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On singular integrals of Calderón-type in $\mathbb{R}^n$, and BMO

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Abstract. We prove L^p (and weighted L^p) bounds for singular integrals of the form

$$\text{p.v.} \int_{\mathbb{R}^n} E\Big(\frac{A(x) - A(y)}{|x - y|}\Big) \frac{\Omega(x - y)}{|x - y|^n} f(y)\, dy\,,$$

where $E(t) = \cos t$ if Ω is odd, and $E(t) = \sin t$ if Ω is even, and where $\nabla A \in \mathrm{BMO}$. Even in the case that Ω is smooth, the theory of singular integrals with "rough" kernels plays a key role in the proof. By standard techniques, the trigonometric function E can then be replaced by a large class of smooth functions F. Some related operators are also considered. As a further application, we prove a compactness result for certain layer potentials.

1. Introduction.

In this note we extend to $\mathbb{R}^n$ some 1-dimensional results of T. Murai (see, *e.g.* [Mu1], [Mu2], [Mu3]). We are concerned with n-dimensional singular integral of "Calderón-type", defined by

$$(1.1) \qquad T[A]f(x) = \text{p.v.} \int_{\mathbb{R}^n} F\Big(\frac{A(x) - A(y)}{|x - y|}\Big) \frac{\Omega(x - y)}{|x - y|^n} f(y)\, dy\,,$$

where F is a suitably smooth function defined on the real line, A is real-valued and belongs to the BMO Sobolev space $I_1(\mathrm{BMO})$ (I_1 denotes the usual fractional integral operator of order 1, suitably defined on BMO), and Ω is homogeneous of degree zero and bounded on the sphere. In applications Ω is usually smooth, but one of the main themes of this paper is that even in the smooth case, singular integrals with rough kernels will arise in a natural way when one extends to $\mathbb{R}^n$ certain 1-dimensional perturbation techniques of G. David (as described, for example, in the survey article of Coifman and Meyer [CM]), and Murai. We recall that BMO is the Banach space of locally integrable functions modulo constants with norm

$$\|b\|_* = \frac{1}{|Q|} \int_Q |b - m_Q(b)| \approx \Big(\frac{1}{|Q|} \int_Q |b - m_Q(b)|^q \Big)^{1/q},$$

where $1 \le q < \infty$,

$$m_Q(b) = \frac{1}{|Q|} \int_Q b,$$

and where the comparability of the various L^q means is a very well known result of John and Nirenberg. Then $A \in I_1(\mathrm{BMO})$ if and only if A is a continuous function with a locally integrable gradient (in the weak sense) and $\nabla A \in \mathrm{BMO}$ (see Strichartz [Stz] for more on the BMO Sobolev spaces). We remark that in fact one can show that A is absolutely continuous in the sense of Tonelli, so that ∇A exists a.e. Since the kernels that we shall consider are anti-symmetric, the case that A is Lipschitz can be reduced to the 1-dimensional setting by the method of rotations, but of course this method is not applicable for $A \in I_1(\mathrm{BMO})$. Furthermore, for at least one of the applications that we have in mind, it will be necessary to prove weighted estimates which can only be obtained by an intrinsically n-dimensional approach.

Statement of results.

We define an operator T by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} E\Big(\frac{A(x) - A(y)}{|x - y|} \Big) \frac{\Omega(x - y)}{|x - y|^n} f(y)\, dy, \tag{1.2}$$

where Ω is homogeneous of degree zero, essentially bounded, and either odd or even; $A \in I_1(\mathrm{BMO})$, and $E(t) = \cos t$ (if Ω is odd) or $E(t) = \sin t$ (if Ω is even). Our main result is the following

Theorem 1.3. *There exists an absolute constant $\mu > 0$ (we shall in fact observe that we may take $\mu = 1$) such that the operator T defined in (1.2) with Ω, A and E as stated, satisfies the norm inequality*

$$\|Tf\|_{p,w} \leq C(n,p,A_p)\,(1+\|\nabla A\|_*)^{\mu}\,\|\Omega\|_\infty\,\|f\|_{p,w}\,, \tag{1.4}$$

for all $1 < p < \infty$ and $w \in A_p$, with constants $C(n,p,A_p)$ depending only on n, p and the A_p constant of w.

Here we shall interpret the principal value in the following weak sense -we shall prove that all (double) truncations of T satisfy (1.4) with a uniform constant independent of the truncation. For anti-symmetric kernels, the principal value limit of the mapping $T : \mathcal{D} \longrightarrow \mathcal{D}'$ exists by a well known device.

A variant of (1.2) which can be treated by the same techniques and which is useful in applications is

$$\begin{aligned}\tilde{T}f(x) = \text{p.v.} \int_{\mathbb{R}^n} (B(x)-B(y))\, E\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \\ \cdot \frac{\Omega(x-y)}{|x-y|^{n+1}}\, f(y)\, dy\,,\end{aligned} \tag{1.5}$$

where B is Lipschitz, $A \in I_1(\mathrm{BMO})$, $\Omega \in L^\infty(S^{n-1})$, but now $E(t) = \cos t$ if Ω is even and $E(t) = \sin t$ if Ω is odd. We have the following

Theorem 1.6. *For $\tilde{T}$ defined in (1.5) (again we make the same weak interpretation of the principal value as in Theorem 1.3), there exists an absolute constant $\tilde{\mu} > 0$ such that for all $1 < p < \infty$, $w \in A_p$, we have*

$$\|\tilde{T}f\|_{p,w} \leq C(n,p,A_p)\,(1+\|\nabla A\|_*)^{\tilde{\mu}}\,\|\nabla B\|_\infty\,\|\Omega\|_\infty\,\|f\|_{p,w}\,. \tag{1.7}$$

Given Theorems 1.3 and 1.6, we will then be able to obtain, by rather standard methods, the following corollaries. The first will be an easy consequence of Theorem 1.3.

Theorem 1.8. *Let μ be the same as in Theorem 1.3, and let $T[A]$ be defined as in (1.1) where $\Omega \in L^\infty(S^{n-1})$ and Ω is odd if F is even or vice versa. Furthermore, we assume that $A \in I_1(\mathrm{BMO})$, that $F \in C^{\mu+2}(\mathbb{R})$ and that F and its first $\mu+2$ derivatives belong to L^1 (so that $\hat{F}(\xi) \leq C\,(1+|\xi|^{-(\mu+2)})$. Then*

$$\|T[A]f\|_{p,w} \leq C\,(1+\|\nabla A\|_*)^{\mu}\,\|\Omega\|_\infty\,\|f\|_{p,w}\,, \qquad 1<p<\infty\,, \tag{1.9}$$

for all $w \in A_p$, *where* C *depends on dimension,* p, F, *and the* A_p *constant of* w.

A corollary of Theorems 1.3, 1.6 and 1.8 is the following

Theorem 1.10. *Let* $\nu = \max\{\mu, \tilde{\mu}\}$ (*with the same* μ, $\tilde{\mu}$ *as in Theorems* 1.3 *and* 1.6). *Set*

$$T_*[A,B]f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} \big(B(x) - B(y) - \nabla B(y)\cdot(x-y)\big) \cdot F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y)\, dy \Big| ,$$

where $A, B \in I_1(\mathrm{BMO})$, $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ *for some* $0 < \alpha \leq 1$, $F \in C^{\nu+2}$, *with* F *and its first* $\nu + 2$ *derivaties belongig to* L^1, *and* F *and* Ω *are either both odd or both even. Furthermore, we now suppose that* $F(t) \leq C\,(1+|t|)^{-1}$. *Then for all* $1 < p < \infty$, $w \in A_p$, *we have*

$$\|T_*[A,B]f\|_{p,w} \leq C(n,p,A_p,F,\Omega)\, \|\nabla B\|_*\, (1 + \|\nabla A\|_*)^\nu\, \|f\|_{p,w} .$$

REMARK. We point out that were we interested only in proving Theorem 1.10, we could have done so without invoking Theorems 1.3, 1.6 and 1.8. The point is that if A and B are Lipschitz, then the operator norm of $T_*[A,B]$ is not changed if A and B are perturbed by a linear function. For B this is obvious, and for A it is a consequence of the method of rotations. The perturbation techniques of David can then be used to extend to the case that $A, B \in I_1(\mathrm{BMO})$, because for $\Omega \in \mathrm{Lip}_\alpha$, and for F satisfying the mild decay assumption, the kernel is almost (although not quite) a "standard" kernel. We are motivated to prove the sharper results given here (especially those for the trigonometric kernels, Theorems 1.3 and 1.6) by analogy to the 1-dimensional work of Murai.

A special case of particular interest is the double layer potential for Laplace's equation. The boundedness of the trace of this operator on the boundary of a BMO_1 domain (*i.e.* a domain whose boundary is locally the graph $(x, A(x))$ of a function $A \in I_1(\mathrm{BMO})$) is a special case of results for singular integrals on surfaces (see, *e.g.,* the papers of Semmes [Se], David [D], or David and Jerison [DJe], but it can also be obtained as a very easy corollary of Theorem 1.10. In fact the boundary double layer potential in (local) graph coordinates equals

$$Kf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{A(x) - A(y) - \nabla A(y) \cdot (x - y)}{\Big(1 + \Big(\dfrac{A(x) - A(y)}{|x - y|}\Big)^2\Big)^{(n+1)/2}} \, \frac{f(y)}{|x - y|^{n+1}} \, dy \, .$$

Thus, if we set $\Omega = 1$, $A = B$, $F(t) = (1 + t^2)^{-(n+1)/2}$, we may deduce that

$$\|Kf\|_{L^p(\Gamma)} \le C(n, p, \|\nabla A\|_*) \, \|f\|_{L^p(\Gamma)} \, , \tag{1.11}$$

where Γ is the hypersurface $(x, A(x))$, and where $C(n, p, \|\nabla A\|_*) \to 0$ as $\|\nabla A\|_* \to 0$. The inequality (1.11) is an immediate corollary of Theorem 1.10 and the following observation:

Lemma 1.12. *Surface measure on Γ equals an A_p weight times Lebesgue measure, if $\nabla A \in$ BMO, i.e., the weight $w = \sqrt{1 + |\nabla A|^2}$ belongs to $\bigcap_{1<p<\infty} A_p$, and furthermore, the A_p constant of w is nor larger than $C\,(1 + \|\nabla A\|_*)$.*

PROOF OF THE LEMMA. Since $\omega \approx 1 + |\nabla A|$, it is enough to show that

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q (1 + |b|)\Big) \Big(\frac{1}{|Q|} \int_Q (1+|b|)^{-1/(p-1)}\Big)^{p-1} \le C\,(1 + \|b\|_*) \, , \tag{1.13}$$

for all $1 < p < \infty$ and $b \in$ BMO.

