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Interpolation of infinite order entire functions

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Introduction.

In [2] and [3], Berenstein and Struppa studied the relation between Dirichlet series and solutions of convolution equations. In [3], they considered the equation

$$\mu * f = 0,$$

where f is an analytic function on the upper half plane and μ is an analytic functional on the complex plane $\mathbb{C}$. It turns out that if μ satisfies a "slowly decreasing" condition, then f can be represented by a series

$$f(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{J_k} P_{k,l}(z)\, e^{a_{k,l} z},$$

where $\hat{\mu}(a_{k,l}) = 0$, $P_{k,l}$ are polynomials and $\hat{\mu}$ represents the Fourier transform of μ. This is a representation of a generalized Dirichlet series. Under certain conditions, "gap" theorems similar to the Fabry gap theorem for Dirichlet series may be proven. We refer the reader to [3] for further remarks.

In [2], they consider the case where f is holomorphic in a cone Γ contained in the right half plane with vertex at the origin and μ can be described by integration against a measurable function with compact support in that cone. In this case, with μ slowly decreasing, f can be represented as a Dirichlet series, in its simplest form

$$f(w) = \sum_{k=1}^{\infty} c_k(w)\, e^{-z_k w}, \qquad w \in \Gamma,$$

where c_k is a polynomial of degree less than m_k and $\hat{\mu}$ vanishes at z_k with multiplicity m_k. We again refer the interested reader to [2] for further details.

The point of this approach to Dirichlet series is that, while the clasical Fabry gap theorem requires that the frequencies

$$0 < |z_1| < |z_2| < \dots$$

satisfy the addititional finite density condition

$$\overline{\lim_{k\to\infty}} \frac{k}{|z_k|} < +\infty,$$

and $m_k = 1$, and other convergence theorems require finite density, in [2] and [3], one can alow for the existence of a constant $\alpha > 0$ such that

$$\overline{\lim_{k\to\infty}} \frac{k}{|z_k|^{\alpha}} < +\infty,$$

(as well as no assumption on the z_k being real or $m_k = 1$). Clearly, this condition is weaker than the previous one when $\alpha > 1$. It leads to the study of interpolation problems for holomorphic functions of finite order α.

On the other hand, a sequence like $z_k = \ln k$ will not satisfy such hypotheses for any $\alpha > 0$. Nevertheless, such sequences are of great interest since the family of ordinary Dirichlet series includes the Riemann ζ-function

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}.$$

In order to study such problems, one must have some knowledge of infinite order functions. This article, which is based on the author's Thesis, provides the framework necessary in order to extend the theorems of [2] and [3] to the infinite order case.

Let f be an entire function which vanishes at the points z_k with multiplicity m_k and nowhere else. Then, given a doubly indexed sequence of complex numbers $\{a_{k,l}\}_{k\geq 1,\, 0\leq l<m_k}$ satisfying

$$\sum_{l=0}^{m_k-1} |a_{k,l}| \leq A\, e^{B\rho(z_k)}\,, \qquad A, B > 0\,,$$

when does there exist an entire function $\lambda(z)$ such that

$$|\lambda(z)| \leq A_1\, e^{B_1\rho(z)}\,, \qquad A_1, B_1 > 0\,,$$

and

$$\frac{\lambda^{(l)}(z_k)}{l!} = a_{k,l}\,?$$

Here ρ is a subharmonic function satisfying other conditions to be explained in the paper. The new contribution is for the case when ρ grows fast enough so that λ is of infinite order.

The plan of this paper is to extend the results of [1] to the infinite order case. One possibility is to allow an extra constant inside of ρ, *i.e.* let

$$|f(z)| \leq A\, e^{B\rho(Cz)}\,, \qquad A, B, C > 0\,.$$

It turns out here that the proofs are the same and only the statements are presented here in Section 2. In Section 1, the main part of this work, we do not allow the extra constant. Theorem 1.1 is the major result, which precisely calculates orders of infinite order functions. The rest of the section presents ramifications of this result which should lead to methods to attack the problems mentioned above.

Throughout this paper, we let $\mathbb{N}$ denote the natural numbers, $M_f(r)$ denote the maximum modulus of f on a circle of radius r, and $D(z;r)$ denote the circle of radius r centered at z.

1. Interpolation for infinite order functions.

Let $\{z_k\}_{k=1}^{\infty}$ be a divergent sequence of complex numbers and $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers. We start with the following definition.

Definition. *$V = \{(z_k, m_k)\}$ is the multiplicity variety for an entire function f if it vanishes precisely at the points z_k, $k \geq 1$, with multiplicity m_k. We write $V = V(f)$ when V is the multiplicity variety* *for f. More generally, $V = V(f_1, \dots, f_m)$ is a multiplicity variety for $f_1, \dots, f_m$ if $\{z_k\}$ is the set of common zeros of the functions $f_1, \dots, f_m$ and the functions vanish at those points with multiplicity at least m_k, and one of them with multiplicity exactly m_k.*

A well known theorem is the following [1, Theorem 3].

Theorem 1.A. *Let $V = \{(z_k, m_k)\}_{k \geq 1}$ be a multiplicity variety. Let $\{a_{k,l}\}$ be any sequence of complex numbers, where $k \geq 1$ and $0 \leq l \leq m_k - 1$. Then there exists an entire function λ such that*

$$\frac{\lambda^{(l)}(z_k)}{l!} = a_{k,l} .$$

By Theorem 4 in [1], in order to study the interpolation problem, we first study the corresponding problem of zeros.

We are interested in putting growth restrictions on the sequence $\{a_{k,l}\}$ and the function λ in Theorem 1.A. We make the following

Definition. *Let f be an entire function. For $r > 0$, $n_f(r)$ is defined to be the number of zeros, counted with multiplicity, of f in the circle of radius r, excluding those at the origin. When the function being considered is clear, we will drop the subscript.*

The growth of f depends upon the growth of n_f. The relationship is provided by Lemma 1.A. First, let us review what happens in the case of finite order functions and discuss the differences and difficulties encountered for infinite order functions. Our source for this discussion is the book of Levin [4, Chap. 1].

Let $\{a_n\}_{n=1}^{\infty}$ be the set of zeros of an entire function (excluding those at the origin), arranged in order of increasing modulus and repeated according to multiplicity. The starting point is the infinite sum

$$\sum_{n=1}^{\infty} \left| \frac{z}{a_n} \right|^{p_n+1} , \tag{1.1}$$

where p_n is a sequence of non-negative integers chosen so that the sum (1.1) converges uniformly on compact sets. Then we can consider the infinite product

$$E(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{i=1}^{p_n} \frac{1}{i}\left(\frac{z}{a_n}\right)^i\right) . \tag{1.2}$$

In the case of finite order functions, the numbers a_n satisfy the following suplemental condition: there exists a positive number λ such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}}$$

converges. In this case, let p denote the smallest integer for which

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} \tag{1.3}$$

converges. Setting $p_n = p$ for all n in (1.1) will be enough to assure convergence of (1.1). Here, then, is the first complication for infinite order functions: no such simplification (1.3) is posible.

The first step in finding an upper bound for the product (1.2) is finding an upper bound for each term. Taking logarithms of both sides of (1.2) will turn the infinite product into an infinite sum. We will obtain an upper bound for each term in the sum. Adding together all the upper bounds will result in another sum which can be written as a Stieltjes integral. It is this integral which must be evaluated to obtain a formula for the growth of (1.2).

The upper bound for each term is given by Lemma 2 in Chapter 1 of Levin [4, p. 11]. We modify it for our purposes and restate it as Lemma 2.A.

Lemma 2.A. *For $p_n \geq 1$ and all complex numbers z,*

$$\ln\left|\left(1-\frac{z}{a_n}\right)\exp\left(\sum_{i=1}^{p_n}\frac{1}{i}\left(\frac{z}{a_n}\right)^i\right)\right| \leq A_{p_n}\frac{\left|\frac{z}{a_n}\right|^{p_n+1}}{1+\left|\frac{z}{a_n}\right|}, \tag{1.4}$$

where

$$A_{p_n} = 3\,e\,(2+\ln p_n).$$

If $p_n = 0$, the sum is empty, so we have

$$\ln\left|\left(1-\frac{z}{a_n}\right)\right| \leq \ln\left(1+\left|\frac{z}{a_n}\right|\right).$$

Summing (1.4) over all n gives

$$\ln|E(z)| \le \sum_{n=1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(|a_n|+r)}\,, \tag{1.5}$$

for $|z| = r$.

In the case of finite order functions, the numerator is independent of n and may be taken outside the sum. Then, writing (1.5) as a Stieltjes integral, we obtain

$$\ln|E(z)| \le A_p\, r^{p+1} \int_0^{+\infty} \frac{dn(t)}{t^p\,(t+r)}\,. \tag{1.6}$$

Integration by parts then gives the formula in Levin [4, p. 12] for finite order functions

$$\ln|E(z)| \le k_p\, r^p \left(\int_0^r \frac{n(t)}{t^{p+1}}\, dt + r \int_r^{+\infty} \frac{n(t)}{t^{p+2}}\, dt \right)\,, \tag{1.7}$$

for $k_p = 3\,e\,(p+1)\,(2+\ln p)$ for $p \ge 1$, $k_0 = 1$, and $|z| = r$.

In the infinite order case, the Stieltjes integral becomes

$$\int_0^{+\infty} \frac{A_{p(t)}\, r^{p(t)+1}\, dn(t)}{t^{p(t)}\,(t+r)}\,, \tag{1.8}$$

where $p(t)$ is a continuous function such that $p(t) = p_n$ at $t = n$.

Such a simple integration by parts procedure is now impossible. There is also the problem of convergence of (1.8) and chosing the correct $p(t)$ which will minimize the integral. We will see that it is easy to chose the correct $p(t)$. The integral is a Laplace-type integral in that most of the contribution to the integral takes place in the region around $t = r$. This leads us to using the Laplace method to evaluate the integral.

Another diference between finite and infinite order functions can be noted here. Inequality (1.7) shows that the order of $E(z)$ is no larger than one of $n(r)$ (although, the type, of course, may be infinite). For example, we know

$$|\sin(\pi z)| \le A\, e^{\pi r}\,, \qquad \text{for } |z| = r\,,$$

and

$$\left|\frac{1}{\Gamma(z)}\right| \le A\, e^{B r \ln r}\,, \qquad \text{for } |z| = r\,.$$

Here $n_{\sin(\pi z)}(r) = 2r$ and $n_{1/\Gamma(z)}(r) = r$.

This behavior leads us to believe that a function with zeros at $\ln n$ should be bounded by e^{e^r}. However, this is not the case. We will see that infinite order functions can grow much faster than $n(r)$. The reason for this is that the integral (1.8) contains the term $dn(t)$. Very loosely speaking, (1.6) and (1.8) show functions grow at the rate $r\,dn(r)$. Of course, for finite order functions,

$$r\,dn(r) = C\,n(r),$$

for the appropriate constant C. For infinite order functions, $dn(r)$ can be much larger than $n(r)$. The dividing line is at $n(r) = e^r$, in which case $n(r) = dn(r)$. The growth rate of the infinite order functions that we will consider can be somewhat sharpened using the Laplace method.

Before continuing, we first show that the choice of $p(t)$ does not cause any problems with convergence of (1.1).

Lemma 1.1. *Convergence of* (1.8) (*and, hence, of* (1.5)) *implies convergence of* (1.1).

PROOF. Assume p_n is chosen so that (1.5) converges. Then, for large enough $n_0 \geq 1$, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(|a_n|+r)} &\geq \sum_{n=1}^{\infty} \frac{r^{p_n+1}}{|a_n|^{p_n}(2|a_n|)} \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1}.\end{aligned}$$

Roughly speaking, our conditions for Theorem 1.1 require $\ln n'(r)$ to be convex (and therefore, of course, $n(r)$) and

$$n(r) \leq e^{r^{\beta}},$$

for some $\beta > 0$.

