

Reconstructing a neural net from its output

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Introduction.

Neural nets were originally introduced as highly simplified models of the nervous system. Today they are widely used in technology and studied theoretically by scientists from several disciplines. (See *e.g.* [N]). However, they remain little understood.

Mathematically, a (feed-forward) neural net consists of

- (1) A finite sequence of positive integers (D_0, D_1, \dots, D_L) ,
- (2) A family of real numbers (ω_{jk}^ℓ) defined for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$, and
- (3) A family of real numbers (θ_j^ℓ) defined for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.

The sequence (D_0, D_1, \dots, D_L) is called the *architecture* of the neural net, while the ω_{jk}^ℓ are called *weights* and the θ_j^ℓ *thresholds*.

Neural nets are used to compute non-linear maps from \mathbb{R}^N to \mathbb{R}^M by the following construction. We begin by fixing a nonlinear function $\sigma(x)$ of one variable. Analogy with the nervous system suggests that we take $\sigma(t)$ asymptotic to constants as t tends to $\pm\infty$; a standard choice, which we adopt throughout this paper, is $\sigma(x) = \tanh(x/2)$. Given an "input" $(t_1, \dots, t_{D_0}) \in \mathbb{R}^{D_0}$, we define real numbers x_j^ℓ for $0 \leq \ell \leq L$, $1 \leq j \leq D_\ell$ by the following induction on ℓ .

- (4) If $\ell = 0$ then $x_j^\ell = t_j$.

(5) If the $x_k^{\ell-1}$ are known, $1 \leq \ell \leq L$, then we set

$$x_j^\ell = \sigma \left(\sum_{1 \leq k \leq D_{\ell-1}} \omega_{jk}^\ell x_k^{\ell-1} + \theta_j^\ell \right), \quad \text{for } 1 \leq j \leq D_\ell.$$

Here $x_1^\ell, \dots, x_{D_\ell}^\ell$ are interpreted as the outputs of D_ℓ “neurons” in the ℓ^{th} “layer” of the net. The *output map* of the net is defined as the map

$$(6) \quad \Phi: (t_1, \dots, t_{D_0}) \mapsto (x_1^L, \dots, x_{D_L}^L).$$

In practical applications, one tries to pick the neural net

$$[(D_0, D_1, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$$

so that the output map Φ approximates a given map about which we have only imperfect information. The main result of this paper is that under generic conditions, perfect knowledge of the output map Φ uniquely specifies the architecture, the weights and the thresholds of a neural net, up to obvious symmetries. More precisely, the obvious symmetries are as follows. Let $(\gamma_0, \gamma_1, \dots, \gamma_L)$ be permutations, with

$$\gamma_\ell: \{1, \dots, D_\ell\} \rightarrow \{1, \dots, D_\ell\};$$

and let $\{\varepsilon_j^\ell: 0 \leq \ell \leq L, 1 \leq j \leq D_\ell\}$ be a collection of ± 1 's. Assume that γ_ℓ is the identity and $\varepsilon_j^\ell = +1$ whenever $\ell = 0$ or $\ell = L$. Then one checks easily that the neural nets

$$(7) \quad [(D_0, D_1, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)] \quad \text{and}$$

$$(8) \quad [(D_0, D_1, \dots, D_L), (\tilde{\omega}_{jk}^\ell), (\tilde{\theta}_j^\ell)]$$

have the same output map if we set

$$(9) \quad \tilde{\omega}_{jk}^\ell = \varepsilon_j^\ell \omega_{[\gamma_\ell j][\gamma_{\ell-1} k]}^\ell \varepsilon_k^{\ell-1} \quad \text{and} \quad \tilde{\theta}_j^\ell = \varepsilon_j^\ell \theta_{[\gamma_\ell j]}.$$

This reflects the facts that the neurons in layer ℓ are interchangeable, $1 \leq \ell \leq L-1$, and that the function $\sigma(x)$ is odd. The nets (7) and (8) will be called *isomorphic* if they are related by (9). Note in particular that isomorphic neural nets have the same architecture. Our main theorem asserts that, under generic conditions, any two neural nets with the same output map are isomorphic.

We discuss the generic conditions which we impose on neural nets. We have to avoid obvious counterexamples such as

- (10) Suppose all the weights ω_{jk}^ℓ are zero. Then the output map Φ is constant. The architecture and thresholds of the neural net are clearly not uniquely determined by Φ .
- (11) Fix ℓ_0, j_1, j_2 with $1 \leq \ell_0 \leq L-1$ and $1 \leq j_1 < j_2 \leq D_{\ell_0}$. Suppose we have $\theta_{j_1}^{\ell_0} = \theta_{j_2}^{\ell_0}$ and $\omega_{j_1 k}^{\ell_0} = \omega_{j_2 k}^{\ell_0}$ for all k . Then (5) gives $x_{j_1}^{\ell_0} = x_{j_2}^{\ell_0}$. Therefore, the output depends on $\omega_{j j_1}^{\ell_0+1}$ and $\omega_{j j_2}^{\ell_0+1}$ only through the sum $\omega_{j j_1}^{\ell_0+1} + \omega_{j j_2}^{\ell_0+1}$. So the output map does not uniquely determine the weights.

Our hypotheses are more than adequate to exclude these counterexamples. Specifically, we assume that

- (12) $\theta_j^\ell \neq 0$, and $|\theta_j^\ell| \neq |\theta_{j'}^\ell|$ for $j \neq j'$.
- (13) $\omega_{jk}^\ell \neq 0$; and for $j \neq j'$, the ratio $\omega_{jk}^\ell / \omega_{j'k}^\ell$ is not equal to any fraction of the form p/q with p, q integers and $1 \leq q \leq 100 D_\ell^2$.

Evidently, these conditions hold for generic neural nets. The precise statement of our main theorem is as follows. *If two neural nets satisfy (12), (13) and have the same output, then the nets are isomorphic.* In Section I we give a slightly different but clearly equivalent statement of our main result. It would be interesting to replace (12), (13) by minimal hypotheses, and to study functions $\sigma(x)$ other than $\tanh(x/2)$.

We now sketch the proof of our main result, sacrificing accuracy for simplicity. After a trivial reduction, we may assume $D_0 = D_L = 1$. Thus, the outputs of the nodes $x_j^\ell(t)$ are functions of one variable, and the output map of the neural net is $t \mapsto x_1^L(t)$. The key idea is to continue the $x_j^\ell(t)$ analytically to complex values of t , and to read off the structure of the net from the set of singularities of the x_j^ℓ . Note that $\sigma(x) = \tanh(x/2)$ is meromorphic, with poles at the points of an arithmetic progression $\{(2m+1)\pi i : m \in \mathbb{Z}\}$. This leads to two crucial observations.

- (14) When $\ell = 1$, the poles of $x_j^\ell(t)$ form an arithmetic progression Π_j^1 , and
- (15) When $\ell > 1$, every pole of any $x_k^{\ell-1}(t)$ is an accumulation point of poles of any $x_j^\ell(t)$.

In fact, (14) is immediate from the formula $x_j^1(t) = \sigma(\omega_{j_1}^1 t + \theta_j^1)$, which is merely the special case $D_0 = 1$ of (5). We obtain

$$(16) \quad \Pi_j^1 = \left\{ \frac{(2m+1)\pi i - \theta_j^1}{\omega_{j_1}^1} : m \in \mathbb{Z} \right\}.$$

To see (15), fix ℓ, j, \bar{k} , and assume for simplicity that $x_{\bar{k}}^{\ell-1}(t)$ has a simple pole at t_0 , while $x_k^{\ell-1}(t)$, $k \neq \bar{k}$, is analytic in a neighborhood of t_0 . Then

$$(17) \quad x_{\bar{k}}^{\ell-1}(t) = \frac{\lambda}{t - t_0} + f(t),$$

with f analytic in a neighborhood of t_0 .

From (17) and (5), we obtain

$$(18) \quad x_j^\ell(t) = \sigma(\omega_{j\bar{k}}^\ell \lambda (t - t_0)^{-1} + g(t)),$$

with

$$(19) \quad g(t) = \omega_{j\bar{k}}^\ell f(t) + \sum_{k \neq \bar{k}} \omega_{jk}^\ell x_k^{\ell-1}(t) + \theta_j^\ell.$$

analytic in a neighborhood of t_0 .

Thus, in a neighborhood of t_0 , the poles of $x_j^\ell(t)$ are the solutions \tilde{t}_m of the equation

$$(20) \quad \frac{\omega_{j\bar{k}}^\ell \lambda}{\tilde{t}_m - t_0} + g(\tilde{t}_m) = (2m+1)\pi i, \quad m \in \mathbb{Z}.$$

There are infinitely many solutions of (20), accumulating at t_0 . Hence, t_0 is an accumulation point of poles of $x_j^\ell(t)$, which completes the proof of (15).

In view of (14), (15), it is natural to make the following definitions. The *natural domain* of a neural net is the largest open subset of the complex plane to which the output map $t \mapsto x_1^\ell(t)$ can be analytically continued. For $\ell \geq 0$ we define the ℓ^{th} *singular set* $\text{Sing}(\ell)$ by setting

$$\begin{aligned} \text{Sing}(0) &= \text{complement of the natural domain in } \mathbb{C}, \quad \text{and} \\ \text{Sing}(\ell + 1) &= \text{the set of all accumulation points of } \text{Sing}(\ell). \end{aligned}$$

These definitions are made entirely in terms of the output map, without reference to the structure of the given neural net. On the other hand, the sets $\text{Sing}(\ell)$ contain nearly complete information on the architecture, weights and thresholds of the net.

This will allow us to read off the structure of a neural net from the analytic continuation of its output map. To see how the sets $\text{Sing}(\ell)$ reflect the structure of the net, we reason as follows.

From (14) and (15) we expect that

- (21) For $1 \leq \ell \leq L$, $\text{Sing}(L - \ell)$ is the union over $j = 1, \dots, D_\ell$ of the set of poles of $x_j^\ell(t)$, together with their accumulation points (which we ignore here), and
- (22) For $\ell \geq L$, $\text{Sing}(\ell)$ is empty.

Immediately, then, we can read off the “depth” L of the neural net; it is simply the smallest ℓ for which $\text{Sing}(\ell)$ is empty.

We need to solve for $D_\ell, \omega_{jk}^\ell, \theta_j^\ell$. We proceed by induction on ℓ .

When $\ell = 1$, (14) and (21) show that $\text{Sing}(L - 1)$ is the union of arithmetic progressions $\Pi_j^1, j = 1, \dots, D_1$. Therefore, from $\text{Sing}(L - 1)$ we can read off D_1 and the Π_j^1 . (We will return to this point later in the introduction.) In view of (16), Π_j^1 determines the weights and thresholds at layer 1, modulo signs. Thus, we have found $D_1, \omega_{jk}^1, \theta_j^1$.

When $\ell > 1$, we may assume that

- (23) The $D_{\ell'}, \omega_{jk}^{\ell'}, \theta_j^{\ell'}$ are already known, for $1 \leq \ell' \leq \ell$.

Our task is to find $D_\ell, \omega_{jk}^\ell, \theta_j^\ell$. In view of (23), we can find a pole t_0 of $x_{\tilde{k}}^{\ell-1}(t)$ for our favorite \tilde{k} . Assume for simplicity that t_0 is a simple pole of $x_{\tilde{k}}^{\ell-1}(t)$, and that the $x_k^{\ell-1}(t), k \neq \tilde{k}$, are analytic in a neighborhood of t_0 . Then $x_{\tilde{k}}^{\ell-1}(t)$ is given by (17) in a neighborhood of t_0 , with λ already known by virtue of (23). Let U be a small neighborhood of t_0 .

We will look at the image Y of $U \cap \text{Sing}(L - \ell)$ under the map $t \mapsto \lambda/(t - t_0)$. Since λ, t_0 and $\text{Sing}(L - \ell)$ are already known, so is Y . On the other hand, we can relate Y to $D_\ell, \omega_{jk}^\ell, \theta_j^\ell$ as follows. From (21) we see that Y is the union over $j = 1, \dots, D_\ell$ of

- (24) $Y_j = \text{image of } U \cap \{ \text{Poles of } x_j^\ell(t) \}$ under $t \mapsto \lambda/(t - t_0)$.

For fixed j , the poles of $x_j^\ell(t)$ in a neighborhood of t_0 are the \tilde{t}_m given

by (20). We write

$$(25) \quad \frac{\omega_{j\mathbb{k}}^\ell \lambda}{\tilde{t}_m - t_0} = \left[\frac{\omega_{j\mathbb{k}}^\ell \lambda}{(\tilde{t}_m - t_0)} + g(\tilde{t}_m) \right] + [g(t_0) - g(\tilde{t}_m)].$$

Equation (20) shows that the first expression in brackets in (25) is equal to $(2m+1)\pi i$. Also, since $\tilde{t}_m \rightarrow t_0$ as $|m| \rightarrow +\infty$ and g is analytic in a neighborhood of t_0 , the second expression in brackets in (25) tends to zero. Hence,

$$\frac{\omega_{j\mathbb{k}}^\ell \lambda}{\tilde{t}_m - t_0} = (2m+1)\pi i - g(t_0) + o(1), \quad \text{for large } m.$$

Comparing this with the definition (24), we see that Y_j is asymptotic to the arithmetic progression

$$(26) \quad \Pi_j^\ell = \left\{ \frac{(2m+1)\pi i - g(t_0)}{\omega_{j\mathbb{k}}^\ell} : m \in \mathbb{Z} \right\}.$$

Thus, the known set Y is the union over $j = 1, \dots, D_\ell$ of sets Y_j , with Y_j asymptotic to the arithmetic progression Π_j^ℓ . From Y , we can therefore read off D_ℓ and the Π_j^ℓ . (We will return to this point in a moment). We see at once from (26) that $\omega_{j\mathbb{k}}^\ell$ is determined up to sign by Π_j^ℓ . Thus, we have found D_ℓ and $\omega_{j\mathbb{k}}^\ell$. With more work, we can also find the θ_j^ℓ , completing the induction on ℓ .

The above induction shows that the structure of a neural net may be read off from the analytic continuation of its output map. We believe that the analytic continuation of the output map will lead to further consequences in the study of neural nets.

Let us touch briefly on a few points which we glossed over above. First of all, suppose we are given a set $Y \subset \mathbb{C}$, and we know that Y is the union of sets Y_1, \dots, Y_D , with Y_j asymptotic to an arithmetic progression Π_j . We assumed above that Π_1, \dots, Π_D are uniquely determined by Y . In fact, without some further hypothesis on the Π_j , this need not be true. For instance, we cannot distinguish $\Pi_1 \cup \Pi_2$ from Π_3 if $\Pi_1 = \{\text{odd integers}\}$, $\Pi_2 = \{\text{even integers}\}$, $\Pi_3 = \{\text{all integers}\}$. On the other hand, we can clearly recognize $\Pi_1 = \{\text{all integers}\}$ and $\Pi_2 = \{m\sqrt{2} : m \text{ an integer}\}$ from their union $\Pi_1 \cup \Pi_2$. Thus, irrational numbers enter the picture. The rôle of our generic hypothesis (13) is to control the arithmetic progressions that arise in our proof.

Secondly, suppose $x_{\bar{k}}^\ell(t)$ has a pole at t_0 . We assumed for simplicity that $x_k^\ell(t)$ is analytic in a neighborhood of t_0 for $k \neq \bar{k}$. However, one of the $x_k^\ell(t)$, $k \neq \bar{k}$, may also have a pole at t_0 . In that case, the $x_j^{\ell+1}(t)$ may all be analytic in a neighborhood of t_0 , because the contributions of the singularities of the x_k^ℓ in $\sigma(\sum_k \omega_{jk}^{\ell+1} x_k^\ell + \theta_j^{\ell+1})$ may cancel. Thus, the singularity at t_0 may disappear from the output map. While this circumstance is hardly generic, it is not ruled out by our hypotheses (12), (13). Because singularities can disappear, we have to make technical changes in our description of $\text{Sing}(\ell)$. For example, in the discussion following (23), Y need not be the union of the sets Y_j . Rather, Y is their “approximate union”, in a sense to be made precise in (II.A.1) below.

Next, we should point out that the signs of the weights and thresholds require some attention, even though we have some freedom to change signs by applying isomorphisms. (See (9).) In effect, we introduce in Section IV.A an extra induction on the number of neurons in the net, in order to show that the signs come out correctly. The induction comes into play in the substantial Lemma IV.B.16 below.