We set $b_Q = \frac{1}{|Q|} \int_Q b$. The left side of (1.13) is no larger than

$$\begin{aligned} &\Big(\frac{1}{|Q|} \int_Q (1 + |b - b_Q|)\Big) \Big(\frac{1}{|Q|} \int_Q (1 + |b|)^{-1/(p-1)}\Big)^{p-1} \\ &\qquad + \Big(\frac{1}{|Q|} \int_Q \Big(\frac{|b_Q|}{1 + |b|}\Big)^{1/(p-1)}\Big)^{p-1} . \end{aligned} \tag{1.14}$$

The first term in (1.14) is no larger than $1 + \|b\|_*$. We split the $1/(p-1)$ power of the second term in (1.14) into

$$\frac{1}{|Q|} \int_{\substack{Q \\ |b| > |b_Q|/2}} + \frac{1}{|Q|} \int_{\substack{Q \\ |b| \le |b_Q|/2}} = \text{I} + \text{II} .$$

Trivially, $\mathrm{I} \le 2^{1/(p-1)}$. Now II equals

$$\frac{1}{|Q|} \int_{\substack{Q \\ |b| \le |b_Q|/2}} \left(\frac{|b_Q|}{1+|b|} \right)^{1/(p-1)} . \tag{1.15}$$

But the restriction on the domain of integration implies that $|b - b_Q| \approx |b_Q|$, so (1.15) is no larger than a constant times

$$\frac{1}{|Q|} \int_Q |b - b_Q|^{1/(p-1)} ,$$

and the lemma follows.

As another corollary of Theorem 1.10, we will use the techniques of Fabes, Jodeit and Riviere [FJR], to prove the compactness of the boundary double layer potential on "VMO_1" domains. That is, domains whose boundary is given in local coordinates by the graph of a function whose gradient belongs to VMO. We define the space VMO $(\mathbb{R}^n)$ by the property that $v \in$ VMO if and only if there exist continuous v_j with compact support such that $\|v - v_j\| \to 0$ (*i.e.* VMO is the BMO closure of the continuous functions with compact support). Let D be bounded VMO_1 domain in $\mathbb{R}^{n+1}$ and let Γ be its boundary. The boundary double layer potential on Γ is defined by

$$KF = \lim_{\varepsilon \to 0} K_\varepsilon f ,$$

where

$$K_\varepsilon f(P) = C_n \int_{\{|P-Q|>\varepsilon\} \cap \Gamma} \frac{\langle P - Q, N_Q \rangle}{|P - Q|^{n+1}} f(Q)\, d\sigma(Q) , \tag{1.16}$$

and N_Q is the unit outer normal. We will prove the following

Theorem 1.17. *Let $D \subset R^{n+1}$ be a bounded* VMO_1 *domain with boundary Γ. Let K_ε be defined as in* (1.16). *Then $Kf = \lim_{\varepsilon \to 0} K_\varepsilon f$ exists almost everywhere on Γ and in $L^p(\Gamma)$ norm, and K is a compact operator on $L^p(\Gamma)$, $1 < p < \infty$.*

The paper is organized as follows. In the next Section we state some known results which be used in the sequel. In Section 3 we prove our main result Theorem 1.3. The proof of Theorem 1.6 is virtually identical to that of Theorem 1.3 and is left to the interested reader. In Section 4 we prove Theorems 1.8 and 1.10 and in Section 5 we discuss the compactness of the double layer potential on VMO_1 domains (Theorem 1.17).

2. Some useful known results.

We first recall a well known corollary, via the method of rotations, of the corresponding theorem in dimension one. Let T be the singular integral in (1.2), *i.e.*

$$(2.1) \qquad Tf(x) = \text{“p.v.”} \int E\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, \frac{\Omega(x-y)}{|x-y|^n}\, f(y)\, dy\,,$$

with E and Ω as in (1.2), but where we now take A to be Lipschitz. Let $\tilde{T}$ be the singular integral defined in (1.5), again with A Lipschitz. We then have

Theorem 2.2. *For T and $\tilde{T}$ as above, and for all $1 < p < \infty$, the following norm inequalites hold:*

$$\text{(i)} \qquad \|Tf\|_p \le C(n,p)\,(1 + \|\nabla A\|_\infty)^\mu\, \|\Omega\|_\infty\, \|f\|_p$$

and

$$\text{(ii)} \qquad \|\tilde{T}f\|_p \le C(n,p)\,(1 + \|\nabla A\|_\infty)^{\tilde{\mu}}\, \|\nabla B\|_\infty\, \|\Omega\|_\infty\, \|f\|_p\,,$$

where μ and $\tilde{\mu}$ are absolute constants.

PROOF. The method of rotations. By invoking the 1-dimensional result of Murai [Mu2], we may take $\mu = 1$ in (i). Alternatively, one could give an intrinsically n-dimensional treatment by invoking the results in [H2] and following the argument in Section 3 below to bootstrap the Lipschitz constant. We also remark that in the Lipschitz case, the principal value exists almost everywhere, but we shall actually use only the fact that Theorem 2.2 holds for all truncations of T and $\tilde{T}$, with bound independent of the truncation.

We shall also use a $T1$ theorem for rough singular integrals proved by the author in [H], although in the present paper we shall require a less general version than that in [H]. In order to state this theorem, we first need to set some notation. Let $\psi \in C_0^\infty(|x| \le 1)$ be radial, non-trivial, real-valued, and have mean value zero, and set (with slight abuse of notation) $\Psi_s(|x|) = s^{-n}\psi(|x|/s)$. We assume that ψ is normalized so that

$$\int_0^{+\infty} \hat{\psi}(s)^2 \frac{ds}{s} = 1 .$$

If $Q_s f = \psi_s * f$, then Q_s satisfies the "Calderón-reproducing formula"

$$\int_0^{+\infty} Q_s^2 \frac{ds}{s} = I , \tag{2.3}$$

where the operator-valued integral converges in the strong operator topology on L^2, as may be verified by Plancherel's Theorem. Choose a non-negative $\varphi \in C_0^\infty(1/4, 1)$ so that φ defines a smooth partition of unity $\sum_{j=-\infty}^{+\infty} \varphi(2^{-j}r) = 1$, $r > 0$. Set

$$K_j(x,y) = K(x,y)\,\varphi(2^{-j}|x-y|) , \tag{2.4}$$

and define

$$T_j f(x) = \int K_j(x,y)\, f(y)\, dy ,$$

where K satisfies the size condition

$$|K(x,y)| \le C_1\, |x-y|^{-n} . \tag{2.5}$$

We also impose the following weak smoothness condition: assume that for all Q_s as above and $s \le 2^j$, we have, for some $\varepsilon > 0$

$$\|Q_s T_j 1\|_\infty \le C_2\, \|\psi\|_1 (2^{-j}s)^\varepsilon \tag{2.6}$$

and

$$\|Q_s T_j\|_{\text{op}} \le C_3\, \|\psi\|_1 (2^{-j}s)^\varepsilon , \tag{2.7}$$

where $\|\cdot\|_{\text{op}}$ denotes the $L^2 \longrightarrow L^2$ operator norm. We then have

Theorem 2.8. *Suppose that $K(x,y)$ is anti-symmetric (i.e. $K(x,y) = -K(y,x)$), that $K(x,y)$ satisfies the size condition* (2.5) *and that T_j satisfies* (2.6) *and* (2.7). *We define truncated operators*

$$T_{N,M} = \sum_{j=N}^{M} T_j ,$$

and assume that $\|T_{N,M}1\|_* \le C_4$ *(uniformly in* N *and* M*). Then for all* $1 < p < \infty$*, and* $w \in A_p$*, we have*

$$\|T_{N,M}f\|_{p,w} \le C(n,p,A_p)\,(C_1 + C_2 + C_3 + C_4)\,\|f\|_{p,w}\ .$$

In particular, the bound is independent of the truncation.

A much more general version of this theorem is proved in [H] (see also [H2], where these ideas were implicit) but the above will be sufficient for our purposes.

For the safe of self-containment, we will sketch the proof of the special case used here. We will argue formally, and refer the reader to [H] for the details. Using the Calderón reproducing formula, and the partition of unity as above, we write

$$T = T_{N,M} = \int_0^{+\infty}\int_0^{+\infty} \sum_{j=N}^{M} Q_s^2 T_j Q_t^2 \,\frac{ds}{s}\,\frac{dt}{t}\ .$$

It is enough to consider the case $s \le t$ (the other case is dual to this one). By the extrapolation Theorem for A_p weights (see, *e.g.* [GR]), it is enough to consider the case $p = 2$. It suffices to show that for all $w \in A_2$ and $f, g \in C_0^\infty$ we have

$$\Big|\int_0^{+\infty}\int_0^{t} \sum_{j=N}^{M} \langle Q_s T_j Q_t^2 f, Q_s g\rangle \,\frac{ds}{s}\,\frac{dt}{t}\Big| \le C(n,A_2)\,(C_1 + C_2 + C_3 + C_4)\,\|f\|_{2,w}\,\|g\|_{2,1/w}\ ,$$

In the left side of this last inequality, we split the sum into $\sum_1 + \sum_2$, where $\sum_1$ runs over j: $2^j \ge s^\theta t^{1-\theta}$, and $0 < \theta < 1$ is to be chosen. By an idea of [DR], it can be shown that (2.5), (2.7) and an interpolation argument imply a weighted version of (2.7) with a smaller ε, *i.e.*

$$\|Q_s T_j f\|_{2,w} \le C(n,A_2)\,(C_1 + C_3)\,(2^{-j}s)^{\varepsilon}\,\|f\|_{2,w}\ .$$

An application of Schwarz's inequality and weighted Littlewood-Payley theory now yield the desired estimate for $\sum_1$.

Next, to handle $\sum_2$, we consider

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = \int_0^{+\infty}\int_0^{t} \langle Q_s \textstyle\sum_2 (T_j - T_j 1) Q_t^2 f, Q_s g\rangle \,\frac{ds}{s}\,\frac{dt}{t}$$

$$+\int_0^{+\infty}\int_0^t \langle Q_s T_{N,M}1\, Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t}$$
$$-\int_0^{+\infty}\int_0^t \langle Q_s {\textstyle\sum_1} T_j 1\, Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t}.$$

The term III can be handled by a straightforward application of (2.6), Schwarz, and weithted Littlewood-Payley theory. In II, we use that

$$P_s = \int_s^{+\infty} Q_t^2 \frac{dt}{t}$$

is "nice", so the fact that $T_{N,M}1 \in \mathrm{BMO}$, combined with (weighted) Carleson measure theory yield the desired estimate in this case. Finally, to treat I, we write the kernel of $Q_s \sum_2 (T_j - T_j 1)$ as

$$\iint \psi_s(x-z) {\textstyle\sum_2} K_j(z,u)\,(\psi_t(u-y) - \psi_t(x-y))\, du\, dz\,,$$

which we claim is dominated in absolute value by

$$C_n C_1 \left(\frac{s}{t}\right)^{\varepsilon} t^{-n} \chi_{\{|x-y|\le C_n t\}}$$

(so that the corresponding operator is controlled by $(s/t)^{\varepsilon}$ times the Maximal function, and the theorem follows. To prove the claim, we first observe that by definition of $\sum_2$, we have $|z-u| \le s^{\theta} t^{1-\theta}$. Thus, the integrand is unchanged if we multiply it by a smooth radial cut-off function $\eta(|x-u|/(s^{\theta}t^{1-\theta}))$, where $\eta = 1$ on $\{|x| < 10\}$ and vanishes if $|x| > 11$. Furthermore, it is well known (see, *e.g,* [DJo]), that (2.5) plus anty-symmetry imply the Weak Boundedness property

$$\text{(WBP)}\quad |\langle h, Tg\rangle| \le C\, C_1\, R^n(\|h\|_\infty + R\,\|\nabla h\|_\infty)\,(\|g\|_\infty + R\,\|\nabla g\|_\infty)\,,$$

for all $h, g \in C_0^\infty$ with support in any ball of radius R. The claim then follows by (WBP), with

$$\theta = \frac{n+1+\varepsilon}{n+2}\,, \qquad 0 < \varepsilon < 1\,,$$

and

$$h(z) = \psi_s(x-z)\,, \qquad g(u) = (\psi_t(u-y) - \psi_t(x-y))\,\eta\Big(\frac{|x-u|}{s^{\theta}t^{1-\theta}}\Big)\,.$$

