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A weighted version of Journé's Lemma

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In this paper we discuss a weighted version of Journé's covering lemma, a substitute for the Whitney decomposition of an arbitrary open set in $\mathbb{R}^2$ where squares are replaced by rectangles. We then apply this result to obtain a sharp version of the atomic decomposition of the weighted Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and a description of their duals when p is close to 1.

A nonnegative locally integrable function $w(x, y)$ on $\mathbb{R}^2$ is called a *weight*. A weight w is said to satisfy *Muckenhoupt's* $A_p(\mathbb{R}\times\mathbb{R})$ *condition on rectangles*, or plainly the A_p *condition*, $1 < p < \infty$, provided that

$$\sup_R \Big(\frac{1}{|R|}\iint_R w(x,y)\,dx\,dy\Big)\Big(\frac{1}{|R|}\iint_R w(x,y)^{-1/(p-1)}dx\,dy\Big)^{p-1} \le c\,,$$

where R runs over all rectangles with sides parallel to the coordinate axes. When $p = 1$ this condition reduces to

$$\frac{1}{|R|}\iint_R w(x,y)\,dx\,dy \le c \operatorname*{ess\,inf}_{(x,y)\in R} w(x,y)\,, \qquad \text{all } R\,.$$

We say that w satisfies the $A_\infty(\mathbb{R}\times\mathbb{R})$ *condition* if it satisfies the A_p condition for some $p < \infty$. The constant c that appears on the right-hand side in the inequalities above is called *the* A_p *constant of* w, and a property is said to be *independent in* A_p provided it depends on c, and not on the particular weight w in A_p involved. By the Lebesgue differentiation theorem it readily follows that if w satisfies the A_p condition, then $w(x,\cdot)$ satisfies Muckenhoupt's $A_p(\mathbb{R})$ condition, uniformly for a.e. x, with constant $\le c$, the A_p constant for w; similarly for $w(\cdot,y)$.

The same holds for A_∞: an A_∞ weight w is uniformly in $A_\infty(\mathbb{R})$ for a.e. x, or y, fixed. By well-known properties of A_∞ weights, if $w(x,\cdot)$ is an $A_\infty(\mathbb{R})$ weight uniformly in x, then the following holds: given $x \in \mathbb{R}$ and $0<\varepsilon<1$, there exists η, $0<\eta<1$, such that if $A \subseteq I$ and

$$(1) \qquad \frac{w(x,A)}{w(x,I)} > \eta\,, \qquad \text{then} \quad \frac{w(x',A)}{w(x',I)} > \varepsilon \quad \text{for a.e. } x' \in \mathbb{R}\,.$$

It is clear that we may always choose $\eta \ge 1/2$ above, and we do so.

Under the assumption that w is uniformly A_∞ for a variable fixed and uniformly doubling for the other variable fixed, the weighted strong maximal operator $M_{S,w}f(x,y)$ given by

$$M_{S,w}f(x,y) = \sup_{(x,y)\in R} \frac{1}{w(R)} \int\!\!\int_R |f(u,v)|\, w(u,v)\, du\, dv\,,$$

is known to be bounded in $L^2_w(\mathbb{R}^2)$, say, *cf.* [JT] and [F1].

Given a bounded open set $\Omega \subset \mathbb{R}^2$, $x \in \mathbb{R}$ and $t>0$, following [J], let

$$E_{x,t} = \{y \in \mathbb{R} :\ [x-t, x+t] \times \{y\} \subseteq \Omega\}\,.$$

Each $E_{x,t}$ is open, because Ω is open, and, for each x, $E_{x,t}$ is decreasing in t.

Let $E_{x,t} = \bigcup_k J^k_{x,t}$ denote the decomposition of $E_{x,t}$ into open interval components, and let $t(k,x)$ be the infimum over those $\tau \ge t$ such that

$$(2) \qquad w(x, J^k_{x,t} \cap E_{x,\tau}) \le \eta\, w(x, J^k_{x,t})\,,$$

where $1/2 \le \eta < 1$ corresponds to the value $\varepsilon = 1/2$ above.

Proposition 1. *Given a bounded open set Ω, let*

$$\hat{\Omega} = \bigcup_{x,t,k} (x - t(k,x), x + t(k,x)) \times J^k_{x,t}\,,$$

and assume that the weight $w(x,y)$ is uniformly $A_\infty(\mathbb{R})$ for a variable fixed, and uniformly doubling for the other variable fixed. Then $w(\hat{\Omega}) \le c\,w(\Omega)$, where c is independent of Ω.

