_1 - y_1| \int_0^{1-|y-z|/|y|} \frac{1}{(1-r^2)^{(n+1)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\
&\quad + |z_1 - y_1| \int_{1-|y-z|/|y|}^1 \frac{1}{(1-r^2)^{(n+1)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\
&= H_3(y,z) + H_4(y,z).
\end{aligned}$$

We estimate H_3 in the same way we estimated K_3 and get

$$H_3(y,z) \le c_n(R)\, \frac{|y|^{1/2}}{|y-z|^{n-5/2}}\,.$$

We also have

$$H_4(y,z) \le |z_1 - y_1| \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|z-(1-r)y|^2/(2r)}\, dr$$

$$\leq e^{2R}\,|z_1 - y_1| \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|z-y|^2/(2r)}\, dr$$

$$\leq c\, e^{2R}\, \frac{|z_1 - y_1|}{|z-y|^{n-1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-2}\, e^{-u^2}\, du$$

$$\leq C(R)\, \frac{1}{|z-y|^{n-2}}\,.$$

In the region $|y| > R$ and $|y-z| < 2R/|y|$ the last kernel is dominated by

$$C(R)\, \frac{|y|^{1/2}}{|y-z|^{n-1/2}}\,.$$

In the region $|y| \leq R$ and $\varepsilon < |y-z| \leq 2R$ we proceed as in the proof of Theorem 1 to get

$$H_1(y,z) \leq C_n(R)\, \frac{1}{|y-z|^{n-1}}\,,$$

and

$$H_2(y,z) \leq C_n(R)\, \frac{1}{|y-z|^{n-3}}\,.$$

Now by arguing as in the proof of Theorem 1 we obtain that

$$H_1^* f(y) = \sup_{\varepsilon>0} \Big| \int\limits_{\substack{N_R^y \\ |y-z|>\varepsilon}} H(y,z)\, f(z)\, dz \Big| \leq C \sum_{i=4}^{6} T_i f(y)\,,$$

and therefore H_1^* is of weak-type (1,1).

We also define

$$H_2^* f(y) = \int\limits_{\mathbb{R}^n \setminus N_R^y} |H(y,z)|\, |f(z)|\, dz\,,$$

and since $\varphi(r) \leq \sqrt{2}$, we have that

$$|H(y,z)| \leq c \int_0^1 \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr\,.$$

The right hand-side of last inequality is the kernel of the operator considered in Lemma 2, Section 3, which restricted to the region $\mathbb{R}^n \setminus N_R^y$ gives an integral operator of weak-type (1,1) for R sufficiently large. This will be proved in Theorem 2.

3. The estimate of $\mathbf{K}_2^*$.

For $z \neq 0$ set $\eta = |z|$. Given $y \in \mathbb{R}^n$ there exist unique ξ and v such that

$$y = \xi \frac{z}{\eta} + v\,.$$

Given $y, z \in \mathbb{R}^n$ by $\alpha(y,z)$ we denote the angle between y, z, $0 \leq \alpha(y,z) \leq \pi$. We shall show that the set $\mathbb{R}^n \times \mathbb{R}^n \setminus N_R$ can be written as a disjoint union of the following sets:

$$D_1 = \{(y,z) \notin N_R : \ \xi \leq \eta, \text{ and } \alpha(y,z) \geq \pi/4\}\,,$$

$$D_2 = \{(y,z) \notin N_R : \ \xi > \eta, \text{ and } |y-z| \geq \beta\,(|y| \vee |z|)\}\,,$$

$$D_3 = \{(y,z) \notin N_R : |y-z| < \beta\,(|y| \vee |z|), \quad \text{or both } \xi \leq \eta \text{ and } \alpha(y,z) < \pi/4\}\,,$$

provided $\beta > 0$ and sufficiently small.

We set

$$D_3^1 = \{(y,z) \notin N_R : \ |y-z| < \beta\,(|y| \vee |z|)\}\,,$$

and

$$D_3^2 = \{(y,z) \notin N_R : \ \xi \leq \eta \text{ and } \alpha(y,z) < \pi/4\}\,.$$

We clearly have that $D_3 = D_3^1 \cup D_3^2$, $D_1 \cap D_2 = \varnothing$, $D_2 \cap D_3 = \varnothing$, and $D_1 \cap D_3^2 = \varnothing$. Observe that if $|y-z| < \beta\,(|y| \vee |z|)$ then $|y| \approx |z|$, with constant in the equivalence only depending on β. Therefore

$$\{(y,z) : \ |y-z| < \beta\,(|y| \vee |z|)\} \subset \{(y,z) : \ |y-z| \leq C_\beta\,|z|\}\,.$$

If β is sufficiently small then $C_\beta < 1/2$ and we have

$$\{(y,z) : \ |y-z| < \beta\,(|y| \vee |z|)\} \subset \{(y,z) : \ \alpha(y,z) < \pi/4\}\,.$$

Then by taking $\beta > 0$ sufficiently small we obtain that $D_1 \cap D_3^1 = \varnothing$, and $\mathbb{R}^n \times \mathbb{R}^n \setminus N_R = D_1 \cup D_2 \cup D_3$.

Lemma 2. *Let K be the kernel defined by* (1.2). *Given $\beta > 0$ sufficiently small, $B_1 > 0$, and $B_2 \geq 2$, there exist $R > 0$ sufficiently large and a constant C depending only on β, R, B_1, B_2 and n such that the following estimates hold.*

a) *If* $(y,z) \in D_1$ *then*

$$|K(y,z)| \le C\, e^{\xi_+^2-\eta^2}, \qquad \xi_+ = \xi \vee 0 .$$

b) *If* $(y,z) \in D_2$ *then*

$$|K(y,z)| \le C .$$

*c.*1) *If* $(y,z) \in D_3$ *and* $|y-z| \ge \beta\,(|y| \vee |z|)$ *then*

$$|K(y,z)| \le C\, e^{\xi^2-\eta^2} .$$

*c.*2) *If* $(y,z) \in D_3$, $|y-z| < \beta\,(|y| \vee |z|)$, *and* $|v| < B_1/\eta$ *then*

$$|K(y,z)| \le C\,\eta^n \left(1 \wedge e^{\xi^2-\eta^2}\right) .$$

*c.*3) *If* $(y,z) \in D_3$, $|y-z| < \beta\,(|y| \vee |z|)$, *and* $|v| \ge B_2/\eta$ *then*

$$|K(y,z)| \le C\, \frac{1}{|v|^n} \left(1 \wedge e^{\xi^2-\eta^2}\right) .$$

PROOF. We define

$$K_1(y,z) = \int_0^1 \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr .$$

Clearly $|K(y,z)| \le K_1(y,z)$ where $K(y,z)$ is defined by (1.2).

Case a). Suppose first $y \cdot z \le 0$. This implies $\xi \le 0$, and consequently $\xi_+ = 0$. We also have

$$|y-z|^2 \ge |y|^2 + |z|^2 - 2y \cdot z \ge (|z|^2 + |y|^2) \ge (\beta\,(|y| \vee |z|))^2 ,$$

and since $(y,z) \notin N_R$, we obtain that $|y-z| \ge \beta R$. In this case we also have $y \cdot (y-z) \ge 0$.