Finally, in the definition of the natural domain, we have assumed that there is a unique maximal open set to which the output map continues analytically. This need not be true of a general real-analytic function on the line—for instance, take $f(t) = (1 + t^2)^{1/2}$. Fortunately, Lemma III.A.1 below shows that the natural domain is well-defined for any function that continues analytically to the complement of a countable set. The defining formula (5) lets us check easily that the output map continues to the complement of a countable set, so the natural domain makes sense. This concludes our overview of the proof of our main theorem.

Both the uniqueness problem and the use of analytic continuation have already appeared in the neural net literature. In particular, it was R. Hecht-Nielsen who pointed out the rôle of isomorphisms and posed the uniqueness problem. His paper with Chen and Lu [CLH] on “equioutput transformations” on the space of all neural nets influenced our work. E. Sontag [So] and H. Sussman [Su] proved sharp uniqueness theorems for one hidden layer. The proof in [Su] uses complex variables.

At this stage, few non-trivial results are known for neural nets with more than one hidden layer, *i.e.* with $L > 1$. However, a recent paper of Macintyre and Sontag [MS] proves finiteness of the VC dimension, a measure of the computing power of a neural net.

I am grateful to R. Crane, S. Markel, J. Pearson, E. Sontag, R. Sverdlove, and N. Winarsky for introducing me to the study of neural nets.

I. Statement of the Main Results.

A. Definitions.

A *neural net* consists of the following:

- (1) A finite sequence of positive integers (D_0, D_1, \dots, D_L) with $L \geq 1$.
 - (2) A collection of real numbers (ω_{jk}^ℓ) , defined for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$.
 - (3) A collection of real numbers (θ_j^ℓ) , defined for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.
- Here, L is called the *depth* of the net, and (D_0, D_1, \dots, D_L) is called the *architecture* of the net. The ω_{jk}^ℓ are called *weights*, while the θ_j^ℓ are called *thresholds*.

Thus, a neural net has the form $[(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$. We denote neural nets by $\mathcal{N}, \mathcal{N}', \tilde{\mathcal{N}}$, etc.

For $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$, we define functions $x_j^\ell(t_1, \dots, t_{D_0}, \mathcal{N})$ by the following induction on ℓ .

$$(4) \quad x_j^0(t_1, \dots, t_{D_0}, \mathcal{N}) = t_j \quad \text{for } 1 \leq j \leq D_0.$$

$$(5) \quad x_j^\ell(t_1, \dots, t_{D_0}, \mathcal{N}) = \sigma \left(\sum_{k=1}^{D_{\ell-1}} \omega_{jk}^\ell x_k^{\ell-1}(t_1, \dots, t_{D_0}, \mathcal{N}) + \theta_j^\ell \right),$$

for $1 \leq j \leq D_\ell$, where

$$(6) \quad \sigma(x) = \tanh \left(\frac{x}{2} \right).$$

We call $(t_0, t_1, \dots, t_{D_0}) \in \mathbb{R}^{D_0}$ the *input* to the neural net; we call $(x_j^L(t_1, \dots, t_{D_0}, \mathcal{N}))_{1 \leq j \leq D_L} \in \mathbb{R}^{D_L}$ the *output*, or the *function computed by the neural net*; and we call $x_j^\ell(t_1, \dots, t_{D_0}, \mathcal{N})$ the *function computed by the j^{th} node of the ℓ^{th} layer*.

When it is clear which neural net we are talking about, we may write $x_j^\ell(t_1, \dots, t_{D_0})$ in place of $x_j^\ell(t_1, \dots, t_{D_0}, \mathcal{N})$. Also, when $D_0 = 1$, we may write $x_j^\ell(t)$ or $x_j^\ell(t, \mathcal{N})$ in place of $x_j^\ell(t_1), x_j^\ell(t_1, \mathcal{N})$.

The *size* of a neural net $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ is defined simply as $D_0 + D_1 + \dots + D_L$.

Next we discuss isomorphisms of neural nets. Let

$$(7) \quad \mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$$

be a neural net. Then let

$$(8) \quad \gamma_\ell: \{1, \dots, D_\ell\} \rightarrow \{1, \dots, D_\ell\}$$

be permutations. Finally, let ε_j^ℓ be a collection of signs,

$$(9) \quad \varepsilon_j^\ell = \pm 1, \quad \text{for } 0 \leq \ell \leq L, 1 \leq j \leq D_\ell.$$

In terms of \mathcal{N} , (γ_ℓ) , (ε_j^ℓ) , we define a new neural net

$$(10) \quad \tilde{\mathcal{N}} = [(D_0, \dots, D_L), (\tilde{\omega}_{jk}^\ell), (\tilde{\theta}_j^\ell)],$$

where

$$(11) \quad \tilde{\omega}_{jk}^\ell = \varepsilon_j^\ell \omega_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell \varepsilon_k^{\ell-1}$$

and

$$(12) \quad \tilde{\theta}_j^\ell = \varepsilon_j^\ell \theta_{(\gamma_\ell j)}^\ell.$$

An easy induction on ℓ shows that

$$(13) \quad x_j^\ell(\tilde{t}_1, \dots, \tilde{t}_{D_0}, \tilde{\mathcal{N}}) = \varepsilon_j^\ell x_{(\gamma_\ell j)}^\ell(t_1, \dots, t_{D_0}, \mathcal{N}),$$

provided $(\tilde{t}_1, \dots, \tilde{t}_{D_0})$ and (t_1, \dots, t_{D_0}) are related by

$$(14) \quad \tilde{t}_j = \varepsilon_j^0 t_{(\gamma_0 j)}, \quad \text{for } 1 \leq j \leq D_0.$$

In particular, if we assume

$$(15) \quad \varepsilon_j^\ell = 1 \text{ when } \ell=0 \text{ or } L \text{ and } \gamma_0, \gamma_L \text{ are the identity permutation,}$$

then (13), (14) show that

$$(16) \quad x_j^L(t_1, \dots, t_{D_0}, \tilde{\mathcal{N}}) = x_j^L(t_1, \dots, t_{D_0}, \mathcal{N}), \quad \text{for } 1 \leq j \leq D_L.$$

Thus, the neural nets $\tilde{\mathcal{N}}$ and \mathcal{N} compute the same function. We say that the nets $\mathcal{N}, \tilde{\mathcal{N}}$ are *isomorphic* if they are related by (7), ..., (12) for some choice of $(\gamma_\ell), (\varepsilon_j^\ell)$ satisfying (15). For fixed

$$[(\gamma_\ell)_{0 \leq \ell \leq L}, (\varepsilon_j^\ell)_{0 \leq \ell \leq L, 1 \leq j \leq D_\ell}]$$

satisfying (15), the map $\mathcal{N} \mapsto \tilde{\mathcal{N}}$ given by (7), (10), (11), (12) is called the *isomorphism induced by* $[(\gamma_\ell), (\varepsilon_j^\ell)]$. One checks easily that compositions and inverses of isomorphisms are again isomorphisms. Note that any two isomorphic neural nets $\mathcal{N}, \tilde{\mathcal{N}}$ have the same architecture.

It is useful to pick out a single representative from an isomorphism class of neural nets. Thus, we say that $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ is in *standard order* if for each $\ell, 1 \leq \ell < L$, we have

$$(17) \quad 0 < \theta_1^\ell < \theta_2^\ell < \dots < \theta_{D_\ell}^\ell.$$

The proof of the following observation is left to the reader.

(18) **Lemma.** *Every neural net $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ satisfying the generic condition*

$$(19) \quad \theta_j^\ell \neq 0, \quad |\theta_j^\ell| \neq |\theta_{j'}^\ell| \quad \text{for } j \neq j',$$

is isomorphic to one and only one neural net in standard order.

B. The Main Theorems.

The main result of this paper is as follows.

(1) **Uniqueness Theorem.** *Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ and $\tilde{\mathcal{N}} = [(\tilde{D}_0, \dots, \tilde{D}_L), (\tilde{\omega}_{jk}^\ell), (\tilde{\theta}_j^\ell)]$ be neural nets in standard order, satisfying the following generic conditions.*

$$(2) \quad \omega_{jk}^\ell \neq 0 \quad \text{and} \quad \left| \frac{\omega_{jk}^\ell}{\omega_{j'k}^\ell} \right| \neq \frac{p}{q},$$

for $j \neq j', p, q \in \mathbb{Z}, 1 \leq q \leq 100 D_\ell^2$,

$$(3) \quad \tilde{\omega}_{jk}^\ell \neq 0 \quad \text{and} \quad \left| \frac{\tilde{\omega}_{jk}^\ell}{\tilde{\omega}_{j'k}^\ell} \right| \neq \frac{p}{q},$$

for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \leq q \leq 100 \tilde{D}_\ell^2$. Assume $D_0 = \tilde{D}_0$, $D_L = \tilde{D}_{\tilde{L}}$, and

$$(4) \quad x_j^L(t_1, \dots, t_{D_0}, \mathcal{N}) = x_j^{\tilde{L}}(t_1, \dots, t_{D_0}, \tilde{\mathcal{N}}),$$

for all $(t_1, \dots, t_{D_0}) \in \mathbb{R}^{D_0}$ and all j , $1 \leq j \leq D_L$.

Then the nets \mathcal{N} and $\tilde{\mathcal{N}}$ are identical:

$$(5) \quad L = \tilde{L},$$

$$(6) \quad D_\ell = \tilde{D}_\ell \quad \text{for } 0 \leq \ell \leq L,$$

$$(7) \quad \omega_{jk}^\ell = \tilde{\omega}_{jk}^\ell \quad \text{for } 1 \leq \ell \leq L, \quad 1 \leq j \leq D_\ell, \quad 1 \leq k \leq D_{\ell-1},$$

$$(8) \quad \theta_j^\ell = \tilde{\theta}_j^\ell \quad \text{for } 1 \leq \ell \leq L, \quad 1 \leq j \leq D_\ell.$$

The Uniqueness Theorem 1 reduces immediately to the special case. $D_0 = 1$, $D_L = 1$. To see this, we fix j_0 , $1 \leq j_0 \leq D_L$, and k_0 , $1 \leq k_0 \leq D_0$. Then we restrict attention to the j_0^{th} outputs $x_{j_0}^L(\cdot, \tilde{\mathcal{N}})$ for inputs of the form $(0, \dots, 0, t, 0, \dots, 0)$, where the t occurs in the k_0^{th} coordinate. Thus we obtain functions $x^L(t, \mathcal{N})$, $x^{\tilde{L}}(t, \tilde{\mathcal{N}})$ of a single variable t . These functions are computed by neural nets $\mathcal{N}_{\text{reduced}}$, $\tilde{\mathcal{N}}_{\text{reduced}}$ obtained from \mathcal{N} , $\tilde{\mathcal{N}}$ by deleting irrelevant input and output nodes. The special case $D_0 = D_L = 1$ of Theorem (1), applied to $\mathcal{N}_{\text{reduced}}$ and $\tilde{\mathcal{N}}_{\text{reduced}}$, shows that $\mathcal{N}_{\text{reduced}}$ and $\tilde{\mathcal{N}}_{\text{reduced}}$ are identical. Since j_0 and k_0 were arbitrary, it follows that \mathcal{N} and $\tilde{\mathcal{N}}$ are identical. Thus, Theorem (1) is reduced to the special case $D_0 = D_L = 1$.

From now on, we change the definition of neural nets to include the requirement $D_0 = D_L = 1$. Thus, a neural net computes a single function of one variable. In view of the elementary Lemma A.18, our uniqueness theorem is reduced to the following statement.

(9) **Uniqueness Theorem.** Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ and $\tilde{\mathcal{N}} = [(\tilde{D}_0, \dots, \tilde{D}_{\tilde{L}}), (\tilde{\omega}_{jk}^\ell), (\tilde{\theta}_j^\ell)]$ be neural nets satisfying the generic conditions

$$(10) \quad \omega_{jk}^\ell \neq 0, \quad \text{and} \quad \left| \frac{\tilde{\omega}_{jk}^\ell}{\tilde{\omega}_{j',k}^\ell} \right| \neq \frac{p}{q},$$

for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \leq q \leq 100 D_\ell^2$, and

$$(11) \quad \tilde{\omega}_{jk}^\ell \neq 0, \quad \text{and} \quad \left| \frac{\tilde{\omega}_{jk}^\ell}{\tilde{\omega}_{j',k}^\ell} \right| \neq \frac{p}{q},$$

for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \leq q \leq 100 \tilde{D}_\ell^2$.

If $x_1^L(t, \mathcal{N}) = x_1^L(t, \tilde{\mathcal{N}})$ for all real t , then \mathcal{N} and $\tilde{\mathcal{N}}$ are isomorphic.

The rest of this paper is devoted to the proof of Theorem 9.

C. A Small Technical Lemma.

The following observation on isomorphic neural nets will be used much later, in the proof of Theorem B.9.

(1) **Lemma.** Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ and $\tilde{\mathcal{N}} = [(D_0, \dots, D_L), (\tilde{\omega}_{jk}^\ell), (\tilde{\theta}_j^\ell)]$ be isomorphic neural nets. Assume that

(2) $\omega_{jk}^\ell \neq 0$ for all ℓ, j, k , $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$,

(3) $|\omega_{jk}^\ell| \neq |\omega_{j'k}^\ell|$ for all $\ell, j \neq j', k$, $1 \leq \ell \leq L$, $1 \leq j, j' \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$,

(4) $\omega_{jk}^\ell = \tilde{\omega}_{jk}^\ell$ for $1 \leq \ell \leq L-1$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$.

(Note: We do not assume (4) for $\ell = L$).

Then \mathcal{N} and $\tilde{\mathcal{N}}$ are identical.

PROOF. Since $\mathcal{N}, \tilde{\mathcal{N}}$ are isomorphic, there are permutations γ_ℓ and signs ε_j^ℓ such that

$$(5) \quad \omega_{jk}^\ell = \varepsilon_j^\ell \tilde{\omega}_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell \varepsilon_k^{\ell-1},$$

$$(6) \quad \theta_j^\ell = \varepsilon_j^\ell \tilde{\theta}_{(\gamma_\ell j)}^\ell,$$

$$(7) \quad \gamma_0 = \text{identity}, \gamma_L = \text{identity}, \varepsilon_1^0 = 1, \varepsilon_1^L = 1.$$

Since $\tilde{\omega}_{jk}^\ell = \omega_{jk}^\ell$ for $\ell \leq L-1$, (5) implies

$$(8) \quad \omega_{jk}^\ell = \varepsilon_j^\ell \omega_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell \varepsilon_k^{\ell-1}, \quad \text{for } 1 \leq \ell \leq L-1,$$

so that

$$(9) \quad |\omega_{jk}^\ell| = |\omega_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell|, \quad \text{for } 1 \leq \ell \leq L-1.$$

From (7) we have $\gamma_0 = \text{identity}$. By (3) and (9), $\gamma_{\ell-1} = \text{identity}$ implies $\gamma_\ell = \text{identity}$ for $1 \leq \ell \leq L-1$. Hence $\gamma_\ell = \text{identity}$ for all $\ell \leq L-1$. Since $\gamma_L = \text{identity}$ by (7), we know that all the $\gamma_\ell = \text{identity}$. Thus, (8) becomes

$$(10) \quad \omega_{jk}^\ell = \varepsilon_j^\ell \omega_{jk}^\ell \varepsilon_k^{\ell-1}, \quad \text{for } 1 \leq \ell \leq L-1.$$

From (7) we have $\varepsilon_k^0 = 1$, since $D_0 = 1$. By (2) and (10), $\varepsilon_k^{\ell-1} = 1$ (all k) implies $\varepsilon_j^\ell = 1$ (all j) for $1 \leq \ell \leq L-1$. Hence $\varepsilon_j^\ell = 1$ whenever $\ell \leq L-1$. Since also (7) gives $\varepsilon_j^L = 1$ because $D_L = 1$, we know that $\varepsilon_j^\ell = 1$ for all $\ell, j, 0 \leq \ell \leq L, 1 \leq j \leq D_\ell$. Since $\varepsilon_j^\ell = 1$ and $\gamma_\ell = \text{identity}$, (5) and (6) show that the nets $\mathcal{N}, \tilde{\mathcal{N}}$ are identical.

II. Approximate Arithmetic Progressions.

A. Preliminaries.

(1) **Definition.** Let $E, E_1, \dots, E_n \subset \mathbb{C}$ be given. We say that E is the approximate union of E_1, \dots, E_n if the following conditions hold.

(2) $E \subset E_1 \cup \dots \cup E_n$, and

(3) Any point belonging to exactly one of the E_1, \dots, E_n belongs to E .