Theorem 1.1. *Let $f(z)$ be an entire function of infinite order with prescribed zeros at $z = a_n$, excluding possible zeros at the origin. Let $n(r)$ be a majorant of the number of zeros in the annulus $0 < |z| < r$ such that*

$$\begin{aligned}dn_f[r_k, r_{k+1}) &= n_f(r_{k+1}) - n_f(r_k) \\ &\leq n(r_{k+1}) - n(r_k) \\ &= dn[r_k, r_{k+1}),\end{aligned} \tag{1.9}$$

for every $k \geq 1$ *where* $0 < r_1 < r_2 < \ldots$ *is the sequence of increasing moduli of the zeros not at the origin. Assume for large enough* $r \geq r_0$*,*

$$\ln n'(r) \text{ is increasing and convex,} \tag{1.10}$$

$$\ln r = o(\ln n'(r)) \quad \text{as } r \to \infty\,, \tag{1.11}$$

$$(\ln n')''((1 \pm \delta)r) \leq B_1\,(\ln n')''(r)\,, \tag{1.12}$$

$$|(\ln n')'''((1 \pm \delta)r)| \leq B_2\,|(\ln n')'''(r)|\,, \tag{1.13}$$

where (1.12) *and* (1.13) *hold for any* δ *such that* $0 < \delta < 1$ *and for some constants* B_1*,* $B_2 > 0$.

Then there exists an infinite product E *asociated with those zeros such that*

$$\ln|E(z)| \leq \frac{C \ln\left(r\dfrac{n'(r)}{n''(r)}\right) n'(r)}{\left(\dfrac{1}{r}\dfrac{n''(r)}{n'(r)} + \dfrac{n'(r)\,n'''(r) - (n''(r))^2}{(n'(r))^2}\right)^{1/2}}\,,$$

for $|z| = r$ *and large enough* $r > 0$.

PROOF. We will use the following notations and lemma for the proof of Theorem 1.1.

$$p(t) = t\,(\ln n')'(t)\,,$$

$$\psi(t) = (\ln r - \ln t)\,p(t) + \ln n'(t)\,,$$

$$\varphi(t) = \frac{6\,e + 3\,e\ln(p(t))}{t+r}\,.$$

Before we proceed, we establish a technical lemma. The conditions (1.10), (1.11), (1.12) and (1.13) of Theorem 1.1 imply

Lemma 1.2. *For the* δ *chosen in the proof of the theorem, and for large enough* $r \geq r_0$*,*

a) *For every* $0 < C_1 < 1$*, there exists* $0 < C_2 < (1 + C_1)/2$ *such that*

$$\ln n'(C_1 r) \leq C_2 \ln n'(r)\,.$$

b) *For some* $\varepsilon > 0$ *and for every* $t > e^2 r$*,* $\psi(t) \leq -\varepsilon \ln n'(t)$.

c) *For* $|t - r| \leq \delta r$*,* $|(\ln n')'(t) - (\ln n')'(r)| \leq \delta^{1/2} B_1^2\,|(\ln n')'(r)|$*, for some* $B_1 > 0$.

d) *For* $|t-r| \le \delta r$, $|(\ln n')''(t) - (\ln n')''(r)| \le \delta^{1/2} B_2^2 \, |(\ln n')'(r)|$, *for some* $B_2 > 0$.

e) ψ *is increasing on* $(r_0, r]$ *and decreasing on* $[r, +\infty)$.

f) $(\ln n')''(r) < O(r^N)$ *and* $(\ln n')'(r) < O(r^N)$, *for some* $N > 0$.

g) $\displaystyle\int_0^{e^2 r} \varphi(t)\, dt = O(1)$.

h) $\varphi(t) \in L^\infty(2r, +\infty)$.

i) $\displaystyle\sup_{|t-r|\le\delta r} \varphi(t) = O(\ln p(r))$.

j) $\displaystyle\frac{r}{(n'(r))^\eta} = O(1-\psi''(r))$, *for any* $\eta > 0$.

PROOF. a) Follows from (1.9).

b) We have

$$\begin{aligned}
\psi(t) &\le (\ln r - \ln e^2 r)\, t\, (\ln n')'(t) + \ln n'(t) \\
&\le -2t\, (\ln n')'(\xi) + \ln n'(t) \\
&= -\frac{2}{\delta} (\ln n'(t) - \ln n'(t - \delta t) + \ln n'(t) \\
&= \Big(1 - \frac{2}{\delta}\Big) \ln n'(t) + \frac{2}{\delta} \ln n'(t-\delta) \\
&\le \Big(1 - \frac{2}{\delta}\Big) \ln n'(t) + \frac{2}{\delta} C \ln n'(t) \\
&= \Big(1 - \frac{2}{\delta}\Big) \ln n'(t) + \frac{2}{\delta} \Big(\frac{2-\delta}{2} - \varepsilon\Big) \ln n'(t) \\
&= -\frac{2\varepsilon}{\delta} \ln n'(t).
\end{aligned}$$

The second line is the mean value theorem. The fifth and sixth lines follow from a) with $\varepsilon = (1 + C_1)/2 - C_2$.

c) We have

$$\begin{aligned}
|(\ln n')'(t) - (\ln n')'(r)| &= |(t-r)\,(\ln n')''(r \pm \delta_1 r)| \\
&\le \delta\, r\, B_1\, |(\ln n')''(r \pm \delta_2 r)| \\
&= \delta^{1/2} B_1\, |(\ln n')'(r \pm \delta^{1/2} r) - (\ln n')'(r)| \\
&\le \delta^{1/2} B_1^2\, |(\ln n')'(r)|.
\end{aligned}$$

The first and third lines follow from the mean value theorem. The second and fourth lines follow from (1.12).

d) The proof is identical to a), using (1.13) instead of (1.12).

e) Obvious.

f) Condition (1.12) implies that $(\ln n')''(r) < O(r^N)$, for some $N \in \mathbb{N}$. Then

$$\begin{aligned} O(r^N) &> (1-\delta)\, r \max_{s\in[(1-\delta)r,r]} (\ln n')''(s) \qquad \text{(by (1.12))} \\ &\geq \int_{(1-\delta)r}^{r} (\ln n')''(t)\, dt\,. \end{aligned}$$

Evaluating the integral,

$$\begin{aligned} \int_{(1-\delta)r}^{r} (\ln n')''(t)\, dt &= (\ln n')'(r) - (\ln n')'(1-\delta)r) \\ &\geq (\ln n')'(r) - C\, (\ln n')'(r) \\ &= (1-C)\, (\ln n')'(r)\,. \end{aligned}$$

The second line follows from integrating (1.12) for $C = B_1/(1+B_1)$. The proof is complete since $1 - C > 0$.

g) We have

$$\begin{aligned} \int_0^{e^2 r} \varphi(t)\, dt &< C \int_0^{e^2 r} \frac{\ln t}{t+r}\, dt \qquad \text{(from part f))} \\ &\leq C_1\, r\,. \end{aligned}$$

h) For $t \geq 2\,r$,

$$|\varphi(t)| \leq \left|\frac{C \ln r}{r}\right| \leq C\,.$$

i) We have

$$\int_{r-\delta r}^{r+\delta r} |\varphi(t) - \varphi(r)|\, dt \leq \delta\, C\, r\, \varphi(r) \leq \delta\, C\, \ln p(r)\,.$$

j) We have

$$\begin{aligned} n'(r) &\geq C\, r^{\nu_0(r)} && \text{(by (1.1))} \\ &> C\, r^N && \text{(for any } N \in \mathbb{N}) \\ &\geq (-\psi''(r))^{1/(2\eta)} && \text{(for some } \eta > 0 \text{ and part f))} \\ &\geq \left(\frac{(-\psi''(r))^{1/2}}{\ln p(r)}\right)^{1/\eta}. \end{aligned}$$

The desired conclusion follows upon rearranging.

Note that (1.10) implies that both $(\ln n')'(r)$ and $(\ln n')''(r)$ are non-negative. By (1.5),

$$\ln|E(z)| \leq \sum_{n=1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n} (r+|a_n|)} . \tag{1.14}$$

We write (1.14) as two sums

$$\sum_{n=1}^{n_0-1} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n} (r+|a_n|)} + \sum_{n_0-1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n} (r+|a_n|)} ,$$

where n_0 is the first integer such that $|a_{n_0}| \geq r_0$. The first sum is finite so we may chose $\varphi_n \equiv 1$. Then, for $r > 2|a_{n_0}|$, the sum is bounded above by

$$C \sum_{n=1}^{n_0-1} \left|\frac{r}{a_n}\right| \leq C \sum_{n=1}^{\infty} \left|\frac{r}{a_n}\right|^{p_n+1}$$

The second sum can be written as a Stieltjes integral

$$\int_{|a_{n_0}|}^{+\infty} \frac{A_{p(t)}\, r^{p(t)+1}\, dn_f(t)}{t^{p(t)}\,(t+r)} .$$

This is bounded above by

$$\int_{|a_{n_0}|}^{+\infty} \frac{A_{p(t)}\, r^{p(t)+1}\, dn(t)}{t^{p(t)}\,(t+r)} \tag{1.15}$$

by (1.9). We rewrite (1.15) in the form

$$r \int_{|a_{n_0}|}^{+\infty} \frac{A_{p(t)}\, e^{(\ln r - \ln t)p(t) + \ln n'(t)}}{t+r}\, dt \tag{1.16}$$

and we ned to minimize this integral. Treat the exponent as a function of two variables, $p(t)$ and t, and let

$$\psi_0(p(t), t) = (\ln r - \ln t)\, p(t) + \ln n'(t) .$$

Taking partials,

$$\frac{\partial \psi_0}{\partial p(t)} = \ln r - \ln t\,, \qquad \frac{\partial \psi_0}{\partial t} = -\frac{1}{t}\,p(t) + \frac{n''(t)}{n'(t)}\,.$$

The partial derivatives simultaneously vanish at

$$t = r\,, \qquad p(t) = t\,\frac{n''(t)}{n'(t)}\,.$$

By the second derivative test for partials, we find that this critical point is a saddle point. Accordingly, we use a generalization of the Laplace method to evaluate the integral (1.16). (See [11, p. 27] for a detailed proof of the Laplace method).

We need to evaluate

$$\int_{|a_{n_0}|}^{+\infty} \varphi(t)\, e^{\psi(t)}\, dt = e^{\psi(r)} \int_{|a_{n_0}|}^{+\infty} \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt = e^{\psi(r)} I\,.$$

Then

$$\begin{aligned} I &= \Big(\int_{|a_{n_0}|}^{r-\delta r} + \int_{r-\delta r}^{r+\delta r} + \int_{r+\delta r}^{e^2 r} + \int_{e^2 r}^{+\infty} \Big)\, \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where $0 < \delta < 1/2$.

For I_1, by Hölder's inequality,

$$\begin{aligned} I_1 &= \int_{|a_{n_0}|}^{r-\delta r} \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt \\ &\le \|\varphi(t)\|_1\, \|e^{\psi(t)-\psi(r)}\|_\infty \\ &\le C\, r\, (e^{\psi(r-\delta r)-\psi(r)})\,, \qquad \text{(by Lemma 1.2.g))}. \end{aligned}$$

Now

$$\begin{aligned} \psi(r-\delta r) - \psi(r) &= (\ln r - \ln(r-\delta r))\,(r-\delta r)\, \frac{n''(r-\delta r)}{n'(r-\delta r)} \\ &\quad + \ln n'(r-\delta r) - \ln n'(r) \\ &\le \ln\Big(\frac{1}{1-\delta}\Big)\, (1-\delta)\, r\, \frac{n''(\xi)}{n'(\xi)} \end{aligned}$$

$$
\begin{aligned}
&+ \ln n'(r-\delta r) - \ln n'(r) \\
&= (1-\delta)\ln\Big(\frac{1}{1-\delta}\Big)\, r\, \frac{\ln n'(r) - \ln n'(r-\delta r)}{\delta r} \\
&+ \ln n'(r-\delta r) - \ln n(r)\,.
\end{aligned}
$$

The ξ in the fourth line is obtained from the mean value theorem, and using the fact that $(\ln n')'(r)$ is increasing. It follows that

$$
\begin{aligned}
\psi(r-\delta r) - \psi(r) = \Big(\frac{1-\delta}{\delta}\ln\Big(\frac{1}{1-\delta}\Big) - 1\Big) \\
\cdot (\ln n'(r) - \ln n'(r-\delta r))\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\psi(r-\delta r) - \psi(r) &\le -\eta_1 (\ln n'(r) - \ln n'(r-\delta r)) \\
&\le -\eta_1 (1-C) \ln n'(r)\,.
\end{aligned}
$$

In the last two lines, η_1 and C are constants such that $\eta_1 > 0$ and $0 < C < 1$. The last line follows from Lemma 1.2.a) and e).