There is one more result which we shall find useful in the sequel. It is an unpublished theorem of Mary Weiss, and the proof can be found in a paper of C. Calderón [CC, Lemma 1.4]. We define a maximal operator

$$D_*A(x) = \sup_{h\neq 0} \frac{|A(x+h) - A(x)|}{|h|} . \tag{2.9}$$

We have the following

Lemma 2.10 (M. Weiss). *Suppose that $q > n$, and $\nabla A \in L^q_{\mathrm{loc}}$ (the gradiend being defined in the weak sense). Then, for all $\gamma > 1$,*

$$\text{(i)} \quad \frac{|A(x) - A(y)|}{|x-y|} \le C_{q,\gamma} \Big(\frac{1}{|x-y|^n} \int_{|x-z|\le\gamma|x-y|} |\nabla A(z)|^q \, dz \Big)^{1/q},$$

and

$$\text{(ii)} \quad \|D_*A\|_q \le C_q \, \|\nabla A\|_q \, .$$

Remark. Since BMO $\subseteq L^q_{\mathrm{loc}}$, a standard argument involving Lemma 2.10.ii shows that if $A \in I_1(\mathrm{BMO})$, then ∇A exists almost everywhere.

Proof of Theorem 1.3.

Using the same notation as in Theorem 2.8, we set

$$K(x,y) = \cos\Big(\frac{A(x) - A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n} , \tag{3.1}$$

where Ω is odd and bounded, so that the size condition (2.5) and the anti-symmetry condition $K(x,y) = -K(y,x)$ are immediate (we will prove explicitly only the case $E(t) = \cos t$, Ω odd, as the proof in the other case is identical). We will prove that the truncated operators $T_{N,M}$ satisfy (1.4) with a bound independent of the truncation. Without loss of generality we may take $\|\Omega\|_\infty = 1$, so by Theorem 2.8 it is enough to verify the smoothness conditions (2.6) and (2.7) with constants $C_2, C_3 \le C_n(1 + \|\nabla A\|_*)$, and to show that $T_{N,M}1 \in$ BMO with

$$\|T_{N,M}1\|_* \le C_n \, (1 * \|\nabla A\|_*)^\mu , \tag{3.2}$$

uniformly in N and M, with the same μ as in Lemma 2.2.i. The proof of (2.7) will be deferred until the end of this section. We note in passing that in contrast to the case that A is Lipschitz, the kernel $K(x,y)$ need not, in general, satisfy "standard" smoothness estimates when $A \in I_1(\mathrm{BMO})$, even if $\Omega \in C^\infty$. It turns out that when one perturbs A by an appropriate linear term in order to get nice (local) estimates for A, one introduces a factor that is homogeneous of degree zero and which may have uncontrollably large regularity estimates. We will be forced then, to prove (2.7) with a bound depending only on the size of Ω. We shall return to this point below.

We now proceed to prove (3.2), and in the process we will also verify (2.6).

In order to prove (3.2), we first recall a characterization of BMO which appears in a paper of Stromberg [Sbg, Lemma 3.1 and its corollary] where the idea is attributed to F. John.

Lemma 3.3 (John-Stromberg). *Let b be measurable and assume that there exist $\alpha > 0$ and $0 < \gamma \leq 1/2$ such that for every cube Q there is a constant C_Q with*

$$|\{x \in Q : \ |b(x) - C_Q| > \alpha\}| \leq \gamma\, |Q|\,.$$

Then $b \in \mathrm{BMO}$ and $\|b\|_* \leq C_\gamma\, \alpha\,.$

We will prove (3.2) by verifying the conditions of Lemma 3.3 for $b = T_{N,M}1$, and with $\alpha = C\,(1 + \|\nabla A\|_*)^\mu$, α, γ independent of N, M. We note that the use of Lemma 3.3 to treat $T1$ via bootstrapping has appeared previously in [CM]. We fix a cube Q with center x_0 and side length s. Let $\eta \in C_0^\infty[-11\sqrt{n}, 11\sqrt{n}\,]$, and suppose $\eta = 1$ on $[-10\sqrt{n}, 10\sqrt{n}\,]$. We write

$$\begin{aligned} T_{N,M}1 = & \int_{\mathbb{R}^n} \cos\Big(\frac{A(x) - A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n} \\ & \qquad \cdot \Phi_N^M(|x-y|)\,\Big(1 - \eta\Big(\frac{|x-y|}{s}\Big)\Big)\,dy \\ & + \int_{\mathbb{R}^n} \cos\Big(\frac{A(x) - A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n} \\ & \qquad \cdot \Phi_N^M(|x-y|)\,\eta\Big(\frac{|x-y|}{s}\Big)\,dy \\ = & \; U(x) + V(x)\,, \end{aligned} \tag{3.4}$$

where $\Phi_N^M(r) = \sum_{j=N}^M \varphi(2^{-j}r)$, with the same φ as in the definition of K_j (see (2.4)).

We treat U first. Let Q_j be a cube with center x_0 and side length $10 \cdot 2^j$. Set

$$A_j(x) = A(x) - \vec{a}_j \cdot x\,,$$

where $\vec{a}_j = \frac{1}{|Q_j|}\int_{Q_j} \nabla A$. By an elementary trigonometric identity, $U(x)$ equals (in polar coordinates)

$$\tag{3.5} \sum_{j=N}^{M} \int_{S^{n-1}} \Omega_j(\theta) \int_0^{+\infty} \varphi(2^{-j}r) \cdot \cos\left(\frac{A_j(x) - A_j(x - r\theta)}{r}\right) \left(1 - \eta\left(\frac{r}{s}\right)\right) \frac{dr}{r}\, d\theta\,,$$

where $\Omega_j(\theta) = \cos(\vec{a}_j \cdot \theta)\,\Omega(\theta)$, minus another term with sine in place of cosine in (3.5) and in the definition of Ω_j. We discuss only (3.5), as the other term can handle by the same argument, and for simplicity we again designate (3.5) as $U(x)$. We now claim that

$$\tag{3.6} \int_Q |U(x) - U(x_0)|\, dx \leq C_n \, \|\nabla A\|_* \, |Q|\,.$$

Note that the integrand in (3.5) is zero unless $2^j \geq 10\sqrt{n}\, s$. But by Lemma 2.10, for $r \approx 2^j$ and $|x - x_0| \leq \sqrt{n}\, s < 2^j/10$, we have for any $q > n$,

$$\begin{aligned} &|A_j(x) - A_j(x_0)| + |A_j(x - r\theta) - A_j(x_0 - r\theta)| \\ &\quad \leq C_n \, |x - x_0| \left(|x - x_0|^{-n} \int_{Q_j} |\nabla A|^q\right)^{1/q} \\ &\quad \leq C_n \, s^{1-n/q}\, 2^{jn/q} \, \|\nabla A\|_*\,. \end{aligned}$$

The claim now follows by a straightforward computation (the reader should bear in mind that the j-th summand is vacuous unless $2^j \geq C_n\, s$; the same argument also proves (2.6) -in that case we treat what is essentially a single j term in (3.5)).

Thus, by Tchebychev's inequality and (3.6), we have

$$\tag{3.7} |\{x \in Q : \; |U(x) - U(x_0)| > \beta_1 \,(1 + \|\nabla A\|_*)^{\mu}\}| \leq \frac{1}{10}\, |Q|\,,$$

for β_1 large enough and depending only upon dimension.

We now consider $V(x)$. For $x \in Q$, we write

$$V(x) = \int \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n}\, \Phi_N^M(|x-y|)\, \chi_{10\,Q}(y)\, dy$$
$$+ \int \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n}$$
$$\cdot\, \Phi_N^M(|x-y|) \Big(\eta\Big(\frac{|x-y|}{s}\Big) - \chi_{10\,Q}(y)\Big)\, dy$$
$$= W(x) + Y(x)\,.$$

The domain of integration in $Y(x)$ is contained in an annulus centered at x with inner and outer radii comparable to s, so

$$|Y(x)| \leq \beta_2 = \beta_2(n)\,, \tag{3.8}$$

uniformly in $x \in Q$.

The term W will be treated by a variant of the perturbation technique of G. David. We set

$$A_Q(x) = A(x) - \vec{a}_Q \cdot x\,,$$

where

$$\vec{a}_Q = \frac{1}{20\,Q} \int_{20\,Q} \nabla A\,.$$

Then

$$K(x,y) = \cos\Big(\frac{A_Q(x)-A_Q(y)}{|x-y|}\Big) \frac{\Omega_{1,Q}(x-y)}{|x-y|^n} = K_{1,Q}(x,y)\,, \tag{3.9}$$

where $\Omega_{1,Q}(x) = \cos(\vec{a}_Q \cdot x/|x|)\,\Omega(x)$, minus another term $K_{2,Q}$ with sine in place of cosine. This splitting of K into $K_{1,Q} - K_{2,Q}$ give rise to a splitting of W into $W_1 - W_2$. Since each term can be handled in the same way, we concentrate on

$$W_1(x) = \int K_{1,Q}(x,y)\, \Phi_N^M(|x-y|)\, \chi_{10\,Q}(y)\, dy\,.$$

Since $x \in Q$ and $y \in 10\,Q$, we can replace A_Q in this last expression by a function A_1 which agrees (modulo and additive constant) with A_Q on $10\,Q$, is supported in $20\,Q$, and satisfies

$$\int |\nabla A_1|^q \leq C_{n,q}\, \|\nabla A\|_*^q \tag{3.10}$$

(see Cohen [Co, p. 698] for the details of the construction of such an A_1; see also $\tilde{A}$ in Corollary 5.6 below). Now by (3.10) and Lemma 2.10, we have that for $q > n$ and L a large number to be chosen

$$\begin{aligned} &|\{x \in (10\,Q)^\circ : \; D_* A_1(x) > L\,\|\nabla A\|_*\}| \\ &\qquad \le \frac{1}{(L\,\|\nabla A\|_*)^q} \int |D_* A_1|^q \\ &\qquad \le \frac{C_n\,|Q|}{L} \le \frac{6^{-n}|Q|}{100}\,, \end{aligned} \tag{3.11}$$

for $L = L_n$ large enough and depending only on dimension. The Set

$$G = \{x \in (10\,Q)^\circ : \; D_* A_1 > L\,\|\nabla A\|_*\}$$

is open, and $A_1|_{G^c}$ is Lipschitz with constant $L\,\|\nabla A\|_*$. We can therefore make a Whitney extension [S, Chapter 6] of A_1, call it $\tilde{A}_1$, such $\tilde{A}_1$ and A_1 agree on G^c and $\tilde{A}$ is Lipschitz on $\mathbb{R}^n$ with constant $C_n\,L\,\|\nabla A\|_*$. We set

$$\tilde{K}_{1,Q} = \cos\Big(\frac{\tilde{A}(x) - \tilde{A}(y)}{|x-y|}\Big)\,\Omega_{1,Q}(x-y)$$

($\Omega_{1,Q}$ as in (3.9)), define

$$\tilde{W}_1(x) = \int \tilde{K}_{1,Q}(x,y)\,\Phi_N^M(|x-y|)\,\chi_{10\,Q}(y)\,dy\,, \tag{3.12}$$

and let $\tilde{W}_2$ be the analogous term with sine in place of cosine (the cosine in $\Omega_{1,Q}$ is of course also replaced by a sine). Then

$$\begin{aligned} \tilde{W}_1(x) &= \tilde{T}_1(\chi_{10\,Q})(x)\,, \\ \tilde{W}_2(x) &= \tilde{T}_2(\chi_{10\,Q})(x)\,, \end{aligned}$$

for appropriate (truncated) singular integral operators $\tilde{T}_1$, $\tilde{T}_2$, each with L^2 operator norm no larger than $C_n\,(1 + \|\nabla A\|_*)^\mu$ (see Theorem 2.2.i).