(4) **Definition.** For $\omega, \beta \in \mathbb{C}$ with $\omega \neq 0$, define $\Pi(\omega, \beta) = \{\omega k + \beta : k \in \mathbb{Z}\}$. We say that $E \subset \mathbb{C}$ approximates $\Pi(\omega, \beta)$ if for every $\varepsilon > 0$ the following conditions hold.

(5) All but finitely many points of E lie within distance ε of some point in $\Pi(\omega, \beta)$, and

(6) All but finitely many points of $\Pi(\omega, \beta)$ lie within distance ε of some point in E .

Note that

(7) $\Pi(\omega, \beta) = \Pi(\omega', \beta')$ if and only if $\omega' = \pm\omega$ and $\beta' = \beta + \omega m$ for some $m \in \mathbb{Z}$.

(8) **Definition.** Let H be a set of integers. We define the upper and lower densities $\Delta^*(H)$, $\Delta_*(H)$ by setting

$$(9) \quad \Delta^*(H) = \limsup_{\substack{N \rightarrow \infty \\ M \rightarrow -\infty}} \frac{\text{Number of integers in } [M, N] \cap H}{N - M}$$

and

$$(10) \quad \Delta_*(H) = \liminf_{\substack{N \rightarrow \infty \\ M \rightarrow -\infty}} \frac{\text{Number of integers in } [M, N] \cap H}{N - M}.$$

If $\Delta^*(H) = \Delta_*(H)$, then we write $\Delta(H)$ for their common value, and we say that H has density $\Delta(H)$.

We will need the following special case of H. Weyl's Theorem on the equidistribution mod 1 of arithmetic progressions (See [W]).

(11) **Theorem.** Suppose $\theta \in \mathbb{R}$ is irrational and $0 < \varepsilon < 1/2$. Let $\beta \in \mathbb{R}$. Then $\Delta(\{k \in \mathbb{Z}: |\theta k + \beta - m| < \varepsilon \text{ for some } m \in \mathbb{Z}\}) = 2\varepsilon$.

(12) **Corollary.** Let $\omega, \omega', \beta, \beta'$ be complex numbers, with $\omega, \omega' \neq 0$. Assume that ω'/ω is real and irrational. Then we can make the density

$$\Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, \Pi(\omega', \beta')\} < \varepsilon\})$$

arbitrarily small, by taking $\varepsilon > 0$ small enough.

B. The Deconstruction Lemma.

Suppose $E \subset \mathbb{C}$ is the approximate union of sets E_1, \dots, E_D ; and suppose that each E_j approximates an arithmetic progression $\Pi(\omega_j, \beta_j)$. We want to know that the progressions $\Pi(\omega_j, \beta_j)$ are uniquely determined by E . Also, for each j_0 , we want to pick out infinitely many points $(x_\nu)_{\nu \geq 1}$ that belong to E_{j_0} but not to any E_j , $j \neq j_0$. The following result provides this information.

Deconstruction Lemma. Let $E, E_1, \dots, E_D, \tilde{E}_1, \dots, \tilde{E}_{\tilde{D}}$ be subsets of \mathbb{C} , and let $\Pi(\omega_1, \beta_1), \dots, \Pi(\omega_D, \beta_D), \Pi(\tilde{\omega}_1, \tilde{\beta}_1), \dots, \Pi(\tilde{\omega}_{\tilde{D}}, \tilde{\beta}_{\tilde{D}})$ be arithmetic progressions. Assume the following conditions.

- (1) E is the approximate union of E_1, \dots, E_D .
- (2) Each E_j approximates the arithmetic progression $\Pi(\omega_j, \beta_j)$.
- (3) For $j \neq j'$ and p, q integers with $1 \leq q \leq 100 D^2$, we have $|\omega_j/\omega_{j'}| \neq p/q$.
- (4) E is the approximate union of $\tilde{E}_1, \dots, \tilde{E}_{\tilde{D}}$.
- (5) Each \tilde{E}_j approximates the arithmetic progression $\Pi(\tilde{\omega}_j, \tilde{\beta}_j)$.
- (6) For $j \neq j'$, and p, q integers with $1 \leq q \leq 100 \tilde{D}^2$, we have $|\tilde{\omega}_j/\tilde{\omega}_{j'}| \neq p/q$.

Then $D = \tilde{D}$, and there is a permutation

$$\gamma: \{1, \dots, D\} \rightarrow \{1, \dots, D\}$$

with the following properties:

- (7) $\Pi(\omega_j, \beta_j) = \Pi(\tilde{\omega}_{\gamma j}, \tilde{\beta}_{\gamma j})$ for $1 \leq j \leq D$.
- (8) Given j_0 , $1 \leq j_0 \leq D$, there is a sequence $(x_\nu)_{\nu \geq 1}$ in \mathbb{C} such that
- (9) $|x_\nu| \rightarrow \infty$ as $\nu \rightarrow \infty$,
- (10) Each x_ν belongs to E_{j_0} but not to E_j for $j \neq j_0$,
- (11) Each x_ν belongs to $\tilde{E}_{\gamma j_0}$, but not to \tilde{E}_j for $j \neq \gamma j_0$.

C. Preparation for the proof of the Deconstruction Lemma.

We begin with a definition. We say that $\Pi(\omega, \beta)$ fits into $E \subset \mathbb{C}$ if for any $\varepsilon > 0$ we have

$$(1) \quad \Delta_*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, E\} < \varepsilon\}) \geq \frac{9}{10}.$$

Note that (1) is phrased in terms of ω and β , but in fact depends only on $\Pi(\omega, \beta)$. (See (A.7)).

(2) **Lemma.** *If $E, E_j, \Pi(\omega_j, \beta_j)$ are as in the Deconstruction Lemma, then $\Pi(\omega_j, \beta_j)$ fits into E .*

PROOF. Fix $j' \neq j$. For small enough $\varepsilon > 0$ we will estimate

$$(3) \quad \Delta_\varepsilon(j, j') = \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega_j k + \beta_j, \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon\}).$$

Let $\ell_j, \ell_{j'}$ denote the lines $\omega_j \mathbb{R} + \beta_j, \omega_{j'} \mathbb{R} + \beta_{j'}$ in \mathbb{C} . We distinguish several cases.

CASE 1: $\ell_j \neq \ell_{j'}$. Then $\text{dist}\{x, \ell_{j'}\}$ is bounded below by a positive constant as $x \in \ell_j$ tends to infinity. Hence, $\Delta_\varepsilon(j, j') = 0$ for $\varepsilon > 0$ small enough.

CASE 2: $\ell_j = \ell_{j'}$ and $\omega_j/\omega_{j'}$ is irrational. Then by (A.12), we can make $\Delta_\varepsilon(j, j')$ arbitrarily small by taking $\varepsilon > 0$ small enough.

CASE 3: $\ell_j = \ell_{j'}$ and $\omega_j/\omega_{j'} = p/q$ in lowest terms, with $p, q \in \mathbb{Z}$ and $q > 0$. In view of (B.3), we have $q > 100D^2$.

Assume we are given distinct integers $k_1, k_2 \in \mathbb{Z}$ with

$$(4) \quad \text{dist}\{\omega_j k_1 + \beta_j, \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon,$$

and

$$(5) \quad \text{dist}\{\omega_j k_2 + \beta_j, \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon.$$

Thus, for integers m_1 and m_2 , we have

$$(6) \quad |(\omega_j k_1 + \beta_j) - (\omega_{j'} m_1 + \beta_{j'})| < \varepsilon,$$

and

$$(7) \quad |(\omega_j k_2 + \beta_j) - (\omega_{j'} m_2 + \beta_{j'})| < \varepsilon.$$

Subtracting (6) from (7), and recalling that $\omega_j/\omega_{j'} = p/q$, we get

$$(8) \quad \left| \frac{p}{q} - \frac{m_2 - m_1}{k_2 - k_1} \right| < \frac{2\varepsilon}{|\omega_{j'}| |k_2 - k_1|}.$$

If $p/q \neq (m_2 - m_1)/(k_2 - k_1)$, then

$$\left| \frac{p}{q} - \frac{m_2 - m_1}{k_2 - k_1} \right| \geq \frac{1}{q |k_2 - k_1|},$$

which contradicts (8) provided we take $\varepsilon < |\omega_{j'}|/(2q)$. Hence, $p/q = (m_2 - m_1)/(k_2 - k_1)$. Since p/q is in lowest terms, it follows that $k_2 - k_1$

is a multiple of q . So we have shown that (4), (5) imply $k_2 \equiv k_1 \pmod q$. It follows at once that $\Delta_\varepsilon(j, j') \leq 1/q$, for $\varepsilon > 0$ small enough.

Since $q > 100 D^2$ in *CASE 3*, our analysis of the above cases gives $\Delta_\varepsilon(j, j') < 1/(100 D^2)$ for $j' \neq j$, if $\varepsilon > 0$ is small enough.

Summing over all $j' \neq j$ and recalling (3), we get

$$(9) \quad \begin{aligned} \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega_j k + \beta_j, \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon \text{ for some } j' \neq j\}) \\ < \frac{1}{100 D}. \end{aligned}$$

Now suppose $k \in \mathbb{Z}$ satisfies

$$(10) \quad \text{dist}\{\omega_j k + \beta_j, \Pi(\omega_{j'}, \beta_{j'})\} > \varepsilon, \quad \text{for all } j' \neq j.$$

Since E_j approximates $\Pi(\omega_j, \beta_j)$, we can find $x_k^j \in E_j$ for all but finitely many k so that

$$(11) \quad |x_k^j - (\omega_j k + \beta_j)| \leq \frac{\varepsilon}{10}.$$

In particular, $|x_k^j| \rightarrow \infty$ as $k \rightarrow \infty$.

On the other hand, since $E_{j'}$ approximates $\Pi(\omega_{j'}, \beta_{j'})$, we have $E_{j'} \subset F_{j'} \cup \{z \in \mathbb{C}: \text{dist}\{z, \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon/10\}$ with $F_{j'}$ finite. Hence, (10) implies

$$(12) \quad \text{dist}\{\omega_j k + \beta_j, E_{j'}\} \geq \frac{9}{10} \varepsilon, \quad \text{for all } j' \neq j,$$

for all but finitely many k . Comparing (11) and (12), we see that $x_k^j \notin E_{j'}$, $j' \neq j$. Thus, all but finitely many k satisfying (10) have the property $x_k^j \in E_j \setminus \cup_{j' \neq j} E_{j'}$. Since E is the approximate union of E_1, \dots, E_D , it follows that $x_k^j \in E$. Hence, (11) implies

$$(13) \quad \text{dist}\{\omega_j k + \beta_j, E\} < \varepsilon.$$

So (13) holds for all but finitely many of the k that satisfy (10). Therefore, by (9), we have $\Delta_*(\{k \in \mathbb{Z}: \text{dist}\{\omega_j k + \beta_j, E\} < \varepsilon\}) \geq 1 - 1/(100 D)$, which shows that $\Pi(\omega_j, \beta_j)$ fits into E .

(14) **Lemma.** *Let $\Pi(\omega, \beta)$ and $\Pi(\omega', \beta')$ be arithmetic progressions, with $\Pi(\omega, \beta) \not\subset \Pi(\omega', \beta')$. Fix $D > 0$, and define*

$$(15) \quad q(\omega, \omega') = \begin{cases} q, & \text{if } \frac{\omega}{\omega'} = \frac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z}, q \geq 2. \\ 100 D^2, & \text{if } \frac{\omega}{\omega'} \text{ is irrational, an integer, or non-real.} \end{cases}$$

Then

$$(16) \quad \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, \Pi(\omega', \beta')\} < \varepsilon\}) \leq \frac{1}{q(\omega, \omega')},$$

for $\varepsilon > 0$ small enough.

PROOF. As in the previous lemma, we set $\ell = \omega\mathbb{R} + \beta$, $\ell' = \omega'\mathbb{R} + \beta'$, and we distinguish several cases.

CASE 1: $\ell \neq \ell'$. As in the proof of the previous lemma, the left-hand side of (16) is equal to zero if ε is small enough.

CASE 2: $\ell = \ell'$ and ω/ω' irrational. As in the proof of the previous lemma, we can make the left-hand side of (16) arbitrarily small by taking ε small enough.

CASE 3: $\ell = \ell'$ and $\omega/\omega' = p/q$ in lowest terms, with $p, q \in \mathbb{Z}$ and $q \geq 2$. As in the proof of the previous lemma,

$$\text{dist}\{\omega k_1 + \beta, \Pi(\omega', \beta')\} < \varepsilon, \quad \text{dist}\{\omega k_2 + \beta, \Pi(\omega', \beta')\} < \varepsilon,$$

imply $k_2 = k_1 \bmod q$, so that (16) is obvious.

CASE 4: $\ell = \ell'$ and $\omega/\omega' = p$ for some integer p . Then $\beta' - \beta$ is not a multiple of ω' , since $\Pi(\omega, \beta) \not\subset \Pi(\omega', \beta')$.

Take $\varepsilon < \min_{k \in \mathbb{Z}} |\beta' - \beta - k\omega'|$. Then for all $k, m \in \mathbb{Z}$ we have

$$|(\omega k + \beta) - (\omega' m + \beta')| = |\beta - \beta' - (m - pk)\omega'| > \varepsilon,$$

so that $\text{dist}\{\omega k + \beta, \Pi(\omega', \beta')\} > \varepsilon$, and the left-hand side of (16) equals zero.

(17) **Lemma.** Assume the hypotheses of the Deconstruction Lemma, and suppose

$$(18) \quad |\omega_1| < |\omega_2| < \cdots < |\omega_D|.$$

Fix an integer s , $1 \leq s \leq D$. Let $\Pi(\omega, \beta)$ be an arithmetic progression with the following properties:

$$(19) \quad \Pi(\omega, \beta) \text{ fits into } E,$$

$$(20) \quad \omega \neq p\omega_j/q \quad \text{for } p, q \in \mathbb{Z}, 1 \leq j \leq s, \text{ if } j < s.$$

Then either $\Pi(\omega, \beta) = \Pi(\omega_s, \beta_s)$, or else $|\omega| > |\omega_s|$.