The above argument used the fact that

$$
\frac{1-\delta}{\delta}\ln\Big(\frac{1}{1-\delta}\Big) < 1\,.
$$

This follows directly from the inequality $\ln x < x-1$ for $x > 0$, $x \ne 1$. Thus, by Lemma 1.2.j), we have

$$
I_1 = O\Big(\frac{\ln p(r)}{(-\psi''(r))^{1/2}}\Big)\,.
$$

In order to make our calculations more precise, we will further split I_2.

$$
\begin{aligned}
I_2 &= \int_{r-\delta r}^{r+\delta r} \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt \\
&= \varphi(r)\int_{r-\delta r}^{r+\delta r} e^{\psi(t)-\psi(r)}\, dt + \int_{r-\delta r}^{r+\delta r} (\varphi(t)-\varphi(r))\, e^{\psi(t)-\psi(r)}\, dt \\
&= I_2' + I_2''\,.
\end{aligned}
$$

From Taylor's formula with remainder, for $|t-r| \le \delta r$,

$$\psi(t) - \psi(r) = \psi'(t')\,\frac{(t-r)^2}{2}\,,$$

where $t' = t'(t)$, and $|t'-r| \le \delta r$. Using c) and d) in Lemma 1.2 above, we can make the following calculations.

$$\begin{aligned}
|\psi'(t) - \psi'(r)| &= \Big|(\ln r - \ln t)\, p''(t) - \frac{1}{t}\,(\ln n')'(t) - (\ln n')''(t) \\
&\qquad + \frac{1}{r}\,(\ln n')'(r) + (\ln n')''(r)\Big| \\
&= \big|2\,(\ln r - \ln t)\,(\ln n')''(t) + (\ln r - \ln t)\, t\,(\ln n')'''(t) \\
&\qquad - \frac{1}{t}\,(\ln n')'(t) - (\ln n')''(t) \\
&\qquad + \frac{1}{r}\,(\ln n')'(r) - (\ln n')''(r)\big|\,.
\end{aligned}$$

This can be estimated by

$$\begin{aligned}
|\psi'(t) - \psi'(r)| &\le \Big|2\,\ln\Big(\frac{1}{1+\delta}\Big)\,(\ln n')''(t)\Big| \\
&\qquad + \Big|\ln\Big(\frac{1}{1+\delta}\Big)\, t\,(\ln n')'''(t)\Big| \\
&\qquad + |(\ln n')''(t) - (\ln n')''(r)| \\
&\qquad + \Big|\frac{1}{r}\,((\ln n')'(t) - (\ln n')'(r))\Big| \\
&\le \delta\, C\,(\ln n')''(r) + \varepsilon_3\,(\ln n')''(r) \\
&\qquad + \varepsilon_1\,\frac{1}{r}\,(\ln n')'(r) + \varepsilon_2\,(\ln n')''(r) \\
&\le \varepsilon\,\Big|\frac{1}{r}\,(\ln n')'(r) + (\ln n')''(r)\Big| \\
&= \varepsilon\,|\psi''(r)|\,.
\end{aligned} \tag{1.17}$$

In (1.17), ε_3 was obtained using the same method of proof in Lemma 1.2.c). Also, in the first term, the fact that

$$\Big|\ln\frac{1}{1+\delta}\Big| \approx \frac{\delta}{1+\delta}$$

was used. For I_2', let $\varepsilon > 0$ be arbitrary. By the argument above, we may choose δ such that

$$|\psi''(t) - \psi''(r)| < \varepsilon\, |\psi''(r)|, \qquad \text{for } |t-r| < \delta\, r. \tag{1.18}$$

So by (1.18),

$$\varphi(r) \int_r^{r+\delta r} e^{(1+\varepsilon)\,\psi''(r)\,(t-r)^2/2}\, dt \le I_2' \le \varphi(r) \int_r^{r+\delta r} e^{(1-\varepsilon)\,\psi''(r)\,(t-r)^2/2}\, dt.$$

Using the fact that $\int_0^{+\infty} e^{-u^2}\, du = \sqrt{\pi}/2$, we conclude that

$$I_2' \sim \frac{\varphi(r)}{-\psi''(r)^{1/2}}.$$

We will now consider I_2''.

$$\begin{aligned} |I_2''| &\le \sup_{|t-r|\le \delta r} |\varphi(t) - \varphi(r)| \int_{r-\delta r}^{r+\delta r} e^{\psi(t)-\psi(r)}\, dt \\ &\le C\, \ln p(r) \int_{r-\delta r}^{r+\delta r} e^{(1-\varepsilon)\psi''(r)\,(t-r)^2/2}\, dt, \end{aligned}$$

by Lemma 1.2.i), the mean value theorem, and (1.18).

Making the substitution

$$u = \left(\frac{-\psi''(r)\,(1-\varepsilon)}{2}\right)^{1/2} (t-r),$$

we obtain

$$|I_2''| \le \frac{C\, \ln p(r)}{(-\psi''(r))^{1/2}(1-\varepsilon)^{1/2}} \int_0^{+\infty} e^{-u^2}\, du.$$

Since the integral converges, we have

$$|I_2''| = O\left(\frac{\ln p(r)}{(-\psi''(r))^{1/2}}\right).$$

The estimate of I_3 is similar to I_1, with ψ decreasing on $[r, +\infty)$, so

$$I_3 = O\left(\frac{\ln p(r)}{(-\psi''(r))^{1/2}}\right).$$

For I_4, by Hölder's inequality,

$$e^{\psi(r)} I_4 \le \int_{e^2 r}^{+\infty} \varphi(t)\, e^{\psi(t)}\, dt \le \|\varphi(t)\|_\infty \int_{e^2 r}^{+\infty} e^{-\eta \ln n'(t)}\, dt,$$

for some $\eta > 0$, by Lemma 1.2.b). Thus $e^{\psi(r)} I_4$ converges by Lemma 1.2.h) and (1.1) above.

Putting everything together, we have

$$\ln|E(z)| \le C \frac{\ln p(r)}{(-\psi''(r))^{1/2}} n'(r).$$

Corollary 1.1. *The conditions of Theorem* 1.1 *imply that* $\ln n'(r) = O(r^N)$ *for some* $N \in \mathbb{N}$. *Hence*

$$n'(r) \le e^{r^N}, \qquad \textit{for some } N \in \mathbb{N}.$$

PROOF.

$$\begin{aligned}
O(r^N) &= \frac{1}{2} r\, (\ln n')'(r) && \text{(by Lemma 1-2 f))}\\
&\ge \int_{r/2}^{r} (\ln n')'(t)\, dt && \text{(since } (\ln n')'(t) \text{ is non-decreasing)}\\
&= \ln n'(r) - \ln n'\left(\frac{r}{2}\right)\\
&\ge \ln n'(r) - C \ln n'(r) && \text{(for some } 0 < C < 1, \\
& && \text{using Lemma 1.2.a))}\\
&= (1 - C) \ln n'(r).
\end{aligned}$$

The proof is complete since $1 - C > 0$.

EXAMPLE 1.1. Let $f_0(z)$ be the function with zeros at $a_n = \ln n$ for each $n \in \mathbb{N}$. Then $n_{f_0}(r) = e^r$. So

$$\ln n'_{f_0}(r) = r, \qquad (\ln n'_{f_0})'(r) = 1, \qquad (\ln n'_{f_0})''(r) = 0.$$

It is clear that the conditions of Theorem 1.1 are satisfied. Thus

$$\ln|f_0(z)| \le \frac{C\,\ln r\,e^r}{r^{-1/2}} = C\,r^{1/2}\ln r\,e^r\,.$$

Note that the only estimation of any kind occurred in the use of Lemma 1.A. The integral in Theorem 1.1 was evaluated fairly explicitly (*i.e.* up to the multiplicative constant term). Thus, the growth obtained should be best possible.

EXAMPLE 1.2. Theorem 1.1 is independent of the argument of the zeros. Let $g_0(z)$ be the function with zeros at $a_n = (\ln n)^{1/\alpha}\,e^{i\theta_n}$ with $\alpha > 1$, $0 \le \theta_n < 2\pi$ and $n \in \mathbb{N}$. Then $n_{g_0}(r) = e^{r^\alpha}$. It follows that

$$\begin{aligned}\ln n'_{g_0}(r) &= \ln(\alpha\,r^{\alpha-1}) + r^\alpha\,,\\ (\ln n'_{g_0})'(r) &= \frac{\alpha-1}{r} + \alpha\,r^{\alpha-1}\,,\\ (\ln n'_{g_0})''(r) &= \frac{1-\alpha}{r^2} + \alpha(\alpha-1)\,r^{\alpha-2}\,.\end{aligned}$$

Notice that for all $\alpha > 1$, $(\ln n'_{g_0})'(r)$ is eventually positive, so $\ln n'_{g_0}(r)$ is eventually convex. The rest of the conditions of Theorem 1.1 hold, so

$$\begin{aligned}\ln|g_0(z)| &\le \frac{C\,\ln(r\ln(\alpha\,r^{\alpha-1}) + r^{\alpha+1})\,\alpha\,r^{\alpha-1}\,e^{r^\alpha}}{\Big(\dfrac{\alpha-1}{r^2} + \alpha\,r^{\alpha-2} + \dfrac{1-\alpha}{r^2} + \alpha(\alpha-1)\,r^{\alpha-2}\Big)^{1/2}}\\ &\le C\,r^{\alpha/2}\,e^{r^\alpha}\ln r\,.\end{aligned}$$

Let $\rho(z)$ be a radial non-negative subharmonic function. Then we make the following definition.

Definition. *A_ρ is the space of all entire functions f satisfying the growth condition*

$$|f(z)| \le A\,e^{B\rho(z)}\,, \qquad A = A_f,\; B = B_f > 0\,.$$

In order to further develop the interpolation theory (see [1]), we now turn to the question of minimum modulus. Again there is a complication. We can no longer obtain the lower bound

$$|f(z)| \ge \varepsilon\,e^{-A\rho(r)}\,, \qquad \text{for } |z| = r\,,$$

outside some family of circles, as was done for finite order functions. The reason is simple. This was obtained in the case of finite order functions by [4, p. 21] and using the fact that $\rho(2r) = O(\rho(r))$. This is no longer true for infinite order functions (see Example 1.1).

To handle this, and in order to provide the necessary Fréchet space with which to study the interpolation theory, we make the following definition.

Definition. *Define $h(r)$ to be some non-increasing function which satisfies the condition*

$$\rho(r + h(r)) = O(\rho(r)). \tag{1.19}$$

We will also assume that $\lim_{r\to\infty} h(r)/r = 0$. This condition is not a restriction for our purposes (*i.e.* infinite order functions) since $h(r) = C$ for some $C > 0$, implies $f(z)$ is a finite order function and this case has been dealt with already (see [1], [7] and [8]). As an example,

$$\rho(r) = \mu(r)\, e^{r^\beta} \quad \text{implies} \quad h(r) = \frac{1}{r^{\beta-1}}$$

and assume that $\mu(r)$ is non-decreasing, $\mu(2r) = O(\mu(r)$, and $\mu(r) = O(r^N)$ for some $N \in \mathbb{N}$. This is easily seen. First, it is clear that

$$\mu\Big(r + \frac{1}{r^{\beta-1}}\Big) = O(\mu(r))$$

because of the upper bound on $\mu(r)$. Since $(1+x)^\beta = 1 + O(x)$, as $x \to 0$, we have that

$$\Big(r + \frac{1}{r^{\beta-1}}\Big)^\beta = r^\beta\Big(1 + \frac{1}{r^\beta}\Big)^\beta = r^\beta\Big(1 + O\Big(\frac{1}{r^\beta}\Big)\Big) = r^\beta + O(1).$$

Thus

$$e^{(r+r^{1-\beta})^\beta} \le e^{r^\beta + C} = O(e^{r^\beta}).$$

This example leads to

Corollary 1.2. *Assumming the hypotheses of Theorem 1.1,*

$$h(r) \ge \frac{1}{r^{\beta-1}}, \qquad \textit{for some } \beta > 0. \tag{1.20}$$

PROOF. Follows from Corollary 1.1 and the above calculation.

