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The sharp Poincaré inequality for free vector fields: an endpoint result

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1. Introduction.

The main purpose of this paper is to show a Poincaré type inequality of the following form:

$$\begin{aligned}(1.1)\qquad &\Big(\frac{1}{|B(r)|}\int_{B(r)}|f-f_{B(r)}|^q\Big)^{1/q}\\ &\qquad\le C\,r\Big(\frac{1}{|B(r)|}\int_{B(r)}\Big(\sum_{i=1}^m|X_if|\Big)^p\Big)^{1/p},\end{aligned}$$

for all $f\in C^\infty(\overline{B(r)})$ with $q=pQ/(Q-p)$, $1<p<Q$, where Q is a positive integer which will be specified later, and $X_1,\dots,X_m$ are C^∞ vector fields on $\mathbb{R}^d$ satisfying Hörmander's condition, $B(r)$ denotes a metric ball of radius r associated to the natural metric induced by the vector fields,

$$f_{B(r)}=\frac{1}{|B(r)|}\int_{B(r)}f\,.$$

The first Poincaré type inequality for the general vector fields satisfying Hörmander's condition was derived by D. Jerison. In 1986, D. Jerison [J] proved the above type inequality in the case $q=p$. When $q=p=2$, this inequality is equivalent to finding a lower bound $C^{-1}r^{-2}$ on the least nonzero eigenvalue in the Newman problem for $L = \sum_{i=1}^{m} X_i^* X_i$ on the ball $B(r)$. Soon after, D. Jerison and A. Sánchez-Calle in [JS] proved, among other things, the Poincaré inequality associated to the subelliptic operators.

Recently, the author has shown in [L1] the weighted Poincaré type inequalities for vector fields. One of the main ingredients of [L1] is the pointwise estimate for functions (without compact support) over the metric balls controlled by the fractional integral of certain maximal function (see (1.2) below). A nonweighted Poincaré type inequality (1.1) was also obtained, as a byproduct, in [L1] for all $q < pQ/(Q-p)$ except the endpoint $q = pQ/(Q-p)$.

Jerison's work [J] deserves some more explaination here. In [J] (see also [JS]), he first showed an inequality of the following form:

$$\int_B |f - f_B|^p \le C\rho(B)^p \int_{2B} \left(\sum_{i=1}^{m} |X_i f| + |f| \right)^p ,$$

where C is independent of f and B, $2B$ is the double of the metric ball B. Then he got rid of 2 in the limit of the integral on the right side (replacing $2B$ by B) and obtained the desired result by a covering argument based on the Whitney decomposition. This argument was motivated by an argument due to R. Kohn [K] in the case of the bilipschitzian image of a ball. This Jerison-Kohn type of covering argument is by now fairly well-known.

In [L1], we were able to prove a pointwise estimate for any $f \in C^\infty(\overline{B})$ of the following type (for any $\xi \in B$):

$$|f(\xi) - C(f,B)| \le \int_{cB} \frac{M(\sum_{i=1}^{m} |\tilde{X}_i f| + |f|)}{\rho(\xi,\eta)^{Q-1}} d\eta , \tag{1.2}$$

where $\tilde{X}_i$ are the lifted vector fields of X_i by the Rothschild-Stein lifting theorem [RS], and B is the metric ball associated to $\{\tilde{X}_i\}_{i=1}^m$ in the space with extra variables, and M is the Hardy-Littlewood maximal operator in this metric space with metric $\rho(\xi,\eta)$, and $c \ge 1$ is an absolute constant. Having obtained this pointwise estimate, we showed in [L1] for $f \in C^\infty(\overline{cB})$ that

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |f(x) - f_B|^q dx \right)^{1/q} \\ &\qquad \le c\,\rho(B) \left(\frac{1}{|B|} \int_{cB} \left(\sum_{i=1}^{m} |X_i f| + |f| \right)^p \right)^{1/p} , \end{aligned} \tag{1.3}$$

for all $1 \leq q \leq pQ/(Q-p)$, $1 < p < Q$, where B is the ball associated to the original vector fields $X_1, \dots, X_m$ and $\rho(B)$ is the radius of B. The advantage of such pointwise estimates is that it also leads to weighted Poincaré inequalities and sharper unweighted inequality (see [L1]).