PROOF. Assume the lemma is false. Thus,

$$(21) \quad \Pi(\omega, \beta) \neq \Pi(\omega_s, \beta_s),$$

$$(22) \quad |\omega| \leq |\omega_s|.$$

Suppose for the moment that $\Pi(\omega, \beta) \subset \Pi(\omega_j, \beta_j)$ for some $j, 1 \leq j \leq D$. Then

$$(23) \quad \omega = p\omega_j, \quad \text{for an integer } p \neq 0.$$

If $j < s$, then (23) contradicts (20). Hence, $j \geq s$ and (18), (23) yield

$$(24) \quad |\omega| = |p| |\omega_j| \geq |\omega_j| \geq |\omega_s|.$$

Moreover, at least one of the inequalities in (24) will be strict, unless $p = \pm 1$ and $j = s$. Therefore, (22) yields $p = \pm 1$ and $j = s$. Since $\Pi(\omega, \beta) \subset \Pi(\omega_j, \beta_j) = \Pi(\omega_s, \beta_s)$ and $\omega = p\omega_j = \pm\omega_s$, it follows that $\Pi(\omega, \beta) = \Pi(\omega_s, \beta_s)$, contradicting (21). This contradiction proves that

$$(25) \quad \Pi(\omega, \beta) \not\subset \Pi(\omega_j, \beta_j), \quad \text{for } 1 \leq j \leq D.$$

Next, we apply (19) and (1) to conclude that

$$(26) \quad \Delta_*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, E\} < \varepsilon\}) \geq \frac{9}{10}, \quad \text{for any } \varepsilon > 0.$$

Since E is the approximate union of E_1, \dots, E_D , we have $E \subset E_1 \cup \dots \cup E_D$, so that $\text{dist}\{\omega k + \beta, E\} < \varepsilon$ implies $\text{dist}\{\omega k + \beta, E_j\} < \varepsilon$ for some j . Hence, (26) yields

$$(27) \quad \sum_{j=1}^D \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, E_j\} < \varepsilon\}) \geq \frac{9}{10}, \quad \text{for any } \varepsilon > 0.$$

Moreover, each E_j approximates $\Pi(\omega_j, \beta_j)$. Hence, given $\varepsilon > 0$, we have

$$E_j \subset F_j \cup \{z \in \mathbb{C}: \text{dist}\{z, \Pi(\omega_j, \beta_j)\} < \varepsilon\}$$

with F_j finite. So, for large integers k , $\text{dist}\{\omega k + \beta, E_j\} < \varepsilon$ implies $\text{dist}\{\omega k + \beta, \Pi(\omega_j, \beta_j)\} < 2\varepsilon$. Thus, (27) implies

$$(28) \quad \sum_{j=1}^D \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, \Pi(\omega_j, \beta_j)\} < 2\varepsilon\}) \geq \frac{9}{10},$$

for any $\varepsilon > 0$. Taking ε small enough, and using (25) to apply Lemma (14), we obtain

$$(29) \quad \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, \Pi(\omega_j, \beta_j)\} < 2\varepsilon\}) \leq \frac{1}{q(\omega, \omega_j)},$$

where

$$(30) \quad q(\omega, \omega_j) = \begin{cases} q, & \text{if } \frac{\omega}{\omega_j} = \frac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z}, q \geq 2, \\ 100D^2, & \text{otherwise.} \end{cases}$$

From (28), (29), we obtain

$$(31) \quad \sum_{j=1}^D \frac{1}{q(\omega, \omega_j)} \geq \frac{9}{10}.$$

On the other hand, we can prove an upper bound for the left-hand side of (31). Immediately from (20) and (30), we have $q(\omega, \omega_j) \geq 10s$ for $j < s$, so that

$$(32) \quad \sum_{1 \leq j < s} \frac{1}{q(\omega, \omega_j)} \leq \frac{1}{10}.$$

To control $q(\omega, \omega_j)$ for $j \geq s$, we prove

$$(33) \quad q(\omega, \omega_j) \geq 2,$$

and

$$(34) \quad q(\omega, \omega_j) \leq 10D, \quad \text{for at most one } j \geq s.$$

In fact, (33) is immediate from (30). If (34) were false, then we would have

$$(35) \quad \frac{\omega}{\omega_j} = \frac{p}{q}, \quad \frac{\omega}{\omega_{j'}} = \frac{p'}{q'},$$

with $p, q, p', q' \in \mathbb{Z}$, $j, j' \geq s$, $j \neq j'$, $1 \leq q, q' \leq 10D$. From (18), (22), (35) we have $|\omega| \leq |\omega_s| \leq |\omega_j|$ so that $|p| \leq |q|$; and similarly, $|p'| \leq |q'|$. In particular,

$$(36) \quad 0 < |p|, |p'|, |q|, |q'| \leq 10D,$$

by another application of (35). A final application of (35) gives

$$(37) \quad \frac{\omega_j}{\omega_{j'}} = \frac{q p'}{p q'} \equiv \frac{P}{Q}.$$

Since $1 \leq |Q| \leq 100 D^2$ by (36), equation (37) contradicts hypothesis (B.3) of the Deconstruction Lemma. This contradiction proves (34). Immediately from (33), (34), we obtain

$$(38) \quad \sum_{s \leq j \leq D} \frac{1}{q(\omega, \omega_j)} \leq \frac{1}{2} + \frac{D-s}{10D} \leq \frac{1}{2} + \frac{1}{10}.$$

Together, (32) and (38) yield

$$\sum_{j=1}^D \frac{1}{q(\omega, \omega_j)} \leq \frac{7}{10},$$

contradicting (31). Thus, assuming our lemma to be false, we arrived at a contradiction.

(39) **Lemma.** *Assume the hypotheses of the Deconstruction Lemma. Then there is no arithmetic progression $\Pi(\omega, \beta)$ with the following properties:*

(40) $\Pi(\omega, \beta)$ fits into E ,

(41) $\omega \neq \frac{p}{q} \omega_j$ for $p, q \in \mathbb{Z}$, $1 \leq q \leq 10D$, whenever $1 \leq j \leq D$.

PROOF. Assume $\Pi(\omega, \beta)$ satisfies (40), (41). By (40) and (1), we have $\Delta_*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, E\} < \varepsilon\}) \geq 9/10$, for any $\varepsilon > 0$. As in the proof of the previous lemma, this implies that

$$(42) \quad \sum_{j=1}^D \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega k + \beta, \Pi(\omega_j, \beta_j)\} < 2\varepsilon\}) \geq \frac{9}{10},$$

for any $\varepsilon > 0$. Moreover, (41) shows that $\Pi(\omega, \beta) \not\subset \Pi(\omega_j, \beta_j)$ for any j , $1 \leq j \leq D$, so that Lemma (14) applies. We obtain from (14) and (42) the estimate

$$(43) \quad \sum_{j=1}^D \frac{1}{q(\omega, \omega_j)} \geq \frac{9}{10},$$

with

$$(44) \quad q(\omega, \omega_j) = \begin{cases} q, & \text{if } \frac{\omega}{\omega_j} = \frac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z}, q \geq 2, \\ 100D^2, & \text{otherwise.} \end{cases}$$

On the other hand, (41) and (44) imply $q(\omega, \omega_j) \geq 10D$ for all j , so that

$$\sum_{j=1}^D \frac{1}{q(\omega, \omega_j)} \leq \frac{1}{10},$$

contradicting (43).

D. Proving the Deconstruction Lemma.

Let $E, E_1, \dots, E_D, \tilde{E}_1, \dots, \tilde{E}_{\tilde{D}}, \Pi(\omega_j, \beta_j), \Pi(\tilde{\omega}_j, \tilde{\beta}_j)$ be as in the Deconstruction Lemma. Hypothesis (B.3) shows that the $|\omega_j|$ are all distinct. Without loss of generality, we may therefore permute the E_j and $\Pi(\omega_j, \beta_j)$ to reduce matters to the case

$$(1) \quad |\omega_1| < \dots < |\omega_D|.$$

Similarly, we may assume

$$(2) \quad |\tilde{\omega}_1| < \dots < |\tilde{\omega}_{\tilde{D}}|.$$

Also, we may assume

$$(3) \quad D \leq \tilde{D}.$$

For the rest of the proof, we will assume (1), (2), (3). We will prove that

$$(4) \quad \Pi(\omega_j, \beta_j) = \Pi(\tilde{\omega}_j, \tilde{\beta}_j), \quad \text{for } 1 \leq j \leq D.$$

To see this, fix $s, 1 \leq s \leq D$, and suppose

$$(5) \quad \Pi(\omega_j, \beta_j) = \Pi(\tilde{\omega}_j, \tilde{\beta}_j), \quad \text{for } 1 \leq j < s.$$

(This assumption is vacuous for $s = 1$). We will see that (5) implies

$$(6) \quad \Pi(\omega_s, \beta_s) = \Pi(\tilde{\omega}_s, \tilde{\beta}_s).$$

In fact, (5) and (A.7) show that

$$(7) \quad \omega_j = \pm \tilde{\omega}_j \quad \text{for } 1 \leq j < s.$$

The analogue of Lemma C.2 for the \tilde{E}_j and $\Pi(\tilde{\omega}_j, \tilde{\beta}_j)$ shows that

$$(8) \quad \Pi(\tilde{\omega}_s, \tilde{\beta}_s) \quad \text{fits into } E.$$

Since $s \leq \min\{D, \tilde{D}\}$, equation (7) and hypothesis (B.6) show that

$$(9) \quad \tilde{\omega}_s \neq \frac{p}{q} \omega_j, \quad \text{for } 1 \leq j < s, p, q \in \mathbb{Z}, 1 \leq q \leq 10s.$$

Conditions (8), (9) are the hypotheses of Lemma (C.17), which tells us that

$$(10) \quad \text{either } \Pi(\tilde{\omega}_s, \tilde{\beta}_s) = \Pi(\omega_s, \beta_s), \quad \text{or else } |\tilde{\omega}_s| > |\omega_s|.$$

The same argument works with the rôles of the $\Pi(\omega_j, \beta_j)$ and $\Pi(\tilde{\omega}_j, \tilde{\beta}_j)$ interchanged, so we have also

$$(11) \quad \text{either } \Pi(\omega_s, \beta_s) = \Pi(\tilde{\omega}_s, \tilde{\beta}_s), \quad \text{or else } |\omega_s| > |\tilde{\omega}_s|.$$

Since at least one of the inequalities $|\omega_s| > |\tilde{\omega}_s|$, $|\tilde{\omega}_s| > |\omega_s|$ must be false, (10) and (11) imply (6). Thus, (5) implies (6), completing the proof of (4).

Next we show that

$$(12) \quad D = \tilde{D}.$$

If (12) were false, then by (3) we would have

$$(13) \quad \tilde{D} \geq D + 1.$$

By the analogue of Lemma C.2 for the \tilde{E}_j and $\Pi(\tilde{\omega}_j, \tilde{\beta}_j)$, we know that

$$(14) \quad \Pi(\tilde{\omega}_{D+1}, \tilde{\beta}_{D+1}) \quad \text{fits into } E.$$

Also, by (4) and (A.7) we have

$$(15) \quad \tilde{\omega}_j = \pm \omega_j, \quad \text{for } 1 \leq j \leq D.$$

Hence, hypothesis (B.6) shows that

$$(16) \quad \tilde{\omega}_{D+1} \neq \frac{p}{q} \omega_j, \quad \text{for } 1 \leq j \leq D, \ p, q \in \mathbb{Z}, \ 1 \leq q \leq 10D.$$

Together, conditions (14) and (16) contradict Lemma C.39. This contradiction completes the proof of (12).

Next, fix $j_0, 1 \leq j_0 \leq D$. We will construct a sequence $(x_\nu)_{\nu \geq 1}$ such that

$$(17) \quad |x_\nu| \rightarrow \infty,$$

$$(18) \quad x_\nu \in E_{j_0} \setminus \bigcup_{j \neq j_0} E_j,$$

$$(19) \quad x_\nu \in \tilde{E}_{j_0} \setminus \bigcup_{j \neq j_0} \tilde{E}_j.$$

To construct (x_ν) with these properties, we first note that

$$(20) \quad \Pi(\omega_{j_0}, \beta_{j_0}) \not\subset \Pi(\omega_j, \beta_j), \quad \text{for } j \neq j_0,$$

since ω_{j_0} is not an integer multiple of ω_j . Hence for $\varepsilon > 0$ small enough, Lemma C.14 shows that

$$(21) \quad \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega_{j_0} k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon\}) \leq \frac{1}{q(\omega_{j_0}, \omega_j)},$$

where

$$q(\omega_{j_0}, \omega_j) = \begin{cases} q, & \text{if } \frac{\omega_{j_0}}{\omega_j} = \frac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z}, \ q \geq 2, \\ 100D^2, & \text{otherwise.} \end{cases}$$

Hypothesis (B.3) shows that $q(\omega_{j_0}, \omega_j) \geq 100D^2$ for $j \neq j_0$, so that (21) implies

$$\Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega_{j_0} k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon\}) \leq \frac{1}{100D^2}, \quad \text{for } j \neq j_0.$$

Summing over j , we obtain

$$(22) \quad \begin{aligned} \Delta^*(\{k \in \mathbb{Z}: \text{dist}\{\omega_{j_0} k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon, \text{ for some } j \neq j_0\}) \\ \leq \frac{1}{100D}. \end{aligned}$$

Let $K = \{k \in \mathbb{Z}: \text{dist}\{\omega_{j_0}k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} \geq \varepsilon \text{ for all } j \neq j_0\}$. Recall that E_j approximates $\Pi(\omega_j, \beta_j)$, so that

$$E_j \subset F_j \cup \{z \in \mathbb{C}: \text{dist}\{z, \Pi(\omega_j, \beta_j)\} < \varepsilon/3\}$$

with F_j finite. Therefore, for all but finitely many $k \in \mathbb{Z}$ we have

$$\text{dist}\{\omega_{j_0}k + \beta_{j_0}, E_j\} < \frac{2\varepsilon}{3} \quad \text{implies} \quad \text{dist}\{\omega_{j_0}k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon.$$

It follows that

$$(23) \quad \text{dist}\{\omega_{j_0}k + \beta_{j_0}, E_j\} \geq \frac{2}{3}\varepsilon, \quad \text{for all } j \neq j_0,$$

for all but finitely many $k \in K$.

Similarly, \tilde{E}_j approximates $\Pi(\tilde{\omega}_j, \tilde{\beta}_j) = \Pi(\omega_j, \beta_j)$ by (4) and (12), so the proof of (23) yields also

$$(24) \quad \text{dist}\{\omega_{j_0}k + \beta_{j_0}, \tilde{E}_j\} \geq \frac{2}{3}\varepsilon, \quad \text{for all } j \neq j_0,$$

for all but finitely many $k \in K$.

On the other hand, E_{j_0} approximates $\Pi(\omega_{j_0}, \beta_{j_0})$. Hence, for all but finitely many $k \in K$ we can find

$$(25) \quad \hat{x}_k \in E_{j_0} \quad \text{satisfying}$$

$$(26) \quad |\hat{x}_k - (\omega_{j_0}k + \beta_{j_0})| < \frac{\varepsilon}{3}.$$

Comparing (26) with (23), (24), we conclude that all but finitely many $k \in K$ satisfy

$$(27) \quad \hat{x}_k \notin E_j \quad \text{for } j \neq j_0, \quad \text{and}$$

$$(28) \quad \hat{x}_k \notin \tilde{E}_j \quad \text{for } j \neq j_0.$$

Since E is the approximate union of E_1, \dots, E_D , we know from (25) and (27) that $\hat{x}_k \in E$. This in turn gives $\hat{x}_k \in \tilde{E}_1 \cup \dots \cup \tilde{E}_D$, since E is the approximate union of the \tilde{E}_j . In view of (28), we obtain

$$(29) \quad \hat{x}_k \in \tilde{E}_{j_0},$$

for any k satisfying (25)-(28).

Thus, for all but finitely many $k \in K$, the following hold:

$$(30) \quad |\hat{x}_k - (\omega_{j_0} k + \beta_{j_0})| < \varepsilon,$$

$$(31) \quad \hat{x}_k \in E_{j_0} \setminus \bigcup_{j \neq j_0} E_j,$$

$$(32) \quad \hat{x}_k \in \tilde{E}_{j_0} \setminus \bigcup_{j \neq j_0} \tilde{E}_j.$$

Finally, let $(x_\nu)_{\nu \geq 1}$ be an enumeration of the \hat{x}_k for $k \in K$ that satisfy (30), (31), (32). Estimate (22) and the definition of K show that there are indeed infinitely many such \hat{x}_k , so we get an infinite sequence. Estimate (30) shows that $|x_\nu| \rightarrow \infty$ as $\nu \rightarrow \infty$. Hence, (17), (18), (19) follow at once from (30), (31), (32). We have proven the conclusions of the Deconstruction Lemma, with $\gamma = \text{identity}$.

We conclude this section with a simple special case of the Deconstruction Lemma.

(33) **Corollary.** *Suppose E_j approximates $\Pi(\omega_j, \beta_j)$ for $1 \leq j \leq D$. Assume that $|\omega_j/\omega_{j'}| \neq p/q$ for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \leq q \leq 100D^2$. Then for each j_0 , $1 \leq j_0 \leq D$, we can find a sequence $(x_\nu)_{\nu \geq 1}$ of complex numbers, such that*

$$(34) \quad |x_\nu| \rightarrow \infty \text{ as } \nu \rightarrow \infty, \text{ and}$$

$$(35) \quad x_\nu \in E_{j_0} \setminus \bigcup_{j \neq j_0} E_j \text{ for each } \nu.$$

PROOF. Set $\tilde{E}_j = E_j$, $\Pi(\tilde{\omega}_j, \tilde{\beta}_j) = \Pi(\omega_j, \beta_j)$, $E = E_1 \cup \dots \cup E_D$. Then the Deconstruction Lemma applies, and it gives a sequence $(x_\nu)_{\nu \geq 1}$ satisfying (34) and (35).

III. Analytic Continuation of Neural Nets.

A. Preliminaries.

Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{j_k}^\ell), (\theta_j^\ell)]$ be a neural net. We will show that the functions $x_j^\ell(t, \mathcal{N})$, defined initially for t real, continue analytically to an open subset of \mathbb{C} with countable complement. We will analyze the largest domain Ω to which we can analytically continue the output $x_1^L(t, \mathcal{N})$. The point-set topology of Ω leads us to define a

hierarchy of singular sets $\text{Sing}(\ell, \mathcal{N})$ in the complex plane. The sets $\text{Sing}(\ell, \mathcal{N})$ are defined entirely in terms of the output function $t \mapsto x_1^t(t, \mathcal{N})$ ($t \in \mathbb{R}$), yet they carry a lot of information on the architecture, weights and thresholds of \mathcal{N} .

We begin our discussion with a simple, general result on analytic functions defined in the complement of a countable set.