Let $\tilde{\rho}(r) = r\,\rho(r)/h(r)$. Clearly we also have

$$\tilde{\rho}(r + h(r)) = O(\tilde{\rho}(r)) \tag{1.21}$$

by (1.19) and (1.20) since $r/h(r)$ is non-decreasing and $1 \leq r/h(r) \leq r^{\beta}$.

Further analogs of results in [1] require $f^{m_k}(a_k)/m_k! \leq A\,e^{B\rho(r_k)}$. Since it is easily possible that $h(r) < 1$, Cauchy's formula provides the following bound on the multiplicities m_k. We have

$$\left|\frac{f^{(m_k)}(a_k)}{m_k!}\right| = \left|\frac{1}{2\pi i}\int\limits_{|z-z_k|=h(r_k)} \frac{f(z)\,dz}{(z-a_k)^{m_k+1}}\right| \leq \frac{A\,e^{B\rho(r_k)}}{h(r_k)^{m_k}}\,.$$

If $h(r) < 1$, we must have

$$\left(\frac{1}{h(r_k)}\right)^{m_k} \leq A\,e^{B\rho(r_k)}$$

or

$$m_k \ln\frac{1}{h(r_k)} \leq C + D\,\rho(r_k)\,,$$

which implies

$$m_k \leq \frac{C + D\,\rho(r_k)}{-\ln h(r_k)}\,. \tag{1.22}$$

We refer to (1.22) as the *automatic bound.*

We make the following definition of a slowly decreasing function in A_ρ.

Definition. *A function $f \in A_\rho$ is called slowly decreasing if the following two conditions hold:*

i) *There exist $\varepsilon > 0$, $A > 0$ such that each connected component S_α of the set*

$$S(f;\varepsilon,A) = \{z:\ |f(z)| < \varepsilon\,e^{-A\tilde{\rho}(z)}\}$$

is relatively compact.

ii) *There exists a constant $B > 0$ independent of α such that*

$$\tilde{\rho}(\zeta) \le B\,\tilde{\rho}(z) + B\,, \qquad \text{for any } z, \zeta \in S_\alpha, \text{ any } \alpha\,.$$

Before we proceed, we need a lemma which gives an upper bound on $n(r)$. This is a modification of the proof given in [4, p. 15] to the case of infinite order functions. Recall $\tilde{\rho}(r) = r\,\rho(r)/h(r)$.

Lemma 1.3. *Let $f \in A_\rho$ of infinite order. Assume*

a) $|f(0)| \ge 1$,

b) $\rho(r + h(r)) = O(\rho(r))$,

c) $\displaystyle\lim_{r\to\infty} \frac{h(r)}{r} = 0$.

Then

$$n_f(r) = O(\tilde{\rho}(r))\,.$$

PROOF.

$$\begin{aligned}
C\,\rho(r + h(r)) &\ge \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r + h(r))\,e^{i\theta})|\,d\theta \\
&\ge \int_r^{r+h(r)} \frac{n(t)}{t}\,dt && \text{(by Jensen's inequality)} \\
&\ge n(r)\int_r^{r+h(r)} \frac{dt}{t} && \text{(since } n(t) \text{ is increasing)} \\
&= n(r)\,(\ln(r + h(r)) - \ln r) \\
&= n(r)\,\ln\left(1 + \frac{h(r)}{r}\right) \\
&\sim \frac{h(r)}{r}\,n(r) && \text{(by condition c))}
\end{aligned}$$

which implies the statement of the lemma.

Notice that since $\tilde{\rho}(r+h(r)) = O(\tilde{\rho}(r))$, we also have $n_f(r+h(r)) = O(\tilde{\rho}(r))$.

The following theorem also makes use of a theorem of Momm's [6] for lower bounds of an entire function.

Theorem 2.A. *Let f be entire with $f(0)=1$. Then, for each $0<r<r+h(r)$, there is a Jordan curve Γ in $r<|z|<r+h(r)$ around the origin such that*

$$\ln|f(z)| \geq \frac{-C}{h(r)}\left(\int_0^{r+h(r)} \sqrt{\frac{\ln M_f(t)}{r+h(r)-t}}\,dt\right)^2, \quad \text{for } z\in\Gamma. \tag{1.23}$$

Theorem 1.2. *Let $f\in A_\rho$ satisfy the conditions of Theorem 1.1 and assume $f(0)=1$. Then f is slowly decreasing and (f), the ideal generated by f in $A_{\tilde\rho}$ is closed in the space $A_{\tilde\rho}$ (i.e. g/f entire implies that $g/f\in A_\rho$).*

PROOF. To prove slowly decreasing, it suffices to show that in every annulus $r\leq|z|\leq r+h(r)$, there exists a Jordan curve around the origin in that annulus such that on that curve, $f(z)$ attains the appropriate minimum modulus.

Let $\sigma(r)=\ln\rho(r)$. Consider the integral in (1.23). With our notation, it is estimated from above by

$$\int_0^{r+h(r)} \frac{e^{\sigma(t)/2}}{\sqrt{r+h(r)-t}}\,dt.$$

Upon integrating by parts, we have

$$\begin{aligned}\int_0^{r+h(r)} \frac{e^{\sigma(t)/2}}{\sqrt{r+h(r)-t}}\,dt &= -2\sqrt{r+h(r)-t}\,e^{\sigma(t)/2}\Big|_0^{r+h(r)} \\ &\quad + \int_0^{r+h(r)} \sqrt{r+h(r)-t}\,\sigma'(t)\,e^{\sigma(t)/2}\,dt \\ &\leq \sqrt{r+h(r)}\int_0^{r+h(r)} \sigma'(t)\,e^{\sigma(t)/2}\,dt \\ &= 2\sqrt{r+h(r)}\,e^{\sigma(t)/2}\Big|_0^{r+h(r)} \\ &\leq 2\sqrt{r+h(r)}\,e^{\sigma(r+h(r))/2} \\ &\leq 2\sqrt{(r+h(r))\,\rho(r+h(r))} \\ &\leq B\sqrt{r\,\rho(r)}.\end{aligned}$$

Plugging into (1.23), we obtain

$$\ln|f(z)| \geq \frac{-Cr}{h(r)}\,\rho(r).$$

This proves that f is slowly decreasing in $A_{\tilde{\rho}}$. By Proposition 3 of [1], (f) is closed in $A_{\tilde{\rho}}$.

REMARK. Notice that both the minimum modulus theorem for analytic functions with no zeros and Lemma 1.3 was used in the proof. Independently, a multiplicative factor of $r/h(r)$ appeared in both cases. This leads us to believe that $A_{\tilde{\rho}}$ is the correct space in which to study interpolation theory for infinite order functions.

EXAMPLE 1.3. Consider the $f_0(z)$ in Example 1.1. By Theorem 1.2, f_0 is slowly decreasing and (f_0) is closed in the space

$$\{f(z): \ln|f(z)| \le C\, r^{3/2} \ln r\, e^r\}.$$

EXAMPLE 1.4. Consider the $g_0(z)$ in Example 1.2. By Theorem 1.2, g_0 is slowly decreasing and (g_0) is closed in the space

$$\{g(z): \ln|g(z)| \le C\, r^{3\alpha/2} e^{r^\alpha} \ln r\}.$$

As a consequence, in view of [1, Proposition 3], if f is slowly decreasing in A_ρ, it is only invertible in the space $A_{\tilde{\rho}}$.

More generally, any theorem in [1] that mentions

$$|f(z)| \ge \varepsilon\, e^{-A\rho(r)}, \qquad |z| = r,$$

must be changed in the infinite order case to

$$|f(z)| \ge \varepsilon\, e^{-A\tilde{\rho}(r)}$$

to reflect the above facts.

Asociated to a multiplicity variety V in $\mathbb{C}$, there is a unique closed ideal in $A(\mathbb{C})$,

$$I = I(V) = \{F \in A(\mathbb{C}) : F \text{ vanishes at } z_k \text{ with multiplicity} \ge m_k\}.$$

Two functions g and h in $A(\mathbb{C})$ can be identified modulo I if and only if

$$(1.24) \qquad \frac{g^{(l)}(z_k)}{l!} = \frac{h^{(l)}(z_k)}{l!} = a_{k,l}, \qquad 0 \le l \le m_k - 1,\ k = 1, 2, \ldots$$

Hence Theorem 1.A above states that the quotient space $A(\mathbb{C})/I$ can be identified to the space of al sequences $\{a_{k,l}\}$. We will describe them as *analytic functions on* V and denote that space by $A(V)$. The map $\varrho_V = \varrho$,

$$\varrho : A(\mathbb{C}) \to A(V),$$

which takes $g \in A(\mathbb{C})$ to $\{a_{k,l}\} \in A(V)$ via (1.24) above, is called the *restriction map.*

Before we proceed, we need some definitions. In what follows, let $h_1(r) = \min\{h(r), 1\}$.

Definition. *Let* $V = \{(z_k, m_k)\}$ *be a multiplicity variety. Then* $A_\rho(V)$ *is the space of all functions* $\{a_{k,l}\} \in A(V)$ *such that for some constants* $A,\ B > 0$

$$\sum_{l=0}^{m_k-1} |a_{k,l}|\, h_1(r_k)^l \le A\, e^{B\rho(r_k)}, \qquad k \ge 1,\ 0 \le l \le m_k. \tag{1.25}$$

Note that when $m_k = O(e^{B\rho(r_k)})$, then (1.25) is equivalent to

$$|a_{k,l}|\, h_1(r_k)^l \le A_1\, e^{B_1\rho(r_k)}, \qquad k \ge 1,\ 0 \le l \le m_k. \tag{1.26}$$

Notice that because of the automatic bound, we may take (1.26) as the definition for $A_\rho(V)$ whenever $h(r) < 1$.

We now define $\rho(r; s) = \rho(r + h_1(r)s)$.

Definition. *The space* $A_{\rho,\infty}(V)$ *consists of those* $\{a_{k,l}\} \in A(V)$ *such that for some* $A,\ B > 0$ *and all* $s \ge 1$

$$\sum_{l=0}^{m_k-1} |a_{k,l}|\, (h_1(r_k)s)^l \le A\, e^{B\rho(r_k;s)}, \qquad k \ge 1,\ 0 \le l \le m_k. \tag{1.27}$$

Note that because of the definition of $h_1(r)$, we can be more precise about what is meant by $\rho(r_k; s)$. By repeating (1.19) $[s]+1$ times, where $[\cdot]$ denotes greatest integer, we obtain

$$\rho(r_k; s) \le A\, e^{Bs}(\rho(r_k)),$$

for some constants $A,\ B > 0$.

Definition. *If ϱ maps A_ρ onto $A_\rho(V)$, we will say that V is an interpolating variety for A_ρ. If ϱ maps A_ρ onto $A_{\rho,\infty}$, then we will say that V is a weak interpolating variety for A_ρ.*

Squires, in his Thesis, [7, Theorem 2] and [8, Theorem 3], provides a purely geometric condition for a finite order function to be interpolating. That is, whether or not $f \in A_\rho$ interpolated depended only upon the geometry of $V(f)$. That theorem has an analog in the infinite order case, and we present that here.

Theorem 1.3. *Let $f \in A_\rho$, where $n_f(r)$ satisfies the conditions of Theorem 1.1 and assume $V = V(f)$. Then V is an interpolating variety in $A_{\tilde\rho}$ if and only if there exists constants C, $D > 0$ such that*

$$m_k \le \frac{C\,\tilde\rho(|a_k|) + D}{|\ln h(|a_k|)|}\,, \tag{1.28}$$

$$\int_0^{h(|a_k|)} \frac{n(a_k,t,V)}{t}\,dt \le C\,\tilde\rho(|a_k|) + D\,. \tag{1.29}$$

PROOF. By [1, Theorem 4], if V is an interpolating variety, then there exist constants $\varepsilon, C > 0$ such that

$$\frac{|f^{(m_k)}(a_k)|}{m_k!} \ge \varepsilon e^{-C\,\tilde\rho(|a_k|)}\,. \tag{1.30}$$

(Recall f is only invertible in the space $A_{\tilde\rho}$). Now (1.28) follows from Cauchy's formula. (If $h(r) < 1$, this is precisely the automatic bound.) We first show that (1.30) implies

$$\prod_{\substack{|a_n - a_k| < h(|a_k|)\\ a_n \ne a_k}} \frac{|a_n - a_k|}{|a_k|} \ge \varepsilon\, e^{-C\tilde\rho(|a_k|)}\,. \tag{1.31}$$

Let $f(z) = (z - a_k)^{m_k} g(z)$ and write $1 = |g(a_k)|/|g(a_k)|$.