PROOF. As it is readily seen by the containment relation between the sets involved, we have

$$(3)\ w\big(((x-s,x+s)\times J^k_{x,t})\cap\Omega\big)\geq w\big((x-s,x+s)\times(J^k_{x,t}\cap E_{x,s})\big)\,.$$

Now, if $s < t(k,x)$, from (2) and (1) it follows that

$$(4)\qquad w(x',J^k_{x,t}\cap E_{x,s})>\frac{1}{2}\,w(x',J^k_{x,t})\,,\qquad \text{a.e. } x'\in\mathbb{R}\,.$$

Thus, integrating (4) over $(x-s,x+s)$, combining the resulting expression with (3), and setting $R=(x-s,x+s)\times J^k_{x,t}$, we obtain

$$(5)\qquad \iint_R \chi_\Omega(x,y)\,w(x,y)\,dx\,dy>\frac{1}{2}\iint_R w(x,y)\,dx\,dy\,.$$

Now, if $(x',y')\in\hat{\Omega}$, there exist x,t,k such that $x'\in(x-t(k,x),x+t(k,x))$, and also $s<t(k,x)$ so that $(x',y')\in(x-s,x+s)\times J^k_{x,t}=R$. Whence, by (5),

$$\hat{\Omega}\subseteq\{M_{S,w}(\chi_\Omega)>\frac{1}{2}\}\,,$$

and by the continuity of $M_{S,w}$ in $L^2_w(\mathbb{R}^2)$,

$$w(\hat{\Omega})\leq c\,w(\Omega)\,,$$

with c independent of Ω.

Proposition 2. *Suppose Ω and w are as in Proposition 1, and that ϕ is a nondecreasing function with $\phi(0)=0$. Then*

$$\int_0^{+\infty}\int_{\mathbb{R}}\sum_k\int_{J^k_{x,t}}\phi\Big(\frac{t}{t(k,x)}\Big)\,w(x,y)\,dy\,dx\,\frac{dt}{t}\leq c\,w(\Omega)\int_0^1\phi(s)\,\frac{ds}{s}\,,$$

where c is a constant independent of Ω and ϕ.

PROOF. From (2) it readily follows that

$$w(x,J^k_{x,t})\leq\frac{1}{1-\eta}\,w(x,J^k_{x,t}\setminus E_{x,t(k,x)})\,.$$

Thus, save for the factor $1/(1-\eta)$, the left-hand side of the above expression does not exceed

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty}\sum_k \chi_{J^k_{x,t}\setminus E_{x,t(k,x)}}(y)\,\phi(\frac{t}{t(k,x)})\,w(x,y)\,\frac{dt}{t}\,dy\,dx$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} B(x,y)\,w(x,y)\,dx\,dy\,,$$

say. We want to show that

$$B(x,y) \le c\,\chi_\Omega(x,y)\int_0^1 \phi(s)\,\frac{ds}{s}\,.$$

Clearly if $(x,y)\notin\Omega$, then $B(x,y)=0$. Also, if $(x,y)\in\Omega$, at most one summand in the above sum does not vanish, the one corresponding to the index k, say. Thus,

$$B(x,y) = \chi_\Omega(x,y)\int_0^{+\infty} \chi_{J^k_{x,t}\setminus E_{x,t(k,x)}}(y)\,\phi(\frac{t}{t(k,x)})\,\frac{dt}{t}\,.$$

Let $T(x,y) = \sup\{s : \ [x-s, x+s]\times\{y\}\subseteq\Omega\}$ Since $J^k_{x,t}$ is an interval component of $E_{x,t}$, from this definition it readily follows that $t\le T(x,y)$. We may also assume that $T(x,y)\le t(k,x)$, for if $t(k,x) < T(x,y)$, then it follows that $y\in E_{x,t(k,x)}$, and the integrand above vanishes. Whence

$$\begin{aligned} B(x,y) &\le \chi_\Omega(x,y)\int_0^{T(x,y)} \chi_{J^k_{x,t}\setminus E_{x,t(k,x}}(y)\,\phi(\frac{t}{t(k,x)})\,\frac{dt}{t} \\ &\le \chi_\Omega(x,y)\int_0^{T(x,y)} \phi(\frac{t}{T(x,y)})\,\frac{dt}{t} \\ &= \chi_\Omega(x,y)\int_0^1 \phi(s)\,\frac{ds}{s}\,. \end{aligned}$$

Replacing this in the expression above gives the desired estimate.

Now we pass to discuss the discrete version of Journé's covering lemma. For Ω as before, let $M_2(\Omega)$ denote the collection of those rectangles (dyadic) $R = I\times J$ so that I, J are dyadic and J is maximal with respect to the inclusion property in Ω.