(1) **Lemma.** *Let $f(t)$ be a function on \mathbb{R} , and suppose that f continues analytically to an open set $\Omega \subset \mathbb{C}$, with $\mathbb{R} \subset \Omega$ and $\mathbb{C} \setminus \Omega$ countable. Then there is one and only one open set $\Omega_* \subset \mathbb{C}$ with the following properties:*

- (2) $\mathbb{R} \subset \Omega_*$,
- (3) $\mathbb{C} \setminus \Omega_*$ is countable,
- (4) Let $\Omega' \subset \mathbb{C}$ be any connected open set that meets \mathbb{R} . Then f continues analytically into Ω' if and only if $\Omega' \subset \Omega_*$.

We call Ω_* the natural domain of f .

PROOF. We start with the following remark.

- (5) Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open sets, with Ω_1 connected and $\mathbb{C} \setminus \Omega_2$ countable. Let F_1, F_2 be analytic on Ω_1, Ω_2 respectively, and assume $F_1 = F_2$ to infinite order at some point of $\Omega_1 \cap \Omega_2$. Then $F_1 = F_2$ on all of $\Omega_1 \cap \Omega_2$.

Indeed, (5) is immediate from the fact that $\Omega_1 \cap \Omega_2$ is the complement of a countable set in Ω_1 , and thus $\Omega_1 \cap \Omega_2$ is connected.

Now let W be the collection of all open sets $\Omega' \subset \mathbb{C}$ such that $\mathbb{R} \subset \Omega'$, $\mathbb{C} \setminus \Omega'$ is countable, and f continues analytically to Ω' . If $\Omega', \Omega'' \in W$, and if F, G denote the analytic continuations of f to Ω', Ω'' respectively, then $F = G$ in $\Omega' \cap \Omega''$ by (5). It follows that f continues analytically to $\Omega_* = \cup_{\Omega' \in W} \Omega'$. Since $\Omega \in W$ by hypothesis, properties (2) and (3) are obvious, and we know that

- (6) f continues analytically to any open set $\Omega' \subset \Omega_*$.

Next, suppose $\Omega' \subset \mathbb{C}$ is open and connected, and meets \mathbb{R} ; and assume f continues analytically to an analytic function G on Ω' . Let F denote the analytic continuation of f to Ω_* . Then $F = G$ on $\Omega' \cap \Omega_*$ by (5), so that f continues analytically to $\Omega' \cup \Omega_*$. We have shown that

- (7) If $\Omega' \subset \mathbb{C}$ is open, connected and meets \mathbb{R} , and if f continues analytically to Ω' , then $\Omega' \subset \Omega_*$.

Assertions (6) and (7) complete the proof of (4). It remains only to prove the uniqueness of Ω_* . Thus, suppose Ω_*^1 and Ω_*^2 both have properties (2), (3), (4). Then Ω_*^1 is an open, connected set that meet \mathbb{R} , and f continues analytically to Ω_*^1 . Since Ω_*^2 has property (4), it follows that $\Omega_*^1 \subset \Omega_*^2$. Similarly, $\Omega_*^2 \subset \Omega_*^1$, which proves that Ω_* is unique.

(8) **Definition.** Let f be a function on \mathbb{R} . If f continues analytically to an open set with countable complement, then we define the sets $\text{Sing}(\ell, f) \subset \mathbb{C}$ for $\ell \geq 0$ by the following induction:

(9) $\text{Sing}(0, f)$ is the complement of the natural domain of f ,

(10) $\text{Sing}(\ell + 1, f)$ is the set of accumulation points of $\text{Sing}(\ell, f)$.

We will take f to be the output of a neural net. The next two lemmas help us to show that f continues analytically to an open set with countable complement, and to understand the sets $\text{Sing}(\ell, f)$.

(11) **Lemma.** Let F be analytic on a connected, open set $\Omega \subset \mathbb{C}$. Let $\Pi(\omega, \beta)$ be an arithmetic progression. Suppose that F is either non-constant, or else identically equal to a constant not belonging to $\Pi(\omega, \beta)$. Then the set $E = \{t \in \Omega: F(t) \in \Pi(\omega, \beta)\}$ has no accumulation points in Ω . In particular, E is countable.

PROOF. Suppose $t_\nu \rightarrow t_*$ as $\nu \rightarrow \infty$, with $t_\nu \in E$ and $t_* \in \Omega$. Then $F(t_\nu) \rightarrow F(t_*)$ and $F(t_\nu) \in \Pi(\omega, \beta)$. It follows that $F(t_\nu)$ is eventually constant: $F(t_\nu) = b$ for all $\nu \geq \nu_0$, with $b \in \Pi(\omega, \beta)$. Since $\{t_\nu\}$ accumulate at t_* and Ω is connected, it follows in turn that $F(t) = b$ for all $t \in \Omega$, contradicting our hypothesis on F . Thus, E has no accumulation points in Ω . This implies that $E_N = \{t \in E: |t| \leq N \text{ and } \text{dist}\{t, \mathbb{C} \setminus \Omega\} \geq 1/N\}$ is a bounded set without accumulation points. Hence E_N is finite, so that $E = \cup_{N \geq 1} E_N$ is countable.

(12) **Lemma.** Let U be a disc centered at $z_0 \in \mathbb{C}$. Suppose Φ is meromorphic and Ψ analytic in a neighborhood of \bar{U} . Assume Φ has a single pole at z_0 (not necessarily simple). Let $\Pi(\omega, \beta)$ be an arithmetic progression. Then the set

$$(13) \quad E = \{\Phi(t) + \Psi(t): t \in U \setminus \{z_0\} \text{ and } \Phi(t) \in \Pi(\omega, \beta)\}$$

approximates the arithmetic progression

$$(14) \quad \Pi(\omega, \beta + \Psi(z_0)).$$

PROOF. Suppose that Φ has a pole of order m at z_0 . Then for large enough $\zeta \in \mathbb{C}$, the solutions of

$$(15) \quad \Phi(z) = \zeta$$

are given by a Puiseux expansion. That is, the solutions to (15) are

$$(16) \quad z = H(\zeta^{-1/m}),$$

where H is analytic in a neighborhood of the origin, and $\zeta^{-1/m}$ runs over all the m^{th} roots of ζ^{-1} . Also, $H(0) = z_0$ and $H'(0) \neq 0$ (see [H]).

Let $\varepsilon > 0$ be given. We have to verify

(17) All but finitely many $\xi \in E$ lie within distance ε of $\Pi(\omega, \beta + \Psi(z_0))$, and

(18) All but finitely many $\xi \in \Pi(\omega, \beta + \Psi(z_0))$ lie within distance ε of E .

Pick $\delta > 0$ so small that

$$(19) \quad |z - z_0| \leq \delta \text{ implies } z \in U \text{ and } |\Psi(z) - \Psi(z_0)| < \varepsilon.$$

To verify (17), suppose

$$(20) \quad 0 < |z - z_0| \leq \delta \text{ and } \Phi(z) \in \Pi(\omega, \beta).$$

Then $\text{dist}\{\Phi(z) + \Psi(z), \Pi(\omega, \beta + \Psi(z_0))\} \leq |\Psi(z) - \Psi(z_0)| < \varepsilon$. On the other hand, Φ is analytic on a neighborhood of the closure of $\hat{U} = \{z \in U : |z - z_0| > \delta\}$, and Φ is non-constant on \hat{U} since Φ has a pole at z_0 . Hence, Φ is bounded on \hat{U} , and $\{z \in \hat{U} : \Phi(z) = \xi\}$ is finite for any ξ . It follows that $\{z \in \hat{U} : \Phi(z) \in \Pi(\omega, \beta)\}$ is finite. Thus, all but finitely many points of E arise as $\xi = \Phi(z) + \Psi(z)$ for some z satisfying (20), and therefore satisfy $\text{dist}\{\xi, \Pi(\omega, \beta + \Psi(z_0))\} < \varepsilon$. This proves (17).

To verify (18), let $\xi \in \Pi(\omega, \beta + \Psi(z_0))$ be sufficiently large, and let $z = H((\xi - \Psi(z_0))^{-1/m})$ for any choice of the m^{th} root. Thus, $z \in U \setminus \{z_0\}$ and $\Phi(z) = \xi - \Psi(z_0) \in \Pi(\omega, \beta)$, so that $\zeta = \Phi(z) + \Psi(z) = \xi - \Psi(z_0) + \Psi(z)$ belongs to E . Moreover, if ξ is large enough, then $(\xi - \Psi(z_0))^{-1/m}$ will be so small that $|z - z_0| = |H((\xi - \Psi(z_0))^{-1/m}) - H(0)| < \delta$. Therefore, $|\Psi(z) - \Psi(z_0)| < \varepsilon$ by (19), so that $|\zeta - \xi| = |\Psi(z) - \Psi(z_0)| < \varepsilon$, and thus $\text{dist}\{\xi, E\} < \varepsilon$. This completes the proof of (18).

B. Continuing the Output to the Complement of a Countable Set.

Fix a neural net $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$. Recall that $D_0 = D_L = 1$. By induction on ℓ ($0 \leq \ell \leq L$) we will define for $1 \leq j \leq D_\ell$ a set $\Omega_j^\ell \subset \mathbb{C}$ and a function $x_j^\ell(t, \mathcal{N})$ on Ω_j^ℓ . For $\ell = 0$, we set

- (1) $\Omega_1^0 = \mathbb{C}$, and
- (2) $x_1^0(t, \mathcal{N}) = t$.

Assume we have defined the $\Omega_j^{\ell-1}$ and $x_j^{\ell-1}(t, \mathcal{N})$ for a fixed ℓ , $1 \leq \ell \leq L$. Then set

$$(3) \quad \Omega_*^{\ell-1} = \bigcap_{1 \leq j \leq D_{\ell-1}} \Omega_j^{\ell-1},$$

$$(4) \quad E_j^\ell = \left\{ t \in \Omega_*^{\ell-1} : \sum_{k=1}^{D_{\ell-1}} \omega_{jk}^\ell x_k^{\ell-1}(t, \mathcal{N}) + \theta_j^\ell \in \Pi(2\pi i, \pi i) \right\},$$

$$(5) \quad \Omega_j^\ell = \Omega_*^{\ell-1} \setminus E_j^\ell,$$

and

$$(6) \quad x_j^\ell(t, \mathcal{N}) = \sigma \left(\sum_{k=1}^{D_{\ell-1}} \omega_{jk}^\ell x_k^{\ell-1}(t, \mathcal{N}) + \theta_j^\ell \right), \quad \text{for } t \in \Omega_j^\ell.$$

Note that (6) makes sense because we need not evaluate $\sigma(\cdot)$ at one of its poles. The poles of σ are precisely $\Pi(2\pi i, \pi i)$. Note also that $\mathbb{R} \subset \Omega_j^\ell$, since $\sum_{k=1}^{D_{\ell-1}} \omega_{jk}^\ell x_k^{\ell-1}(t, \mathcal{N}) + \theta_j^\ell$ is real for t real, so that $\mathbb{R} \cap E_j^\ell = \emptyset$. For real t , our formulas (2) and (6) agree with the definition of $x_j^\ell(t, \mathcal{N})$ given in Section I. Hence we have extended the outputs of the nodes from \mathbb{R} to subsets Ω_j^ℓ of the complex plane.

Note that (3) leaves Ω_*^L undefined. We make the natural definition

$$(7) \quad \Omega_*^L = \Omega_1^L$$

(recall that $D_L = 1$ and compare with (3)). Our definitions have the obvious consequences

$$(8) \quad \Omega_*^\ell = \Omega_*^{\ell-1} \setminus \bigcup_{j=1}^{D_\ell} E_j^\ell, \quad 1 \leq \ell \leq L,$$

and

$$(9) \quad \mathbb{C} \setminus \Omega_*^\ell = \bigcup_{1 \leq \ell' \leq \ell} \bigcup_{j=1}^{D_{\ell'}} E_j^{\ell'}, \quad 1 \leq \ell \leq L.$$

(10) **Lemma.** *The following properties hold for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.*

(11) Ω_j^ℓ is open, and $\mathbb{C} \setminus \Omega_j^\ell$ is countable.

(12) E_j^ℓ is a countable subset of $\Omega_*^{\ell-1}$, with no accumulation points in $\Omega_*^{\ell-1}$.

(13) $x_j^\ell(t, \mathcal{N})$ is analytic on Ω_j^ℓ .

(14) $x_j^\ell(t, \mathcal{N})$ has poles at the points of E_j^ℓ .

PROOF. We use induction on ℓ . Fix ℓ , $1 \leq \ell \leq L$, and assume

(15) $\Omega_j^{\ell-1}$ is open, and $\mathbb{C} \setminus \Omega_j^{\ell-1}$ is countable, $1 \leq j \leq D_{\ell-1}$, and

(16) $x_j^{\ell-1}(t, \mathcal{N})$ is analytic on $\Omega_j^{\ell-1}$, $1 \leq j \leq D_{\ell-1}$.

Note that (15), (16) are obvious for $\ell = 1$ by (1), (2).

We will show that (15), (16) imply (11)-(14). This will imply Lemma (10). From (15), (16) and (3), we see that $\Omega_*^{\ell-1}$ is open, that $\mathbb{C} \setminus \Omega_*^{\ell-1}$ is countable, and that

$$(17) \quad X_j^\ell(t) = \sum_{k=1}^{D_{\ell-1}} \omega_{jk}^\ell x_j^{\ell-1}(t, \mathcal{N}) + \theta_j^\ell, \quad 1 \leq j \leq D_\ell,$$

is analytic on $\Omega_*^{\ell-1}$. Moreover, $\mathbb{R} \subset \Omega_*^{\ell-1}$, and $X_j^\ell(t)$ is real for $t \in \mathbb{R}$. Hence, if $X_j^\ell(t)$ is constant on $\Omega_*^{\ell-1}$, then that constant is real. In particular, $X_j^\ell(t)$ is either non-constant on $\Omega_*^{\ell-1}$, or else identically equal to a constant not in $\Pi(2\pi i, \pi i)$. Therefore, by Lemma A.11 and (4), the set $E_j^\ell \subset \Omega_*^{\ell-1}$ is countable and has no accumulation points in $\Omega_*^{\ell-1}$. This proves (12), from which (11) follows at once by virtue of (5), since $\Omega_*^{\ell-1}$ is open and has countable complement. Assertion (13) follows from the formula $x_j^\ell(t, \mathcal{N}) = \sigma(X_j^\ell(t))$, since $X_j^\ell(t) \notin \Pi(2\pi i, \pi i)$ for $t \notin E_j^\ell$. To verify (14), let $t_0 \in E_j^\ell \subset \Omega_*^{\ell-1}$. By (12), we can find a disc $U_\delta = \{t \in \mathbb{C} : |t - t_0| < \delta\} \subset \Omega_*^{\ell-1}$ such that $U_\delta \setminus \{t_0\}$ does not meet E_j^ℓ . Thus, $U_\delta \setminus \{t_0\} \subset \Omega_*^\ell$, so $X_j^\ell(t) \notin \Pi(2\pi i, \pi i)$ and $x_j^\ell(t, \mathcal{N}) = \sigma(X_j^\ell(t))$ for $t \in U_\delta \setminus \{t_0\}$. Moreover, $X_j^\ell(t)$ is analytic on $\Omega_*^{\ell-1}$, hence on U_δ ;

and we have $X_j^\ell(t_0) \in \Pi(2\pi i, \pi i)$ since $t_0 \in E_j^\ell$. These remarks show that $x_j^\ell(t, \mathcal{N})$ has a pole at t_0 , proving (14). The proof of (11)-(14) is complete.

Lemma (10) shows in particular that the output of the neural net $t \mapsto x_1^L(t, \mathcal{N})$ continues analytically from \mathbb{R} to an open subset of \mathbb{C} with countable complement. Hence, the natural domain of $x_1^L(t, \mathcal{N})$ and the sequence of singular sets $\text{Sing}(\ell, x_1^L(t, \mathcal{N}))$ are well-defined. We write $\text{Sing}(\ell, \mathcal{N})$ for $\text{Sing}(\ell, x_1^L(t, \mathcal{N}))$, and note that

- (18) The sets $\text{Sing}(\ell, \mathcal{N})$, $\ell \geq 0$, are determined completely by the output $t \mapsto x_1^L(t, \mathcal{N})$ ($t \in \mathbb{R}$) of the neural net \mathcal{N} .