The covering lemma argument in [J] also applies to the case $p < q$ (see [L1]). Thus by employing this covering lemma argument we were able to get rid of the constant c in the limit of the integral on the right hand side of (1.3) (replacing c by 1) and we proved in [L1]

$$\left(\frac{1}{|B|}\int_B |f(x)-f_B|^q dx\right)^{1/q} \leq c\,\rho(B)\left(\frac{1}{|B|}\int_B \left(\sum_{i=1}^m |X_i f|\right)^p\right)^{1/p},$$

for small $\rho(B)$ and $1 \leq q < pQ/(Q-p)$. However, it seems that such covering lemma argument does not lead to the endpoint result for $q = pQ/(Q-p)$ (see the discussion in [L1] and the remark at the end of this paper). Thus the endpoint result still remains unproved though it is expected to be true.

Thus the purpose of this paper is two fold. One is to show the endpoint result for $q = pQ/(Q-p)$, the other is that we carry out a different covering argument from the one in [J]. We like to point out that the method we used here is motivated by B. Bojarski's work [B] in which he proved the Sobolev embedding theorems on domains satisfying certain chain condition in the setting of euclidean space. This argument can be extended to the weighted version for doubling weights, see for example, Chua [C].

The following remarks are in order. First of all, when the vector fields are free (see [RS] or below for definition), *e.g.* on graded nilpotent group like the Heisenberg group, this endpoint result is sharp. Our nonweighted Poincaré inequality proved in [L1] for all $1 < p < Q$ and $1 \leq q < pQ/(Q-p)$, and the sharp form for $q = pQ/(Q-p)$ here in this special case have recently been used to prove a compensated compactness result on the Heisenberg group by Grafakos-Rochberg [GR]. Secondly, as Professor M. Christ pointed out to us, for non-free vector fields, even the exponent $q = pQ/(Q-p)$ may not be the optimal one. We like to thank Professor M. Christ for bringing this to our attention. However, we do not discuss further here. Thirdly, we should mention that the Sobolev type inequality (for functions with compact support) for vector fields can be obtained easily (see [L1]) thanks to the fundamental solution estimates of the sums of squares of vector fields by Sánchez-Calle [Sa] and Nagel-Stein-Wainger [NSW] (for the more general case by C. Fefferman and A. Sánchez-Calle, see [FeS]), see also [L1] for the weighted versions of Sobolev embedding theorem and [L2] for the Rellich-Kondrachov compact embedding theorem with applications to the estimates for the fundamental solutions of degenerate subelliptic operators. The weighted Poincaré-Sobolev inequalities have been used in [L1] to establish the Harnack inequalities for degenerate subelliptic operators of divergence form, and as a continuation in [L4], for strongly degenerate Schrödinger's operators which contains the result in [CGL]. Fourthly, when $1 < p < \infty$, both weighted Poincaré and Sobolev inequalities hold for $1 \leq q \leq pQ/(Q-1) + \delta_p$ for certain $\delta_p > 0$ in the case of equal weights in the A_p class (see Theorem B in [L1]). Thus in the case of no weights, Poincaré and Sobolev inequalities hold as well with such exponents $p < q$ as a special case.

2. Some preliminaries and the statement of the theorem.

Let Ω be a bounded, open and pathconnected domain in $\mathbb{R}^n$, and let $X_1, \ldots, X_m$ be a collection of C^∞ real vector fields defined in a neighbourhood of the closure $\overline{\Omega}$ of Ω. For a multi-index $\alpha = (i_1, \ldots, i_k)$, denote by X_α the commutator $[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]] \ldots]$ of length $k = |\alpha|$. Throughout this paper we assume that the vector fields satisfy Hörmander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha| \leq s}$ span the tangent space of $\mathbb{R}^d$ at each point of Ω. We can define a metric as follows: An admissible path γ is a Lipschitz curve $\gamma : [a, b] \to \Omega$ such that there exist functions $c_i(t)$, $a \leq t \leq b$, satisfying $\sum_{i=1}^m c_i(t)^2 \leq 1$ and $\gamma'(t) = \sum_{i=1}^m c_i(t) X_i(\gamma(t))$ for almost every $t \in [a, b]$. Then a natural metric on Ω associated to $X_1, \ldots, X_m$ is defined by