Given arbitrary intervals I, J, not necessarily dyadic, let

$$J^I = \{y \in J : \ I \times \{y\} \subseteq \Omega\}.$$

If by rI we denote the interval concentric with I with sidelength r times that of I, we define $\hat{I}$ as follows: it is the smallest interval I' concentric with I, $I' \supset (1/8)I$, such that

$$w(x, J^{I'}) \leq \frac{1}{2}\, w(x, J) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Proposition 3. *Suppose the open set Ω, weight w and the function ϕ are as in Proposition 2. Then*

$$\sum_{R \in M_2(\Omega)} w(R)\, \phi\Big(\frac{|I|}{|\hat{I}|}\Big) \leq c \left(\int_0^1 \phi(8s)\, \frac{ds}{s} \right) w(\Omega).$$

PROOF. Let $\mathcal{I}_n$ denote the collection of those dyadic intervals I such that $R = I \times J \in M_2(\Omega)$ for some dyadic interval J, and $|I| = 2^n$, $n = 0, \pm 1, \pm 2, \ldots$ Then, since $J^I \supset J$ for $R = I \times J \in M_2(\Omega)$, the sum we want to estimate does not exceed

$$\sum_n \sum_{I \in \mathcal{I}_n} \int_I \sum_{J^I} \int_{J^I} \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, w(x, y)\, dx\, dy$$
$$\leq \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_I \sum_{J^I} \int_{J^I} w(x, y)\, dx\, dy\, \phi\Big(\frac{|I|}{|\hat{I}|}\Big) \frac{dt}{t}.$$

Fix now n, and $I \in \mathcal{I}_n$. Let $S = \{x \in I : \ [x - t, x + t] \in I\}$, and note that for $t \in (2^{n-3}, 2^{n-2})$, since $|I| = 2^n$, $2S \supseteq I$. Thus by the uniform doubling property of $w(\cdot, y)$, the above expression does not exceed

$$c \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_S \sum_{J^I} \int_{J^I} \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, w(x, y)\, dy\, dx\, \frac{dt}{t}.$$

Furthermore, since $t \geq 2^{n-3} = |(1/8)I|$, and since $x \in S$, it readily follows that $y \in J_{x,t}^k$, one of the components of $E_{x,t}$, and the above expression is dominated by

$$c \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_S \sum_k \int_{J_{x,t}^k} \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, w(x, y)\, dy\, dx\, \frac{dt}{t}.$$

Since in the above expression $|I| \leq 8t$, and since $[x-t, x+t] \subseteq I$ and consequently $J = J^I \subseteq J^k_{x,t}$, we see from the definitions of $t(k,x)$ and $|\hat{I}|$ (recall that $1/2 \leq \eta < 1$) that these quantities are essentially the same. Moreover, since in the definition of $t(k,x)$ the right-hand side is larger, so must be the left-hand side, and consequently $t(k,x) \leq |\hat{I}|$. Thus we may continue our estimation by

$$\begin{aligned} c \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_S \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{8t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t} \\ \leq c \sum_n \int_{2^{n-3}}^{2^{n-2}} \int_{\mathbb{R}} \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{8t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t} \\ \leq c \int_0^{+\infty} \int_{\mathbb{R}} \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{8t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t}\,. \end{aligned}$$

Then the proof proceeds exactly as that of Proposition 2.

Proposition 4. *Under the conditions of Proposition* 3, *we have*

$$w\Big(\bigcup_{R \in M_2(\Omega)} \hat{I} \times J\Big) \leq c\, w(\Omega)\,, \qquad c \text{ independent of } \Omega\,.$$

Because the proof is similar to that of Proposition 1 it is omitted.

As a first application of the weighted version of Journé's lemma we discuss the atomic decomposition of the weighted Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $0 < p \leq 1$.

Given a smooth function ψ supported in $(-1,1)$ with nonvanishing integral, put

$$\psi_{s,t}(x,y) = \frac{1}{s}\,\psi\big(\frac{x}{s}\big)\,\frac{1}{t}\,\psi\big(\frac{y}{t}\big)\,, \qquad s,t > 0\,,$$

and for a distribution f in $\mathbb{R}^2$, let

$$f^*(x,y) = \sup_{\varepsilon_1,\varepsilon_2 > 0} |f * \psi_{\varepsilon_1,\varepsilon_2}(x,y)|\,.$$

Then $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ consists of those distributions f such that $f^* \in L^p_w(\mathbb{R}^2)$, and we set $\|f\|_{H^p_w} = \|f^*\|_{L^p_w}$. We would like to discuss the so-called *atomic decomposition* of elements of these spaces when $w \in A_r(\mathbb{R} \times \mathbb{R})$, $1 \leq r \leq 2$.