Let $G = \cup I_i$ be a truncated Whitney decomposition of G into non-overlapping cubes I_i with

$$\operatorname{diam} I_i \le \operatorname{dist}\{I_i, G^c\} \le 4 \operatorname{diam} I_i\,,$$

and set $G^* = \cup 6I_i$. For $x \in G^{*c} \cap Q$, we have that

$$W_1(x) - \tilde{W}_1(x) = \sum_i \int_{I_i} \Big(\cos\Big(\frac{A_1(x) - A_1(y)}{|x-y|}\Big) - \cos\Big(\frac{A_1(x) - \tilde{A}(y)}{|x-y|}\Big)\Big) \cdot \frac{\Omega_{1,Q}(x-y)}{|x-y|^n}\, \Phi_N^M(|x-y|)\, dy\,,$$

where we have used that $A_1 = \tilde{A}$ on G^c. By the Whitney construction, we can select $y_i \in G^c$ such that dist $\{y_i, I_i\} \le 4\, \mathrm{diam}\, I_i$, so that the first term in the integral is bounded in absolute value by

$$\begin{aligned} \frac{|A_1(y) - \tilde{A}(y)|}{|x-y|} &\le \frac{|A_1(y) - A(y_i)|}{|x-y|} + \frac{|\tilde{A}(y_i) - \tilde{A}(y)|}{|x-y|} \\ &\le (D_* A_1(y) + C_n L \, \|\nabla A\|_*) \frac{\mathrm{diam}\, I_i}{|x-y|}\,. \end{aligned} \tag{3.13}$$

If we set

$$R_1 = |W_1 - \tilde{W}_1|\,,$$

then by (3.12), for $x \in G^{*c}$, we have

$$R_1(x) \le \sum_i \frac{d_i}{(d_i + |x - y_i|)^{n+1}} \int_{I_i} (C_n L\, \|\nabla A\|_* + D_* A_1(y))\, dy\,,$$

where $d_i = \mathrm{diam}\, I_i$, and we have used that for $x \in (6I_i)^c$, we have $|x-y| \approx |x - y_i|$. Then

$$\begin{aligned} \int_{G^{*c}\cap Q} R_1(x)\, dx &\le C_n \int_{\cup I_i} (L\, \|\nabla A\|_* + D_* A_1(y))\, dy \\ &\le C_n \Big(L\, \|\nabla A\|_* |Q| + \int_{10\, Q} D_* A_1(y)\, dy \Big)\,. \end{aligned}$$

By Hölder's inequality, (3.10) and Lemma 2.10, we have

$$\int_{G^{*c}\cap Q} R_1(x)\, dx \le C_n L\, \|\nabla A\|_*\, |Q|\,, \tag{3.14}$$

and the same estimate holds for $\|R_2\|_{L^1(G^{*c}\cap Q)}$, where $R_2 = |W_2 - \tilde{W}_2|$.

By the definition of G^* and (3.11), we have that $|G^*| \le |Q|/100$. We now take $C_Q = U(x_0)$ in Lemma 3.3. Then, by (3.7), for β_3 to be chosen, and with β_1, β_2 as in (3.7) and (3.8) respectively,

$$\begin{aligned}|\{x \in Q:\ &|T_N^M 1 - U(x_0)| \ge (\beta_1+\beta_2+\beta_3)\,(1 * \|\nabla A\|_*^\mu\}| \\ &\le \frac{|Q|}{100} + \frac{|Q|}{10} + |\{x \in Q \cap G^{*c}:\ |\tilde W_1(x)| + |\tilde W_2(x)| \\ &\qquad + |R_1(x)| + |R_2(x)| > \beta_3(1 + \|\nabla A\|_*^\mu\}|\,.\end{aligned}$$

If we take β_3 large enough and depending only on dimension, the conclusion of Lemma 3.3 will follow by Tchebychev's inequality, (3.14) and its equivalent for R_2, and the fact that for $i = 1,2$, $\tilde W_i = \tilde T_i(\chi_{10\,Q})$ where $\|\tilde T_i\|_{\rm op} \le C_n\,(1 + \|\nabla A\|_*)\mu$. The details are left to the reader. This concludes the proof of (3.2).

To finish the proof of Theorem 1.3, we need to verify (2.7); *i.e.* with Q_s as in (2.3), we will prove that for $s \le 2^j$ and for some $\varepsilon > 0$,

$$\|Q_s T_j\|_{\rm op} \le C_n\,\|\psi\|_1\,(2^{-j}s)^\varepsilon\,(1 + \|\nabla A\|_*)\,, \tag{3.15}$$

where we recall that $T_j f(x) = \int K_j(x,y)\,f(y)\,dy$, and

$$K_j(x,y) = \cos\Big(\frac{A(x) - A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n}\,\varphi(2^{-j}|x-y|)\,.$$

The proof of (3.15) follows some ideas of Christ and Journé [CJ, estimate (5.4) and its proof, and also personal communication]. More generally, we shall consider kernels $K(x,y)$ which have the following property: we assume that for every cube I (with sides parallel to the coordinates axes), the kernel has the following interpretation:

$$K(x,y) = \frac{\Omega_I(x-y)}{|x-y|^n}\,\sigma_I(x,y)\,, \qquad (x,y) \in I \times I\,, \tag{3.16}$$

where Ω_I is homogeneous of degree zero and essentially bounded on the sphere and where σ_I has compact support and belongs to the Sobolev space $L^2_\varepsilon(\mathbb{R}^n \times \mathbb{R}^n)$, for some $\varepsilon > 0$ and with L^2_ε norm $C\,|I|^{1-\varepsilon/n}$. We observe that for $(x,y) \in I\times I$, the kernels of Theorem 1.3 can be written in the form (3.16) as a difference of two terms, the first of which has

$$\Omega_I(\theta) = \cos(\vec a_I \cdot \theta)\,\Omega(\theta)\,,$$

and

$$\sigma_I(x,y) = \cos\Big(\frac{A_I(x)-A_I(y)}{|x-y|}\Big)\,\eta_I(x,y)\,, \tag{3.17}$$

where

$$\vec{a}_I = \frac{1}{|I|}\int_I |\nabla A|\,, \qquad A_I(x) = A(x) - \vec{a}_I\cdot x\,,$$

and where for $(x,y)\in I\times I$ we are permitted to multiply by a smooth cut-off function η_I which equals 1 on $I\times I$ and vanishes on the complement of $2I\times 2I$. The second term is the same as this one but with sines in place of cosines. By Lemma 2.10, it is routine to verify that the particular σ_I in (3.17) satisfies, for all $0<\varepsilon<1$,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\sigma_I(x+u,y+v)-\sigma_I(x,y)|^2}{(|u|+|v|)^{2n+2\varepsilon}}\,dx\,dy\,du\,dv& \\ \le C_\varepsilon\,|I|^{2(1-\varepsilon/n)}(1+\|\nabla A\|_*)^2\,,&\end{aligned} \tag{3.18}$$

with a constant C_ε independent of I (as long as η_I is defined in terms of translates and dilates of some fixed "mother" η). We remark that in particular, (3.18) holds if

$$\begin{aligned}\iiint_{\substack{x,y,h\in I\\ |h|<s}} (|\sigma_I(x+h,y)-\sigma_I(x,y)|^2 + |\sigma_I(x,y+h)-\sigma_I(x,y)|^2)\,dx\,dy\,dh& \\ \le C\,|I|^2\Big(\frac{s}{|I|^{1/n}}\Big)^{2\delta}\,,&\end{aligned}$$

for some $\delta>\varepsilon$ and C independent of I. The latter estimate is [CJ, (5.4)]. It is now enough to prove the following

Lemma 3.19. *Let (as usual) $K_j(x,y) = K(x,y)\,\varphi(2^{-j}|x-y|)$, where for all I, $K(x,y)$ has the representation* (3.16)*, with $\|\Omega_I\|_\infty \le 1$, and*

$$\|\sigma_I\|_{L^2_\varepsilon(\mathbb{R}^n\times\mathbb{R}^n)} \le C_0\,|I|^{1-\varepsilon/n}\,,$$

for some $\varepsilon>0$ and C_0 independent of I. Then, for

$$T_jf(x) = \int K_j(x,y)\,f(y)\,dy\,,$$

we have that for some $\tilde{\varepsilon} > 0$,

$$\|Q_s T_j\|_{\text{op}} \le C_n C_0 \|\psi\|_1 \Big(\frac{s}{2^j}\Big)^{\tilde{\varepsilon}}, \qquad s < 2^j .$$

PROOF OF LEMMA 3.19. By dilation invariance, it is enough to consider the case $j = 0$. We decompose $\mathbb{R}^n$ into a mesh of non-overlapping unit cubes, $\mathbb{R}^n = \cup I_i$, so that $f = \sum_i f \chi_{I_i}$ almost everywhere. Then for $s \le 1$, $Q_s T_0 f \chi_{I_i}$ is supported in $10\sqrt{n}\, I_i$, so that the terms $Q_s T_0 f \chi_{I_i}$ have "bounded overlaps". Thus we have the orthogonality property

$$\int \Big| \sum_i Q_s T_0 f \chi_{I_i} \Big|^2 \le C_n \sum_i \int |Q_s T_0 f \chi_{I_i}|^2 .$$

It is therefore enough to prove that for f supported in any unit cube I_0

$$\int |Q_s T_0 f|^2 \le C_n C_0 \|\psi\|_1 s^{\tilde{\varepsilon}} \int |f|^2 .$$

We now fix such a unit cube I_0, assume that f is supported there, and write $K_0(x,y) = K(x,y)\,\varphi(|x-y|)$, where $K(x,y)$ has the representation (3.16) with $I = 5\sqrt{n}\, I_0$. By the Sobolev space estimate for σ_I, we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\sigma}(\xi,\tau)|^2 (1 + |\xi| + |\tau|)^{2\varepsilon} \, d\xi \, d\tau \le C_0^2 , \tag{3.20}$$

where for simplicity of notation we will now write $\sigma = \sigma_I$, since I is fixed. We can then decompose $\sigma(x,y) = g(x,y) + h(x,y)$, where

$$g(x,y) = (\chi_{\{|\xi|+|\tau|>s^{-\delta}\}} \hat{\sigma}(\xi,\tau))^{\vee}(x,y)$$

and

$$h(x,y) = (\chi_{\{|\xi|+|\tau|\le s^{-\delta}\}} \hat{\sigma}(\xi,\tau))^{\vee}(x,y) ,$$

$\delta > 0$ to be chosen. then by Plancherel and (3.24),

$$\|g\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \le C_0 \, s^{\delta\varepsilon} . \tag{3.21}$$

We are now ready to estimate $\|Q_sT_0 f\|_2^2$. By Schwarz, the part of this expression with g in place of σ is dominated by

$$\begin{aligned}(3.22)\qquad &\int \Big(\iint |\psi_s(x-z)|\,|g(z,u)|^2\,dz\,du\Big)\\ &\qquad\cdot \Big(\iint |\psi_s(x-z)|\,|f(u)|^2\,du\,dz\Big)\,dx\,.\end{aligned}$$

The second factor in brackets equals $\|\psi\|_1\|f\|_2^2$. By (3.21), we then have that (3.22) is no larger than $C\,C_0\|\Omega\|_\infty\|\psi\|_1^2\,s^{2\delta\varepsilon}\,\|f\|_2^2$, which is the desired estimate for this term.