For the numerator, note $f^{(m_k)}(a_k) = m_k!\, g(a_k)$. For the denominator, write the canonical factorization of $g(a_k)$. We have

$$\prod_{\substack{|a_n - a_k| < h(|a_k|)\\ a_n \ne a_k}} \frac{|a_n - a_k|}{a_k} = \frac{1}{P_1P_2P_3}\left|\frac{f^{(m_k)}(a_k)}{m_k!}\right|$$

with

$$P_1 = \prod_{|a_n - a_k| \geq h(|a_k|)} \left| \left(1 - \frac{a_k}{a_n}\right) E_{p_n}\left(\frac{a_k}{a_n}\right) \right|,$$

$$P_2 = \prod_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \left| E_{p_n}\left(\frac{a_k}{a_n}\right) \right|,$$

$$P_3 = \prod_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \left| \frac{a_k}{a_n} \right|.$$

Here, E_{p_n} represents the exponential part of the n^{th} term in the factorization of $f(z)$ from Theorem 1.1.

We need only obtain appropriate upper bounds for the three products in the denominator. Note that $P_1 = f_k(a_k)$, where

$$f_k(z) = \prod_{|a_n - a_k| \leq h(|a_k|)} \left(1 - \frac{a_k}{a_n}\right) E_{p_n}\left(\frac{a_n}{a_k}\right).$$

We show that $f(z)$ has the same growth as $f_k(z)$. First, it is clear that $dn_{f_k}(t) = dn_f(t)$. Second, letting $\{b_j\}_{j=1}^{\infty} = Z(f_k)$, we have $|b_j| \geq |a_j|$. Then

$$\sum_{j=1}^{\infty} \left| \frac{z}{b_j} \right|^{p_k} \leq \sum_{j=1}^{\infty} \left| \frac{z}{a_j} \right|^{p_k},$$

so by the argument in Theorem 1.1, $|f_k(a_k)| \leq A\, e^{B\rho(|a_k|)}$.

For the second product, we have

$$\begin{aligned} \ln P_2 &\leq \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \sum_{j=1}^{p_n} \frac{1}{j} \left| \frac{a_k}{a_n} \right|^j \\ &\leq \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \sum_{j=1}^{p_n} \frac{1}{j} \left(\frac{|a_k| + h(|a_k|)}{|a_n|} \right)^j. \end{aligned}$$

It follows that

$$\ln P_2 \leq \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} C \ln p(|a_n|) \left(\frac{|a_k| + h(|a_k|)}{|a_n|} \right)^{p(|a_n|)}$$

$$\leq \sum_{\substack{|a_n-a_k|<h(|a_k|)\\ a_n\neq a_k}} C\ln p(|a_n|)\Big(\frac{|a_k|+h(|a_k|)}{|a_n|}\Big)^{p(|a_n|)+1}$$

$$\leq C\,\rho(|a_k|+h(|a_k|))$$

$$\leq C\rho(|a_k|)\,,$$

since $|a_n|<|a_k|+h(|a_k|)$ and using the arguments in Theorem 1.1 and the assumptions on ρ.

For the last product, we have

$$\prod_{\substack{|a_n-a_k|<h(|a_k|)\\ a_n\neq a_k}} \frac{|a_n|}{a_k} \leq \Big(\frac{|a_k|}{|a_k|-h(|a_k|)}\Big)^{n_f(|a_k|+h(|a_k|))}$$

$$\leq \Big(\frac{|a_k|}{|a_k|-h(|a_k|)}\Big)^{C\,|a_k|\rho(|a_k|)/h(|a_k|)}$$

$$\leq \Big(1+\frac{h(|a_k|)}{|a_k|-h(|a_k|)}\Big)^{C\,|a_k|\rho(|a_k|)/h(|a_k|)}$$

$$\leq \Bigg(\Big(1+\frac{h(|a_k|)}{\frac{|a_k|}{h(|a_k|)}-1}\Big)^{|a_k|/h(|a_k|)}\Bigg)^{C\rho(|a_k|)}$$

$$\leq e^{2\,C\,\rho(|a_k|)}\,.$$

We have just shown (1.31). Let $n(a_k,t,V)$ be the number of points in V in a disk of radius t centered at a_k, excluding a_k itself. Inequality (1.31) also implies

$$\prod_{\substack{|a_n-a_k|<h(|a_k|)\\ a_n\neq a_k}} \frac{|a_n-a_k|}{a_k} \geq \varepsilon\, e^{-C\,\tilde\rho(|a_k|)}\,.$$

Taking logarithms,

$$\sum_{\substack{|a_n-a_k|<h(|a_k|)\\ a_n\neq a_k}} \ln|a_n-a_k| - n(a_k,h(|a_k|),V)\,\ln|a_k| \geq -C\,\tilde\rho(|a_k|)-D\,. \tag{1.32}$$

Upon writing a Stieltjes integral and integrating by parts, we have

$$\int_0^{h(|a_k|)} \frac{n(a_k,t,V)}{t}\,dt = \sum_{\substack{|a_n-a_k|<h(|a_k|)\\ a_n\neq a_k}} \ln\frac{h(|a_k|)}{|a_n-a_k|}$$

$$= n(a_k, h(|a_k|), V) - \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \ln |a_n - a_k| . \tag{1.33}$$

Now, (1.31) and (1.32) together give

$$\int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t}\, dt \le C\, \tilde{\rho}(|a_k|) + D + n(a_k, h(|a_k|), V) \ln \left(\frac{h(|a_k|)}{|a_k|}\right),$$

which gives (1.29) for $|a_k| > h(|a_k|)$.

Since f attains the desired minimum modulus on some Jordan curve around the origin in every annulus $r \le |z| \le r + h(|a_k|)$, by Jensen's Theorem,

$$\begin{aligned} \ln \left| \frac{f^{(m_k)}(a_k)}{m_k!} \right| &= \frac{1}{2\pi} \int_0^{2\pi} \ln |f(a_k + h(|a_k|)\, e^{i\theta})|\, d\theta \\ &\quad - \int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t}\, dt - m_k \ln h(|a_k|) \\ &\ge -C\, \tilde{\rho}(|a_k|) - D - \int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t}\, dt \\ &\quad - m_k \ln h(|a_k|) . \end{aligned}$$

Conditions (1.28) and (1.29) then imply

$$\ln \left| \frac{f^{(m_k)}(a_k)}{m_k!} \right| \ge -C\, \tilde{\rho}(|a_k|) - D .$$

By [1, Theorem 4], V is an interpolating variety.

It should be aparent that the concept of a slowly decreasing function is very important in interpolation theory. Since the definition of slowly decreasing means that $f(z)$ attains a minimum modulus, that implies that groups of zeros of f are "well" separated, in some sense. The idea now is that convexity of $n(r)$ should already imply that individual zeros of f are well separated, since the growth of $n(r)$ is so regular. That is, we should be able to show that, under certain conditions, if $n(r)$ is convex, then not only is f slowly decreasing, but each component of $S(f; \varepsilon, A)$ contains only one zero of f.

As a starting point, consider the diference quotient

$$\frac{n(r_{k+1}) - n(r_k)}{r_{k+1} - r_k},$$

where $\{r_k\}$ is the increasing sequence of the moduli of the zeros. The numerator is simply the multiplicity of the zero at $z = a_k$. The idea is that if the denominator has a "nice"lower bound, then each distinct zero of the function f should be trapped in its own component of $S(f; \varepsilon, A)$. To take a concrete example, consider the function $f_0(z)$ in Example 1.1. There,

$$r_{k+1} - r_k = \ln(k+1) - \ln k = \ln\left(1 + \frac{1}{k}\right) \sim \frac{1}{k} \sim \frac{1}{n'_{f_0}(r_k)}.$$

This is the kind of condition we would like for Theorem 1.4.

We introduce the following notation for Theorem 1.4:

$$\nu(r) = \frac{\rho(r)}{n'(r)},$$

i.e., the factor multiplied by $n'(r)$ in Theorem 1.1,

$$\tilde{\nu}(r) = \frac{\nu(r)\, r}{h(r)\, \ln(r\, n'(r))},$$

$$\tilde{h}(r) = \min\{h(r), \tilde{\nu}(r)\}.$$

Notice that since $\tilde{h}(r)$ is always bounded above by $h(r)$, (1.25) holds, with $\tilde{h}(r)$ replacing $h(r)$. Also, (1.25) holds with $n'(r)$ replacing $\rho(r)$ since $n'(r)$ is bounded above by e^{r^β} for some $\beta > 0$.

Lemma 1.4. *For $n(r)$ satisfying the conditions of Theorem* 1.1,

$$\tilde{\nu}(r) \geq \frac{C}{r^N},$$

for some constants C, $N > 0$.

PROOF. Since $h(r) \leq r$, we have

$$\begin{aligned}\tilde{\nu}(r) &= \frac{\nu(r)\, r}{h(r)\, \ln(r\, n'(r))} \\ &\geq \frac{\nu(r)}{\ln(r\, n'(r))} \\ &= \frac{C\, \ln(r\, (\ln n')'(r))}{\left(\frac{1}{r}\, (\ln n')'(r) + (\ln n')''(r)\right)^{1/2} \ln(r\, n'(r))}.\end{aligned}$$

For the numerator,

$$\ln(r(\ln n')'(r)) \geq C, \tag{1.34}$$

since $\ln n'(r)$ is increasing by (1.10). For the denominator,

$$\left(\frac{1}{r}(\ln n')'(r) + (\ln n')''(r)\right)^{1/2} \leq C\,r^N, \tag{1.35}$$

for some $N > 0$ by Lemma 1.2.e); and

$$\ln(r\,n'(r)) \leq C\,r^N, \tag{1.36}$$

by Corollary 1.1 for some (possibly different) $N > 0$. Now, (1.34), (1.35) and (1.36) imply

$$\tilde{\nu}(r) \geq \frac{C}{r^N}.$$

Theorem 1.4. *Let $f(z) \in A_\rho$, where $n_f(r)$ satisfies the conditions of Theorem 1.1. Assume*

$$m_k \leq r_k^m, \qquad \text{for some uniform } m > 0, \tag{1.37}$$

$$\frac{1}{n'(r_{k+1})} \leq r_{k+1} - r_k \leq \frac{r_k^m}{n'(r_k)}, \qquad \text{for all } k. \tag{1.38}$$

Here r_k is the modulus of a zero of f and m_k its multiplicity. Then there exists a family of circles C_k, each circle centered at a zero of $f(z)$ such that

$$C_j \cap C_k = \varnothing, \qquad \text{for } j \neq k, \tag{1.39}$$

$$\ln|f(z)| \geq -C\,\tilde{\rho}(r) - D, \tag{1.40}$$

for $|z| = r$ and some constants C, $D > 0$ for al z outside the set of exceptional circles $\cup_k C_k$.

Proof. Recall we are assumming that the moduli of distinct zeros of f are separated per the discussion prior to this theorem. We must establish the circles satisfying (1.39). We first show that the definition of $\tilde{h}(r)$ is reasonable, in the sense that it is possible to have several zeros of $f(z)$ in any circle of radius $2\,\tilde{h}(r)$.

From (1.10), we have

$$\frac{1}{n'(r)} \le \frac{1}{r^{\nu_0(r)}}, \tag{1.41}$$

for some $\nu_0(r)$ such that $\lim_{r\to\infty} \nu_0(r) = \infty$. Thus it will be sufficient if we can show

$$\tilde{h}(r) \ge \frac{C}{r^\beta}, \tag{1.42}$$

for some $\beta > 0$ and $C > 0$. And $\tilde{h}(r)$ satisfies (1.42) since $h(r)$ and $\tilde{\nu}(r)$ do, by Corollary 1.2 and Lemma 1.4, respectively.