A function $a(x, y)$ is called a (p, w)-*atom*, if

a) the set where $a(x, y) \neq 0$ is contained in a set Ω, with

$$\|a\|_{L^2_w} \leq w(\Omega)^{1/2-1/p} < +\infty,$$

b) $a = \sum a_R$, where the subatoms a_R have the following properties:

i) if $a_R(x, y) \neq 0$, then $(x, y) \in \tilde{R} = 3I \times 3J$, and $\tilde{R} \subseteq \Omega$,

ii) $R = I \times J$ is a dyadic rectangle, and no rectangle is repeated,

iii) for all integers $\alpha \leq [r/p - 1]$,

$$\int_{\tilde{I}} x^\alpha a_R(x, y)\, dx = \int_{\tilde{J}} y^\alpha a_R(x, y)\, dy = 0,$$

iv) $\left(\sum \|a_R\|^2_{L^2_w}\right)^{1/2} \leq w(\Omega)^{1/p-1/2}$.

The atomic decomposition states that $f \in H^p_w(\mathbb{R} \times \mathbb{R})$ if and only if $f = \sum_i \lambda_i a_i$, where the a_i's are (p, w)-atoms, the sum is taken in the sense of distributions and in the norm sense, and $\sum_i \lambda_i^p \leq c\|f\|^p_{H^p_w}$. That $f \in H^p_w$ can be decomposed into such sum is very similar to the unweighted case considered by R. Fefferman in [F2], and the proof is not discussed here.

Thus, we propose to prove the following result

Proposition 5. *Suppose that $w \in A_r$ and that a is a (p, w)-atom. Then $\|a\|_{H^p_w} \leq c$, where c is independent of a and independent in A_r.*

PROOF. Given $R = I \times J \subseteq \Omega$, let $\hat{I}$ now denote the interval which is the largest between $\hat{I}$ from Journé's lemma and $2I$; and similarly for $\hat{J}$. Let $\hat{R} = (\hat{I} \times J) \bigcup (I \times \hat{J}) = \hat{I} \times \hat{J}$. If

$$\hat{\Omega} = \bigcup_{R \subseteq \Omega} \hat{R},$$

then by Proposition 4 above, $w(\hat{\Omega}) \leq c\, w(\Omega)$, where c is independent of Ω and w.

In order to estimate $\|a\|_{H^p_w} = \|a^*\|_{L^p_w}$, we break up the integral that gives the L^p_w norm into $\hat\Omega$ and $\mathbb{R}^2 \setminus \hat\Omega$. The contribution over $\hat\Omega$ is readily handled: indeed, if M_S denotes the strong maximal function, then since $w \in A_2(\mathbb{R}\times\mathbb{R})$, and $a^*(x,y) \le c\, M_S a(x,y)$, by Propositon 4 it follows that

$$\begin{aligned}
&\int_{\hat\Omega} a^*(x,y)^p\, w(x,y)\, dx\, dy \\
&\qquad \le c \int_{\hat\Omega} M_S a(x,y)^p\, w(x,y)^{p/2}\, w(x,y)^{1-p/2}\, dx\, dy \\
&\qquad \le c \left(\int_{\hat\Omega} M_S a(x,y)^2\, w(x,y)\, dx\, dy \right)^{p/2} w(\hat\Omega)^{1-p/2} \\
&\qquad \le c\, \|a\|^p_{L^2_w}\, w(\hat\Omega)^{1-p/2} \\
&\qquad \le c\, w(\Omega)^{p(1/2-1/p)}\, w(\Omega)^{1-p/2} \\
&\qquad \le c\,.
\end{aligned}$$

Next, if $a = \sum_r a_R$, we consider each subatom a_R separately; by translation if necessary we may assume that a_R is centered at the origin, and if $R = I \times J$, we estimate the larger expression

$$\int_{\mathbb{R}\setminus\hat I} \int_{\mathbb{R}} a_R{}^*(x,y)^p\, w(x,y)\, dx\, dy\,.$$

For this purpose we show that the following two estimates hold:

$$\int_{\mathbb{R}\setminus\hat I} \int_{\mathbb{R}\setminus\hat J} a_R{}^*(x,y)^p\, w(x,y)\, dx\, dy \le c \left(\frac{|R|}{|\hat R|} \right)^p, \tag{6}$$

and

$$\int_{\mathbb{R}\setminus\hat I} \int_{\hat J} a_R{}^*(x,y)^p\, w(x,y)\, dx\, dy \le c \left(\frac{|I|}{|\hat I|} \right)^p. \tag{7}$$