We write

$$\begin{aligned} K_1(y,z) &= \int_0^{1/2} \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\ &\quad + \int_{1/2}^1 \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\ &= K_2(y,z) + K_3(y,z) . \end{aligned}$$

Since $y \cdot z \le 0$, we have

$$\begin{aligned} K_2(y,z) &= \int_0^{1/2} \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}} \\ &\qquad \cdot e^{-(r^2|y|^2 + r^2|z|^2 - 2ry\cdot z)/(1-r^2)}\, dr\, e^{-\eta^2} \\ &\le \Big(\frac{4}{3}\Big)^{(n+3)/2} \Big(|z_1| \int_0^{1/2} e^{-(rz_1)^2/2}\, dr \\ &\qquad + |y_1| \int_0^{1/2} e^{-(ry_1)^2/2}\, dr \Big)\, e^{-\eta^2} \\ &\le c_n \Big(\int_0^{+\infty} e^{-u^2}\, du \Big)\, e^{-\eta^2} . \end{aligned}$$

Also, since $y \cdot (y-z) \ge 0$, we have

$$\begin{aligned} K_3(y,z) &= \int_0^{1/2} \frac{|z_1 - (1-r)y_1|}{((2-r)r)^{(n+3)/2}}\, e^{-|(1-r)y - z|^2/((2-r)r)}\, dr \\ &\le |z_1 - y_1| \int_0^{1/2} \frac{1}{((2-r)r)^{(n+3)/2}} \\ &\qquad \cdot e^{-|y-z|^2/((2-r)r)}\, e^{2\, y\cdot(y-z)/(2-r)}\, dr \\ &\quad + \int_0^{1/2} \frac{|y_1|}{((2-r))^{(n+3)/2}}\, e^{-r|y|^2/(2-r)}\, \frac{1}{r^{(n+1)/2}} \\ &\qquad \cdot e^{-|y-z|^2/((2-r)r)}\, e^{2\, y\cdot(y-z)/(2-r)}\, dr \\ &\le 2\, e^{(4/3)\, y\cdot(y-z)} \Big(|z_1 - y_1| \int_0^{1/2} \frac{1}{((2-r)r)^{n/2}} \\ &\qquad \cdot e^{-|y-z|^2/((2-r)r)} \\ &\qquad \cdot \frac{(1-r)}{((2-r)r)^{3/2}}\, dr \\ &\quad + \int_0^{1/2} \frac{1}{((2-r)r)^{(n-1)/2}} \\ &\qquad \cdot e^{-|y-z|^2/((2-r)r)}\, \frac{(1-r)}{((2-r)r)^{3/2}}\, dr \Big) . \end{aligned}$$

By making the change of variables

$$u = \frac{|y-z|}{((2-r)r)^{1/2}}$$

the last expression equals to

$$\begin{aligned} C\, e^{(4/3)\, y\cdot(y-z)} \Big(\frac{|z_1 - y_1|}{|y-z|^{n+1}} \int_{2\,|y-z|/\sqrt{3}}^{+\infty} u^n\, e^{-u^2}\, du \\ + \frac{1}{|y-z|^n} \int_{2\,|y-z|/\sqrt{3}}^{\infty} u^{n-1}\, e^{-u^2}\, du \Big) \\ \leq C\, e^{(4/3)\, y\cdot(y-z)} \frac{1}{|y-z|^n}\, e^{-4\,|y-z|^2/3} \\ \cdot \big(P_{n-1}(|y-z|) + P_{n-2}(|y-z|)\big)\,, \end{aligned}$$

where P_{n-1} and P_{n-2} are polynomials of degree $n-1$ and $n-2$ respectively. Therefore

$$\begin{aligned} K_3(y,z) &\leq C_n(R)\, e^{(4/3) y\cdot(y-z)}\, e^{-4\,|y-z|^2/3} \\ &= C_n(R)\, e^{(4/3) y\cdot z} e^{-4\,|z|^2/3} \\ &\leq C_n(R)\, e^{-\eta^2}\,, \end{aligned}$$

since $|y-z| \geq \beta R$.

Second, we assume $y\cdot z > 0$. In this case we have $0 < \xi \leq \eta$. Since $\alpha(y,z) \geq \pi/4$, we also have $\xi \leq |y|/\sqrt{2}$. In addition, for $0 < \xi \leq \eta$ and $0 \leq r \leq 1$, we have $|\eta - r\xi| = \eta - r\xi \leq \eta$.

We write

$$\begin{aligned} K_1(y,z) &= \int_0^1 \frac{|(\eta - r\xi)\frac{z_1}{\eta} - r v_1|}{(1-r^2)^{(n+3)/2}}\, e^{-(r|v|)^2/(1-r^2)} \\ &\qquad \cdot e^{-|\xi - r\eta|^2/(1-r^2)}\, dr\, e^{\xi^2-\eta^2} \\ &= I_1\, e^{\xi^2-\eta^2}\,. \end{aligned}$$

We shall show that the integral I_1 is bounded by a constant independent of ξ, η and v.

In fact, let us first assume that $\eta < R/2$. Since $(y,z) \notin N_R$, $|y| > R$, consequently

$$|v| = \Big|y - \xi\frac{z}{\eta}\Big| \geq |y| - \xi > \Big(1 - \frac{1}{\sqrt{2}}\Big)R\,.$$

Therefore

$$\begin{aligned}
I_1 &\le \eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, dr \\
&\quad + \int_0^1 \frac{|rv_1|}{(1-r^2)^{1/2}} \frac{1}{(1-r^2)^{(n+2)/2}}\, e^{-(r|v|)^2/(1-r^2)}\, dr \\
&\le C\,(1+R) \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, dr \\
&\le C\,(1+R)\Big(\int_0^{1/2} \frac{1}{(1-r)^{(n+3)/2}}\, dr \\
&\qquad + \int_{1/2}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|v|^2/(16(1-r))}\, dr \Big) \\
&\le C_n(R) \left(1 + |v|^{-(n+1)} \int_0^\infty u^n\, e^{-u^2}\, du \right) \\
&\le C_n(R)\,.
\end{aligned}$$

In case $\eta \ge R/2$, we have

$$I_1 \le C\,\eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr\,.$$

If ξ is near to η, *i.e.* $(1-\beta/2)\eta < \xi \le \eta$, then $\eta - \xi < \beta\,\eta/2$, and we claim that

$$|y-z| \ge \beta\,\eta\,,$$

for $\beta > 0$ sufficiently small. In fact, we obviously have $|y-z| \ge |y| - \eta$, and by the assumptions $y\cdot z > 0$ and $\alpha(y,z) \ge \pi/4$ we have $\xi \le |y|/\sqrt{2}$. Then

$$|y-z| \ge \left[\sqrt{2}\left(1 - \frac{\beta}{2}\right) - 1\right] \eta\,.$$

If

$$\beta \le \frac{2(\sqrt{2}-1)}{2+\sqrt{2}}$$

then the quantity in brackets is greater than or equal to β, and the claim follows. Consequently

$$|v| \ge |y-z| - |\eta - \xi| > \left(\beta - \frac{\beta}{2}\right)\eta = \beta\,\frac{\eta}{2} > \beta\,\frac{R}{2}\,.$$

Hence,

$$\frac{(r|v|)^2}{4(1-r)} \geq \frac{\beta^2}{4}\frac{\eta^2}{4}\frac{r^2}{1-r},$$

therefore

$$\begin{aligned}
I_1 &\leq C\eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(\beta r\eta)^2/(16(1-r))}\, dr \\
&\leq C\, 2^{(n+3)/2}\, \eta \int_0^{1/2} e^{-(\beta r \eta)^2/(8)}\, dr \\
&\quad + C\eta \int_{1/2}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(\beta\eta)^2/(64(1-r))}\, dr \\
&\leq \frac{C_n}{\beta}\left(1 + \frac{\eta}{\eta^{n+1}}\right) \\
&\leq C_n(\beta, R)\,.
\end{aligned}$$