$$\begin{aligned}\varrho(\xi, \eta) = \min\{b \geq 0 : {} & \text{there exists an admissible path} \\ & \gamma : [0, b] \longrightarrow \Omega \text{ such that} \\ & \gamma(0) = \xi\,, \text{ and } \gamma(b) = \eta\}\,.\end{aligned}$$

The metric ball is defined by $B(\xi, r) = \{\eta : \varrho(\xi, \eta) < r\}$. This metric is equivalent to the various other metrics defined in the work of Nagel-Stein-Wainger [NSW]. Note that the Lebesgue measure is doubling with respect to the metric balls as shown in [NSW]. Thus (Ω, ϱ) is a homogeneous space.

By the Rothschild-Stein lifting theorem (see [RS]), the vector fields $\{X_i\}_{i=1}^m$ on $\Omega \subset \mathbb{R}^d$ can be lifted to vector fields $\{\tilde{X}_i\}_{i=1}^m$ in $\tilde{\Omega} = \Omega \times T \subset$

$\mathbb{R}^d \times \mathbb{R}^{N-d}$, where T is the unit ball in $\mathbb{R}^{N-d}$, by adding extra variables so that the resulting vector fields are free, *i.e.*, the only linear relation between the commutators of order less than or equal to s at each point of $\tilde{\Omega}$ are the antisymmetric and Jacobi's identity. Let $\mathcal{G}(m,s)$ be the free Lie algebra of steps with m generators, that is the quotient of the free Lie algebra with m generators by the ideal generated by the commutators of order at least $s+1$. Then $\{X_\alpha\}_{|\alpha|\leq s}$ are free if and only if $d = \dim \mathcal{G}(m,s)$. We also define $Q = \sum_{j=1}^{s} j m_j$ where m_j is the number of linearly independent commutators of length j.

Here is the statement of the main theorem.

Theorem 2.1. *There exist positive constants r_0 and μ such that for any $f \in C^\infty(\overline{B(r)})$, and $1 < p < Q$, $1 \leq q \leq pQ/(Q-p)$, we have the following*

$$\left(\frac{1}{|B(r)|}\int_{B(r)} |f - f_{B(r)}|^q\right)^{1/q} \leq C\,r\left(\frac{1}{|B(r)|}\int_{B(r)}\left(\sum_{i=1}^{m}|X_i f|\right)^p\right)^{1/p},$$

provided $\mu B(r) \subset \Omega$ and $r \leq r_0$.

2. Boman chain condition for the metric space (Ω, ϱ).

We first introduce the notion of the so-called "Boman chain condition" in the context of homogeneous space. This condition seems slightly different from the corresponding version in the euclidean space. However, it suffices for our purpose here.

Definition. *Let (X, ϱ) be a homogeneous space in the sense of Coifman - Weiss. An open set E in X is said to satisfy the Boman chain condition if there exists a positive constant μ and a family $\mathcal{F}$ of disjoint metric balls B such that*

i) $E = \bigcup_{B\in\mathcal{F}} 2B$.

ii) $\sum_{B\in\mathcal{F}} \chi_{10B}(x) \leq M\,\chi_\Omega(x)$ *for all* $x \in X$.

iii) *There is a so-called "central ball" $B_0 \in \mathcal{F}$ such that each ball $B \in \mathcal{F}$ can be connected to B_0 by a finite chain of balls $B_0, \ldots, B_{k(B)} =$*

B in such a way that $2B_j \bigcap 2B_{j+1} \neq \varnothing$ and $4B_j \bigcap 4B_{j+1}$ contains a metric ball D_j whose volume is comparable to those of both B_j and B_{j+1}.

iv) *Moreover, $B \subset \mu B_j$ for all $j = 0, 1, \ldots, k(B)$.*

The above E is called a Boman chain domain. The explicit numbers $2, 4$ and 10 are not essential here. We define in such a way just for the simplicity.