We do (6) first. Let $p_N(\psi,\cdot)$ denote the Taylor expansion of degree N of ψ. By the moment condition on a_R it readily follows that

$$\begin{aligned}
&|a_R * \psi_{\varepsilon_1,\varepsilon_2}(x,y)| \\
&\quad \le \frac{1}{\varepsilon_1\varepsilon_2} \iint_{\tilde R} |\psi(\frac{x-u}{\varepsilon_1}) - p_N(\psi, -\frac{u}{\varepsilon_1})| \\
&\qquad\qquad \cdot |\psi(\frac{y-v}{\varepsilon_2}) - p_N(\psi, -\frac{v}{\varepsilon_2})|\, |a_R(u,v)|\, du\, dv \\
&\quad \le \frac{c}{\varepsilon_1\varepsilon_2} \iint_{\tilde R} \left(\frac{|u|}{\varepsilon_1}\right)^{N+1} \left(\frac{|v|}{\varepsilon_2}\right)^{N+1} |a_R(u,v)|\, du\, dv\,.
\end{aligned}$$

Notice that if $x \notin 2I$ and $u \in I$, then $|x|/2 \le |x-u| \le 2|x|$, so that if $\varepsilon_1 \le |x|/2$, then $\psi_{\varepsilon_1}(x-u) = 0$. We may thus assume that $\varepsilon_1 \ge |x|/2$, and likewise that $\varepsilon_2 \ge |x_2|/2$. Therefore, since $|u\,v| \le |\tilde{R}|$, the above expression does not exceed

$$\frac{c|\tilde{R}|^{N+1}}{(|x|\,|y|)^{N+2}} \iint_{\tilde{R}} |a_R(u,v)|\, w(u,v)^{1/2} w(u,v)^{-1/2} du\, dv$$
$$\le \frac{c|R|^{N+1}}{(|x|\,|y|)^{N+2}} \|a_R\|_{L^2_w} \left(\iint_{\tilde{R}} w(u,v)^{-1} du\, dv \right)^{1/2}.$$

Now, by the bound on a_R, and since $w \in A_2(\mathbb{R}\times\mathbb{R})$, this expression does not exceed

$$c\, \frac{|R|^{N+1}}{(|x|\,|y|)^{N+2}} \, \frac{1}{w(\Omega)^{1/p-1/2}} \, \frac{|\tilde{R}|}{w(\tilde{R})^{1/2}} \,.$$

Thus

$$\tag{8} \begin{aligned} &\int_{\mathbb{R}\setminus\tilde{I}} \int_{\mathbb{R}\setminus\tilde{J}} a_R^*(x,y)^p\, w(x,y)\, dx\, dy \\ &\quad \le c\, \frac{|R|^{(N+2)p}}{w(\Omega)^{1-p/2} w(\tilde{R})^{p/2}} \int_{\mathbb{R}\setminus\tilde{I}} \int_{\mathbb{R}\setminus\tilde{J}} \frac{w(x,y)}{(|x|\,|y|)^{(N+2)p}}\, dx\, dy \,. \end{aligned}$$

In order to estimate the integral in (8) note that if $w \in A_r(\mathbb{R}\times\mathbb{R})$, then by the choice of N, $N(p+2) \ge r$; the argument proceeds now using well-known estimates in the case of the line, *cf.* [T, Proposition IX, 4.5 (iv)], and the fact that the restrictions of w are uniformly in $A_r(\mathbb{R})$ for each variable fixed. Indeed, the expression in question does not exceed

$$\begin{aligned} &c \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{w(x,y)}{(|x| + |\hat{I}|)^{(N+2)p} (|y| + |\hat{J}|)^{(N+2)p}}\, dx\, dy \\ &\quad \le c \int_{\mathbb{R}} \frac{1}{(|y| + |\hat{J}|)^{(N+2)p}} \int_{\mathbb{R}} \frac{w(x,y)}{(|x| + |\hat{I}|)^{(N+2)p}}\, dx\, dy \\ &\quad \le c\, \frac{1}{|\hat{I}|^{(N+2)p}} \int_{\mathbb{R}} \frac{1}{(|y| + |\hat{J}|)^{(N+2)p}} \int_{\hat{I}} w(x,y)\, dx\, dy \\ &\quad = c\, \frac{1}{|\hat{I}|^{(N+2)p}} \int_{\hat{I}} \int_{\mathbb{R}} \frac{w(x,y)}{(|y| + |\hat{J}|)^{(N+2)p}}\, dy\, dx \end{aligned}$$

$$\le c\frac{1}{|\hat I|^{(N+2)p}}\,\frac{1}{|\hat J|^{(N+2)p}}\int_{\hat I}\int_{\hat J} w(x,y)\,dx\,dy$$
$$= c\frac{w(\hat R)}{(|\hat I|\,|\hat J|)^{(N+2)p}}\,.$$