If $0 < \xi \leq (1-\beta/2)\eta$ then $\eta - \xi \geq \beta\eta/2$, consequently

$$\begin{aligned}
I_1 &\leq C\eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\
&= C\eta \int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\
&\quad + C\eta \int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\
&= \mathrm{I} + \mathrm{II}\,.
\end{aligned}$$

If $0 < r < 1 - (\eta - \xi)/(2\eta)$, then $1 - r > (\eta - \xi)/(2\eta) > \beta/4$. Hence,

$$\mathrm{I} \leq C\left(\frac{4}{\beta}\right)^{(n+3)/2} \eta \int_0^1 e^{-(\xi - r\eta)^2/2}\, dr \leq C_n(\beta)\,.$$

If $1-(\eta-\xi)/(2\eta) < r < 1$, then $r\eta - \xi > (\eta+\xi)/2 - \xi > \beta\eta/4$. Therefore

$$\mathrm{II} \leq C\eta \int_0^1 \frac{1}{r^{(n+3)/2}}\, e^{-(\beta\eta)^2/(32\, r)}\, dr \leq C_n(\beta)\,\frac{1}{\eta^n} \leq C_n(\beta, R)\,.$$

This completes the proof of case a).

Case b). Since $(y,z) \in D_2$, it follows that $\eta < \xi \le |y|$ and $|y-z| \ge \beta\,(|y| \vee |z|) = \beta\,|y|$. Also, $(y,z) \notin N_R$ implies $|y-z| \ge \beta R$. So,

$$\frac{\beta}{2} \le \frac{|y-z|}{2|y|} \le \frac{1}{2} + \frac{\eta}{2|y|} \le 1 .$$

Then we have

$$\begin{aligned}
K_1(y,z) &\le C \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|ry-z|^2/(4(1-r))}\, dr \\
&= C \int_0^1 \frac{1}{r^{(n+2)/2}}\, e^{-|(1-r)y-z|^2/(4r)}\, dr \\
&= \frac{4C}{|y-z|^2}\, e^{(1/2)\, y\cdot(y-z)} \int_0^1 \frac{e^{-r|y|^2/4}}{r^{(n-2)/2}} \cdot \frac{d}{dr}\Big(e^{-|y-z|^2/(4r)}\Big)\, dr \\
&= \frac{4C}{|y-z|^2}\, e^{(1/2)\, y\cdot(y-z)} \Bigg(\frac{e^{-r|y|^2/4}}{r^{(n-2)/2}}\, e^{-|y-z|^2/(4r)} \Bigg|_{r=0}^{r=1} \\
&\qquad - \int_0^1 \frac{d}{dr}\Big(\frac{e^{-r|y|^2/4}}{r^{(n-2)/2}}\Big)\, e^{-|y-z|^2/(4r)}\, dr \Bigg) .
\end{aligned}$$

For $n=2$ the last expression equals to

$$\begin{aligned}
&\frac{4C}{|y-z|^2}\, e^{(1/2)\, y\cdot(y-z)}\, e^{-|y|^2/4}\, e^{-|y-z|^2/4} \\
&\quad + C\,\frac{|y|^2}{|y-z|^2} \int_0^1 e^{-|(1-r)y-z|^2/(4r)}\, dr \le C\,\frac{1+|y|^2}{|y-z|^2} \\
&\quad \le C(\beta, R) ,
\end{aligned}$$

and in case $n>2$ equals to

$$\begin{aligned}
&\frac{4C}{|y-z|^2}\, e^{-|z|^2/4} + \frac{C}{|y-z|^2}\, |y|^2 \int_0^1 \frac{1}{r^{(n-2)/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \\
&\quad + \frac{2C(n-2)}{|y-z|^2} \int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \\
&\qquad \le \frac{C}{|y-z|^2} \Big(1 + (1+|y|^2) \int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \Big) .
\end{aligned}$$

To estimate the last integral we write

$$\begin{aligned}\int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr &= \int_0^{|y-z|/(2|y|)} + \int_{|y-z|/(2|y|)}^1 \\ &\le \int_0^{|y-z|/(2|y|)} \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \\ &\quad + \left(\frac{2\,|y|}{|y-z|}\right)^{n/2}\end{aligned}$$

If $0 < r < |y-z|/(2|y|)$ then $|y-z-ry| \ge |y-z| - r|y| \ge |y-z|/2$, and consequently

$$\begin{aligned}\int_0^{|y-z|/(2|y|)} \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr &\le \int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z|^2/(16\, r)}\, dr \\ &\le \frac{C_n}{|y-z|^{n-2}}\,.\end{aligned}$$

Therefore for $n > 2$

$$K_1(y,z) \le C_n(\beta)\left(\frac{1}{|y-z|^2} + \frac{1+|y|^2}{|y-z|^2} + \frac{1+|y|^2}{|y-z|^n}\right) \le C_n(\beta, R)\,.$$

This completes the proof of case b).

If $(y,z) \in D_3$ then $\eta \ge R/2$ (otherwise $D_3^z = \varnothing$), and since $(y,z) \notin N_R$, we have $|y-z| > R/\eta$.

Case c.1). Since $(y,z) \in D_3$, we have $\xi \le \eta$ and $\alpha(y,z) < \pi/4$. So, $|y-z| \ge \beta\eta$, $0 < \xi \le \eta$, and $\eta > R/2$. Then by arguing as in case a) when $y \cdot z > 0$, and $\eta > R/2$ we obtain

$$K_1(y,z) \le C_n(\beta)\, e^{\xi^2-\eta^2}\,.$$

Case c.2). Let us first assume that $\xi > \eta$. We claim that there exists $R = R(B_1)$ large enough, such that if $\xi > \eta$ then

$$\eta < \xi \le |y| < \frac{\eta}{1-\beta}\,, \tag{3.1}$$

and

$$\xi - \eta > \frac{1}{\eta}\,. \tag{3.2}$$

In fact, $|y| \le \beta|y| + \eta$ implies (3.1). Next, suppose that $\xi - \eta \le 1/\eta$ then

$$|v| \ge |y - z| - (\xi - \eta) \ge \frac{R-1}{\eta} \ge \frac{B_1}{\eta}.$$

If $R \ge B_1 + 1$ we get a contradiction with the fact that $|v| < B_1/\eta$.