We now consider the part of $\|Q_sT_0 f\|_2$ with h in place of σ. By the definition of h and Minkowski's inequality, this term is bounded by

$$\begin{aligned}(3.23)\qquad &\iint_{|\xi|+|\tau|\le s^{-\delta}} |\hat\sigma(\xi,\tau)|\,\Big(\int\Big|\iint \psi_s(x-z)\,e^{-2\pi i z\cdot\xi}k(z-u)\\ &\qquad\qquad\cdot e^{-2\pi i u\cdot\tau} f(u)\,du\,dz\Big|^2 dx\Big)^{1/2} d\xi\,d\tau\,,\end{aligned}$$

where

$$(3.24)\qquad k(x)=\frac{\Omega(x)}{|x|^n}\,\varphi(|x|)\,.$$

(Here we have taken $\Omega=\Omega_I$, since I is fixed). We write

$$\begin{aligned}(3.25)\qquad &\int \psi_s(x-z)\,e^{-2\pi i z\cdot\xi}k(z-u)\,dz\\ &\qquad=\int \psi_s(x-z)\,(e^{-2\pi i z\cdot\xi}-e^{-2\pi i x\cdot\xi})\,k(z-u)\,dz\\ &\qquad\quad+e^{-2\pi i x\cdot\xi}\psi_s * k(x-u)\,.\end{aligned}$$

By Fourier transform estimates of Duoandikoetxea and Rubio de Francia [DR, Section 4], the L^2 operator norm of $f\mapsto \psi_s * k * f$, for k of the form (3.24), is dominated by

$$C_\alpha\,\|\psi\|_1\,s^\alpha\,,\qquad 0<\alpha<1\,.$$

We note that this last estimate does not require that Ω have mean value zero. The desired estimate for the part of (3.23) corresponding to the second term in (3.25) then follows easily if we take δ small enough.

The first term in (3.25) is bounded in absolute value by the pointwise estimate

$$C\,|\xi|\,s\,\|\psi\|_1 \chi_{\{|x-u|\leq 2\}}\,,$$

and a routine computation yields the conclusion of Lemma 3.19.

4. Proofs of Theorems 1.8 and 1.10.

We prove Theorem 1.8 first, by means of a well known technique (see, *e.g.* [CM, pp. 33-34]). We consider only the case that F is even and Ω is odd, the proof in the other case being virtually identical.

Once again, we interpret the principal value in a weak sense, treating truncated operators and obtaining bounds independent of the truncation, and without loss of generality we take $\|\Omega\|_\infty = 1$. We set

$$T_{N,M}[A] = \sum_{j=N}^{M} T_j\,,$$

where T_j has kernel

$$K_j(x,y) = F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n}\,\varphi(2^{-j}|x-y|)\,.$$

Following [CM], we write

$$F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) = C\int_{-\infty}^{+\infty} \hat{F}(\xi)\,\cos\Big(\xi\,\frac{A(x)-A(y)}{|x-y|}\Big)\,d\xi\,,$$

where in the Fourier inversion formula we have used that F is even. But by Theorem 1.3, the operator $T_{N,M,\xi}$ defined by

$$T_{N,M,\xi}f(x) = \sum_{j=N}^{M}\int \cos\Big(\xi\,\frac{A(x)-A(y)}{|x-y|}\,\frac{\Omega(x-y)}{|x-y|^n}\Big)\varphi(2^{-j}|x-y|)\,f(y)\,dy$$

satisfies, for all $1<p<\infty$ and $w\in A_p$,

$$\|T_{N,M,\xi}f\|_{p,w} \leq C(n,p,A_p)\,(1+\|\nabla A\|_*|\xi|)^\mu\,\|f\|_{p,w}\,. \tag{4.1}$$

The Theorem then follows by a straightforward argument involving Fubini's Theorem, Minkowski's integral inequality, (4.1) and the fact that we have imposed enough regularity upon F that $|\hat{F}(\xi)| \le C(1 + |\xi|)^{-\mu-2}$.

PROOF OF THEOREM 1.10. We first state a lemma which can be deduced from Theorem 1.6 in exactly the same way that Theorem 1.8 followed from Theorem 1.3. The proof is left to the reader. We recall that $\Phi_N^M(r) = \sum_{j=N}^M \varphi(2^{-j}r)$.

Lemma 4.2. *Let $\tilde{T}_{N,M}[A,B]$ be defined by*

$$\tag{4.3} \begin{aligned} \tilde{T}_{N,M}[A,B]\,f(x) = \int (B(x) - B(y))\,F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \\ \cdot \frac{\Omega(x-y)}{|x-y|^{n+1}}\,\Phi_N^M(|x-y|)\,f(y)\,dy\,, \end{aligned}$$

where B is Lipschitz, $A \in I_1(\mathrm{BMO})$, and $\Omega \in L^\infty(S^{n-1})$. We also assume that F and Ω are either both odd, or both even and that $F \in C^{\tilde{\mu}+2}$ where F and its first $\tilde{\mu}+2$ derivatives belong to L^1 (for the same $\tilde{\mu}$ as in Theorem 1.6). Then for all $1 < p < \infty$ and $w \in A_p$,

$$\tag{4.4} \begin{aligned} &\|\tilde{T}_{N,M}[A,B]\,f\|_{p,w} \\ &\qquad \le C(n,p,F,A_p)\,\|\nabla B\|_\infty (1 + \|\nabla A\|_*)^{\tilde{\mu}}\,\|\Omega\|_\infty\,\|f\|_{p,w}\,, \end{aligned}$$

uniformly in N and M.

In order to apply G. David "good-λ" techniques to prove Theorem 1.10, we shall want to control certain appropriate maximal singular integrals. We let $T_*[A]$ be the maximal singular integral corresponding to $T[A]$ in (1.1), *i.e.*

$$\tag{4.5} \begin{aligned} T_*[A]f(x) = \sup_{N,M\in\mathbb{Z}} \Big| \int F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y}{|x-y|^n} \\ \cdot \Phi_N^M(|x-y|)\,f(y)\,dy\Big|\,, \end{aligned}$$

where we now suppose that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, that $F(t) \le C(1-|t|)^{-1}$ and otherwise A, F and Ω are as in Theorem 1.8. We also define

$$\tag{4.6} \tilde{T}_*[A,B]\,f = \sup_{N,M} \big|\tilde{T}_{N,M}[A,B]\,f\big|\,,$$

with $\tilde{T}_{N,M}[A, B]$ as in (4.3) but again with $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ and $F(t) \le C\,(1 - |t|)^{-1}$. Even though the kernels of these operators are not quite standard (since A need not be Lipschitz) the decay of F at infinity still permits us to prove

Lemma 4.7. *With $T_*[A]$ and $\tilde{T}_*[A, B]$ defined in* (4.5) *and* (4.6) *respectively, and now assuming that $\Omega \in Lip_\alpha(S^{n-1})$ for some $0 < \alpha \le 1$, and $F(t) \le (1 + |t|)^{-1}$ we have, for all $1 < p < \infty$,*

$$\|T_*[A]\, f\|_p \le C(n, p, F, \Omega)\,(1 + \|\nabla A\|_*)^{\mu}\, \|f\|_p \tag{4.8}$$

and

$$\|T_*[A, B]\, f\|_p \le C(n, p, F, \Omega)\, \|\nabla B\|_\infty\, (1 + \|\nabla A\|_*)^{\tilde{\mu}}\, \|f\|_p\,. \tag{4.9}$$

PROOF OF LEMMA 4.7. We prove only (4.8) as the other inequality has the same proof. The proof will be based on a well known inequality of Cotlar, which can be obtained by a very slight modification of the usual arguments for standard kernels. We begin by observing that for $f \in L^p$, the limit

$$T_N[A]\, f = \lim_{M \to \infty} T_{N,M}[A]\, f$$

exists pointwise as an almost everywhere convergent integral, and by Fatou's Lemma and Theorem 1.8,

$$\|T_N[A]\, f\|_p \le C(n, p)\,(1 + \|\nabla A\|_*)^{\mu}\, \|\Omega\|_\infty\, \|f\|_p\,, \tag{4.10}$$

even without imposing the smootness assumption on Ω or the decay condition $F(t) \le C\,(1 + |t|)^{-1}$. Furthermore, the difference between sharp and smooth truncations is controlled by the maximal function, so it is enough to consider the maximal singular integral

$$T_* f = \sup_{\varepsilon > 0} |T_\varepsilon f|\,,$$

where

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} F\Big(\frac{A(x) - A(y)}{|x - y|}\Big)\, \frac{\Omega(x - y)}{|x - y|^n}\, f(y)\, dy\,. \tag{4.11}$$

Now, by (4.10), T_ε is uniformly bounded on L^p, so by Banach-Alaouglu, there exists a subsequence $\varepsilon_j \to 0$ (depending on f), such that $T_{\varepsilon_j} f$

converges in the weak-$*$ sense to something which we call Tf, and furthermore Tf satisfies the bound (4.10). With this definition of Tf, we claim that the following Cotlar inequality holds almost everywhere:

$$
\begin{aligned}
T_*f(x) \le C(n,\delta,\Omega,F)\Big(&MTf(x) \\
&+ (1+\|\nabla A\|_*)^{\mu}\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}\Big),
\end{aligned}
\tag{4.12}
$$

for all $\delta > 0$. In fact (4.12) can be proved by a small modification of the argument in [Jo, pp. 56-57], once we establish the following lemma. The lemma says essentially that in the present setting our kernels are almost standard.

Lemma 4.13. *Let*

$$K(x,y) = F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\frac{\Omega(x-y)}{|x-y|^n},$$

with F, A and Ω as in Lemma 4.7. Then

$$
\begin{aligned}
&\int_{|x-y|>2|x-x'|} |K(x,y)-K(x',y)|\,|f(y)|\,dy \\
&\qquad \le C(n,\delta,F,\Omega)\,(1+\|\nabla A\|_*)\,\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)},
\end{aligned}
$$

for all $\delta > 0$.