In the case $X = \mathbb{R}^n$ and ϱ is the euclidean metric, it is the standard chain condition. It is known that any euclidean cubes, balls, John's domain, bounded Lipschitz domains and (ϵ, ∞) domains are all Boman chain domains. In the general homogeneous space, it will be hard to verify if some domain is a Boman chain domain. However, we will show below any metric ball in our (Ω, ϱ) associated to the vector fields $X_i, \ldots, X_m$ is indeed a Bomain chain domain.

Lemma 3.1. *Let $B = B(\xi_1, r_1) \subset \Omega$ be a metric ball. Then B is a Boman chain domain.*

We like to point out that Lemma 3.1 is implicit in Jerison's work [J]. Indeed, the only thing we need to check is iv) in the definition above. And, it also follows from Jerison's work. For the completeness of the presentation, we will show the details. We first need the following Whitney decomposition whose proof can be found in Folland-Stein's book [FS] (see also [J]).

Lemma 3.2. *Let $E = B(\xi_1, r_1)$, then there is a pairwise disjoint family of balls $\mathcal{F}$ and a constant M depending only on the doubling constant of the Lebesgue measure with respect to the metric balls such that*

i) $E = \bigcup_{B \in \mathcal{F}} 2B$,

ii) $B \in \mathcal{F}$ *implies that* $10^2 \rho(B) \leq \rho(B, \partial E) \leq 10^3 \rho(B)$,

iii) $\#\{B \in \mathcal{F} : \ \eta \in 10B\} \leq M$.

Here $\rho(B, \partial E)$ is the distance, in the metric ϱ, from B to ∂E. $\#S$ denotes the number of elements in the set S.

Lemma 3.2 already provides more or less the first three conditions in the Boman chain condition. For $B \in \mathcal{F}$, define γ_B as an admissible path from the center η_B of B to ξ_1 (the center of E) of length less or equal than r_1. Denote the subset of E defined by the image of γ_B by γ_B as well. This path may not be unique, but will be fixed throughout this paper. Denote $\mathcal{F}(B) = \{A \in \mathcal{F} : 2A \bigcap \gamma_B \neq \varnothing\}$. The following has been proved by Jerison [J].

Lemma 3.3. *Let $B \in \mathcal{F}$, then there are no elements of $\mathcal{F}(B)$ of radius less than $\rho(B)/100$.*

PROOF OF LEMMA 3.1. We select a central ball $B_0 \in \mathcal{F}$ such that $\xi_1 \in 2B_0$ and will fix it throughout the proof. As proved in [J], $\#\mathcal{F}(B)$, which is equal to the number of elements in $\mathcal{F}(B)$, is finite and with a upper bound $C\log(r_1/\rho(B))$ though it is not uniformly bounded on $B \in \mathcal{F}$. We then order the elements of $\mathcal{F}(B)$ as $\mathcal{F}(B) = \{A_1, \dots, A_{k(B)}\}$ such that $A_1 = B_0$ and $A_{k(B)} = B$, and $2A_k \bigcap 2A_{k+1} \neq \varnothing$ for all k. Thus by the construction in Lemma 3.2, $4A_k \bigcap 4A_{k+1}$ contains a ball D_k whose volume is comparable to those of both A_k and A_{k+1}. Thus, the first three conditions in the definition of chain condition are already verified. We will show $B \subset \mu A_k$, for some $\mu > 0$ and for all k. Now let η_B be the center of the ball B and η_k be the center of the ball A_k. Then

$$\begin{aligned}\varrho(\eta_B, \eta_{A_k}) &\leq \varrho(\eta_B, \partial E) + \varrho(\eta_{A_k}, \partial E)\\ &\leq 10^3\rho(B) + 10^3\rho(A_k) \leq 10^6\rho(A_k),\end{aligned}$$

for all k, by Lemma 3.2. Thus $B \subset 10^8 A_k$, by Lemma 3.2 again. Therefore, we can take $\mu = 10^8$.