In a similar spirit, we see at once from the definitions (1)-(7) that

- (19) For each ℓ , $1 \leq \ell \leq L$, the sets Ω_j^ℓ , Ω_*^ℓ , E_j^ℓ and the functions $x_j^\ell(t, \mathcal{N})$ are determined completely by the $D_{\ell'}$, $\omega_{jk}^{\ell'}$, $\theta_j^{\ell'}$ with $1 \leq \ell' \leq \ell$.

C. The Structure of the Singular Sets.

In this section, we will study the sets $\text{Sing}(\ell, \mathcal{N})$ associated to a neural net $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$, in terms of the sets Ω_*^ℓ , Ω_j^ℓ , E_j^ℓ defined in the previous section.

- (1) **Lemma.** $E_1^L \subset \text{Sing}(0, \mathcal{N}) \subset \bigcup_{\ell=1}^L \bigcup_{j=1}^{D_\ell} E_j^\ell$.

PROOF. Let Ω_* be the natural domain of $x_1^L(t, \mathcal{N})$ and let $X(t)$ be the analytic continuation of $x_1^L(t, \mathcal{N})$ to Ω_* . Lemma B.10 shows that $x_1^L(t, \mathcal{N})$ continues analytically from \mathbb{R} to Ω_1^L . Hence the defining property (A.4) for the natural domain tells us that

- (2) $\Omega_1^L \subset \Omega_*$, and
- (3) $X(t) = x_1^L(t, \mathcal{N})$ for $t \in \Omega_1^L$.

From (2), (B.7), (B.9) and (A.9), we get

$$\text{Sing}(0, \mathcal{N}) = \mathbb{C} \setminus \Omega_* \subset \mathbb{C} \setminus \Omega_1^L = \mathbb{C} \setminus \Omega_*^L = \bigcup_{1 \leq \ell \leq L} \bigcup_{j=1}^{D_\ell} E_j^\ell,$$

which is half of Lemma 1.

To verify the other half of Lemma 1, suppose $t_0 \in E_1^L \cap \Omega_*$. Since $E_1^L \subset \Omega_*^{L-1}$ and E_1^L has no accumulation points in Ω_*^{L-1} , we know that a small enough disc

$$U_\delta = \{t \in \mathbb{C}: |t - t_0| < \delta\}$$

is contained in Ω_*^{L-1} , and that $U_\delta \setminus \{t_0\}$ does not meet E_1^L . Then (3) shows that $X(t) = x_1^L(t, \mathcal{N})$ in $U_\delta \setminus \{t_0\}$, while (B.14) gives

$$\lim_{\substack{t \rightarrow t_0 \\ t \neq t_0}} |x_1^L(t, \mathcal{N})| = \infty.$$

Hence, $\lim_{t \rightarrow t_0, t \neq t_0} |X(t)| = \infty$, which contradicts the fact that $X(t)$ is analytic on an open set Ω_* containing t_0 . Therefore, $E_1^L \cap \Omega_*$ is empty, i.e. $E_1^L \subset \mathbb{C} \setminus \Omega_* = \text{Sing}(0, \mathcal{N})$.

Next we study the E_j^ℓ , as well as

$$(4) \quad \overset{\circ}{E}_j^\ell = E_j^\ell \setminus \bigcup_{j' \neq j} E_{j'}^\ell.$$

To do so, we impose the following hypothesis on the weights (ω_{jk}^ℓ) .

(5) **Assumption.** $\omega_{jk}^\ell \neq 0$, and for $j \neq j'$, the ratio $\omega_{jk}^\ell / \omega_{j'k}^\ell$ is not equal to any fraction of the form p/q with p, q integers and $1 \leq q \leq 100 D_\ell^2$.

Immediately from (B.1)-(B.4), we see that E_j^1 consists of all $t \in \mathbb{C}$ such that $\omega_{j1}^1 t + \theta_j^1 \in \Pi(2\pi i, \pi i)$. In other words,

$$(6) \quad E_j^1 = \Pi \left(\frac{2\pi i}{\omega_{j1}^1}, \frac{\pi i - \theta_j^1}{\omega_{j1}^1} \right), \quad 1 \leq j \leq D_1.$$

From (4), (5), (6) and Corollary II.D.33, we get

$$(7) \quad \overset{\circ}{E}_j^1 \text{ is infinite. } 1 \leq j \leq D.$$

The following lemma shows how $E_j^{\ell+1}$ and $\overset{\circ}{E}_j^{\ell+1}$ look near a point of $\overset{\circ}{E}_j^\ell$.

(8) **Lemma.** Fix $t_0 \in \overset{\circ}{E}_{k_0}^\ell$, $1 \leq \ell \leq L-1$, $1 \leq k_0 \leq D_\ell$. For $\delta > 0$, set

$$(9) \quad U_\delta = \{t \in \mathbb{C}: |t - t_0| < \delta\}.$$

If δ is small enough, then the following properties hold.

$$(10) \quad U_{2\delta} \subset \Omega_*^{\ell-1}.$$

(11) $U_{2\delta} \setminus \{t_0\} \subset \Omega_{k_0}^\ell$; thus, $x_{k_0}^\ell(t, \mathcal{N})$ is analytic on $U_{2\delta} \setminus \{t_0\}$, with a pole at t_0 .

(12) $U_{2\delta} \subset \Omega_k^\ell$ for $k \neq k_0$; thus $x_k^\ell(t, \mathcal{N})$ is analytic on $U_{2\delta}$.

$$(13) \quad \begin{aligned} & (U_\delta \setminus \{t_0\}) \cap E_j^{\ell+1} \\ &= \left\{ t \in U_\delta \setminus \{t_0\}: \sum_{k=1}^{D_\ell} \omega_{jk}^{\ell+1} x_k^\ell(t, \mathcal{N}) + \theta_j^{\ell+1} \in \Pi(2\pi i, \pi i) \right\}, \end{aligned}$$

for $1 \leq j \leq D_{\ell+1}$.

(14) The set $F_j^\ell = \{x_{k_0}^\ell(t, \mathcal{N}): t \in (U_\delta \setminus \{t_0\}) \cap E_j^{\ell+1}\}$ approximates the arithmetic progression $\Pi(2\pi i/\omega_{jk_0}^{\ell+1}, \beta_{jk_0}^{\ell+1})$ for some complex number $\beta_{jk_0}^{\ell+1}$, $1 \leq j \leq D_{\ell+1}$.

(15) For each j_0 , $1 \leq j_0 \leq D_{\ell+1}$, t_0 is an accumulation point of $\overset{\circ}{E}_{j_0}^{\ell+1}$.

PROOF. We know that $t_0 \in \overset{\circ}{E}_{k_0}^\ell \subset E_{k_0}^\ell \subset \Omega_*^{\ell-1}$ by (B.12), so (10) holds simply because $\Omega_*^{\ell-1}$ is open. Another application of (B.12) shows that $U_{2\delta} \setminus \{t_0\}$ meets none of the E_k^ℓ $1 \leq k \leq D_\ell$, if δ is small. Since $t_0 \in \overset{\circ}{E}_{k_0}^\ell$, it follows that $U_{2\delta} \setminus \{t_0\} \subset \Omega_{k_0}^\ell$ and $U_{2\delta} \subset \Omega_k^\ell$ for $k \neq k_0$, by (B.5) and (10). Therefore, (11) and (12) follow from (B.13), (B.14).

Next note that (11), (12) yield $U_{2\delta} \setminus \{t_0\} \subset \Omega_*^\ell$. Hence (13) follows at once from the definition (B.4).

We set $U = U_\delta$,

$$\begin{aligned} \Phi(t) &= \sum_{k=1}^{D_\ell} \omega_{jk}^{\ell+1} x_k^\ell(t, \mathcal{N}) + \theta_j^{\ell+1}, \\ \Psi(t) &= - \sum_{\substack{1 \leq k \leq D_\ell \\ (k \neq k_0)}} \omega_{jk}^{\ell+1} x_k^\ell(t, \mathcal{N}) - \theta_j^{\ell+1}, \end{aligned}$$

$z_0 = t_0$, $\Pi(\omega, \beta) = \Pi(2\pi i, \pi i)$. Then (11) and (12) show that the hypotheses of Lemma A.12 are satisfied. In view of (13), that lemma shows that $\{\omega_{jk_0}^{\ell+1} x_{k_0}^\ell(t, \mathcal{N}) : t \in (U \setminus \{t_0\}) \cap E_j^{\ell+1}\}$ approximates an arithmetic progression of the form $\Pi(2\pi i, \beta_j)$. This yields (14) at once.

It remains to verify (15). By (5), (14) and Corollary II.D.33, we can find a sequence $(x_\nu)_{\nu \geq 1}$ satisfying

- (16) $|x_\nu| \rightarrow \infty$, as $\nu \rightarrow \infty$,
- (17) $x_\nu \in F_{j_0}$,
- (18) $x_\nu \notin F_j$, for $j \neq j_0$, $1 \leq j \leq D_{\ell+1}$.

By definition of F_j , (17) means that

- (19) $x_\nu = x_{k_0}^\ell(t_\nu, \mathcal{N})$, with
- (20) $t_\nu \in (U_\delta \setminus \{t_0\}) \cap E_{j_0}^{\ell+1}$.

If we had $t_\nu \in E_j^{\ell+1}$ for some $n \neq j_0$, then (19), (20) would imply $x_\nu \in F_j$, contradicting (18). Hence $t_\nu \notin E_j^{\ell+1}$, $j \neq j_0$, so that (20) can be sharpened to

$$(21) \quad t_\nu \in (U_\delta \setminus \{t_0\}) \cap \overset{\circ}{E}_{j_0}^{\ell+1}.$$

Also, (11), (19) and (16) show that $t_\nu \rightarrow t_0$ as $\nu \rightarrow \infty$. Therefore, (21) shows that t_0 is an accumulation point of $\overset{\circ}{E}_{j_0}^{\ell+1}$, which is (15).

(22) **Corollary.** *The set $\overset{\circ}{E}_j^\ell$ is infinite, for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.*

PROOF. We use induction on ℓ . For $\ell = 1$, the Corollary is already known (see (7)). If $\overset{\circ}{E}_{k_0}^\ell$ is non-empty, then (15) shows that $\overset{\circ}{E}_j^{\ell+1}$ must be infinite, completing the induction.

(23) **Corollary.** *The output function $x_1^L(t, \mathcal{N})$ is non-constant.*

PROOF. If $x_1^L(t, \mathcal{N})$ were constant, its natural domain would be all of \mathbb{C} , so that $\text{Sing}(0, \mathcal{N})$ would be empty. However, we know that $\overset{\circ}{E}_1^L = E_1^L \subset \text{Sing}(0, \mathcal{N})$ by Lemma C.1, and $\overset{\circ}{E}_1^L$ is infinite, by the preceding corollary.

(24) **Corollary.** *All the functions $x_j^\ell(t, \mathcal{N})$, $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, are non-constant.*

PROOF. Fix $\bar{\ell}$, \bar{j} . Then $x_{\bar{j}}^{\bar{\ell}}(t, \mathcal{N})$ is the output of a simpler neural net $\mathcal{N}_\# = [(D_0^\#, \dots, D_{L_\#}^\#), \omega_{jk}^{\#\ell}, (\theta_j^{\#\ell})]$, defined by

$$\begin{aligned} L_\# &= \bar{\ell}, \quad D_\ell^\# = D_\ell, \quad \text{for } \ell < L_\#, \quad D_{L_\#} = 1, \\ \omega_{jk}^{\#\ell} &= \omega_{jk}^\ell \quad \text{and} \quad \theta_j^{\#\ell} = \theta_j^\ell, \quad \text{for } \ell < L_\#, \\ \omega_{1k}^{\#\bar{\ell}} &= \omega_{\bar{j}k}^{\bar{\ell}} \quad \text{and} \quad \theta_1^{\#\bar{\ell}} = \theta_{\bar{j}}^{\bar{\ell}}. \end{aligned}$$

The net $\mathcal{N}_\#$ again satisfies (5), so Corollary (23) applies to $\mathcal{N}_\#$. Thus $x_1^{L_\#}(t, \mathcal{N}_\#)$ is non-constant, and we observed that $x_{\bar{j}}^{\bar{\ell}}(t, \mathcal{N}) = x_1^{L_\#}(t, \mathcal{N}_\#)$.

Next, we relate $\text{Sing}(\ell, \mathcal{N})$ for $\ell \geq 1$ to the sets $E_j^{\ell'}$.

(25) **Lemma.** $\mathring{E}_j^\ell \subset \text{Sing}(L - \ell, \mathcal{N})$ for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.

PROOF. We use downward induction on ℓ . When $\ell = L$, (25) is contained in (1). For the induction step, fix ℓ ($1 \leq \ell \leq L - 1$), and assume

$$(26) \quad \mathring{E}_j^{\ell+1} \subset \text{Sing}(L - \ell - 1, \mathcal{N}).$$

We shall prove that

$$(27) \quad \mathring{E}_{k_0}^\ell \subset \text{Sing}(L - \ell, \mathcal{N}), \quad \text{for } 1 \leq k_0 \leq D_\ell.$$

In fact, (26) and (15) show that every point of $\mathring{E}_{k_0}^\ell$ is an accumulation point of $\text{Sing}(L - \ell - 1, \mathcal{N})$ and thus belongs to $\text{Sing}(L - \ell, \mathcal{N})$ by Definition A.10. Hence, (26) implies (27), completing the induction.

(28) **Lemma.** $\text{Sing}(L - \bar{\ell}, \mathcal{N}) \subset \bigcup_{1 \leq \ell \leq \bar{\ell}} \bigcup_{1 \leq j \leq D_\ell} E_j^\ell$, for $1 \leq \bar{\ell} \leq L$.

PROOF. Again we use downward induction on $\bar{\ell}$. When $\bar{\ell} = L$, (28) is contained in (1). For the induction step, fix $\bar{\ell}$, $2 \leq \bar{\ell} \leq L$, and assume

$$(29) \quad \text{Sing}(L - \bar{\ell}, \mathcal{N}) \subset \bigcup_{1 \leq \ell \leq \bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell.$$

We shall prove that

$$(30) \quad \text{Sing}(L - \bar{\ell} + 1, \mathcal{N}) \subset \bigcup_{1 \leq \ell < \bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell.$$

In fact, (B.9) shows that the right-hand side of (29) is a closed set. Hence, any accumulation point of $\text{Sing}(L - \bar{\ell}, \mathcal{N})$ is again contained in the right-hand side of (29). By definition (A.10), this implies that

$$\text{Sing}(L - \bar{\ell} + 1, \mathcal{N}) \subset \bigcup_{1 \leq \ell \leq \bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell.$$

Therefore, (30) will follow if we can prove

$$(31) \quad \text{No point of } E_{j_0}^{\bar{\ell}} \text{ is an accumulation point of } \bigcup_{1 \leq \ell \leq \bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell, \\ 1 \leq j_0 \leq D_{\bar{\ell}}.$$

Assertion (31) is equivalent to

$$(32) \quad E_{j_0}^{\bar{\ell}} \text{ contains no accumulation points of } E_j^\ell \quad 1 \leq \ell \leq \bar{\ell}, \quad 1 \leq j_0 \leq D_{\bar{\ell}}, \\ 1 \leq j \leq D_\ell.$$

Thus, (30) follows from (32). To prove (32), we distinguish two cases.

CASE 1: $\ell < \bar{\ell}$. From (B.9) and (B.12) we see that

$$\mathbb{C} \setminus \Omega_*^{\bar{\ell}-1} = \bigcup_{1 \leq \ell < \bar{\ell}} \bigcup_{j=1}^{E_\ell} E_j^\ell$$

and that $E_{j_0}^{\bar{\ell}} \subset \Omega_*^{\bar{\ell}-1}$. Since $\Omega_*^{\bar{\ell}-1}$ is open, these remarks imply (32) for $\ell < \bar{\ell}$.

CASE 2: $\ell = \bar{\ell}$. Then (B.12) shows that $E_{j_0}^{\bar{\ell}} \subset \Omega_*^{\bar{\ell}-1}$, and that $E_j^{\bar{\ell}}$ has no accumulation points in $\Omega_*^{\bar{\ell}-1}$. These remarks imply (32) for $\ell = \bar{\ell}$.

Thus, (32) holds in either case, which completes the proof of (30). We have shown that (29) implies (30), completing the downward induction.

$$(33) \quad \text{Corollary. } \text{Sing}(\ell, \mathcal{N}) \text{ is empty for } \ell \geq L.$$

PROOF. Lemma (28) yields $\text{Sing}(L-1, \mathcal{N}) \subset \bigcup_{j=1}^{D_\ell} E_j^1$. From (6) we see that $\bigcup_{j=1}^{D_1} E_j^1$ has no accumulation points. Hence, $\text{Sing}(L, \mathcal{N})$ is empty, from which (33) is obvious.