Now (1.38), (1.41) and (1.42) imply

$$r_{k+1} - r_k \le C\,\tilde{h}(r_k). \tag{1.43}$$

Writing $r_{k+1} = r_k + (r_{k+1} - r_k)$ and using (1.43), (1.38) implies

$$r_{k+1} - r_k \ge \frac{B_1}{n'(r_k)}, \qquad \text{for some } B_1 > 0. \tag{1.44}$$

Then, a circle of radius, for example, $B_1/(3\,n'(r_k))$ around r_k will guarantee that (1.39) holds. Fix $B_0 = B_1/3$ for the rest of the proof.

The proof of (1.40) is a modification of that given in [4, p. 125]. Assume $|z| = r$ and that z does not lie in the set of exceptional circles just found. Let

$$F(z) = \prod_{|z-a_n|\le 2\,\tilde{h}(r)} \left(1 - \frac{z}{a_n}\right). \tag{1.45}$$

Notice that, since $|z - a_n| \le 2\,\tilde{h}(r)$ and $|a_n| \ge r - 2\,\tilde{h}(r)$, we have, for r big enough,

$$\left|\frac{z - a_n}{a_n}\right| \le \frac{2\,\tilde{h}(r)}{r - 2\,\tilde{h}(r)} < 1$$

and therefore

$$\ln|F(z)| < 0. \tag{1.46}$$

Now, for z outside the exceptional circles,

$$\begin{aligned}|F(z)| &= \prod_{|z-a_n|\leq 2\,\tilde{h}(r)} \left|\frac{z-a_n}{a_n}\right| \\ &\geq \prod_{|z-a_n|\leq 2\,\tilde{h}(r)} \left(\frac{B_0}{(r-2\,\tilde{h}(r))\,n'(r)}\right) \quad \text{(by (1.44))} \\ &\geq \left(\frac{C}{r\,n'(r)}\right)^{n(z,V,2\,\tilde{h}(r))}\end{aligned}$$

so

$$\ln|F(z)| \geq -C\,n(z,V,2\,\tilde{h}(r))\,\ln(r\,n'(r)) - D\,. \tag{1.47}$$

We next obtain an upper bound for $n(z,V,2\tilde{h}(r))$. Notice that, since (1.47) was obtained independent of the argument of the zeros, it must remain true even if all the zeros lie on the same diameter of $D(z;2\,\tilde{h}(r))$, the circle of center z and radius $2\,\tilde{h}(r)$. Thus, by (1.39), the total of the diameters of the excluded circles in $D(z;2\tilde{h}(r))$ must be less than the diameter of $D(z;2\,\tilde{h}(r))$. The total diameter of the excluded circles in $D(z;2\,\tilde{h}(r))$ is bounded below by $n(z,V,2\,\tilde{h}(r))$ times the smallest diameter of the excluded circles in $D(z;2\,\tilde{h}(r))$. The smallest diameter is bounded below by

$$\frac{2B_0}{n'(r+2\,\tilde{h}(r))} \geq \frac{C_0}{n'(r)}\,.$$

Since the diameter of $D(z;2\,\tilde{h}(r)) = 4\,\tilde{h}(r)$,

$$\frac{C_0\,n(z,V,2\,\tilde{h}(r))}{n'(r)} \leq C\,\tilde{h}(r)\,.$$

(The arbitrary constant C accounts for the fact that some of the excluded circles may be only partly inside $D(z;2\,\tilde{h}(r))$. Thus, there exists a constant $B>0$ such that

$$n(z,V,2\,\tilde{h}(r)) \leq B\,\tilde{h}(r)\,n'(r)\,.$$

Now, since $\tilde{h}(r) \leq \tilde{\nu}(r)$, we have

$$n(z,V,2\,\tilde{h}(r)) \leq B\,\tilde{\nu}(r)\,n'(r) = \frac{B\,r\,\nu(r)\,n'(r)}{h(r)\,\ln(r\,n'(r))} = \frac{B\,\tilde{\rho}(r)}{\ln(r\,n'(r))}\,.$$

Plugging into (1.47), we get

$$\ln|F(z)| \geq -C\,\tilde{\rho}(r) - D. \tag{1.48}$$

Let

$$F(z;\zeta) = \prod_{|z-a_n|\leq \tilde{h}(r)} \left(1 - \frac{z+\zeta}{a_n}\right).$$

Fixing z and thinking of $F(z;\zeta)$ as a function of the complex variable ζ, it is clear that (1.48) still holds for $|\zeta| \leq \tilde{h}(r)$. Note $F(z;0) = F(z)$.

Let $f(z;\zeta) = f(z+\zeta)$ and write

$$\varphi(z;\zeta) = \frac{f(z;\zeta)}{F(z;\zeta)}, \qquad \text{for } |\zeta| \leq \tilde{h}(r).$$

By Theorem 1.2, there exists a r_0, $0 \leq r_0 \leq \tilde{h}(r)$, such that

$$\ln|f(z+\zeta)| \geq -C\,\tilde{\rho}(r) - D, \qquad \text{for every } |\zeta| = r.$$

This remains true even if $\tilde{h}(r) = \tilde{\nu}(r)$. Just repeat the proof with $\tilde{\nu}(r)$ replacing $h(r)$, and use

$$n(z, V, \tilde{\nu}(r)) \leq n(z, V, h(r)) \leq C\,\tilde{\rho}(r).$$

Then, using (1.46), we obtain that

$$\ln|\varphi(z;\zeta)| = \ln|f(z;\zeta)| - \ln|F(z;\zeta)| \geq -C\,\tilde{\rho}(r) - D. \tag{1.49}$$

Now, $\varphi(z;\zeta)$ has no zeros in $|\zeta| \leq r_0$ and by the minimum modulus principle, it takes its minimum modulus on $|\zeta| = r_0$. In particular,

$$\ln|\varphi(z;0)| = \ln|f(z)| - \ln|F(z)| \geq -C\,\tilde{\rho}(r) - D.$$

Finaly, (1.48) and (1.49) together give

$$\ln|f(z)| \geq -C\,\tilde{\rho}(r) - D.$$

We close this chapter with propositions regarding the space $A_{\rho,\infty}(V)$.

Proposition 1.1. $A_{\rho,\infty}(V) \subseteq A_\rho(V)$.

PROOF. Let $s = 1$ in (1.27).

Proposition 1.2. *The restriction map* $\varrho : A(\mathbb{C}) \to A(V)$ *maps* A_ρ (*with* m_k *satisfying the automatic bound*) *into* $A_{\rho,\infty}(V)$.

PROOF. From Cauchy's formula, for $0 \le j \le m_k - 1$,

$$\frac{f^{(j)}(z_k)}{j\,!} = \int\limits_{|\zeta - z| = h_1(r_k)s} \frac{f(\zeta)}{(\zeta - z)^{j+1}}\, d\zeta\,.$$

Then, taking the sum,

$$\begin{aligned}\sum_{j=0}^{m_k-1} \frac{|f^{(j)}(z_k)|}{j\,!} s^j &\le m_k \left(\frac{1}{h_1(r_k)}\right)^{m_k} A\, e^{B\,\rho(r_k;s)} \\ &\le \frac{A_1\,\rho(r_k) + B_1}{-\ln r_k} A_2\, e^{B_2\,\rho(r_k)} A\, e^{B\,\rho(r_k;s)} \\ &\le A\, e^{B\,\rho(r_k;s)}\,.\end{aligned}$$

REMARK. (1.27) and the automatic bound give the estimate

$$|a_{k,l}|\, h_1(r_k)^l \le \frac{A\, e^{B\,\rho(r_k;s)}}{s^l}\,.$$

Compare with (1.26). Then $A_{\rho,\infty}(V) = A_\rho(V)$ if and only if

$$m_k = O\left(\inf_{s \ge 1} \frac{1 + B\,\rho(r_k; s) - \rho(r_k)}{\ln s}\right), \tag{1.50}$$

for some constant $B > 0$.

To see this, first assume (1.50). Note that because of Proposition 1.1, we need only show $A_\rho(V) \subseteq A_{\rho,\infty}(V)$. This condition follows directly, taking into account that (1.50) implies $s^{m_k} \le A\, e^{B\,(\rho(r_k;s) - \rho(r_k))}$. Since $A_\rho(V) \subseteq A_{\rho,\infty}(V)$, we must have

$$s^{m_k} A\, e^{B\,\rho(r_k)} \le A_1\, e^{B_1\,\rho(r_k;s)}\,.$$

Solving for m_k gives (1.50).

2. The Spaces $A_{\boldsymbol{\rho}}$.

We will now establish the theory regarding the spaces $A_{\boldsymbol{\rho}}$. The proofs will be omited, as they are basically unchanged from those in [1].

Let $\rho(z)$ be a subharmonic function satisfying the following two conditions:

(2.1.i) $$\rho(z) \geq 0 \text{ and } \log(1+|z|^2) = O(\rho(z));$$

(2.1.ii) there exist constants B_0, C_0, $D_0 > 0$ such that $|\zeta - z| \leq 1$ implies $\rho(\zeta) \leq B_0\, \rho(C_0 z) + D_0$.

We now define the spaces $A_{\boldsymbol{\rho}}$. The bold face $\boldsymbol{\rho}$ is to emphasize the constant inside the argument of ρ.

Definition. *Let f be entire. We say that $f \in A_{\boldsymbol{\rho}}$ if*

$$|f(z)| \leq A\, e^{B\,\rho(Cz)}, \quad \textit{for some } A,\ B,\ C > 0.$$

Condition (2.1.i) provides a minimum growth for the function f, guaranteeing that all polynomials belong to $A_{\boldsymbol{\rho}}$ for any ρ. We note that the only entire functions that grow slower than a polynomial are the constants, and nothing interesting happens there. Condition (2.1.ii) controls the growth of $\rho(z)$ in small discs. We also show below that (2.1.ii) guarantees that $A_{\boldsymbol{\rho}}$ is closed under differentiation for any ρ. Note that if ρ is radial, then subharmonicity implies that ρ is convex. Furthermore, since (2.1.i) implies that ρ approaches infinity, we can assume that ρ is increasing. If ρ is not radial, we can assume that ρ is eventually increasing on any line.

There are two reasons why we would like to study such spaces.

EXAMPLE 2.1. One reason why we may need to consider the space $A_{\boldsymbol{\rho}}$ instead of the case where we do not allow the constant inside the argument of ρ is that the zeros of the function are to dense, causing the function to grow too fast for the later case. Consider a function with simple zeros at $z = \ln\ln n$ for al $n \in \mathbb{N}$. In this case,

$$n(r) = e^{e^r},$$

which grows faster than the infinite order functions considered earlier.

EXAMPLE 2.2. The other reason that we may need to use the space A_ρ is that while the zeros themselves may not be dense, the multiplicities of the zeros may be too large. Consider the function with zeros at $z = \ln n$, for $n \in \mathbb{N}$ and "pile up"the multiplicity in the following manner:

1) At $z = 1$, place a zero with multiplicity $[e^e]$ (where $[\cdot]$ represents the integer part).

2) At $z = e$, place a zero with multiplicity $[e^{e^e}]$.

3) In general, place the zero a_k at $e^{|a_{k-1}|}$ with multiplicity $[e^{e^{|a_k|}}]$.

Here there are very few distinct zeros. However, the multiplicity is so high - $m_k = e^{e^{|a_k|}}$ for every k -that we are forced to deal with the space A_ρ.

Note that arbitrary derivatives belong to A_ρ for any ρ (*i.e.* the space A_ρ is closed under differentiation)

$$\begin{aligned}\frac{|f^{(n)}(z)|}{n!} &= \left|\frac{1}{2\pi i}\int\limits_{|z-\zeta|=1/C}\frac{f(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}\right| \\ &\le A\,e^{B\,\max_{|z-\zeta|=1/C}\rho(C\zeta)}\,C^{n+1} \\ &\le A\,e^{B\,\rho(Cz)}\,.\end{aligned}$$

Since we are only interested in the case where $\rho(Cz) \neq O(\rho(z))$, most of the time, conditions (2.1.i) and (2.1.ii) will always hold. In fact, if $\rho(z) = \rho(|z|)$, it is easy to see that not only are we dealing with infinite order functions, but (2.1.i) holds for arbitrarily rapidly growing functions.