PROOF OF LEMMA 4.13. We set $r = |x-x'|$, and split the integral into

$$\sum_{j=1}^{\infty}\int_{2^j r<|x-y|\le 2^{j+1} r} = \sum_{j=1}^{\infty}\int_{R_j}$$

It suffices to show that

$$
\begin{aligned}
&\int_{R_j} |K(x,y)-K(x',y)|\,|f(y)|\,dy \\
&\qquad \le C(n,\delta,F,\Omega)\,(1+\|\nabla A\|_*)\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}\,2^{-j\theta},
\end{aligned}
$$

for some $\theta > 0$ and depending on δ. Let B be the ball of radius $3r$ and center x, and set

$$m_B(\nabla A) = \frac{1}{|B|}\int_B \nabla A.$$

There are three cases.

Case 1: $|m_B(\nabla A)| \le 2^{j\varepsilon}(\|\nabla A\|_* + 1)$, where we have fixed $0 < \varepsilon < 1$. In this case, Lemma 2.10 implies that, for $q > n$,

$$\begin{aligned}|A(x) - A(x')| &\le C_{q,n}\,|x - x'|^{1-n/q}\,r^{n/q} \\ &\quad\cdot \Big(r^{-n}\int_B |\nabla A - m_B(\nabla A) + m_B(\nabla A)|^q\Big)^{1/q} \\ &\le C_{q,n}\,r\,(1 + 2^{j\varepsilon})\,(1 + \|\nabla A\|_*)\,,\end{aligned}$$

and also

$$\begin{aligned}|A(x) - A(y)| &\le C_{q,n}\,|x - y| \\ &\quad\cdot \Big(\frac{1}{(2^j r)^n}\int_{2^j B} |\nabla A - m_{2^j B}(\nabla A) + m_{2^j B}(\nabla A)|^q\Big)^{1/q} \\ &\le C_{q,n}\,|x - y|\,(1 + j + 2^{j\varepsilon})\,(1 + \|\nabla A\|_*)\,,\end{aligned}$$

where in the last inequality we have used a well known property of BMO to obtain the bound

$$\begin{aligned}m_{2^j B}(\nabla A) &= m_{2^j B}(\nabla A) - m_B(\nabla A) + m_B(\nabla A) \\ &\le C_n\,j\,\|\nabla A\|_* + 2^{j\varepsilon}(1 + \|\nabla A\|_*)\,.\end{aligned}$$

The claim follows in the present case with $\theta = \min\{1 - \varepsilon, \alpha)$ and $\delta = 0$, by a standard argument using the smoothness of Ω and the fact that F' is bounded (since $\xi\,\hat F(\xi) \in L^1$).

Case 2. $|m_B(\nabla A) \ge 2^{j\varepsilon}(1 + \|\nabla A\|_*)$ and

$$\Big|\frac{x - y}{|x - y|}\cdot\frac{m_B(\nabla A)}{|m_B(\nabla A)|}\Big| \ge 2^{-j\varepsilon/2}\,.$$

In this case

$$\Big|\frac{x - y}{|x - y|}\cdot m_B(\nabla A)\Big| \ge 2^{j\varepsilon/2}(1 + \|\nabla A\|_*)\,.$$

But by Lemma 2.10 ,

$$\begin{aligned}\frac{|A(x) - A(y) - m_B(\nabla A)\,(x - y)|}{|x - y|} &\le C_n\,j\,\|\nabla A\|_* \\ &\ll \Big|\frac{x - y}{|x - y|}\cdot m_B(\nabla A)\Big|\,.\end{aligned} \tag{4.15}$$

Thus

$$\frac{|A(x)-A(y)|}{|x-y|} > C_n\, 2^{j\varepsilon/2}(1+\|\nabla A\|_*)\,.$$

The same estimate holds also for $(A(x')-A(y))/|x-y|$, and by the same argument, since

$$\begin{aligned}\Big|\frac{m_B(\nabla A)}{|m_B(\nabla A)|}\cdot\Big(\frac{x-y}{|x-y|}-\frac{x'-y}{|x'-y|}\Big)\Big| &\le C_n\, 2^{-j}\\ &\ll \Big|\frac{x-y}{|x-y|}\cdot\frac{m_B(\nabla A)}{|m_B(\nabla A)|}\Big|\end{aligned} \tag{4.16}$$

if $\varepsilon < 1$. Thus, by the decay assumption of F,

$$\Big|F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\Big| + \Big|F\Big(\frac{A(x')-A(y)}{|x'-y|}\Big)\Big| \le \frac{C_n}{2^{j\varepsilon/2}(1+\|\nabla A\|_*)}\,.$$

We then obtain (4.14) in the present case, with $\delta = 0$ and $\theta = \varepsilon/2$.

Case 3. $\Big|\frac{x-y}{|x-y|}\cdot\frac{m_B(\nabla A)}{|m_B(\nabla A)|}\Big| \le 2^{-j\varepsilon/2}.$

In this case, with $\vec{w}_0 = m_B(\nabla A)/|m_B(\nabla A)|$, by Hölder's inequality and a change to polar coordinates we have

$$\begin{aligned}&\int_{R_j\cap\{|\frac{x-y}{|x-y|}\cdot\vec{w}_0|\le 2^{-j\varepsilon/2}} |K(x,y)\,f(y)|\,dy\\ &\qquad\le C_n\Big(\frac{1}{2^{jn}r^n}\int_{|\theta\cdot\vec{w}_0|\le 2^{-j\varepsilon/2}} d\theta\int_{2^j r}^{2^{j+1}r}\rho^{n-1}\,d\rho\Big)^{\delta/(1+\delta)}\\ &\qquad\quad\cdot\|F\|_\infty\, M\big(|f|^{1+\delta}\big)^{1/(1+\delta)}(x)\,.\end{aligned}$$

By the first inequality in (4.16), a slight variation of this argument can be used to consider $K(x',y)$. This concludes the proof of Lemma 4.13, and in the present case $\theta = \varepsilon\,\delta/(1+\delta)$.

The Cotlar inequality (4.12) can now be deduced by a rather straightforward adaptation of the argument in [Jo, pp. 56-57]. We leave the details to the reader. Lemma 4.7 then follows immediately.

With the maximal singular integrals under control, we are now in position to apply the "good-λ" perturbation techniques of G. David in a rather straightforward way to establish Theorem 1.10. Since the ideas are familiar, we shall be as brief as possible.

As usual we perform a Whitney decomposition of the set

$$E_\lambda = \{T_*[A,B]\,f > \lambda\} = \cup Q_j\ ,$$

where the Q_j's are non-overlapping and

$$\operatorname{diam} Q_j \le \operatorname{dist}\{Q_j, E_\lambda^c\} \le 4 \operatorname{diam} Q_j\ .$$

Here $T_*[A,B]$ is defined in Theorem 1.10, with $A, B \in I_1(\mathrm{BMO})$. We fix $Q = Q_j$, and it is enough to prove

$$\begin{aligned}|\{x \in Q :\ T_*[A,B]\,f > 3\,\lambda,\ M(|f|^{1+\delta})^{1/(1+\delta)} \le \gamma\,\lambda\}| \\ \le C(n,\Omega,F,\delta)\,\varepsilon_0\,|Q|\,,\end{aligned} \tag{4.17}$$

for any fixed $\delta > 0$ and for some suitably small, fixed ε_0, and γ to be chosen depending upon ε_0, where without loss of generality we take $\|\nabla B\|_* = 1$.

We may assume there is an $x_0 \in Q$ such that

$$M(|f|^{1+\delta})^{1/(1+\delta)}(x_0) \le \gamma\,\lambda\,,$$

or else the left side of (4.17) is zero. For an appropriate $\tilde{x} \in E_\lambda^c$, we let $\tilde{\beta}$ be the ball with center $\tilde{x}$ and radius $10 \operatorname{diam} Q$. We write $f = f_1 + f_2$, where

$$f_1 = f\chi_{\tilde{\beta}}\ , \qquad f_2 = f\chi_{\tilde{\beta}^c}\ .$$

If $\tilde{x} \in E_\lambda^c$ is chosen so that

$$\operatorname{dist}\{\tilde{x}, Q\} \le 4 \operatorname{diam} Q\,,$$

then, by essentially the same argument as that used to prove Lemma 4.13,

$$\begin{aligned}&|T_\varepsilon[A,B]\,f_2(x) - T_\varepsilon[A,B]\,f_2(\tilde{x})| \\ &\quad\le C_n\, M(|f|^{1+\delta})^{1/(1+\delta)}(x_0) \\ &\qquad + C(n,\delta,F,\Omega)\,(1 + \|\nabla A\|_*)\, M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)\,,\end{aligned} \tag{4.18}$$

because x, $\tilde{x}$ and x_0 are all far from the support of f_2. The details are left to the reader, but we do point out that when following the proof of Lemma 4.13, we write

$$B(x) - B(y) - \nabla B(y)\cdot(x-y) = (B(x) - B(y) - m_{\tilde{Q}}(\nabla B)\cdot(x-y) \\ + (m_{\tilde{Q}}(\nabla B) - \nabla B(y))\cdot(x-y)\,,$$

where $\tilde{Q}$ is an appropriate dilate of Q. The first term on the right side of the las expression is "locally standard", and the term $(m_{\tilde{Q}}(\nabla B) - \nabla B(y))\,f(y)$ can be handled by Hölder's inequality. We also mention that when treating (4.18), there is an error term, controlled by $M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)$, which arises when integrating over an appropriate symmetric difference. Since $\tilde{x} \in E^c_\lambda$,

$$T_*[A,B]\,f_2(x) \le \lambda + (4.18) \le \lambda\,(1+\gamma\,C\,(1+\|\nabla A\|_*))\,. \tag{4.19}$$

To handle f_1, we set

$$B_Q(x) = B(x) - \Big(\frac{10^{-n}}{|Q|}\int_{10\,Q}\nabla B\Big)\cdot x = B(x) - m_{10\,Q}(\nabla B)\cdot x$$

and we write

$$\begin{aligned} T_\varepsilon[A,B]\,f_1(x) = &\int_{|x-y|>\varepsilon} (B_Q(x) - B_Q(y))\,F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \\ &\qquad\cdot\frac{\Omega(x-y)}{|x-y|^{n+1}}\,f_1(y)\,dy \\ &+ \int_{|x-y|>\varepsilon} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)\,(x-y)}{|x-y|^{n+1}} \\ &\qquad\cdot\big(m_{10\,Q}(\nabla B) - \nabla B(y)\big)\,f_1(y)\,dy \\ = &\;\tilde{T}_\varepsilon[A,B_Q]\,f_1(x) + T_\varepsilon[A]\,\big((m_{10\,Q}(\nabla B) - \nabla B)\,f_1\big)(x)\,. \end{aligned} \tag{4.20}$$

By a direct application of (4.8), the supremum in ε of the absolute value of the second term in (4.20) is bounded in L^p norm by the $1/p$ power of

$$\begin{aligned} &C(n,p,F,\Omega)\,(1+\|\nabla A\|_*)^{\mu p}\int_{10\,Q}(|m_{10\,Q}(\nabla B) - \nabla B|\,|f|)^p \\ &\le C(n,p,F,\Omega)(1+\|\nabla A\|_*)^{\mu p}|Q|\,\|\nabla B\|_*^p\,M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)\,, \end{aligned} \tag{4.21}$$

for p chosen so that $1 < p < 1 + \delta$.