(34) **Lemma.** Fix $t_0 \in \overset{\circ}{E}_{k_0}^\ell$, $1 \leq \ell \leq L-1$, $1 \leq k_0 \leq D_\ell$, and set

$$(35) \quad U_\delta = \{t \in \mathbb{C}: |t - t_0| < \delta\}, \quad \text{for } \delta > 0.$$

If δ is small enough, then $\text{Sing}(L-\ell-1, \mathcal{N}) \cap (U_\delta \setminus \{t_0\})$ is the approximate union of the sets

$$E_j^{\ell+1} \cap (U_\delta \setminus \{t_0\}), \quad j = 1, \dots, D_{\ell+1}.$$

PROOF. We must prove two assertions:

$$(36) \quad \text{Sing}(L-\ell-1, \mathcal{N}) \cap (U_\delta \setminus \{t_0\}) \subset \bigcup_{j=1}^{D_{\ell+1}} E_j^{\ell+1} \cap (U_\delta \setminus \{t_0\})$$

and

$$(37) \quad \text{Any point belonging to exactly one of the sets } E_j^{\ell+1} \cap (U_\delta \setminus \{t_0\}), \\ 1 \leq j \leq D_{\ell+1}, \text{ belongs also to } \text{Sing}(L-\ell-1, \mathcal{N}) \cap (U_\delta \setminus \{t_0\}).$$

However, (37) is immediate from (25), so it remains only to prove (36). From (28) we have

$$(38) \quad \text{Sing}(L-\ell-1, \mathcal{N}) \cap (U_\delta \setminus \{t_0\}) \subset \bigcup_{1 \leq \ell' \leq \ell+1} \bigcup_{j=1}^{D_{\ell'}} (E_j^{\ell'} \cap (U_\delta \setminus \{t_0\})).$$

On the other hand, since $t_0 \in E_{k_0}^\ell$, (32) shows that $E_j^{\ell'} \cap (U_\delta \setminus \{t_0\})$ is empty if $\ell' \leq \ell$ and δ is small enough. Therefore, (38) implies (36).

(39) **Lemma.** $\text{Sing}(L-1, \mathcal{N})$ is the approximate union for the sets E_j^1 for $j = 1, \dots, D_1$.

PROOF. Immediate from Lemmas 25 and 28.

D. Summary.

Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ be a neural net. We make the following

(1) **Assumption.** $\omega_{jk}^\ell \neq 0$, and for $j \neq j'$, the ratio $\omega_{jk}^\ell/\omega_{j'k}^\ell$ is not equal to any fraction of the form p/q with p, q integers and $1 \leq q \leq 100 D_\ell^2$.

(2) **Lemma.** For each ℓ ($1 \leq \ell \leq L$), the sets $\Omega_j^\ell, \Omega_{*}^\ell, E_j^\ell, \mathring{E}_j^\ell$ and the functions $x_j^\ell(t, \mathcal{N})$ are determined entirely by the $D_{\ell'}$, $\omega_{j'k}^{\ell'}$ and $\theta_{j'}^{\ell'}$ for $\ell' \leq \ell$ (see (B.19)).

(3) **Lemma.** For each $\ell \geq 0$, the set $\text{Sing}(\ell, \mathcal{N})$ is determined entirely by the output $t \mapsto x_1^\ell(t, \mathcal{N})$ ($t \in \mathbb{R}$) of the neural net (see (B.18)).

(4) **Lemma.** For $1 \leq \ell \leq L, 1 \leq j \leq D_\ell$, the function $x_j^\ell(t, \mathcal{N})$ is analytic on Ω_j^ℓ , with poles at the points of E_j^ℓ (see (B.13) and (B.14)).

(5) **Lemma.** $\text{Sing}(\ell, \mathcal{N})$ is empty for $\ell \geq L$ (see (C.33)).

(6) **Lemma.** $\text{Sing}(L-1, \mathcal{N})$ is the approximate union of the arithmetic progressions $\Pi(2\pi i/\omega_{j1}^1, (\pi i - \theta_j^1)/\omega_{j1}^1)$ for $j = 1, \dots, D_1$ (see (C.6) and (C.39)).

(7) **Lemma.** For $1 \leq \ell \leq L, 1 \leq j \leq D_\ell$, the set \mathring{E}_j^ℓ is infinite (see (C.22)).

(8) **Lemma.** Fix $t_0 \in \mathring{E}_{k_0}^\ell, 1 \leq \ell \leq L-1, 1 \leq k_0 \leq D_\ell$. For $\delta > 0$, set

$$(9) \quad U_\delta = \{t \in \mathbb{C}: |t - t_0| < \delta\}, \quad V_\delta = U_\delta \setminus \{t_0\}.$$

Then the following properties hold if δ is small enough.

(10) $V_{2\delta} \subset \Omega_{k_0}^\ell$ (see (C.11)).

(11) $U_{2\delta} \subset \Omega_k^\ell$ for $k \neq k_0$ ($1 \leq k \leq D_\ell$) (see (C.12)).

(12) $\text{Sing}(L-\ell-1, \mathcal{N}) \cap V_\delta$ is the approximate union of the sets $E_j^{\ell+1} \cap V_\delta$ for $j = 1, \dots, D_{\ell+1}$, (see (C.34)).

$$(13) \quad E_j^{\ell+1} \cap V_\delta = \left\{ t \in V_\delta: \sum_{k=1}^{D_\ell} \omega_{jk}^{\ell+1} x_k^\ell(t, \mathcal{N}) + \theta_j^{\ell+1} \in \Pi(2\pi i, \pi i) \right\},$$

$1 \leq j \leq D_{\ell+1}$, (see (C.13)).

(14) The set $F_j = \{x_{k_0}^\ell(t, \mathcal{N}): t \in E_j^{\ell+1} \cap V_\delta\}$ approximates an arithmetic progression of the form $\Pi\left(2\pi i/\omega_{jk_0}^{\ell+1}, \beta_{jk_0}^{\ell+1}\right)$ for some complex number $\beta_{jk_0}^{\ell+1}$, (see (C.14)).

(15) **Lemma.** For $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, the function $t \mapsto x_j^\ell(t, \mathcal{N})$ ($t \in \mathbb{R}$) is non-constant (see (C.24)).

IV. Proof of the Uniqueness Theorem.

A. Setting up the Induction.

In this section, we start the proof of Theorem I.B.9. We begin with some preliminary remarks. Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ be a neural net satisfying condition (I.B.10). By (III.D.6) and Corollary II.D.33, the set $\text{Sing}(L-1, \mathcal{N})$ is non-empty. On the other hand, (III.D.5) shows that $\text{Sing}(\ell, \mathcal{N})$ is empty for $\ell \geq L$. Hence, the depth L of the neural net can be inferred from a knowledge of the sets $\text{Sing}(\ell, \mathcal{N})$, $\ell \geq 0$. Lemma III.D.3 therefore shows that L can be inferred from knowledge of the output function $t \mapsto x_1^L(t, \mathcal{N})$. Thus, if two neural nets produce the same output, then they have the same depth.

Now let $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$, $\tilde{\mathcal{N}} = [(\tilde{D}_0, \dots, \tilde{D}_{\tilde{L}}), (\tilde{\omega}_{jk}^\ell), (\tilde{\theta}_j^\ell)]$ be neural nets satisfying the hypotheses of Theorem I.B.9. We must show that \mathcal{N} and $\tilde{\mathcal{N}}$ are isomorphic. If Theorem I.B.9 were false, then we could find a counterexample with $\max\{\text{Size}(\mathcal{N}), \text{Size}(\tilde{\mathcal{N}})\}$ as small as possible. (Recall that the size of \mathcal{N} is defined as the sum $D_0 + \dots + D_L$.) Thus, we may assume that

(1) $\max\{\text{Size}(\mathcal{N}), \text{Size}(\tilde{\mathcal{N}})\} = S$, and that

- (2) Theorem I.B.9 holds for any two neural nets \mathcal{N}' , $\tilde{\mathcal{N}}$ of size strictly less than S .

Also, from the preceding paragraph, we know that

$$(3) \quad L = \tilde{L}.$$

By induction on $\bar{\ell}$, $1 \leq \bar{\ell} \leq L$, we will prove that by subjecting \mathcal{N} , $\tilde{\mathcal{N}}$ to isomorphisms we can achieve

$$(4) \quad D_\ell = \tilde{D}_\ell, \quad \text{for } \ell \leq \bar{\ell},$$

$$(5) \quad \omega_{jk}^\ell = \tilde{\omega}_{jk}^\ell, \quad \text{for } 1 \leq \ell \leq \bar{\ell}, 1 \leq j \leq D_\ell, 1 \leq k \leq D_{\ell-1}$$

$$(6) \quad \theta_j^\ell = \tilde{\theta}_j^\ell, \quad \text{for } 1 \leq \ell \leq \bar{\ell}, 1 \leq j \leq D_\ell.$$

If we can prove this for $\bar{\ell} = L$, then Theorem I.B.9 is established.

We will prove (4), (5), (6) by induction on $\bar{\ell}$. In this section, we treat the case $\bar{\ell} = 1$, while the next section gives the induction step. Thus, suppose $\bar{\ell} = 1$. Lemmas III.D.3 and III.D.6 show that the set $\text{Sing}(L-1, \mathcal{N})$ is the approximate union of the arithmetic progressions $\Pi(2\pi i/\omega_{j1}^1, (\pi i - \theta_j^1)/\omega_{j1}^1)$, $1 \leq j \leq D_1$, and also the approximate union of the progressions $\Pi(2\pi i/\tilde{\omega}_{j1}^1, (\pi i - \tilde{\theta}_j^1)/\tilde{\omega}_{j1}^1)$, $1 \leq j \leq \tilde{D}_1$. The Deconstruction Lemma therefore tells us that $D_1 = \tilde{D}_1$, and that

$$\Pi\left(\frac{2\pi i}{\omega_{j1}^1}, \frac{\pi i - \theta_j^1}{\omega_{j1}^1}\right) = \Pi\left(\frac{2\pi i}{\tilde{\omega}_{(\gamma j)1}^1}, \frac{\pi i - \tilde{\theta}_{(\gamma j)}^1}{\tilde{\omega}_{(\gamma j)1}^1}\right), \quad 1 \leq j \leq D_1,$$

for a permutation $\gamma: \{1, \dots, D_1\} \rightarrow \{1, \dots, D_1\}$. Remark I.A.7 yields

$$(7) \quad \omega_{j1}^1 = \varepsilon_j \tilde{\omega}_{(\gamma j)1}^1, \quad \theta_j^1 = \varepsilon_j \tilde{\theta}_{(\gamma j)}^1 + 2\pi i m_j, \quad 1 \leq j \leq D_1,$$

for $\varepsilon_j = \pm 1$ and integers m_j . Since θ_j^1 and $\tilde{\theta}_j^1$ are real, we must have $m_j = 0$. Also, by subjecting $\tilde{\mathcal{N}}$ to an isomorphism that permutes the nodes of layer 1, we can achieve $\gamma = \text{identity}$ in (7). Thus,

$$(8) \quad \omega_{j1}^1 = \varepsilon_j \tilde{\omega}_{j1}^1, \quad \theta_j^1 = \varepsilon_j \tilde{\theta}_j^1, \quad \varepsilon_j = \pm 1, \quad 1 \leq j \leq D_1.$$

If $L > 1$, then we can subject $\tilde{\mathcal{N}}$ to an isomorphism that changes the signs of the nodes at layer 1, to achieve $\varepsilon_j = 1$, $1 \leq j \leq D_1$, in (8).

Thus, we have achieved (4), (5), (6) with $\bar{\ell} = 1$, unless $L = 1$. If $L = 1$, then there is no isomorphism that changes the signs at layer 1, since layer 1 is the output layer. In this case we argue as follows. For $L = 1$, the outputs of the nets \mathcal{N} , $\tilde{\mathcal{N}}$ are

$$(9) \quad x_1^L(t, \mathcal{N}) = \sigma(\omega_{11}^1 t + \theta_1^1) \quad x_1^L(t, \tilde{\mathcal{N}}) = \sigma(\tilde{\omega}_{11}^1 t + \tilde{\theta}_1^1).$$

Equation (8) says that

$$(10) \quad \omega_{11}^1 = \varepsilon \tilde{\omega}_{11}^1, \quad \theta_1^1 = \varepsilon \tilde{\theta}_1^1, \quad \varepsilon = \pm 1.$$

From (9), (10) we get $x_1^L(t, \mathcal{N}) = \varepsilon x_1^L(t, \tilde{\mathcal{N}})$ for $t \in \mathbb{R}$. On the other hand, hypothesis of Theorem I.B.9 gives $x_1^L(t, \mathcal{N}) = x_1^L(t, \tilde{\mathcal{N}})$, and $x_1^L(t, \mathcal{N})$ is not identically zero. Hence, $\varepsilon = +1$, so that we have achieved (4), (5), (6).

B. The Inductive Step.

Suppose $\bar{\ell}$ is given, $1 \leq \bar{\ell} \leq L - 1$, and the nets \mathcal{N} , $\tilde{\mathcal{N}}$ in the previous section satisfy

- (1) $D_\ell = \tilde{D}_\ell$, for $0 \leq \ell \leq \bar{\ell}$,
- (2) $\omega_{jk}^\ell = \tilde{\omega}_{jk}^\ell$, for $1 \leq \ell \leq \bar{\ell}$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$,
- (3) $\theta_j^\ell = \tilde{\theta}_j^\ell$, for $1 \leq \ell \leq \bar{\ell}$, $1 \leq j \leq D_\ell$.

Then we will prove that

$$(4) \quad D_{\bar{\ell}+1} = \tilde{D}_{\bar{\ell}+1},$$

and that we can subject \mathcal{N} , $\tilde{\mathcal{N}}$ to isomorphisms to achieve

- (5) $\omega_{jk}^\ell = \tilde{\omega}_{jk}^\ell$, for $1 \leq \ell \leq \bar{\ell} + 1$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$,
- (6) $\theta_j^\ell = \tilde{\theta}_j^\ell$, for $1 \leq \ell \leq \bar{\ell}$, $1 \leq j \leq D_\ell$.

This inductive step will complete the proof of Theorem I.B.9.

Let $E_j^\ell, \overset{\circ}{E}_j^\ell, \Omega_j^\ell, \Omega_*^\ell$ be the sets constructed from \mathcal{N} in Section III, and let $\tilde{E}_j^\ell, \tilde{\overset{\circ}{E}}_j^\ell, \tilde{\Omega}_j^\ell, \tilde{\Omega}_*^\ell$ be the analogous sets arising from $\tilde{\mathcal{N}}$. By (1), (2), (3) and Lemma III.D.2, we have

$$(7) \quad E_j^\ell = \tilde{E}_j^\ell, \quad \overset{\circ}{E}_j^\ell = \tilde{\overset{\circ}{E}}_j^\ell, \quad \tilde{\Omega}_j^\ell = \Omega_j^\ell, \quad \tilde{\Omega}_*^\ell = \Omega_*^\ell, \quad \text{for } \ell \leq \bar{\ell},$$

and

$$(8) \quad x_j^\ell(t, \mathcal{N}) = x_j^\ell(t, \tilde{\mathcal{N}}), \quad \text{for } \ell \leq \bar{\ell}, t \in \Omega_j^\ell.$$

Set

$$(9) \quad k_0 = D_{\bar{\ell}}.$$

Lemma III.D.7 shows that $\overset{\circ}{E}_{k_0}^{\bar{\ell}} = \overset{\circ}{\tilde{E}}_{k_0}^{\bar{\ell}}$ is infinite. Fix any $t_0 \in \overset{\circ}{E}_{k_0}^{\bar{\ell}}$. For $\delta > 0$, set

$$(10) \quad U_\delta = \{t \in \mathbb{C}: |t - t_0| < \delta\}, \quad V_\delta = U_\delta \setminus \{t_0\},$$

and take δ so small that (III.D.10)-(III.D.14) hold with $\bar{\ell}$ in place of ℓ , both for \mathcal{N} and $\tilde{\mathcal{N}}$. Lemma III.D.3 gives

$$(11) \quad \text{Sing}(L - \bar{\ell} - 1, \mathcal{N}) = \text{Sing}(L - \bar{\ell} - 1, \tilde{\mathcal{N}}).$$

We will check that

$$F = \{x_{k_0}^{\bar{\ell}}(t, \mathcal{N}): t \in V_\delta \cap \text{Sing}(L - \bar{\ell} - 1, \mathcal{N})\}$$

is the approximate union of the sets F_j defined in (III.D.14). This amounts to showing that

$$(12) \quad F \subset \bigcup_{j=1}^{D_{\bar{\ell}+1}} F_j, \quad \text{and}$$

(13) Any point x belonging to exactly one of the F_j must belong to F .