4. Proof of the main theorem.

We will need two technical lemmas.

Lemma 4.1. *Given $1 \leq p < \infty$. Let $\{B_\alpha\}$ be an arbitrary family of open metric balls in (Ω, ϱ) with $\mu B_\alpha \subset \Omega$ and $\{a_\alpha\}_{\alpha \in I}$ be nonnegative numbers. Then*

$$\Big\| \sum_\alpha a_\alpha \chi_{\mu B_\alpha} \Big\|_{L^p(\Omega)} \leq C \Big\| \sum_\alpha a_\alpha \chi_{B_\alpha} \Big\|_{L^p(\Omega)},$$

where C is independent of $\{a_\alpha\}$ and $\{B_\alpha\}$.

The proof is standard. For completeness we include a detailed proof.

PROOF. Let $\phi \in L^{p}(\Omega)$, with $1/p + 1/p' = 1$, $1 < p' < \infty$. Let

$$M\phi(\xi) = \sup_{\xi \in B} \frac{1}{|B|} \int_B |\phi(\eta)|\, d\eta$$

be the Hardy-Littlewood maximal function of $\phi(\xi)$ with respect to the metric balls in Ω and the supremum is taken over all balls B including ξ. Then it is known that $\|M\phi\|_{L^{p'}(\Omega)} \le C\,\|\phi\|_{L^{p'}(\Omega)}$. Hence

$$\begin{aligned}
\Big|\int_\Omega \sum_\alpha a_\alpha \chi_{\mu B_\alpha}(\xi)\,\phi(\xi)\,d\xi\Big| &= \Big|\sum_\alpha |\mu B_\alpha| \frac{1}{|\mu B_\alpha|}\int_{\mu B_\alpha} \phi(\xi)\,d\xi\Big| \\
&\le C\Big|\sum_\alpha a_\alpha \int_{B_\alpha} M\phi(\xi)\,d\xi\Big| \\
&\le C\int_\Omega \sum_\alpha a_\alpha \chi_{B_\alpha}(\xi)\,M\phi(\xi)\,d\xi \\
&\le C\Big\|\sum_\alpha a_\alpha \chi_{B_\alpha}\Big\|_{L^p(\Omega)} \|M\phi\|_{L^{p'}(\Omega)} \\
&\le C\Big\|\sum_\alpha a_\alpha \chi_{B_\alpha}\Big\|_{L^p(\Omega)} \|\phi\|_{L^{p'}(\Omega)}\,.
\end{aligned}$$

Thus the lemma follows.

Lemma 4.2. *Assume $p > 1$, then for any metric balls I and B with $I \subset B \subset \Omega$ we have*

$$\left(\frac{\rho(I)}{\rho(B)}\right)\left(\frac{|I|}{|B|}\right)^{1/q-1/p} \le C,$$

provided that $1 \le q \le pQ/(Q-p)$ and $\rho(B) \le r_0$ for some $r_0 > 0$.

This lemma is proved in [L1]. It is Lemma 6.12 in [L1].

We also note that by an easy covering argument we can reduce (1.3) to (this is just for simplicity)

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q dx\right)^{1/q} \\
&\qquad \le c\,\rho(B)\left(\frac{1}{|B|}\int_{2B}\Big(\sum_{i=1}^m |X_i f| + |f|\Big)^p\right)^{1/p}
\end{aligned} \tag{4.3}$$

We note that (4.3) is equivalent to

$$\int_B |f - f_B|^q \le C\,\rho(B)^q\,|B|^{1-q/p}\left(\int_{2B}\Big(\sum_{i=1}^m |X_i f| + |f|\Big)^p\right)^{q/p}, \tag{4.4}$$

for all $1 \le q \le pQ/(Q-p)$.