We write

$$K_1(y,z) = \int_0^1 \frac{|(\eta - r\xi)\frac{z_1}{\eta} - rv_1|}{(1-r^2)^{(n+3)/2}} e^{-(r|v|)^2/(1-r^2)} e^{-|r\xi-\eta|^2/(1-r^2)}\, dr$$

$$\le \int_0^1 \frac{|(\eta - r\xi)|}{(1-r^2)^{1/2}} \frac{1}{(1-r^2)^{(n+2)/2}} \cdot e^{-(r|v|)^2/(1-r^2)} e^{-|r\xi-\eta|^2/(1-r^2)}\, dr$$

$$+ \int_0^1 \frac{r|v|}{(1-r^2)^{1/2}} \frac{1}{(1-r^2)^{(n+2)/2}} \cdot e^{-(r|v|)^2/(1-r^2)} e^{-|r\xi-\eta|^2/(1-r^2)}\, dr$$

$$\le c \int_0^1 \frac{1}{(1-r)^{(n+2)/2}} e^{-|r\xi-\eta|^2/(4(1-r))}\, dr$$

$$= c\left(\int_0^{1-(\xi-\eta)/(2\xi)} \frac{1}{(1-r)^{(n+2)/2}}\, dr + \int_{1-(\xi-\eta)/(2\xi)}^1 \frac{1}{(1-r)^{(n+2)/2}} e^{-|r\xi-\eta|^2/(4(1-r))}\, dr \right)$$

$$= \mathrm{I} + \mathrm{II}.$$

We have

$$\mathrm{I} \le c_n \left(\frac{\xi}{\xi - \eta}\right)^{n/2} \le c_n(\beta)\, \eta^n.$$

If $1 - (\xi - \eta)/(2\xi) < r < 1$ then $r\xi - \eta > (\xi + \eta)/2 - \eta = (\xi - \eta)/2$. Hence

$$\mathrm{II} \le \int_0^{(\xi-\eta)/(2\xi)} \frac{1}{r^{(n+2)/2}} e^{-|\xi-\eta|^2/(16\,r)}\, dr \le \frac{C_n}{|\xi - \eta|^n} \le C_n\, \eta^n.$$

Therefore in case $\xi > \eta$ we obtain

$$K_1(y,z) \le C_n\, \eta^n.$$

Second, suppose now that $\xi \leq \eta$. We claim that for R sufficiently large, *i.e.* $R \geq B_1 + 1$ and

$$\frac{R^2}{R-1} > 8\left(\frac{1-\beta}{1-2\beta}\right),$$

we have

$$\eta - \xi < \frac{\eta}{2(1-\beta)}, \tag{3.3}$$

and

$$\eta - \xi > \frac{1}{\eta}. \tag{3.4}$$

In fact, first observe that in the region $|y-z| < \beta(|y| \vee |z|)$ we have $\xi > 0$ for $\beta < 1/4$. Then

$$\eta = |z| \leq |y-z| + |y| \leq \frac{\beta}{1-\beta}\eta + \xi + |v| \leq \frac{\beta}{1-\beta}\eta + \xi + \frac{R-1}{\eta},$$

and consequently

$$\begin{aligned}\frac{1-2\beta}{1-\beta}\eta &\leq \xi + \frac{R-1}{\eta} \\ &\leq \xi + 2R\frac{R-1}{R^2} \\ &\leq \xi + \frac{R}{2}\frac{1}{2}\frac{1-2\beta}{1-\beta} \\ &< \xi + \frac{1}{2}\frac{1-2\beta}{1-\beta}\eta,\end{aligned}$$

and (3.3) follows. The proof of (3.4) is similar to that of (3.2). Now we have

$$\begin{aligned}K_1(y,z) \leq c\Bigg((\eta-\xi)&\int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr \\ &+ \xi\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr \\ &+ \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr\Bigg)\, e^{\xi^2-\eta^2} \\ = (\mathrm{III}+\mathrm{IV}+\mathrm{V})\, e^{\xi^2-\eta^2}.\end{aligned}$$

We have

$$\begin{aligned}\mathrm{III} \le (\eta-\xi)\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+3)/2}}\,dr \\ + (\eta-\xi)\int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\,dr\,.\end{aligned}$$

If $0<r<1-(\eta-\xi)/(2\eta)$ then $\eta-\xi<2\eta(1-r)$. Also, if $1-(\eta-\xi)/(2\eta)<r<1$ then $r\eta-\xi=\eta-\xi-(1-r)\eta>(\eta-\xi)/2$. Therefore

$$\begin{aligned}(\eta-\xi)\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+3)/2}}\,dr \\ &\le 2\eta\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+1)/2}}\,dr \\ &\le c_n\,\eta\left(\frac{\eta}{\eta-\xi}\right)^{(n-1)/2} \\ &\le C_n\,\eta^n\,,\end{aligned}$$

and

$$\begin{aligned}(\eta-\xi)\int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\,dr \\ &\le (\eta-\xi)\int_0^{(\eta-\xi)/(2\eta)} \frac{e^{-(\xi-\eta)^2/(8r)}}{r^{(n+3)/2}}\,dr \\ &\le c_n\,\frac{\eta-\xi}{(\eta-\xi)^{n+1}} \\ &\le c_n\,\eta^n\,.\end{aligned}$$

Also

$$\begin{aligned}\mathrm{IV} &\le \eta\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+1)/2}}\,dr \\ &\quad + \eta\int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(\xi-\eta)^2/(8(1-r))}\,dr \\ &\le c_n\left(\eta\left(\frac{\eta}{\eta-\xi}\right)^{(n-1)/2} + \frac{\eta}{(\eta-\xi)^{n-1}}\right) \\ &\le c_n\,\eta^n\,,\end{aligned}$$

and

$$\begin{aligned}\mathrm{V} &\leq \int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+2)/2}}\, dr \\ &\quad + \int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(\xi-\eta)^2/(8(1-r))}\, dr \\ &\leq c_n \left(\left(\frac{\eta}{\eta-\xi}\right)^{n/2} + \frac{1}{(\eta-\xi)^n} \right) \\ &\leq c_n\, \eta^n .\end{aligned}$$

This completes the proof of case $c.2$).

Case $c.3$). We shall first assume that $|v| > B_2|\xi-\eta|$ and consider two cases: $\eta < \xi$ and $\eta \geq \xi$. Let us first consider the case $\eta < \xi$ and $|v| > B_2(\xi-\eta)$. We claim that

$$\frac{|v|}{\xi} < \frac{2\beta}{1-\beta},$$

in particular by taking $0 < \beta < 1/5$, we obtain $|v|/\xi < 1/2$. In fact,

$$\eta < \xi \leq |y| < \frac{1}{1-\beta}\,\eta < \frac{1}{1-\beta}\,\xi$$

which implies

$$|v| \leq |y-z| + |\xi-\eta| < \beta\,(|y|+\xi) \leq 2\,\beta\,|y| \leq \frac{2\beta}{1-\beta}\,\eta \leq \frac{2\beta}{1-\beta}\,\xi,$$

and the claim is proved.

We have

$$\begin{aligned}K_1(y,z) &\leq c \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|r\xi-\eta|^2/(4(1-r))}\, dr \\ &= c \left(\int_0^{1-|v|/\xi} + \int_{1-|v|/\xi}^1 \right) \\ &= \mathrm{VI} + \mathrm{VII} .\end{aligned}$$

If $0 < r < 1 - |v|/\xi$, then

$$\eta - r\xi = (1-r)\xi - (\xi - \eta) > |v| - (\xi - \eta) > \left(1 - \frac{1}{B_2}\right)|v|,$$

(we take $B_2 > 1$), therefore

$$\mathrm{VI} \le \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-c|v|^2/(1-r)}\, dr \le c_n \frac{1}{|v|^n}.$$

On the other hand, since $1 - |v|/\xi > 1/2$ we obtain

$$\mathrm{VII} \le \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|v|^2/(16(1-r))}\, dr \le c_n \frac{1}{|v|^n}.$$

Second, let us consider the case that $\xi \le \eta$ and $|v| > B_2(\eta - \xi)$. Then we have

$$\frac{|v|}{\eta} < \frac{2\beta}{1-\beta}.$$

In fact,

$$|v| \le |y - z| + |\xi - \eta| < \frac{\beta}{1-\beta}\eta + (\eta - \xi) \le \frac{\beta}{1-\beta}\eta + \frac{1}{B_2}|v|.$$