Thus, replacing this estimate in the right-hand side of (8), and by Proposition 4, we obtain that the left-hand side there does not exceed

$$c\,\frac{|R|^{(N+2)p}}{w(\Omega)^{1-p/2}w(R)^{p/2}}\,\frac{w(\hat R)}{(|\hat I|\,|\hat J|)^{(N+2)p}}$$
$$\le c\left(\frac{|R|}{|\hat R|}\right)^{(N+1)p}\left(\frac{|R|}{|\hat I|\,|\hat J|}\right)^{p}\left(\frac{w(R)}{w(\Omega)}\right)^{1-p/2}\le c\left(\frac{|R|}{|\hat R|}\right)^{p},$$

which, of course, gives (6).

We show now estimate (7). By the moment condition on a_R we get

$$|a_R * \psi_{\varepsilon_1,\varepsilon_2}(x,y)| \le \frac{1}{\varepsilon_1\varepsilon_2}\left|\int_{\tilde I}\int_{J}\Big(\psi\big(\frac{x-u}{\varepsilon_1}\big)-p_N\big(\psi,-\frac{u}{\varepsilon_1}\big)\Big)\cdot\psi\big(\frac{y-v}{\varepsilon_2}\big)\,a_R(u,v)\,du\,dv\right|$$
$$\le \frac{c}{\varepsilon_1}\int_{\tilde I}\left(\frac{|u|}{\varepsilon_1}\right)^{N+1} M^2a_R(u,y)\,du\,,$$

where M^2 denotes the Hardy maximal operator in the second variable only. Thus

$$a_R^*(x,y)\le c\,\frac{|\tilde I|^{N+1}}{|x|^{N+2}}\int_{\tilde I}M^2a_R(u,y)\,du\,,$$

and consequently,

$$\int_{\mathbb{R}\setminus\hat I}\int_{\hat J}a_R^*(x,y)\,w(x,y)\,dx\,dy$$
$$\le c\,|I|^{(N+1)p}\int_{\mathbb{R}\setminus\hat I}\frac{1}{|x|^{(N+2)p}}\int_{\hat J}\left(\int_{\tilde I}M^2a_R(u,y)\,du\right)^{p}w(x,y)\,dy\,dx$$
$$\le c\,|I|^{(N+2)p}\int_{\hat J}\left(\frac{1}{|\tilde I|}\int_{\tilde I}M^2a_R(u,y)\,du\right)^{p}\int_{R^1\setminus\hat I}\frac{w(x,y)}{|x|^{(N+2)p}}\,dx\,dy$$
$$\le c\,|I|^{(N+2)p}\int_{\hat J}M^1(M^2a_R)(x,y)^p\int_{\mathbb{R}\setminus\hat I}\frac{w(x,y)}{|x|^{(N+2)p}}\,dx\,dy\,,$$

where M^1 denotes the Hardy maximal operator in the first variable only.

As before, by the usual $A_r(\mathbb{R})$ properties it follows that for y-a.e.

$$\int_{\mathbb{R}\setminus\hat{I}} \frac{w(x,y)}{|x|^{(N+2)p}}\,dx \le \frac{c}{|\hat{I}|^{(N+2)p}} \int_{\hat{I}} w(x,y)\,dx\,,$$

and consequently,

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\hat{J}} a_R^*(x,y)^p\, w(x,y)\,dx\,dy$$
$$\le c\left(\frac{|I|}{|\hat{I}|}\right)^{(N+2)p} \int_{\hat{I}}\int_{\hat{J}} M^1(M^2 a_R)(x,y)^p\, w(x,y)\,dy\,dx\,.$$

Note that the above integral looks similar to the first expression we estimated, and, in fact, since $w \in A_2(\mathbb{R}\times\mathbb{R})$, it does not exceed

$$c\,\|a_R\|_{L^2_w}^p\, w(\hat{R})^{1-p/2} \le c\, w(\Omega)^{p/2-1} w(\hat{R})^{1-p/2} \le c\,,$$

which completes the proof of (7).