We now start to prove Theorem 2.1. Fix the central ball B_0 as in Lemma 3.1. We have

$$\begin{aligned}\|f - f_{2B_0}\|^q_{L^q(E)} &\le 2^{q-1}\sum_{B\in\mathcal{F}}\|f - f_{2B}\|^q_{L^q(2B)}\\ &\quad + 2^{q-1}\sum_{B\in\mathcal{F}}\|f_{2B} - f_{2B_0}\|^q_{L^q(2B)}\\ &= \mathrm{I} + \mathrm{II}.\end{aligned} \tag{4.5}$$

Replacing f by $f - f_{2B_0}$ in the inequality (4.4) we will get

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q dx\right)^{1/q}\\ &\qquad \le c\,\rho(B)\left(\frac{1}{|B|}\int_{2B}\Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}|\Big)^p\right)^{1/p},\end{aligned} \tag{4.6}$$

for any given $B \in \mathcal{F}$. Now fix temporarily $B \in \mathcal{F}$ and consider the chain $\mathcal{F}(B) = \{A_1, \ldots, A_{k(B)}\}$ constructed in the proof of Lemma 3.1. Thus

$$\begin{aligned}\|f_{2B} - f_{2B_0}\|_{L^q(2B)} &\le C\sum_{j=1}^{k(B)-1}\|f_{2A_j} - f_{2A_{j+1}}\|_{L^q(2B)}\\ &\le C\sum_{j=1}^{k(B)-1}\Big(\frac{|B|}{|4A_j \bigcap A_{j+1}|}\Big)^{1/q}\\ &\qquad\cdot\|f_{2A_j} - f_{2A_{j+1}}\|_{L^q(4A_j\cap 4A_{j+1})}\\ &\le C\sum_{j=1}^{k(B)-1}\Big(\frac{|B|}{|A_j|}\Big)^{1/q}\|f - f_{2A_j}\|_{L^q(4A_j)}\\ &\quad + C\sum_{i=1}^{k(B)}\Big(\frac{|B|}{|A_{j+1}|}\Big)^{1/q}\|f - f_{2A_{j+1}}\|_{L^q(4A_{j+1})}\end{aligned}$$

$$\leq 2C \sum_{j=1}^{k(B)-1} \left(\frac{|B|}{|A_j|}\right)^{1/q} \|f - f_{2A_j}\|_{L^q(4A_j)} \, .$$

We observe that

$$\|f - f_{2A_j}\|_{L^q(4A_j)} \leq \|f - f_{4A_j}\|_{L^q(4A_j)} + \|f_{4A_j} - f_{2A_j}\|_{L^q(4A_j)} \, ,$$

and

$$\begin{aligned}\|f_{4A_j} - f_{2A_{j+1}}\|_{L^q(4A_j)} &\leq |4A_j|^{1/q} \left(\frac{1}{|2A_j|} \int_{2A_j} |f - f_{4A_j}|\right) \\ &\leq C \left(\int_{4A_j} |f - f_{4A_j}|^q\right)^{1/q} .\end{aligned}$$

Therefore, we get

$$\|f_{2B} - f_{2B_0}\|_{L^q(2B)} \leq C \sum_{j=1}^{k(B)-1} \left(\frac{|B|}{|A_j|}\right)^{1/q} \|f - f_{4A_j}\|_{L^q(4A_j)} \, .$$

Since, by the chain condition, $B \subset \mu A_j$ for each $A_j \in \mathcal{F}(B)$, we then have

$$\begin{aligned}\|f_{2B} - f_{2B_0}\|_{L^q(2B)} \frac{\chi_{2B}(\xi)}{|B|^{1/q}} &\leq C \sum_{A \in \mathcal{F}} \left(\frac{1}{|A|}\right)^{1/q} \|f - f_{4A}\|_{L^q(4A)} \, \chi_{2\mu A}(\xi) \\ &= C \sum_{A \in \mathcal{F}} a_A \, \chi_{2\mu A} \, .\end{aligned}$$

In the above expression, a_A is notationally defined in an obvious way. For the term II in (4.5), we have

$$\text{II} \leq C \sum_{B \in \mathcal{F}} \int_\Omega \|f_{2B} - f_{2B_0}\|^q_{L^q(2B)} \, \frac{\chi_{2B}(\xi)}{|B|} \, .$$