If we choose $B_2 \ge 2$ then

$$\frac{|v|}{2} < \frac{\beta}{1-\beta}\eta.$$

Consequently

$$\begin{aligned} K_1(y,z) \le \Bigg(&\int_0^1 \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\ &+ \int_0^1 \frac{|rv_1|}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))} \\ &\qquad \cdot e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \Bigg) e^{\xi^2 - \eta^2} \\ \le C \Bigg(&(\eta - \xi) \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))} \\ &\qquad \cdot e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \end{aligned}$$

$$+\xi\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(r|v|)^2/(2(1-r))} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\, dr$$

$$+\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\, dr\Big)\, e^{\xi^2-\eta^2},$$

and

$$(\eta-\xi)\int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr$$
$$\le (\eta-\xi)\Big(\int_0^{1-|v|/\eta} \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr + \int_{1-|v|/\eta}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, dr\Big).$$

If $0<r<1-|v|/\eta$ then

$$\xi - r\,\eta = (1-r)\,\eta - (\eta-\xi) > \Big(1-\frac{1}{B_2}\Big)\,|v|\,.$$

Also, $1-|v|/\eta > 1/2$ by taking $\beta < 1/5$. Hence, both summands on the right hand side of the last inequality are bounded by

$$(\eta-\xi)\int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|v|^2/(c(1-r))}\, dr \le c_n\,\frac{\eta-\xi}{|v|^{n+1}} \le c_n\,\frac{1}{|v|^n}\,.$$

The conditions $0<r<1-|v|/\eta$, and $|v|>B_2(\eta-\xi)$ imply that $\eta-\xi<((1-r)\eta))/B_2$, therefore

$$\xi - r\,\eta = (1-r)\,\eta - (\eta-\xi) > \Big(1-\frac{1}{B_2}\Big)\,(1-r)\,\eta \ge \frac{(1-r)\eta}{2}\,,$$

since $B_2\ge 2$. Then we have

$$\xi\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr$$
$$\le \eta\int_0^{1-|v|/\eta} \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(1-r)\,\eta^2/8}\, dr$$

$$
\begin{aligned}
&+\eta \int_{1-|v|/\eta}^{1} \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|v|^2/(8(1-r))}\, dr \\
&\le \eta\, e^{-\eta|v|/8} \int_{|v|/\eta}^{1} \frac{1}{r^{(n+1)/2}}\, dr \\
&\quad + c_n \frac{\eta}{|v|^{n-1}} \int_{\sqrt{\eta|v|/4}}^{\infty} u^{n-2}\, e^{-u^2}\, du \\
&\le C_n \left(\eta \left(\frac{\eta}{|v|} \right)^{(n-1)/2} e^{-\eta|v|/8} + \frac{\eta}{|v|^{n-1}} \frac{e^{-\eta|v|/8}}{\sqrt{\eta|v|}} \right) \\
&\le C_n \left((\eta|v|)^{(n+1)/2} + (\eta|v|)^{1/2} \right) \frac{e^{-\eta|v|/8}}{|v|^n} \\
&\le C_n \frac{1}{|v|^n}\, .
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int_0^1 & \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr \\
&\le \int_0^{1-|v|/\eta} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr \\
&\quad + \int_{1-|v|/\eta}^{1} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, dr \\
&\le C \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-c|v|^2/(1-r)}\, dr \\
&\le c_n \frac{1}{|v|^n}\, .
\end{aligned}
$$

Therefore, if $|v| > B_2|\xi-\eta|$ we then have

$$
K_1(y,z) \le C_n \frac{1}{|v|^n}\, e^{\xi^2-\eta^2}\, .
$$

From now on we may assume $|v| \le B_2|\xi-\eta|$. We may also assume that

$$
|\xi-\eta| > \frac{1}{\eta} \qquad \text{and} \qquad |v| > \frac{B_2}{\eta}\, .
$$

In fact, if $|\xi-\eta| \le 1/\eta$ then $|v| > B_2\,|\xi-\eta|$, which falls into the case previously considered.

Let us first assume $\xi-\eta > 1/\eta$ and $|v| > B_2/\eta$. We have $\xi-\eta < \beta\,\xi$, this is because

$$|y| \le |y-z| + |z| \le \beta\,(|y| \vee |z|) + |z| = \beta\,|y| + \eta$$

and so $|y| < (1-\beta)^{-1}\eta$ which implies $\eta < \xi < |y| < (1-\beta)^{-1}\eta$. If $0 < r < 1/2$ then

$$\eta - r\,\xi = (1-r)\,\xi - (\xi-\eta) > \Big(\frac{1}{2}-\beta\Big)\xi > \frac{1}{B_2}\,\Big(\frac{1}{2}-\beta\Big)\,|v|\,,$$

and by arguing as in the case when $\eta \le \xi$ and $|v| > B_2(\xi-\eta)$, we obtain

$$\begin{aligned} K_1(y,z) &\le C\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|r\xi-\eta|^2/(4(1-r))}\,dr \\ &\le C\int_0^{1/2} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|r\xi-\eta|^2/(4(1-r))}\,dr \\ &\quad + C\int_{1/2}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\,dr \\ &\le C\int_0^1 \frac{1}{r^{(n+2)/2}}\, e^{-c\,|v|^2/(r)}\,dr \\ &\le C_n\,\frac{1}{|v|^n}\,. \end{aligned}$$

Second, let us now assume $\eta-\xi > 1/\eta$ and $|v| > B_2/\eta$. We have

$$\eta - \xi \le |y-z| < \frac{\beta}{1-\beta}\,\eta$$

(this follows because $|y-z| \le \beta\,(|y| \vee |z|)$ and by analizing $|y| > \eta$ or $|y| \le \eta$) which implies

$$\frac{\eta-\xi}{\eta} < \frac{\beta}{1-\beta} < \frac{1}{4}\,,$$

by taking $\beta < 1/3$. We write

$$\begin{aligned} K_1(y,z) \le C\Big(\int_0^1 \frac{|\eta-r\xi|}{(1-r)^{(n+3)/2}}\, & e^{-|\xi-r\eta|^2/(2(1-r))} \\ & \cdot e^{-(r|v|)^2/(2(1-r))}\,dr \end{aligned}$$

$$+\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))} \cdot e^{-(r|v|)^2/(4(1-r))}\, dr\Big) e^{\xi^2-\eta^2}$$

$$= C\bigg(\int_0^{1-((1-\beta)/(2\beta))(\eta-\xi)/\eta} + \int_{1-((1-\beta)/(2\beta))(\eta-\xi)/\eta}^{1-\beta(\eta-\xi)/\eta} + \int_{1-\beta(\eta-\xi)/\eta}^{1} \frac{|\eta-r\xi|}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))} \cdot e^{-(r|v|)^2/(2(1-r))}\, dr$$

$$+\int_0^{1/2} + \int_{1/2}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))} \cdot e^{-(r|v|)^2/(4(1-r))}\, dr\bigg) e^{\xi^2-\eta^2}.$$

If $0<r<1/2$ then

$$\xi - r\,\eta = (1-r)\,\eta - (\eta-\xi) > \frac{\eta}{2} - \frac{\beta}{1-\beta}\,\eta > \frac{\eta}{4} > \frac{|v|}{4B_2},$$

therefore

$$\int_0^{1/2} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr + \int_{1/2}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, dr \le C\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-c|v|^2/(1-r)}\, dr \le \frac{C_n}{|v|^n}.$$