We would like now to improve on these estimates; following R. Fefferman, put

$$b_R(x,y) = \frac{w(R)^{1/2-1/p}}{\|a_R\|_{L^2_w}}\, a_R(x,y)\,,$$

and observe that $b_R(x,y)$ is an atom supported on R, and that the above estimate applied to b_R gives

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\mathbb{R}} b_R^*(x,y)^p\, w(x,y)\,dy\,dx \le c\left(\frac{|I|}{|\hat{I}|}\right)^p + c\left(\frac{|R|}{|\hat{R}|}\right)^p \le c\left(\frac{|I|}{|\hat{I}|}\right)^p.$$

Thus, replacing b_R by its expression in terms of a_R, it readily follows that

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\mathbb{R}} a_R^*(x,y)^p\, w(x,y)\,dx\,dy \le c\,\|a_R\|_{L^2_w}^p\, w(R)^{1-p/2}\left(\frac{|I|}{|\hat{I}|}\right)^p.$$

This is all we need, as we are now ready to sum over the collection of all the maximal dyadic rectangles R contained in Ω. In fact, by Hölder's inequality and the properties of atoms, it follows that

$$\sum_R \int_{\mathbb{R}\setminus\hat{I}} \int_{\mathbb{R}} a_R^*(x,y)^p\, w(x,y)\,dx\,dy$$
$$\leq c \sum_R \|a_R\|_{L_w^2}^p\, w(R)^{1-p/2} \left(\frac{|I|}{|\hat{I}|}\right)^p$$
$$\leq c \left(\sum_R \|a_R\|_{L_w^2}^2\right)^{p/2} \left(\sum_R w(R)^{(1-p/2)(2/p)'} \left(\frac{|I|}{|\hat{I}|}\right)^{p(2/p)'}\right)^{1/(2/p)'}$$
$$\leq c\, w(\Omega)^{p/2-1} \left(\sum_R w(R) \left(\frac{|I|}{|\hat{I}|}\right)^{p(2-p)/2}\right)^{1-p/2}$$

We now invoke Journé's lemma with $\phi(s) = s^{p(2-p)/2}$, and note that the above expression is then dominated by

$$c\, w(\Omega)^{p/2-1}\, w(\Omega)^{1-p/2} \leq c\,,$$

and the proof is complete.

To complete the results discussed here we consider a description of the duals to the Hardy spaces $H_w^p(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$, $r/2 < p \leq 1$, when $w \in A_r(\mathbb{R}\times\mathbb{R})$; by known properties of weights the case $H_w^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ when $w \in A_2(\mathbb{R}\times\mathbb{R})$ is included.

Given a real-valued function b on $\mathbb{R}^2$, and a weight $v \in A_r(\mathbb{R}\times\mathbb{R})$, consider the following expression: if Ω is a bounded open set in $\mathbb{R}^2$, and R runs over the collection of the maximal dyadic rectangles contained in Ω, then set

$$\|b\|_{\eta,v} = \sup_\Omega \inf_{b_R} \left(\frac{1}{w(\Omega)^\eta} \sum_R \|b - b_R\|_{L_v^2(R)}^2\right)^{1/2},$$

where b_R runs over the family of functions of the form

$$b_R(x,y) = c_1\, b_1(y) + c_2\, b_2(x)\,,$$
$$\operatorname{supp} b_1 \subseteq J\,, \quad \operatorname{supp} b_2 \subseteq I\,, \quad R = I \times J\,.$$

We then have

Proposition 6. *$H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)^*$, the dual of the Hardy space $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $r/2 < p \leq 1$ can be identified with $B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$, the collection of those square integrable functions b such that $\|b\|_{2/p-1,1/w} < +\infty$.*

PROOF. We begin by showing that each $b \in B_{2/p-1,1/w}(\mathbb{R} \times \mathbb{R})$ induces a bounded linear functional on $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, with norm less than or equal to $c\,\|b\|_{2/p-1,1/w}$.

Suppose, then, that $a = \sum_R a_R$ is a (p, w)-atom in $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, and let $b \in B_{2/p-1,1/w}(\mathbb{R} \times \mathbb{R})$. Then, by the properties of atoms, a judicious choice of the b_R's, and Cauchy's inequality,

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^2} a(x,y)\, b(x,y)\, dx\, dy\right| \\
&\leq \sum_R \left|\iint_R a_R(x,y)\, b(x,y)\, dx\, dy\right| \\
&\leq \sum_R \iint_R |a_R(x,y)|\, |b(x,y) - b_R(x,y)|\, w(x,y)^{1/2}\, w(x,y)^{-1/2}\, dx\, dy \\
&\leq \Big(\sum_R \|a_R\|^2_{L^2_w}\Big)^{1/2} \Big(\sum_R \|b - b_R\|^2_{L^2_{1/w}}\Big)^{1/2} \\
&\leq w(\Omega)^{(1-2/p)/2} \Big(\sum_R \|b - b_R\|^2_{L^2_{1/w}}\Big)^{1/2} \\
&\leq \|b\|_{2/p-1,1/w}\,.
\end{aligned}$$