To handle the first term in (4.20), we repeat the argument used in the proof of Theorem 1.3 to approximate B_Q by a Lipschitz function $\tilde{B}$ with

$$\|\nabla \tilde{B}\|_\infty \le C_n\, L\, \|\nabla B\|_* = C_n\, L\,,$$

where L depends only on ε_0 and dimension, and such that

$$|\{x \in 10\,Q :\ \tilde{B} \ne B_Q\}| \le \frac{6^{-n}|Q|}{100}\, \varepsilon_0\,.$$

(See (3.12) and the related discussion). Then, for x in a subset G^{*c} of Q with measure at least $(100-\varepsilon_0)|Q|/100$, we have

$$\sup_\varepsilon |\tilde{T}_\varepsilon[A, B_Q]\, f_1(x)| \le Rf(x) + \tilde{T}_*[A, \tilde{B}]\, f_1(x)\,,$$

where by (4.9)

$$\int_Q (\tilde{T}_*[A,\tilde{B}]\, f_1)^{1+\delta} \le C(n,p,F,\Omega)\,(1+\|\nabla A\|_*)^{\tilde{\mu}(1+\delta)} L\, |Q|\, M(|f|^{1+\delta})(x_0)\,,$$

and, for $x \in Q \cap G^{*c}$,

$$Rf(x) \le C(n, \|F\|_\infty) \sum_i \frac{d_i}{d_i + |x-y|^{n+1}} \cdot \int_{I_i} (C_n\, L + D_* B_Q(y))\, |f(y)|\, dy\,,$$

where $d_i = \operatorname{diam} I_i$, and $\cup I_i$ is the Whitney decomposition of the set

$$G = \{x \in 10\,Q :\ \tilde{B} \ne B_Q\}\,.$$

Thus

$$\int_{Q\cap G^{*c}} Rf(x)\, dx \le C_n\, L\, \|F\|_\infty\, M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)\, |Q|\,. \tag{4.23}$$

By combining the estimates (4.19), (4.21), (4.22) and (4.23), and using the fact that $M(|f|^{1+\delta})^{1/(1+\delta)}(x_0) \le \gamma\, \lambda$, we can deduce (4.17) by standard arguments; we take

$$\gamma = \frac{\varepsilon_0}{(1+\|\nabla A\|_*)^\nu\, L}$$

for a suitable ε_0.

5. Compactness of the Boundary Double Layer Potential on bounded VMO_1 domains: Proof of Theorem 1.17.

Given Theorem 1.10, the proof is a relatively straightforward modification of the techniques of Fabes, Jodeit and Riviere [FJR]. We will essentially follow their argument, except for some small technical differences which arise in the VMO_1 case.

We begin by proving a series of elementary lemmas, which say essentially that a VMO_1 function A can be approximated (locally) by $C_0^{1,\alpha}$ functions, $0 < \alpha < 1$. Here $C_0^{1,\alpha} = \{$compactly supported $f \in L^1$ with $|\nabla f(x) - \nabla f(y)| \le C\,|x-y|^\alpha\}$. Although by definition there exists a sequence of compactly supported continuous vector fields converging to ∇A in BMO norm, and each term in this sequence is in turn uniformly approximable by C_0^∞ functions, it is not immediately evident that these smooth vector fields are conservative. That is why we take this more circuitous route.

Lemma 5.1. *Let $b \in \mathrm{BMO}$. Fix a cube I and choose a smooth function η such that $\eta = 1$ on I, $\eta = 0$ on $(2I)^c$, $0 \le \eta \le 1$ and $\|\nabla \eta\|_\infty \le C_n\,(\operatorname{diam} I)^{-1}$. Then $\eta\,(b - m_I b) \in \mathrm{BMO}$, with*

$$\|\eta\,(b - m_I b)\|_* \le C_n\,\|b\|_*\,,$$

where $m_I b = |I|^{-1} \int_I b$.

PROOF. Let Q be a cube which meets $2I$. There are two cases. The first is trivial: if $|Q| \ge 2^n |I|$, then

$$\frac{1}{|Q|}\int_Q |\eta\,(b - m_I b)| \le \frac{1}{|2I|}\int_{2I} |b - m_I b| \le C_n\,\|b\|_*\,.$$

Case 2. $|Q| < 2^n |I|$. We set $b_I = b - m_I b$. Then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\eta\, b_I - m_Q(\eta\, b_I)| \le{}& \frac{1}{|Q|}\int_Q |\eta\,(b_I - m_Q b_I)| \\ &+ \frac{1}{|Q|}\int_Q |\eta\, m_Q b_I - m_Q(\eta\, b_I)|\,.\end{aligned}$$

Since $m_Q b_I = m_Q b - m_I b$, and $0 \le \eta \le 1$, the first term in the last expression is trivially bounded by $\|b\|_*$. We re-write the second term as

$$\begin{aligned}(5.2)\qquad \frac{1}{|Q|}\int_Q \big|(\eta\, m_Q b_I - (m_Q\eta)(m_Q b_I)) \\ + ((m_Q\eta)(m_Q b_I) - m_Q(\eta\, b_I))\big|\,.\end{aligned}$$

The absolute value of the first expression in brackets is equal to

$$\Big|\frac{1}{|Q|}\int_Q (\eta(x)-\eta(y))\,dy\,(m_Q b - m_I b)\Big| \le C_n\,\frac{\operatorname{diam} Q}{\operatorname{diam} I}\,\log\frac{|I|}{|Q|}\,\|b\|_*\,,$$

where in the inequality we have used our assumption about $\|\nabla\eta\|_*$ and also a well known property of BMO. Since $|Q| < 2^n|I|$, the desired estimate follows for this part of (5.2). The second expression in brackets is dominated in absolute value by

$$\frac{1}{|Q|}\int_Q |\eta(y)|\,|m_Q b - m_I b - (b - m_I b)| \le \|b\|_*\,.$$

Corollary 5.3. *Let I and η be as in Lemma* 5.1. *Then $v \in$ VMO implies that $\eta(v - m_I v) \in$ VMO.*

Proof. Choose v_j continuous such that $v_J \to v$ in BMO norm. Now apply Lemma 5.1 to $\eta((v - v_j) - m_I(v - v_j))$.

Corollary 5.4. *Let $v \in$ VMO, and let η, I, be as in Lemma* 5.1. *Then there exists $\{u_j\}$, $u_j \in C_0^\infty$, such that $\|\eta(v - m_I v) - u_j\|_* \to 0$.*

Proof. By the previous corollary, $\eta(v - m_I v)$ can be approximated in BMO norm by continuous functions with compact support, which in turn are uniformly approximable by functions in C_0^∞.

Corollary 5.5. *Let η, I, v and u_j be as in Corollary* 5.4. *Let S be any standard Calderón-Zygmund type convolution singular integral operator with a smooth kernel. Then*

$$\|S u_j - S(\eta\,(v - m_I v))\|_* \to 0\,.$$

Proof. Immediate by standard Calderón-Zygmund theory and the previous corollary.

Corollary 5.6. *Suppose* $\nabla A \in \mathrm{VMO}(\mathbb{R}^n)$*. Let* I *and* η *be as above, so that* $\eta(\nabla A - m_I(\nabla A)) \in \mathrm{VMO}$*. Set*

$$A_I(x) = A(x) - m_I(\nabla A) \cdot x\,.$$

Following [Co, p. 698], *let* x_0 *be a point on the boundary of* $5\sqrt{m}\,I$*, and set*

$$\tilde{A} = \eta(x)\,(A_I(x) - A_I(x_0))\,.$$

Then there exists a sequence $\{A_j\} \subset I_1(\mathrm{BMO})$ *such that* $\nabla A_j \in \mathrm{Lip}_\alpha$*, for any given* $0 < \alpha < 1$*, and* $\|\nabla A_j - \nabla A\|_* \to 0$*.*

PROOF. Let $\vec{R} = (R_1, R_2, \ldots, R_n)$, where R_j denotes the j-th Riesz transform. A well known classical identity says that

$$\tilde{A} = -\vec{R} \cdot \vec{R}\tilde{A} = -I_1(\vec{R} \cdot \nabla\tilde{A})\,.$$

By definition,

$$\begin{aligned} \nabla\tilde{A}(x) &= \nabla\eta(x)\,(A_I(x) - A_I(x_0)) + \eta(x)\,(\nabla A(x) - m_I(\nabla A)) \\ &= \vec{a}(x) + \vec{b}(x)\,. \end{aligned}$$

By Corollary 5.5, there exist $\vec{u}_j \in C_0^\infty$ such that $\|S\vec{u}_j - S\vec{b}\|_* \to 0$, for any classical convolution type singular integral with a smooth kernel. Furthermore $S\vec{u}_j \in \mathrm{Lip}_\alpha$, $0 < \alpha < 1$, for all such S (see, *e.g.* Taibleson [T]) (here $\mathrm{Lip}_\alpha = \{f : |f(x) - f(y)| \le C\,|x - y|^\alpha\}$). By Lemma 2.10,

$$\|\vec{a}\|_\infty \le C_n\,\|\nabla A\|_*\,,$$

and also, for $x, y \in 2I$, and for all $q > n$,

$$|A_I(x) - A_I(y)| \le C_q\,|x - y|^{1-n/q}|I|^{1/q}\|\nabla A\|_*\,.$$

Thus $\vec{a}$ is continuous with compact support and belongs to Lip_α, $0 < \alpha < 1$, so again $S\vec{a} \in \mathrm{Lip}_\alpha$ by [T]. We now define

$$A_j = -I_1(\vec{R} \cdot (\vec{a} + \vec{u}_j))\,.$$

The conclusion of the lemma then follows by Corollary 5.5, the result of [T] and the identity

$$\frac{\partial}{\partial x_j}\,I_1\vec{R} = R_j\vec{R}\,,$$

where $S = R_j\vec{R}$ is a classical singular integral.

Remark. We have thus shown that $\tilde{A}$ can be approximated by $C^{1,\alpha}$ functions. The improvement to $C_0^{1,\alpha}$ will arise in the proof of Theorem 1.17.

We are now in a position to follow [FJR] and prove the compactness of

$$Kf = \lim_{\varepsilon\to 0} K_\varepsilon f\,,$$

where

$$K_\varepsilon f(P) = \int\limits_{\{|P-Q|>\varepsilon\}\cap\Gamma} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^{n+1}}\, f(Q)\, d\sigma(Q)\,.$$

(Here we have dropped the dimensional constant).