If

$$0 < r < 1 - \frac{1-\beta}{2\beta}\frac{\eta-\xi}{\eta}$$

then

$$|\eta - r\xi| = \eta - \xi + (1-r)\,\xi < \left(1 + \frac{2\beta}{1-\beta}\right)(1-r)\,\eta,$$

and

$$\xi - r\,\eta = (1-r)\,\eta - (\eta - \xi) > \left(1 - \frac{2\beta}{1-\beta}\right)(1-r)\,\eta\,.$$

Therefore

$$\begin{aligned}
&\int_0^{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta} \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\
&\quad \le C\,\eta \int_0^{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta} \frac{1}{(1-r)^{(n+1)/2}}\, e^{-c(1-r)\eta^2}\, dr \\
&\quad \le C\,\eta \int_{\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta}^{1} \frac{1}{r^{(n+1)/2}}\, dr\; e^{-c(\eta-\xi)\eta} \\
&\quad \le C_n\,\eta \left(\frac{\eta}{\eta-\xi}\right)^{(n-1)/2} e^{-c(\eta-\xi)\eta} \\
&\quad \le \frac{C_n}{(\eta-\xi)^n} \\
&\quad \le \frac{C_n}{|v|^n}\,.
\end{aligned}$$

The case

$$1 - \frac{1-\beta}{2\beta}\,\frac{\eta-\xi}{\eta} < r < 1 - \beta\,\frac{\eta-\xi}{\eta}$$

is equivalent to

$$\beta\,\frac{\eta-\xi}{\eta} < 1 - r < \frac{1-\beta}{2\beta}\,\frac{\eta-\xi}{\eta}\,.$$

Then

$$|\eta - r\xi| = \eta - \xi + (1-r)\,\xi \le \left(\frac{1}{\beta} + 1\right)(1-r)\,\eta\,,$$

and consequently

$$\int_{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta}^{1-\beta(\eta-\xi)/\eta} \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\, e^{-(r|v|)^2/(2(1-r))}\, dr$$

$$\leq C\,\eta \int_{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta}^{1-\beta(\eta-\xi)/\eta} \frac{1}{(1-r)^{(n+1)/2}} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\,dr\; e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\leq C\,\eta \left(\frac{\eta}{\eta-\xi}\right)^{(n+1)/2} \cdot \int_{\beta(\eta-\xi)/\eta}^{\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta} e^{-c\,\eta|r\eta-(\eta-\xi)|^2/(\eta-\xi)}\,dr\; e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\leq C \left(\frac{\eta}{\eta-\xi}\right)^{(n+1)/2} \cdot \int_{-(1-\beta)(\eta-\xi)}^{\left((1-\beta)/(2\beta)-1\right)(\eta-\xi)} e^{-c\,\eta u^2/(\eta-\xi)}\,du\; e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\leq C \left(\frac{\eta}{\eta-\xi}\right)^{(n+1)/2} \left(\frac{\eta-\xi}{\eta}\right)^{1/2} e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\leq C \left(\frac{\eta}{\eta-\xi}\right)^{n/2} e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\leq \frac{C_n}{|v|^n}\,.$$

If $1-\beta\,(\eta-\xi)/\eta < r < 1$ then

$$|\eta - r\xi| = \eta - r\xi = \eta - \xi + (1-r)\,\xi \leq (1+\beta)(\eta-\xi)$$

and

$$r\eta - \xi = \eta - \xi - (1-r)\,\eta > (1-\beta)(\eta-\xi)\,.$$

Therefore

$$\int_{1-\beta\,(\eta-\xi)/\eta}^{1} \frac{|\eta-r\xi|}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, e^{-(r|v|)^2/(2(1-r))}\,dr$$

$$\leq (1+\beta)\,(\eta-\xi) \int_{1-\beta\,(\eta-\xi)/\eta}^{1} \frac{1}{(1-r)^{(n+3)/2}}\, e^{-c(\xi-\eta)^2/(1-r)}\,dr$$

$$\leq \frac{C_n}{(\eta-\xi)^n}$$

$$\leq \frac{C_n}{|v|^n}\,.$$

Hence

$$K_1(y,z) \le \frac{C_n}{|v|^n}\, e^{\xi^2-\eta^2}\,.$$

This completes the proof of the case $c.3$) and therefore Lemma 2 is complete.

Theorem 2. *The operator K_2^* is of weak-type (1,1) with respect to the measure $\gamma(z)\,dz$, provided that R is sufficiently large.*

PROOF. We decompose K_2^* in the following way

$$\begin{aligned} K_2^* f(y) &= \int_{\mathbb{R}^n \setminus N_R^y} |K(y,z)|\,|f(z)|\,dz \\ &= \int_{D_1^y} |K(y,z)|\,|f(z)|\,dz + \int_{D_2^y} |K(y,z)|\,|f(z)|\,dz \\ &\quad + \int_{D_3^y} |K(y,z)||f(z)|\,dz \\ &= K_R^1 f(y) + K_R^2 f(y) + K_R^3 f(y)\,. \end{aligned}$$

We shall first show that K_R^i, $i=1,2$, are bounded operators in $L^1_\gamma(\mathbb{R}^n)$.
Let $f \in L^1_\gamma(\mathbb{R}^n)$, we may assume $f \ge 0$. We have

$$\|K_R^1 f\|_{L^1_\gamma} = \int_{\mathbb{R}^n} f(z) \int_{D_1^z} |K(y,z)|\,\gamma(y)\,dy\,dz\,,$$

and by Lemma 2, part a)

$$\begin{aligned} \int_{D_1^z} |K(y,z)|\,\gamma(y)\,dy &\le C_n \int_{\{y:\ \xi \le \eta,\, \alpha(y,z) \ge \pi/4\}} e^{\xi_+^2-\eta^2}\, e^{-|y|^2}\,dy \\ &\le C_n\, e^{-\eta^2} \Bigg(\int_{\{y:\ \xi \le 0\}} e^{-|y|^2}\,dy \\ &\qquad + \int_{\{y:\ 0<\xi\le |y|/\sqrt{2}\}} e^{\xi^2}\, e^{-|y|^2}\,dy \Bigg) \\ &\le C_n e^{-|z|^2}\,. \end{aligned}$$

Hence K_R^1 is bounded in $L^1_\gamma(\mathbb{R}^n)$. Also

$$\begin{aligned}\|K_R^2 f\|_{L^1_\gamma} &= \int_{\mathbb{R}^n} f(z) \int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy\,dz \\ &= \int_{|z|\le R} f(z) \int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy\,dz \\ &\quad + \int_{|z|>R} f(z) \int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy\,dz\,,\end{aligned}$$

and by Lemma 2, part *b*)

$$\begin{aligned}\int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy &\le C \int_{\{y:\ \xi>\eta\}} e^{-|y|^2}\,dy \\ &= C \int_{\mathbb{R}^{n-1}} e^{-|v|^2} \int_\eta^\infty e^{-\xi^2}\,d\xi\,dv \\ &\le C\left(\frac{1}{R}\,\chi_{B_R^c(0)}(z) + e^{R^2}\chi_{B_R(0)}(z)\right) e^{-|z|^2}\,.\end{aligned}$$

Therefore K_R^2 is bounded in $L^1_\gamma(\mathbb{R}^n)$.