Since $\sum_{B \in \mathcal{F}} \chi_{2B}(\xi) \leq C$, we derive

$$\text{II} \leq C \int_\Omega \Big| \sum_{A \in \mathcal{F}} a_A \, \chi_{2\mu A} \Big|^q .$$

By lemma (4.1), we then get

$$\mathrm{II} \le C \int_\Omega \Big| \sum_{A\in\mathcal{F}} a_A \chi_A \Big|^q .$$

Since $\sum_{A\in\mathcal{F}} \chi_A(\xi) \le C$, we will have

$$\mathrm{II} \le C \sum_{A\in\mathcal{F}} a_A^q \int_\Omega \chi_A(\xi) \le C \sum_{A\in\mathcal{F}} \|f - f_{4A}\|^q_{L^q(4A)} .$$

Therefore, by the inequality (4.4)

$$\begin{aligned}
\mathrm{II} &\le C \sum_{A\in\mathcal{F}} |A|^{1-q/p}\rho(A)^q \left(\int_{8A} \Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}| \Big)^p \right)^{q/p} \\
&\le C\,|E|^{1-q/p}\rho(E)^q \sum_{A\in\mathcal{F}} \left(\int_{8A} \Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}| \Big)^p \right)^{q/p} \\
&\le C\,|E|^{1-q/p}\rho(E)^q \left(\int_{E} \Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}| \Big)^p \right)^{q/p} .
\end{aligned}$$

In the last inequality we used the fact $q \ge p$, $8A \subset E$ and $\sum_{A\in\mathcal{F}} \chi_{8A}(\xi) \le C$, and in the one next to the last we used Lemma 4.2.

For the term I in (4.5), the estimate is the same by replacing $4A$ by $2A$ in the estimate of II. Therefore we have shown the main theorem if we use the Minkowski inequality on the right side above and letting $\rho(E)$ be very small in order to bootstrap the second term on the right.

Finally, we remark out that if we use the covering argument in [J] (see [L1]), there will be a factor like

$$\log\left(\frac{\rho(E)}{\rho(B)}\right)\left(\frac{\rho(B)}{\rho(E)}\right)^q \left(\frac{|B|}{|E|}\right)^{1-q/p} .$$

In order to bound this term uniformly independent of $B \subset E$, we need to require that $q < pQ/(Q-p)$. This will cause to lose the endpoint. However, the argument in this paper does not produce the above factor (without the logarithm term).

Added in proof: The first draft of the paper was done and circulated in September of 1992. After this paper was accepted for publication, there have been some new embedding theorems established on various function spaces associated with the general vector fields of Hörmander type. Here are the details:

1) We mention here that when $p = 1$ and $q = Q/(Q-1)$ the nonweighted Poincaré inequality has been proved by Franchi, Wheeden and the author [FLW]. It was proved by Jerison [J] when $p = q = 1$. Weighted inequalities are also obtained in [FLW] which improve the previous two-weighted results of the author in [L1].

2) When $p = Q$, besides the Poincaré inequality for $1 \leq q \leq pQ/(Q-1)+\delta_p$, we can even show that the exponential integrability of the function (without assuming the compact support for the function). This has been done by the author [L3] together with the embedding theorems on the Campanato-Morrey spaces $M^{p,\lambda}$ (for $1 < p < \lambda \leq Q$). The embedding theorems on the Campanato-Morrey spaces allow the gap larger than that in the Poincaré inequality, *i.e.* $1/Q$.

3) When $p > Q$, then we can show the function itself belongs to the local Lischitz spaces $C^{0,\gamma}_{\rm loc}$ for $\gamma = 1 - Q/p$. When $\sum_{i=1}^m |X_i f|$ is in certain Morrey spaces $M^{p,\lambda}$ (for $p \geq 1$), we can also show the function itself in certain local Lipschitz spaces or of exponential integrability. These have been shown by the author in [L5].

4) The Sobolev inequality for $p = 1$ and $q = Q/(Q-1)$ for functions with compact support has been proved in [FGaW] for general vector fields. Weighted Sobolev inequalities for $p = 1$ are also proved in [FGaW]. The Sobolev type inequality has been established on the group earlier, see [VSCC] and numerous references therein.

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