By a partition of unity argument we may change to local graph coordinates and treat the Euclidean operator

$$\tilde{K}f = \lim_{\varepsilon\to 0} \tilde{K}_\varepsilon f\,, \tag{5.7}$$

where

$$\tilde{K}_\varepsilon f(x) = \int\limits_{|x-y|^2+(A(x)-A(y))^2>\varepsilon^2} \frac{A(x)-A(y)-\nabla A(y)\cdot(x-y)}{(|x-y|^2+(A(x)-A(y)^2)^{(n+1)/2}}\, f(y)\, dy\,, \tag{5.8}$$

with $\nabla A \in \mathrm{VMO}(\mathbb{R}^n)$. Since surface measure is an A_p weight ($1 < p < \infty$) times Lebesgue measure (see Lemma 1.12), and since we have localized, it is enough to show that $\tilde{K}$ is compact as an operator $L^p_w(I)$, for a cube I and $w \in A_p$. The almost everywhere existence of the principal value in (5.7) will be shown in the course of the proof. As a first approximation, we consider

$$T[A,A]\, f = \lim_{\varepsilon\to 0} T_\varepsilon[A,A]\, f\,, \tag{5.9}$$

where

$$T_\varepsilon[A,B]\, f(x) = \int\limits_{|x-y|>\varepsilon} \frac{B(x)-B(y)-\nabla B(y)\cdot(x-y)}{(|x-y|^2+(A(x)-A(y)^2)^{(n+1)/2}}\, f(y)\, dy\,, \tag{5.10}$$

Theorem 5.11. *Fix a cube $I \subseteq \mathbb{R}^n$, let $A \in I_1(\text{BMO})$, with $\nabla A \in$ VMO. Then for $\operatorname{supp} f \subset I$, the principal value $T[A,A]f$ exists almost everywhere in I and in $L^p_w(I)$ norm, and furthermore $T[A,A]$ is a compact operator on $L^p_w(I)$, $1 < p < \infty$, $w \in A_p$.*

PROOF. Suppose $\operatorname{supp} f \subseteq I$, and let $x, y \in I$. Then for $\tilde{A}$ defined as in Corollary 5.6, we have

$$A(x) - A(y) - \nabla A(y) \cdot (x-y) = \tilde{A}(x) - \tilde{A}(y) - \nabla \tilde{A}(y) \cdot (x-y) .$$

It is therefore enough to consider

$$T[A, \tilde{A}]\, f = \lim_{\varepsilon \to 0} T_\varepsilon[A, \tilde{A}]\, f ,$$

where $T_\varepsilon[A,B]$ is defined in (5.10). We will use the techniques of [FJR], but with Theorem 1.10 in place of Calderón's Theorem.

We begin by observing that for $B \in C_0^{1,\alpha}$, $0 < \alpha < 1$, the pointwise existence of $T[A,B]\, f = \lim_{\varepsilon \to 0} T_\varepsilon[A,B]\, f$ is trivial, because the extra smoothness of B weakens the singularity. Furthermore, we claim that for $B \in C_0^{1,\alpha}$, $T_\varepsilon[A,B]$ converges to $T[A,B]$ in the operator norm of L^p_w. To see this, we use the smoothness of B to write

$$|T[A,B]\, f(x) - T_\varepsilon[A,B]\, f(x)| \leq C \int_{|x-y|>\varepsilon} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \leq C\, \varepsilon^\alpha M f(x) ,$$

where the last inequality is implied by a well-known result for approximate identities (see [S, pp. 61-63]). The claim follows. Thus the compactness of $T[A,B]$ on $L^p_w(I)$, for $B \in C_0^{1,\alpha}$ follows immediately from the compactness of $T_\varepsilon[A,B]$ for each $\varepsilon > 0$. But the latter fact may be deduced by a standard argument (see, *e.g.* [Tor, pp. 429-430]) and the fact that

$$\int_{|x-y|>\varepsilon} |x-y|^{-n} |f(y)|\, dy \leq C_{p,w,\varepsilon}\, \|f\|_{p,w}$$

(see [GR, p. 416]). The details are left to the reader.

We now proceed to prove the compactness of $T[A, \tilde{A}]$. By Corollary 5.6, there exists a sequence $A_j \in I_1(\text{BMO})$ with $\nabla A_j \in \text{Lip}_\alpha$, and such that $\nabla A_j \to \nabla \tilde{A}$ in BMO norm. If we define $\tilde{A}_j$ in the same way as $\tilde{A}$

(see the statement of Corollary 5.6), then $\tilde{A}_j \in C_0^{1,\alpha}$, and for $x \in I$ and $\operatorname{supp} f \subset I$,

$$T[A, A_j]\, f(x) = T[A, \tilde{A}_j]\, f(x)\,.$$

By our previous remarks, the principal value operator $T[A, \tilde{A}_j]$ exists and is compact on $L^p_w(I)$, so the same holds for $T[A, A_j]$. Let us assume for the moment that the principal value $T[A, \tilde{A}]$ exists. The compactness of $T[A, \tilde{A}]$ is then a consequence of the fact that $T[A, A_j] \to T[A, \tilde{A}]$ in the operator norm of L^p_w, and the latter fact may be deduced by writing

$$T[A, \tilde{A}] - T[A, A_j] = T[A, \tilde{A} - A_j]\,,$$

applying Theorem 1.10 with $B = \tilde{A} - A_j$, and using the fact that $\|\nabla\tilde{A} - \nabla A_j\|_* \to 0$.

To see that

$$T[A, \tilde{A}]\, f(x) = \lim_{\varepsilon\to 0} T_\varepsilon[A, \tilde{A}]\, f(x)$$

exists almost everywhere in I, for $\operatorname{supp} f \subset I$, we write

$$T_\varepsilon[A, \tilde{A}]\, f(x) = T_\varepsilon[A, \tilde{A} - A_j]\, f(x) + T_\varepsilon[A, A_j]\, f(x)\,.$$

But we have already observed that $\lim_{\varepsilon\to 0} T_\varepsilon[A, A_j]$ exists, and by Theorem 1.10, the operator

$$T_*[A, \tilde{A} - A_j]\, f = \sup_{\varepsilon>0} |T_\varepsilon[A, \tilde{A} - A_j]\, f|$$

goes to zero in operator norm on L^p_w.

The almost everywhere (and norm) convergence of $T_\varepsilon[A, \tilde{A}]\, f$ as $\varepsilon \to 0$ now follows by standard arguments. This concludes the proof of Theorem 5.11.

We now finish the proof of Theorem 1.17. Let $\operatorname{supp} f \subseteq I$. We write, for $x \in I$,

$$\tilde{K}_\varepsilon f(x) = (\tilde{K}_\varepsilon f(x) - T_\varepsilon[A, A]\, f(x)) + T_\varepsilon[A, A]\, f(x)\,. \tag{5.12}$$

By Theorem 5.11, $T[A, A]\, f = \lim_{\varepsilon\to 0} T_\varepsilon[A, A]\, f$ exists almost everywhere and defines a compact operator on $L^p_w(I)$. Thus, it is enough to show that $\tilde{K}_\varepsilon f - T_\varepsilon[A, A]\, f \to 0$ almost everywhere and in $L^p_w(I)$ operator norm, as $\varepsilon \to 0$.

To control this error term, we will consider a more general expression. Set

$$R_\varepsilon[A,B]\,f(x) = \int\limits_{|x-y|^2+(A(x)-A(y))^2\geq\varepsilon^2\geq|x-y|^2} \frac{B(x)-B(y)-\nabla B(y)\cdot(x-y)}{(|x-y|^2+(A(x)-A(y))^2)^{(n+1)/2}}\, f(y)\,dy\,.$$

Lemma 5.13. *Let* $R_*[A,B]\,f = \sup_{\varepsilon>0}|R_\varepsilon[A,B]\,f|$. *Then for all* $\delta>0$

$$R_*[A,B]\,f(x) \leq C_{n,\delta}\,\|\nabla B\|_*\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}$$

PROOF. Let $Q = Q(\varepsilon, x)$ be the cube with center x and side length ε. Then in R_ε we may replace B by B_Q, where

$$B_Q(x) = B(x) - \Big(\frac{1}{|Q|}\int_Q \nabla B\Big)\cdot x = B(x) - m_Q(\nabla B)\cdot x\,.$$

By Lemma 2.10,

$$|B_Q(x) - B_Q(y)| \leq C_n\,|x-y|\,\log\frac{\varepsilon}{|x-y|}\,\|\nabla B\|_*\,.$$

Thus, with $U_\varepsilon = \{y : |x-y|^2 \leq \varepsilon^2 < |x-y|^2 + (A(x)-A(y))^2\}$, we have

$$\begin{aligned}\sup_{\varepsilon>0}\Big|\int_{U_\varepsilon}&\frac{B_Q(x)-B_Q(y)}{(|x-y|^2+(A(x)-A(y))^2)^{(n+1)/2}}\,f(y)\,dy\Big|\\ &\leq \sup_{\varepsilon>0}\Big(\frac{C_n}{\varepsilon^n}\int\limits_{|x-y|<\varepsilon}\|\nabla B\|_*\,\log\frac{\varepsilon}{|x-y|}\,|f(y)|\,dy\Big)\\ &\leq C_n\,\|\nabla B\|_*\,Mf(x)\,,\end{aligned}$$

where the last inequality follows by [S, pp. 61-63]. By Hölder's inequality

$$\Big|\int_{U_\varepsilon}\frac{\nabla B_Q(y)\cdot(x-y)}{(|x-y)^2+(A(x)-A(y))^2)^{(n+1)/2}}\,f(y)dy\Big|$$

$$\leq \Big(\frac{1}{\varepsilon^n} \int_{|x-y|\leq \varepsilon} |\nabla B_Q(y)|^{(1+\delta)/\delta}\Big)^{\delta/(1+\delta)} \Big(\frac{1}{\varepsilon^n} \int_{|x-y|\leq \varepsilon} |f(y)|^{1+\delta}\Big)^{1/(1+\delta)},$$

and the lemma follows.

We now return to the matter of showing that $R_\varepsilon[A,A]f \to 0$ almost everywhere and in operator norm. As it was the case for $T[A,A]$ in the proof of Theorem 5.11, we have that as an operator on $L^p_w(I)$, $R_\varepsilon[A,A] = R_\varepsilon[A,\tilde{A}]$, with $\tilde{A}$ as in Corollary 5.6. Furthermore there exist $A_j \in C^{1,\alpha}$ with $\|\nabla A_j - \nabla \tilde{A}\|_* \to 0$. Now

$$R_\varepsilon[A,\tilde{A}] = R_\varepsilon[A,\tilde{A} - A_j] + R_\varepsilon[A,A_j]\,.$$

Since $R_\varepsilon[A,A_j] = R_\varepsilon[A,\tilde{A}_j]$ (as operators on $L^p_w(I)$), where $\tilde{A}_j \in C^{1,\alpha}_0$, it is easy to see that for each j,

$$R_\varepsilon[A,A_j]\,f(x) \to 0\,, \qquad \text{a.e. in } I, \text{ as } \varepsilon \to 0\,,$$

and in $L^p_w(I)$ operator norm, by virtue of Hölder's continuity of ∇A_j (in fact, for almost everywhere x,

$$|R_\varepsilon[A,A_j]\,f(x)| \leq C\,\|\nabla A_j\|_{\mathrm{Lip}_\alpha}\,\varepsilon^\alpha\,Mf(x)\,)\,.$$

Finally, by Lemma 5.13,

$$\|R_*[A,\tilde{A} - A_j]\,f\|_{p,w} \leq C(n,p,A_p)\,\|\nabla \tilde{A} - \nabla A_j\|_* \|f\|_{p,w}\;,$$

and Theorem 1.17 follows by letting $j \to \infty$.

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