To show that K_R^3 is of weak-type (1,1) we shall dominate its kernel by a kernel which is constant on certain cubes. We split $\mathbb{R}^n$ into a non-overlapping sequence of open cubes Q_i with center x_i, $i = 1, 2, \ldots$ such that

$$c_n\left(1 \wedge \frac{1}{|x_i|}\right) \le \operatorname{diam}(Q_i) \le 1 \wedge \frac{1}{|x_i|}\,.$$

The sequence of cubes can be chosen such that $\{|x_i|\}$ is a non-decreasing sequence. We set

$$K_3(y,z) = |K(y,z)|\,\chi_{D_3}(y,z)\,,$$

and we define

$$\overline{K}_3(y,z) = \sum_{i=1}^\infty \chi_{Q_i}(y) \sup_{y'\in Q_i} K_3(y',z)\,.$$

We claim that $\overline{K}_3(y,z)$ satisfies the estimates of Lemma 2, part *c.i*), $i = 1, 2, 3$, with new constants.

In fact, first observe that for $y, y' \in Q_i$ we have

$$y = \xi \frac{z}{\eta} + v, \qquad y' = \xi' \frac{z}{\eta} + v',$$

with

$$|\xi - \xi'| \leq |y - y'| \leq \operatorname{diam}(Q_i) \leq \left(1 \wedge \frac{1}{|x_i|}\right),$$

and

$$|\xi + \xi'| \leq |y| + |y'| \leq 2\left(|x_i| + \left(1 \wedge \frac{1}{|x_i|}\right)\right).$$

Therefore,

$$|\xi'^2 - \xi^2| \leq 2\left[\left(1 \wedge \frac{1}{|x_i|}\right)^2 + |x_i|\left(1 \wedge \frac{1}{|x_i|}\right)\right] \leq 4,$$

and consequently

$$e^{\xi'^2 - \eta^2} \leq e^4 e^{\xi^2 - \eta^2}.$$

For given β and R we set $D_3 = D_3(\beta, R)$, where D_3 is defined at the beginning of this section. Observe that if $\beta_1 \leq \beta_2$ then $D_3(\beta_1, R) \subset D_3(\beta_2, R)$; and if $R_1 \leq R_2$ then $D_3(\beta, R_2) \subset D_3(\beta, R_1)$. The fact $(y, z) \in D_3$ implies that $|z| \geq R/2$. We want to show that $\overline{K}_3(y, z)$ satisfies an estimate like in Lemma 2, part c.1). Suppose first that $(y, z) \in D_3(\beta, R)$, $|y - z| \geq \beta(|y| \vee |z|)$, and $y, y' \in Q_i$. In case $|y' - z| \geq \beta(|y'| \vee |z|)$ we may apply Lemma 2, part c.1), to obtain

$$K_3(y', z) \leq c\, e^{\xi'^2 - \eta^2}. \tag{3.1}$$

In case $|y' - z| < \beta(|y'| \vee |z|)$ we shall show that for R sufficiently large we have

$$|y' - z| \geq (1 - \beta)\left(\beta - \frac{2}{R}\right)(|y'| \vee |z|). \tag{3.2}$$

In fact, in such case we have

$$(1 - \beta)|y'| < |z| < \frac{1}{1 - \beta}|y'|,$$

and

$$|y' - z| \geq |y - z| - |y - y'| \geq \beta(|y| \vee |z|) - \frac{2}{R}|z| \geq \left(\beta - \frac{2}{R}\right)|z|.$$

Now by analizing the cases $|y'| > |z|$ and $|y'| \le |z|$ we have that

$$\left(\beta - \frac{2}{R}\right)|z| > \left(\beta - \frac{2}{R}\right)(1-\beta)\,(|y'| \vee |z|)$$

and (3.2) follows. Consequently, if $(y', z) \in D_3((\beta - 2/R)(1-\beta), R)$ then we may apply Lemma 2, part $c.1$), to obtain (3.1). If $(y', z) \notin D_3((\beta - 2/R)(1-\beta), R)$ then by taking β small and R large we get

$$(y', z) \in D_1((\beta - \frac{2}{R})(1-\beta), R) \cup D_2((\beta - \frac{2}{R})(1-\beta), R).$$

If $(y', z) \in D_1((\beta - 2/R)(1-\beta), R)$, since $|y' - z| < \beta\,(|y'| \vee |z|)$, we then have $\xi' > 0$, and by Lemma 2, part a), we get (3.1). If $(y', z) \in D_2((\beta - 2/R)(1-\beta), R)$ then $\xi' > \eta$ and by Lemma 2, part b), we obtain (3.1).

Next, let us assume that $(y, z) \in D_3(\beta, R)$, $|y - z| < \beta\,(|y| \vee |z|)$, and $y, y' \in Q_i$. In this case we have

$$(1-\beta)\,|z| < |y| < \frac{1}{1-\beta}\,|z|,$$

and since $|z| > R/2$, we have $|y| > (1-\beta)R/2 > 3$ for R sufficiently large. This implies that if $y \in Q_i$ then $|x_i| > \sqrt{2}$, and consequently $|y| \approx |x_i|$. It is easy to see that

$$|y' - z| < \left(2\beta + \frac{2}{R}\right)(|y'| \vee |z|).$$

Therefore $(y', z) \in D_3(2\beta + 2/R, R)$. If $|v| < B/\eta$ then

$$|v'| \le |y - y'| + |v| \le \frac{c}{\eta} + |v| < \frac{B + c}{\eta} = \frac{B_1}{\eta}.$$

Then taking β small and R large, by Lemma 2, part $c.2$), we have

$$|K(y', z)| \le C\,\eta^n \left(1 \wedge e^{\xi'^2 - \eta^2}\right) \le \overline{C}\,\eta^n \left(1 \wedge e^{\xi^2 - \eta^2}\right).$$

If $|v| \ge B/\eta$ then $|v'| \ge |v| - |y - y'| \ge |v| - c/\eta$ and so

$$|v'| \ge \frac{B - c}{\eta} = \frac{B_2}{\eta} \qquad \text{and} \qquad |v'| \ge \left(1 - \frac{c}{B}\right)|v|.$$

By Lemma 2, part $c.3$), we obtain

$$|K(y',z)| \le \frac{C_n}{|v'|^n}\left(1 \wedge e^{\xi'^2-\eta^2}\right) \le \frac{\overline{C}_n}{|v|^n}\left(1 \wedge e^{\xi^2-\eta^2}\right).$$

This proves the claim.

Clearly

$$K_R^3 f(y) \le \int_{D_3^y} \overline{K}_3(y,z)\, f(z)\, dz = \overline{K}_R^3 f(y),$$

and $\overline{K}_R^3 f(y)$ is constant on every Q_i.

In order to prove that $\overline{K}_R^3$ is of weak-type (1,1), given $\lambda > 0$ we shall construct a set E with the following properties:

a) $E \subset E_\lambda = \{y \in \mathbb{R}^n : \overline{K}_R^3 f(y) > \lambda\}$.

b) $\displaystyle\int_{E_\lambda} e^{-|y|^2}\, dy \le c \int_E e^{-|y|^2}\, dy$.

c) If $U(z) = \displaystyle\int_E \overline{K}_3(y,z)\, e^{-|y|^2}\, dy$, then $U(z) \le c\, e^{-|z|^2}$ in $\mathbb{R}^n$.