Our question is to study, for given ρ, the range of $A_\rho(\mathbb{C})$ under the restriction map ϱ. The addition of the extra constant in the argument of ρ allows for a lot of "room to maneuver". Putting it another way, we are not putting a very precise growth rate on the functions in A_ρ. However, this does have the advantage of simplifying the calculations to the point that the proofs in this chapter vary only slightly from those in [1]. In Section 2, we will study some clases of infinite order functions in more detail.

We will begin with the A_ρ version of the Semi-Local Interpolation Theorem.

Definition. *For* $f_1, \ldots, f_m \in A_\rho$, $\|\mathbf{f}\| = (\sum_{i=1}^m |f_i|^2)^{1/2}$,

$$S(\mathbf{f};\varepsilon, B, C) = \{z \in \mathbb{C} : \ |\mathbf{f}(z)| < \varepsilon\, e^{-B\,\rho(Cz)}\}\,.$$

We will use a bold $\mathbf{f}$ when we are dealing with a collection of functions $f_1, \ldots, f_m$.

Semi-Local Interpolation Theorem. *Let $\tilde{\lambda}(z)$ be analytic on*

$$S(\mathbf{f}; \varepsilon, B, C)$$

and satisfy

$$|\tilde{\lambda}(z)| \leq A' e^{B' \rho(C' z)}, \qquad \text{for } z \in S(\mathbf{f}; \varepsilon, B, C).$$

Then there exists an entire function $\lambda(z) \in A_{\rho}$, constants ε_1, B_1, $C_1 > 0$ and functions $\alpha_1, \ldots, \alpha_m$ analytic on $S(\mathbf{f}; \varepsilon_1, B_1, C_1)$ such that for all $z \in S(\mathbf{f}; \varepsilon_1, B_1, C_1)$

$$\lambda(z) = \tilde{\lambda}(z) + \sum_{i=1}^{m} \alpha_i(z) f_i(z), \quad \text{and} \quad \frac{\lambda^{(l)}(z_k)}{l!} = \frac{\tilde{\lambda}^{(l)}(z_k)}{l!}$$

and $|\alpha_i(z)| \leq A\, e^{B\rho(Cz)}$ for some new constants A, B, and $C > 0$.

Following [1], we now define the space of analytic functions with growth conditions on a multiplicity variety V.

Definition. *Let $V = \{(z_k, m_k)\}$ be a multiplicity variety. Then $A_{\rho}(V)$ is the space of all functions $\{a_{k,l}\} \in A(V)$ such that for some constants A, B, $C > 0$,*

$$\sum_{l=0}^{m_k - 1} |a_{k,l}| \leq A\, e^{B\,\rho(C z_k)}, \qquad k \in \mathbb{N}. \tag{2.2}$$

If $m_k = O(e^{B\,\rho(C z_k)})$ then (2.2) is equivalent to

$$|a_{k,l}| \leq A\, e^{B\,\rho(C z_k)}. \tag{2.3}$$

We show below that $\varrho(A_{\rho}) \subseteq A_{\rho}(V)$. In general $A_{\rho}(V)$ is much bigger than $\varrho(A_{\rho})$. We give an example here to show what can happen.

EXAMPLE 2.3. To take an extreme case, let $n_f(z)$ satisfy the conditions of Theorem 1.1 with all the multiplicities one, $a_k = \ln k$ and $a_{k,0} = a_k =$

0. Then (2.2) is certainly satisfied for any ρ, but we have seen (Example 1.1) that such a function f must satisfy

$$|f(z)| \le e^{r^{1/2} \ln r\, e^r}, \qquad \text{for } |z| = r .$$

One reason is that the growth (2.2) depends on purely local conditions. Allowing more global conditions will give a sharper bound which in some cases may be more appropriate. Therefore, we now let

$$\rho(C; z; r) = \max_{|\zeta| \le r} \rho(C\,(z + \zeta))$$

and make the following definition.

Definition. *The space $A_{\boldsymbol{p},\infty}(V)$ consists of those $\{a_{k,l}\} \in A(V)$ such that for some A, B, $C > 0$ and al $r \ge 1$*

$$\sum_{l=0}^{m_k - 1} |a_{k,l}|\, r^l \le A\, e^{B\, \rho(C; z_k; r)} . \tag{2.4}$$

We then have the following propositions.

Proposition 2.1. $A_{\boldsymbol{p},\infty}(V) \subseteq A_{\boldsymbol{p}}(V)$.

Proposition 2.2. *$\varrho : A(\mathbb{C}) \to A(V)$ maps $A_{\boldsymbol{p}}$ into $A_{\boldsymbol{p},\infty}(V)$.*

We now seek growth conditions on $\{a_{k,l}\}$ for $A_{\boldsymbol{p},\infty}(V)$ analogous to (2.3).

Proposition 2.3. *If $\rho(z) = \rho(|z|)$ with $r = |z|$, then $\rho(C; z; D_1\, r + D_2) \le \rho(C'; r)$ for some C', D_1, $D_2 > 0$.*

EXAMPLE 2.4. Let $\kappa(r)$ be some function of r tending toward infinity faster than any linear function. Then (2.4), with $r = \kappa(r)$, gives the estimate

$$|a_{k,l}| \le \frac{A\, e^{B \rho(C; z_k; \kappa(r))}}{\kappa(r)^l} . \tag{2.5}$$

Compare with (2.3). Then $A_{\boldsymbol{p},\infty}(V) = A_{\boldsymbol{p}}(V)$ if and only if

$$m_k = O\left(\inf_{r \ge 1} \frac{1 + B\, \rho(C; z_k; \kappa(r) - \rho(D z_k)}{\ln \kappa(r)} \right) , \tag{2.6}$$

for some $B > 0$. To see this, first assume (2.6). Note that by Proposition 2.1, we need only show that $A_{\boldsymbol{p}}(V) \subseteq A_{\boldsymbol{p},\infty}(V)$. Assume (2.2). Since (2.2) holds for the sum, it certainly holds for each term. (2.6) then implies

$$A\, e^{B\,\rho(Cr)} \leq \frac{A'\, e^{B'\,\rho(C';z_k;\kappa(r))}}{(\kappa(r))^{m_k}} \leq \frac{A'\, e^{B'\,\rho(C;z_k;\kappa(r))}}{(\kappa(r))^{l}} \,.$$

So,

$$m_k |a_{k,l}|\, \kappa(r)^l \leq m_k\, A\, e^{B\,\rho(C;z_k;\kappa(r))} \,. \tag{2.7}$$

Since (2.6) implies $m_k = O(e^{B\,\rho(C;z_k;\kappa(r))})$, and (2.7) is true for each l, we conclude

$$\sum_{l=0}^{m_k-1} |a_{k,l}|\, \kappa(r)^l \leq A\, e^{B\,\rho(C;z_k;\kappa(r))} \,.$$

Now assume (2.2). Then

$$\sum_{l=0}^{m_k-1} |a_{k,l}|\, \kappa(r)^l \leq \kappa(r)^{m_k} A\, e^{B\,\rho(Cz_k)} \,.$$

Since $A_{\boldsymbol{p}}(V) \subseteq A_{\boldsymbol{p},\infty}(V)$, we must also have

$$\kappa(r)^{m_k}\, A\, e^{B\,\rho(Cz_k)} \leq A_1\, e^{B_1\,\rho(C_1;z_k;\kappa(r))} \,.$$

Solving for m_k gives (1.10).

We now make the following definitions.

Definition. *If ϱ maps $A_{\boldsymbol{p}}$ onto $A_{\boldsymbol{p}}(V)$, we will say that V is an interpolating variety for $A_{\boldsymbol{p}}$. If ϱ maps $A_{\boldsymbol{p}}$ onto $A_{\boldsymbol{p},\infty}(V)$, then we will say that V is a weak interpolating variety for $A_{\boldsymbol{p}}$.*

Definition. *If $f_1, \ldots, f_m \in A_{\boldsymbol{p}}$ then*

$$I_{\text{loc}}(f_1, \ldots, f_m) \,,$$

the local ideal generated by $(f_1, \ldots, f_m)$, is the set of all functions $g \in A_{\boldsymbol{p}}$ such that, for any $z \in \mathbb{C}$, there is an open neighborhood U of z and functions $g_1, \ldots, g_m \in A(U)$ with the property

$$g = \sum_{j=1}^{m} f_j\, g_j \quad \text{in } U$$

If $V = V(f_1, \dots, f_m)$ is the variety of common zeros of $f_1, \dots, f_m$ (with multiplicity) then $I_{\mathrm{loc}}(f_1, \dots, f_m) = I(V) \cap A_\rho$. Since $I(V)$ is closed in $A(\mathbb{C})$ (with the topology of uniform convergence on compacta) and $A_\rho \hookrightarrow A(\mathbb{C})$ is continuous, it follows that $I_{\mathrm{loc}}(f_1, \dots, f_m)$ is closed in A_ρ. By $(f_1, \dots, f_m)$ we will denote the *ideal generated in* A_ρ by those same functions.

Definition. *We say that* $f_1, \dots, f_m$ *as above are jointly invertible if*

$$I_{\mathrm{loc}}(f_1, \dots, f_m) = (f_1, \dots, f_m)\,.$$

For a single function f, we say that f is invertible if $I_{\mathrm{loc}}(f) = (f)$; in particular, the principal ideal generated by f is closed. In general, we do not expect these two ideals to coincide. We will see later that if $\rho(z) = \rho(|z|)$ then (f) is always closed. Se also [8, Theorem 7.1].

Hence, f invertible in A_ρ implies that if $g \in A_\rho$ and $g/f \in A(\mathbb{C})$ then $g/f \in A_\rho$. It also implies that (f) is closed and, consequently, the map $g \to fg$ is an open map from A_ρ onto (f).

Theorem 2.1. *Let* $f_1, \dots, f_m \in A_\rho$ *and* $V = V(f_1, \dots, f_m)$. *If, for some* ε, B, $C > 0$, *we have for all* $(z_k, m_k) \in V$

$$\sum_{j=1}^{m} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geq \varepsilon\, e^{-B\,\rho(Cz)}\,, \tag{2.8}$$

then V *is an interpolating variety. In the converse direction, if* V *is an interpolating variety and the functions* $f_1, \dots, f_m$ *are jointly invertible, then* (2.8) *holds for some* ε, B, $C > 0$ *at every point* $(z_k, m_k) \in V$.

From the proof of Theorem 2.1, we obtain

Corollary 2.1. *With the same hypotheses of joint invertibility as in Theorem* 2.1, *the multiplicity variety* $V = V(f_1, \dots, f_m)$ *is an interpolating variety if and only if there exist* ε, B, $C > 0$ *such that*

i) *each* $z_k \in V$ *is contained in a bounded component of* $S(\mathbf{f}; \varepsilon, B, C)$ *with diameter at most* 1.

ii) *No two points of* V *lie in the same component.*

We now consider analogous results for weak interpolating varieties. Recall that

$$\rho(C;z;r) = \max_{|\zeta|\le r} \rho(C\,(z+\zeta)).$$

For $B > 0$, $0 \le l \le m_k - 1$, and $\{(z_k, m_k)\} = V$, let

$$\gamma_{k,l} = \gamma_{k,l}(B) = \inf_{r>0} \frac{e^{B\,\rho(C;z;r)}}{r^l}$$

and

$$\gamma_k = \gamma_k(B) = \gamma_{k,m_k-1}(B)\,. \tag{2.9}$$

Notice the $\gamma_{k,l}$ come basically from (2.4), *i.e.*

$$\{a_{k,l}\} \in A_{\mathbf{p},\infty} \quad \text{implies} \quad |a_{k,l}| \le A\,\gamma_{k,l}(B)\,,$$

for some A, $B > 0$.

The following theorem gives a necessary condition for $V = V(f_1, \dots, f_m)$ to be a weak interpolating variety when $f_1, \dots, f_m$ are jointly invertible.

Theorem 2.2. *Let $V = V(f_1, \dots, f_m)$, $f_j \in A_{\mathbf{p}}$. Supose that V is a weak interpolating variety and that $f_1, \dots, f_m$ are jointly invertible. Then for each $B > 0$, there exist constants ε, C_1, $C_2 > 0$ such that*

$$\sum_{j=1}^{m} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \ge \varepsilon\,\gamma_k(B)\,e^{-C_1\,\rho(C_2 z_k)}\,.$$

Next we give sufficient conditions. For each $B > 0$, let $R_k = R_k(B) \ge 1$ denote a point at which

$$\frac{e^{B\,\rho(C;z_k;R_k)}}{R_k^{m_k-1}} \le 2\,\gamma_k\,,$$

(recall (2.9), the definition of γ_k).