Next, if $f \in H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, then it admits an atomic decomposition $f = \sum_j \lambda_j a_j$, where the a_j's are (p, w)-atoms and $\|f\|_{H^p_w} \sim (\sum_j |\lambda_j|^p)^{1/p}$. Thus,

$$\begin{aligned}
\left|\iint_{\mathbb{R}^2} f(x,y)\, b(x,y)\, dx\, dy\right| &\leq \sum_j |\lambda_j| \left|\iint_{\mathbb{R}^2} a_j(x,y)\, b(x,y)\, dx\, dy\right| \\
&\leq \Big(\sum_j |\lambda_j|^p\Big)^{1/p} \|b\|_{2/p-1,1/w}\,,
\end{aligned}$$

and the assertion follows.

Conversely, suppose that $L \in H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)^*$. Then on a dense subset there, consisting of smooth functions, L can be represented by

$b(x,y)$ in the form

$$L(f) = \iint_{\mathbb{R}^2} f(x,y)\, b(x,y)\, dx\, dy\,.$$

Let now Ω be a bounded open set in $\mathbb{R}^2$, and suppose that $\Omega = \bigcup_R R$, where the R's are the maximal dyadic rectangles contained in Ω. Now, given a function $g \in L^2_w(\mathbb{R}^2)$ and $R = I \times J$, set

$$g_R(x,y) = \frac{1}{|I|}\int_I g(u,y)\, du + \frac{1}{|J|}\int_J g(x,v)\, dv - \frac{1}{|R|}\iint_R g(u,v)\, du\, dv\,.$$

Then

$$\int_I (g(x,y) - g_R(x,y))\, dx = \int_J (g(x,y) - g_R(x,y))\, dy = 0\,,$$

and

$$\|g - g_R\|_{L^2_w(R)} \leq c\, \|g\|_{L^2_w(R)}\,.$$

The first assertion is readily verified, and to see the second we consider the first term in g_R, the others being handled analogously. Note that since $w(\cdot, y) \in A_2(\mathbb{R})$ uniformly in y,

$$\begin{aligned}
&\int_J\int_I \left(\frac{1}{|I|}\int_I g(u,y)\, du\right)^2 w(x,y)\, dx\, dy \\
&\leq \int_I\int_J \left(\frac{1}{|I|}\int_I g(u,y)^2 w(u,y)\, du\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(u,y)}\, du\right) w(x,y)\, dx\, dy \\
&= \int_J\int_I g(u,y)^2 w(u,y) \left(\frac{1}{|I|}\int_I \frac{1}{w(u,y)}\, du\right)\left(\frac{1}{|I|}\int_I w(x,y)\, dx\right) du\, dy\,.
\end{aligned}$$

Now, since $w(\cdot, y) \in A_2(\mathbb{R})$, uniformly in y, the above expression involving the inner integrals does not exceed the A_2 constant of w, and the whole expression is less than or equal to $c\, \|g\|_{L^2_w}$, as claimed. The other terms are dealt with in a similar fashion.

Suppose now that Ω is a bounded open subset in $\mathbb{R}^2$, and that $f \in L^2(\Omega)$ is such that

$$\left(\sum_R \|f\|^2_{L^2_w(R)}\right)^{1/2} = 1\,.$$

Then, by the above remark, there is a constant c such that

$$a(x,y) = c\,\frac{1}{w(\Omega)^{1/p-1/2}} \sum_R \big(f(x,y) - f_R(x,y)\big)\chi_R(x,y),$$

is a (p,w)-atom of norm 1, and consequently,

$$\begin{aligned} \|L\| &\geq \|L(a)\| \\ &= c\,\frac{1}{w(\Omega)^{1/p-1/2}} \sum_R \iint_R \big(f(x,y) - f_R(x,y)\big)\, b(x,y)\, dx\, dy \\ &= c\,\frac{1}{w(\Omega)^{1/p-1/2}} \iint_R f(x,y)\,\big(b(x,y) - b_R(x,y)\big)\, dx\, dy\,. \end{aligned}$$

Since this estimate holds for all such f's, by duality it readily follows that

$$c\left(\frac{1}{w(\Omega)^{2/p-1}} \sum_R \|b - b_R\|^2_{L^2_{1/w}(R)}\right)^{1/2} \leq \|L\|,$$

which is precisely what we wanted to show.

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