The last two properties imply the weak-type inequality, in fact

$$\begin{aligned}\int_{E_\lambda} e^{-|y|^2}\, dy &\le c\int_E e^{-|y|^2}\, dy \\ &\le \frac{c}{\lambda}\int_E \overline{K}_R^3\, f(y) e^{-|y|^2}\, dy \\ &= \frac{c}{\lambda}\int_E\int_{\mathbb{R}^n} \overline{K}_3(y,z)\, f(z)\, dz\; e^{-|y|^2}\, dy \\ &= \frac{c}{\lambda}\int_{\mathbb{R}^n} f(z)\, U(z)\, dz \\ &\le \frac{c}{\lambda}\, \|f\|_{L^1_\gamma}.\end{aligned}$$

The construction of the set E is done as in [Sj]. We recall the construction here.

Given a positive integer j we define the cone

$$K_j = \{x : \ \alpha(x,y) \le \pi/4 \ \text{ for some } y \in Q_j\}.$$

To each cube Q_j we associate a forbidden region F_j defined by

$$F_j = \bigcup\{Q_i : \ i > j \ \text{ and } \ Q_i \cap (Q_j + K_j) \neq \varnothing\} .$$

The set E is constructed as follows. Let Q_{i_1} the first cube that intersects E_λ. Since $\overline{K}^3_R f(y)$ is constant on each cube Q_i we have that $Q_{i_1} \subset E_\lambda$. Next we pick Q_{i_2}, $i_1 < i_2$, the first cube that intersects E_λ and is not contained in the forbidden region F_{i_1}. Continuing in this way, Q_{i_j} is the first cube that intersects E_λ and is not contained in the forbidden regions F_{i_k} for any Q_{i_k} already selected. The set E is by definition the union of the cubes Q_{i_j}, $j = 1, 2, \ldots$ Property *a*) above is obvious. The proof of property *b*) can be found in [Sj].

Let us prove *c*). Let S_z denote the support of $\overline{K}_3(\cdot, z)$. The set S_z consists of the union of those Q_i that intersect D_3^z. Let l_v be the line parallel to z and passing through v, with $v \perp z$. We have

$$\begin{aligned} U(z) &= \int_{\mathbb{R}^n} \overline{K}_3(y, z)\, e^{-|y|^2}\, dy \\ &= \int_{\mathbb{R}^{n-1}} e^{-|v|^2} \int_{l_v \cap E \cap S_z} \overline{K}_3(s\frac{z}{\eta} + v, z)\, e^{-s^2}\, ds\, dv , \end{aligned}$$

and we set

$$I(v) = \int_{l_v \cap E \cap S_z} \overline{K}_3(s\frac{z}{\eta} + v, z)\, e^{-s^2}\, ds .$$

Let $w \in l_v \cap E \cap S_z$, then there exists a unique i such that $w \in Q_i \cap D_3^z$. The angle between w and z is less than $\pi/4$, and therefore $z \in K_i$, K_i being the cone defined before. In fact, if $\alpha(w, z) > \pi/4$ and $(w, z) \in D_3$ then $|w - z| < \beta\,(|w| \vee |z|)$ and for β small this implies $\alpha(w, z) \leq \pi/4$.

Every element of l_v is of the form $s\,z/\eta + v$, and if $Q_i \cap l_v \neq \varnothing$ then every cube Q_j with $j > i$ that intersects l_v is included in F_i. Therefore, for every fixed v, $l_v \cap E \cap S_z$ is the line segment I determined by the intersection between Q_i and the line l_v. We shall estimate the size of I. Let y, y' be the endpoints of such segment, with

$$y = \xi\,\frac{z}{\eta} + v\,, \qquad y' = \xi'\,\frac{z}{\eta} + v\,, \qquad \text{and} \qquad 0 < \xi < \xi' .$$

Hence

$$I = \{s\,\frac{z}{\eta} + v : \ \xi < s < \xi'\}, \qquad |\xi - \xi'| \leq \left(1 \wedge \frac{1}{|x_i|}\right) ,$$

where x_i is the center of Q_i. If $|x_i| > 2$ then

$$\xi \le |y| \le |y - x_i| + |x_i| \le \frac{5}{4}|x_i|,$$

and

$$\xi' \le \xi + \frac{1}{|x_i|} \le \xi + \frac{5}{4\xi}.$$

If $|x_i| \le 2$ then $\xi \le |y - x_i| + 2 \le 3$ and $\xi' \le \xi + 1 \le \xi + 3/\xi$. Therefore, I is included in the line segment

$$J = \left\{ s\frac{z}{\eta} + v : \ \xi \le s \le \xi + c\Big(1 \wedge \frac{1}{\xi}\Big) \right\}.$$

Now we shall estimate $I(v)$. Let us first assume that $w \in I$ and $|w - z| \ge \beta(|w| \vee |z|)$. We have

$$\overline{K}_3(w, z) = \overline{K}_3(y, z) \le \overline{C}\, e^{\xi^2 - \eta^2}.$$

If Q_i intersects part of the circles

$$|y - z| \le \beta\eta \qquad \text{or} \qquad |y - z| \le \frac{\beta}{1-\beta}\eta,$$

then one can get the same estimates, *i.e.* $\overline{C}\, e^{\xi^2-\eta^2}$, by applying lemma 2, part *c*.1), with $|y - z| > \delta\eta$ and $\delta < \beta$.

Hence

$$I(v) \le C\left(1 \wedge \frac{1}{\xi}\right) e^{\xi^2-\eta^2} e^{-\xi^2} \le e^{-|z|^2}.$$

Second, we assume $w \in I$, $|w - z| < \beta(|w| \vee |z|)$ and $|v| < B/\eta$. We have

$$\overline{K}_3(w, z) = \overline{K}_3(y, z) \le C\,\eta^n \left(1 \wedge e^{\xi^2-\eta^2}\right).$$

Consequently

$$I(v) \le \frac{c}{\xi}\eta^n \left(1 \wedge e^{\xi^2-\eta^2}\right) e^{-\xi^2} \le c\,\eta^{n-1} e^{-\eta^2},$$

since $\xi > \eta$ (see proof of Lemma 2, part *c*.1)).

Third, we assume $w \in I$, $|w - z| < \beta(|w| \vee |z|)$ and $|v| \ge B/\eta$. Then

$$\overline{K}_3(w, z) = \overline{K}_3(y, z) \le \frac{C}{|v|^n}\left(1 \wedge e^{\xi^2-\eta^2}\right).$$

Consequently

$$I(v) \leq \frac{c}{\xi} \frac{1}{|v|^n} \left(1 \wedge e^{\xi^2-\eta^2}\right) e^{-\xi^2} \leq c \frac{c}{\eta} \frac{1}{|v|^n} e^{-\eta^2},$$

since $\xi > \eta$ (see proof of Lemma 2, part $c.3$)).

Therefore, in the first case above we have

$$U(z) \leq c \left(\int_{\mathbb{R}^{n-1}} e^{-|v|^2} \, dv \right) e^{-|z|^2},$$

and in the second and third cases above we obtain

$$\begin{aligned} U(z) &= \int_{|v|_{n-1} < B/\eta} e^{-|v|^2} I(v) \, dv + \int_{|v|_{n-1} \geq B/\eta} e^{-|v|^2} I(v) \, dv \\ &\leq c_n \left(\eta^{n-1} \int_{|v|_{n-1} < B/\eta} dv + \frac{1}{\eta} \int_{|v|_{n-1} \geq B/\eta} \frac{1}{|v|^n} \, dv \right) e^{-|z|^2} \\ &\leq c_n \, e^{-|z|^2}. \end{aligned}$$

This completes the proof of property c), therefore, Theorem 2 is complete.

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Eugene B. Fabes*
School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455, U.S.A.

Cristian E. Gutiérrez*
Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122, U.S.A.

Roberto Scotto
School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455, U.S.A.

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