Theorem 2.3. *Let $f_1, \dots, f_m \in A_{\mathbf{p}}$ and $V = V(f_1, \dots, f_m)$. Suppose that for each $B > 0$, there exist constants ε_1, C_1, C_2, C_3, C_4, $C_5 > 0$ such that for all $(z_k, m_k) \in V$,*

i) $m_k \leq C_1\,\rho(C_2 z_k) + C_3$,

ii) $\rho(z; 2R_k) \leq C_1\,\rho(C_2 z_k) + C_3$, *for all* $|z - z_k| \leq 2R_k$,

iii) $\displaystyle\sum_{j=1}^{m} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geq \varepsilon_1\,\gamma_k(B)\,e^{-C_4\,\rho(C_5 z_k)}$.

Then V is a weak interpolating variety.

There is also an analogue to Corollary 2.1, with the same notation.

Corollary 2.2. *If the hypotheses of Theorem 2.3 hold, then for some constants ε, B, C, C_1, C_2, C_3, $C_4 > 0$ we have*

i) *Each $z_k \in V$ belongs to a bounded component of $S(\mathbf{f}; \varepsilon, B, C)$ and $\rho(z)$ satisfies*

$$\rho(C_1 z) \leq C_2\,\rho(C_3 \zeta) + C_4\ ,$$

for any z,ζ of that component.

ii) *No two distinct points of V lie in the same bounded component of $S(\mathbf{f}; \varepsilon, B, C)$.*

Since $A(V)$ is generally much larger than the range of the restriction map $\varrho : A_\rho \to A(V)$, the next question is to try and find a description of the subspace of $A(V)$ which is the range of ϱ. We start with the concept of a slowly decreasing function.

Definition. *A function $f \in A_\rho$ is caled slowly decreasing if the following two conditions hold.*

i) *There exist constants ε, B, $C > 0$ such that each connected component S_α of the set*

$$S(f; \varepsilon, B, C) = \{z :\ |f(z)| < \varepsilon\,e^{-B\,\rho(Cz)}\}$$

is relatively compact.

ii) *There exist constants D_1, D_2, D_3, $D_4 > 0$, independent of α, such that*

$$\rho(D_1 z) \leq D_2\,\rho(D_3 \zeta) + D_4\ ,$$

for any z,$\zeta \in S_\alpha$ and any α.

Proposition 2.4. *If f is slowly decreasing, then f is invertible.*

Proposition 2.5. *If $\rho(z) = \rho(|z|)$, then any $f \in A_{\boldsymbol{\rho}}$, not identically zero, is slowly decreasing.*

We need the following lemma. Most of the proofs are immediate consequences of the definition of slowly decreasing.

Lemma 2.3. *If $f \in A_{\boldsymbol{\rho}}$ is slowly decreasing, then there are rectifiable Jordan curves Γ_α with the following properties:*

a) *The sets U_k are pairwise disjoint and $V = V(f) \subset \bigcup_\alpha U_\alpha$ where $U_\alpha = \operatorname{int}\Gamma_\alpha$.*

b) *For some constant $A > 0$, we have, for all α,*

$$|f(z)| \geq \frac{1}{A}\, e^{-A\,\rho(Az)}\,,$$

for $z \in \Gamma_\alpha$.

c) *For some constants B, $B_1 > 0$, we have, for any α, and any pair $z,\zeta \in \overline{U}_\alpha$,*

$$\rho(B_1 z) \leq B\,\rho(B\zeta) + B\,.$$

d) *If d_α is the diameter of Γ_α, then for some constant $C > 0$, we have*

$$d_\alpha \leq C\,e^{C\,\rho(Cz)}\,,$$

for any $z \in \overline{U}_\alpha$.

e) *For some constant $D > 0$,*

$$\operatorname{length}(\Gamma_\alpha) \leq D\,e^{D\,\rho(Dz)}\,,$$

for any $z \in \overline{U}_\alpha$.

f) *If n_α denotes the number of points in $V_\alpha = V \cap U_\alpha$, counted with multiplicity, then*

$$n_\alpha \leq N\,e^{N\,\rho(Nz)}\,,$$

for some constant $N > 0$ and any $z \in \overline{U}_\alpha$.

For the U_α obtained in Lemma 2.3 we make the following definition.

Definition. *Let $\{a_{k,l}^{(\alpha)}\} = \{a_{k,l}\} \cap A(V_\alpha)$. Then $A_{\boldsymbol{\rho},g}(V)$ consists of those functions $\{a_{k,l}\} \in A(V)$ such that, for $\varphi \in A(U_\alpha)$ and*

$$\|a_{k,l}^{(\alpha)}\|_\alpha = \inf\{\|\varphi\|_\infty :\ \varphi \in A(U_\alpha) \text{ and } \varrho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\}\}\,,$$

there exist constants C_1, C_2, $C_3 > 0$ independent of α such that

$$\|a_{k,l}^{(\alpha)}\|_\alpha \le C_1\, e^{C_2\,\rho(C_3 z)}\,, \qquad \text{for any } z \in U_\alpha\,. \tag{2.10}$$

See also [5] for a discussion of these spaces.

Theorem 2.4. *If the function $f \in A_{\boldsymbol{\rho}}$ is slowly decreasing, the map ϱ_V induces a linear topological isomorphism between the spaces $A_{\boldsymbol{\rho}}/((f))$ and $A_{\boldsymbol{\rho},g}(V)$ for $V = V(f)$.*

Theorem 2.4 shows that even though $A_{\boldsymbol{\rho},g}(V)$ was defined in terms of a specific family of curves Γ_α, $A_{\boldsymbol{\rho},g}(V)$ is actually a subspace of $A(V)$ independent of the family $\{\Gamma_\alpha\}$.

We can obtain a characterization of $A_{\boldsymbol{\rho},g}(V)$ in terms of polynomial interpolation if the following lemma holds.

Lemma 2.4. *If $\rho(z) = \rho(|z|)$ then* c) *in Lemma* 2.3 *can be replaced by*

c$'$) *Let $W_\alpha = \{z \in \mathbb{C} :\ \mathrm{dist}\,\{z,\, U_\alpha\} \le 2d_\alpha\}$, then for some constant $B > 0$, we have*

$$\rho(\zeta) \le B\,\rho(Bz) + B\,,$$

for any α and any $z, \zeta \in W_\alpha$.

To continue, we first restate some facts about the Newton interpolation formula and divided differences (see [10, p. 326]).

Let $\zeta_1, \dots, \zeta_n$, $n = n_\alpha$ stand for the points in V_α, counted with multiplicity. Then the polynomials

$$\begin{gathered} P_0 = 1\,, \\ P_1(z) = z - \zeta_1\,, \\ \vdots \\ P_{n-1}(z) = \prod_{j=1}^{n-1} (z - \zeta_j)\,, \end{gathered}$$

form a basis of the space of polynomials of degre $n-1$. There is a unique polynomial $Q = Q_\alpha$ of degre at most $n-1$ such that

$$\varrho_{V_\alpha}(Q_\alpha) = \{a_{k,l}^{(\alpha)}\}$$

and it can be written as

$$Q(z) = \sum_{j=0}^{n-1} \Delta^{(j)} P_j(z) .$$

$Q(z)$ is called the *Newton interpolation polynomial.* The coefficients

$$\Delta^{(j)} = \Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})$$

are the j^{th} divided differences of the $a_{k,l}^{(\alpha)}$'s. They can be computed recursively. For example, if $\zeta_1 = z_k$ then $\Delta^{(0)} = a_{k,0}$. If $\zeta_1, \dots, \zeta_m$ are distinct points, then

$$\Delta^{(m)} = \sum_{k=1}^{m} \frac{a_k}{\prod_{j \neq k}(\zeta_k - \zeta_j)} .$$

Higher multiplicities are handled by taking appropriate limits. For example, if

$$\zeta_1 = \cdots = \zeta_l = z_k ,$$

then $\Delta^{(l)} = a_{k,l}$. If $Q(z)$ satisfies

$$|Q(z)| \leq K_1 \, e^{K_2 \, \rho(K_3 z)} \tag{2.11}$$

for some K_1, K_2, $K_3 > 0$, the above discussion shows that (2.10) holds and hence there exists a function $\varphi \in A_{\boldsymbol{p}}$ such that $\varrho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\}$. This can be done even if we know the estimate for a single α since $\{b_{k,l}\} \in A(V)$, defined by

$$b_{k,l} = \begin{cases} a_{k,l} , & \text{if } z_k \in V_\alpha , \\ 0 , & \text{if } z_k \notin V_\alpha , \end{cases}$$

is in $A_{\boldsymbol{p},g}(V)$. We can estimate the $\Delta^{(j)}$ by the following lemma [10, p. 329].

Lemma 2.A. *Let φ be holomorphic in the open set $W \subseteq \mathbb{C}$, $|\varphi(z)| \leq M$ in W, and $\zeta_1, \dots, \zeta_n$ be given such that for some $\delta > 0$, $\bigcup_{j=1}^{n} D(\zeta_j; \delta) \subseteq W$, then*

$$|\Delta^{(j)}| \leq \left(\frac{2}{\delta}\right)^j M , \qquad 0 \leq j \leq n-1 .$$

(Here the $\Delta^{(j)}$ *are computed with respect to* $\varrho_V(Q)$*,* V *the multiplicity variety associated to* $\zeta_1, \dots, \zeta_n$*).*

Hence, assumming c′) holds, if either $Q = Q_\alpha$ satisfies (2.1) or (2.10) holds, we have, for some constants A_1, B_1, $C_1 > 0$,

$$\|\{a_{k,l}^{(\alpha)}\}\|'_\alpha = \max_{0 \le j \le n-1} |\Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})\, d_\alpha^j| \le A_1\, e^{B_1\, \rho(C_1 z)}, \tag{2.12}$$

for all $z \in U_\alpha$, $n = n_\alpha$. This follows from Lemma 2.A with $W = W_\alpha$ and $\delta = 2\, d_\alpha$. In particular, if

$$\{a_{k,l}\} \in A_{\boldsymbol{\rho},g}(V),$$

then (2.12) holds for every α with the constants independent of α. Conversely, if (2.12) holds for a given α, then it is obvious from the definition of the polynomials P_j and Q_α that, for every $z \in U_\alpha$ and some new constants A_2, B_2, $C_2 > 0$,

$$\begin{aligned} |Q_\alpha(z)| &\le A_1\, e^{B_1\, \rho(C_1 z)} \sum_{j=0}^{n-1} \frac{|(z-\zeta_1)\cdots(z-\zeta_j)|}{d_\alpha^j} \\ &\le n_\alpha\, A_1\, e^{B_1\, \rho(C_1 z)} \\ &\le A_2\, e^{B_2\, \rho(C_2 z)}. \end{aligned}$$

The last inequality follows from Lemma 2.3.f). Hence, if (2.12) holds with constants independent of α, then $\{a_{k,l}\} \in A_{\boldsymbol{\rho},g}(V)$. We collect these remarks in Theorem 2.5.

Theorem 2.5. *Let* $f \in A_{\boldsymbol{\rho}}$ *be slowly decreasing and the norms* $\|a_{k,l}^{(\alpha)}\|'_\alpha$ *of* $\{a_{k,l}\} \in A(V)$ *be defined with respect to some grouping* $\{\Gamma_\alpha\}$ *satisfying* a)-c′)-f) *of Lemmas* 2.3 *and* 2.4. *Then* $A_{\boldsymbol{\rho}}/((f))$ *is isomorphic under the restriction map* ϱ *to the subspace of* $A(V)$ *of those* $\{a_{k,l}\}$ *such that* (2.12) *holds for some constants* A_1, B_1, $C_1 > 0$ *independent of* α.

We close this section with some remarks on when each V_α contains only one point of V.

Proposition 2.6. *If* f *is slowly decreasing and there is a grouping* $\{\Gamma_\alpha\}$ *for* $V = V(f)$ *such that every* V_α *contains a single point of* V*, then* $A_{\boldsymbol{\rho},g}(V) = A_{\boldsymbol{\rho},\infty}(V)$.

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