To see (12), let $x \in F$. Then $x = x_{k_0}^{\bar{\ell}}(t, \mathcal{N})$ with $t \in V_\delta \cap \text{Sing}(L - \bar{\ell} - 1, \mathcal{N})$, so that $t \in V_\delta \cap E_j^{\bar{\ell}+1}$ for some j , by (III.D.12). Since $x = x_{k_0}^{\bar{\ell}}(t, \mathcal{N})$ with $t \in V_\delta \cap E_j^{\bar{\ell}+1}$, we have $x \in F_j$, proving (12). To check (13), suppose x belongs to F_{j_0} but not to any other F_j . Then since $x \in F_{j_0}$, we have $x = x_{k_0}^{\bar{\ell}}(t, \mathcal{N})$ with $t \in E_{j_0}^{\bar{\ell}+1} \cap V_\delta$. If $t \in E_j^{\bar{\ell}+1}$ for some $j \neq j_0$, then it would follow that $x \in F_j$, contradicting our assumption. Hence, t belongs to $E_{j_0}^{\bar{\ell}+1}$ but not to $E_j^{\bar{\ell}+1}$ for $j \neq j_0$. Since also $t \in V_\delta$, (III.D.12) implies $t \in V_\delta \cap \text{Sing}(L - \bar{\ell} - 1, \mathcal{N})$, so that $x = x_{k_0}^{\bar{\ell}}(t, \mathcal{N})$ belongs to F . This completes the proof of (13), and shows that F is the approximate union of the F_j , $1 \leq j \leq D_{\bar{\ell}+1}$. In view of (11), an analogous argument shows that F is also the approximate union of the

$\tilde{F}_j, 1 \leq j \leq \tilde{D}_{\bar{\ell}+1}$, where $\{\tilde{F}_j\}$ are the analogues of the $\{F_j\}$ arising from \tilde{N} . Moreover, F_j approximates $\Pi\left(2\pi i/\omega_{jk_0}^{\bar{\ell}+1}, \beta_j\right)$ for suitable β_j , while \tilde{F}_j approximates $\Pi\left(2\pi i/\tilde{\omega}_{jk_0}^{\bar{\ell}+1}, \tilde{\beta}_j\right)$ for suitable $\tilde{\beta}_j$, by (III.D.14).

Since also the $(\omega_{jk_0}^{\bar{\ell}+1})$ and $(\tilde{\omega}_{jk_0}^{\bar{\ell}+1})$ satisfy (I.B.10) and (I.B.11), the Deconstruction Lemma applies. Hence, (4) holds, and for some permutation $\gamma: \{1, \dots, D_{\bar{\ell}+1}\} \rightarrow \{1, \dots, D_{\bar{\ell}+1}\}$ we have

$$\Pi\left(\frac{2\pi i}{\omega_{jk_0}^{\bar{\ell}+1}}, \beta_j\right) = \Pi\left(\frac{2\pi i}{\tilde{\omega}_{(\gamma j)k_0}^{\bar{\ell}+1}}, \tilde{\beta}_{(\gamma j)}\right), \quad 1 \leq j \leq D_{\bar{\ell}+1}.$$

In particular,

$$(14) \quad \omega_{jk_0}^{\bar{\ell}+1} = \varepsilon_j \tilde{\omega}_{(\gamma j)k_0}^{\bar{\ell}+1}, \quad 1 \leq j \leq D_{\bar{\ell}+1}, \quad \text{with } \varepsilon_j = \pm 1.$$

By subjecting \tilde{N} to an isomorphism that permutes the nodes of layer $(\bar{\ell} + 1)$, we can preserve (1)-(4) and (7), (8), (11), and bring about $\gamma = \text{identity}$ in (4). Thus we may assume

$$(15) \quad \omega_{jk_0}^{\bar{\ell}+1} = \varepsilon_j \tilde{\omega}_{jk_0}^{\bar{\ell}+1}, \quad 1 \leq j \leq D_{\bar{\ell}+1},$$

with $\varepsilon_j = \pm 1$ and $\gamma = \text{identity}$. Recall that $k_0 = D_{\bar{\ell}}$ (see (9)).

The next step is to establish the following result.

(16) **Lemma.** $\omega_{jk}^{\bar{\ell}+1} = \varepsilon_j \tilde{\omega}_{jk}^{\bar{\ell}+1}, 1 \leq j \leq D_{\bar{\ell}+1}, 1 \leq k \leq D_{\bar{\ell}}$, and $\theta_j^{\bar{\ell}+1} = \varepsilon_j \tilde{\theta}_j^{\bar{\ell}+1}, 1 \leq j \leq D_{\bar{\ell}+1}$.

Note that the proof of (15) applies to other k , not just k_0 , and shows that $\omega_{jk}^{\bar{\ell}+1} = \varepsilon_{jk} \tilde{\omega}_{(\gamma' j)k}^{\bar{\ell}+1}$ with $\varepsilon_{jk} = \pm 1$, and γ' depending on k . However, (16) gives sharper restrictions on the ω 's and $\tilde{\omega}$'s.

PROOF OF LEMMA 16. We return to the Deconstruction Lemma applied to $F, F_j, \tilde{F}_j, \Pi\left(2\pi i/\omega_{jk_0}^{\bar{\ell}+1}, \beta_j\right), \Pi\left(2\pi i/\tilde{\omega}_{jk_0}^{\bar{\ell}+1}, \tilde{\beta}_j\right)$. Since $\gamma = \text{identity}$, the Deconstruction Lemma yields for each fixed $j_0, 1 \leq j_0 \leq D_{\bar{\ell}+1}$, a sequence $(x_\nu)_{\nu \geq 1}$ with the properties

$$(17) \quad |x_\nu| \rightarrow \infty, \quad \text{as } \nu \rightarrow \infty,$$

$$(18) \quad x_\nu \in F_{j_0} \setminus \bigcup_{j \neq j_0} F_j,$$

$$(19) \quad x_\nu \in \tilde{F}_{j_0} \setminus \bigcup_{j \neq j_0} \tilde{F}_j.$$

Since $x_\nu \in F_{j_0}$, we have

$$(20) \quad x_\nu = x_{k_0}^{\bar{\ell}}(t_\nu, \mathcal{N}),$$

with

$$(21) \quad t_\nu \in V_\delta \cap E_{j_0}^{\bar{\ell}+1}.$$

Observe that

$$(22) \quad t_\nu \notin V_\delta \cap E_j^{\bar{\ell}+1}, \quad \text{for } j \neq j_0.$$

In fact, if (22) were false, then (20) would show that $x_\nu \in F_j$ with $j \neq j_0$, contradicting (18). Similarly, we know from (19) that

$$(23) \quad t_\nu \notin V_\delta \cap \tilde{E}_j^{\bar{\ell}+1}, \quad \text{for } j \neq j_0.$$

From (21), (22) and (III.D.12), we see that $t_\nu \in V_\delta \cap \text{Sing}(L - \bar{\ell} - 1, \mathcal{N})$. Hence also $t_\nu \in V_\delta \cap \text{Sing}(L - \bar{\ell} - 1, \tilde{\mathcal{N}})$ by (11), so that another application of (III.D.12) yields $t_\nu \in V_\delta \cap \tilde{E}_{j_1}^{\bar{\ell}+1}$ for some j_1 . In view of (23), we must have $j_1 = j_0$. Hence,

$$(24) \quad t_\nu \in V_\delta \cap \tilde{E}_{j_0}^{\bar{\ell}+1}.$$

From (17), (20), $t_\nu \in V_\delta$, and (III.D.4), (III.D.10), we see that

$$(25) \quad t_\nu \rightarrow t_0, \quad \text{as } \nu \rightarrow \infty, \quad t_\nu \neq t_0.$$

From (21), (24) and (III.,D.13), we get

$$(26) \quad t_\nu \in V_\delta, \quad \sum_{k=1}^{D_I} \omega_{j_0 k}^{\bar{\ell}+1} x_k^{\bar{\ell}}(t_\nu, \mathcal{N}) + \theta_{j_0}^{\bar{\ell}} \in \Pi(2\pi i, \pi i)$$

and

$$(27) \quad t_\nu \in V_\delta, \quad \sum_{k=1}^{D_I} \tilde{\omega}_{j_0 k}^{\bar{\ell}+1} x_k^{\bar{\ell}}(t_\nu, \tilde{\mathcal{N}}) + \theta_{j_0}^{\bar{\ell}+1} \in \Pi(2\pi i, \pi i).$$

In view of (15), (26), (27), we obtain

$$\begin{aligned} & \left(\sum_{k=1}^{D_{\ell}-1} \omega_{j_0 k}^{\bar{\ell}+1} x_k^{\bar{\ell}}(t_\nu, \mathcal{N}) + \theta_{j_0}^{\bar{\ell}+1} \right) \\ & - \left(\sum_{k=1}^{D_{\ell}-1} (\varepsilon_{j_0} \tilde{\omega}_{j_0 k}^{\bar{\ell}+1}) x_k^{\bar{\ell}}(t_\nu, \tilde{\mathcal{N}}) + (\tilde{\theta}_{j_0}^{\bar{\ell}+1} \varepsilon_{j_0}) \right) \in \Pi(2\pi i, 0). \end{aligned}$$

That is,

$$(28) \quad F(t_\nu) \in \Pi(2\pi i, 0),$$

with

$$(29) \quad \begin{aligned} F(t) = & \left(\sum_{k=1}^{D_{\ell}-1} \omega_{j_0 k}^{\bar{\ell}+1} x_k^{\bar{\ell}}(t, \mathcal{N}) + \theta_{j_0}^{\bar{\ell}+1} \right) \\ & - \left(\sum_{k=1}^{D_{\ell}-1} (\varepsilon_{j_0} \tilde{\omega}_{j_0 k}^{\bar{\ell}+1}) x_k^{\bar{\ell}}(t, \tilde{\mathcal{N}}) + (\varepsilon_{j_0}^{\bar{\ell}+1} \tilde{\theta}_{j_0}^{\bar{\ell}+1}) \right). \end{aligned}$$

Since $t_0 \in \overset{\circ}{E}_{k_0}^{\bar{\ell}}$ with $k_0 = D_{\bar{\ell}}$, (III.D.4) and (III.D.11) show that $F(t)$ is analytic on U_δ . Also, (25) yields $F(t_\nu) \rightarrow F(t_0)$ as $\nu \rightarrow \infty$. This shows that $F(t_\nu)$ is eventually constant, by (28). Another application of (25) shows that $F(t)$ is constant on U_δ . In view of (28), there is an integer m such that $F(t) = 2\pi im$ for all $t \in U_\delta$. However, $F(t)$ is analytic on $\Omega_*^{\bar{\ell}}$ (by (7), (29) and (III.D.4)). Since $\Omega_*^{\bar{\ell}}$ contains V_δ (by (III.D.10), (III.D.11)), and since $\Omega_*^{\bar{\ell}}$ is connected, it follows by analytic continuation that $F(t) = 2\pi im$ for all $t \in \Omega_*^{\bar{\ell}}$. In particular, $F(t) = 2\pi im$ for $t \in \mathbb{R}$. A glance at (29) shows that $F(t)$ is real for $t \in \mathbb{R}$. Hence, $m = 0$, so that $F(t) = 0$ for t real, *i.e.*

$$(30) \quad \sum_{k=1}^{D_{\ell}-1} \omega_{j_0 k}^{\bar{\ell}+1} x_k^{\bar{\ell}}(t, \mathcal{N}) + \theta_{j_0}^{\bar{\ell}+1} = \sum_{k=1}^{D_{\ell}-1} (\tilde{\omega}_{j_0 k}^{\bar{\ell}+1} \varepsilon_{j_0}) x_k^{\bar{\ell}}(t, \tilde{\mathcal{N}}) + (\tilde{\theta}_{j_0}^{\bar{\ell}+1} \varepsilon_{j_0}),$$

for all $t \in \mathbb{R}$.

To complete the proof of (16), we distinguish two cases.

CASE 1: $D_{\bar{\ell}} = 1$. Then already (15) shows that $\omega_{j_0 k}^{\bar{\ell}+1} = \varepsilon_{j_0} \tilde{\omega}_{j_0 k}^{\bar{\ell}+1}$ for $1 \leq k \leq D_{\bar{\ell}}$, and (30) shows that $\theta_{j_0}^{\bar{\ell}+1} = \varepsilon_{j_0} \tilde{\theta}_{j_0}^{\bar{\ell}+1}$. Since j_0 is arbitrary, $1 \leq j_0 \leq D_{\bar{\ell}+1}$, the proof of (16) is complete in *Case 1*.

CASE 2: $D_{\bar{\ell}} > 1$. Then the left and right-hand sides of (30), composed with $\sigma(\cdot)$, are the outputs of auxiliary neural nets $\widehat{\mathcal{N}}$ and $\check{\mathcal{N}}$ respectively. Specifically, we set

$$(31) \quad \widehat{\mathcal{N}} = [(D_0, D_1, \dots, D_{\bar{\ell}-1}, D_{\bar{\ell}} - 1, 1), (\widehat{\omega}_{jk}^\ell), (\widehat{\theta}_j^\ell)], \quad \text{with}$$

$$(32) \quad \widehat{\omega}_{jk}^\ell = \omega_{jk}^\ell, \quad \widehat{\theta}_j^\ell = \theta_j^\ell \quad \text{for } \ell \leq \bar{\ell}, \quad \text{and}$$

$$(33) \quad \widehat{\omega}_{1k}^{\bar{\ell}+1} = \omega_{j_0 k}^{\bar{\ell}+1}, \quad \widehat{\theta}_1^{\bar{\ell}+1} = \theta_{j_0}^{\bar{\ell}+1}; \quad \text{and we set}$$

$$(34) \quad \check{\mathcal{N}} = [(D_0, D_1, \dots, D_{\bar{\ell}-1}, D_{\bar{\ell}} - 1, 1), (\check{\omega}_{jk}^\ell), (\check{\theta}_j^\ell)], \quad \text{with}$$

$$(35) \quad \check{\omega}_{jk}^\ell = \widetilde{\omega}_{jk}^\ell, \quad \check{\theta}_j^\ell = \widetilde{\theta}_j^\ell \quad \text{for } \ell \leq \bar{\ell}, \quad \text{and}$$

$$(36) \quad \check{\omega}_{1k}^{\bar{\ell}+1} = \varepsilon_{j_0} \widetilde{\omega}_{j_0 k}^{\bar{\ell}+1}, \quad \check{\theta}_1^{\bar{\ell}+1} = \varepsilon_{j_0} \widetilde{\theta}_{j_0}^{\bar{\ell}+1}.$$

That is, $\widehat{\mathcal{N}}$ is made from \mathcal{N} by deleting the following nodes:

- (a) Node k_0 at level $\bar{\ell}$;
- (b) All nodes except node j_0 at level $\bar{\ell} + 1$;
- (c) All nodes at levels higher than $\bar{\ell} + 1$.

For the surviving nodes in $\widehat{\mathcal{N}}$, the weights and thresholds are the same as those of \mathcal{N} . Thus, $x_{j_0}^{\bar{\ell}+1}(t, \widehat{\mathcal{N}})$ is the output of the net $\widehat{\mathcal{N}}$.

Similarly, $\check{\mathcal{N}}$ is made from $\widetilde{\mathcal{N}}$ by deleting the same nodes as in (a), (b), (c) above. For the surviving nodes in $\check{\mathcal{N}}$, the weights and thresholds are the same as those of $\widetilde{\mathcal{N}}$, except that we multiply the weights and thresholds at the output level of $\check{\mathcal{N}}$ by ε_{j_0} . Thus, $\varepsilon_{j_0} x_{j_0}^{\bar{\ell}+1}(t, \check{\mathcal{N}})$ is the output of the net $\check{\mathcal{N}}$. Note that our assumption $D_{\bar{\ell}} > 1$ was needed to define $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$ as neural nets.

Equation (30) shows that the nets $\widehat{\mathcal{N}}$ and $\check{\mathcal{N}}$ produce the same output. Also, $\widehat{\mathcal{N}}$ and $\check{\mathcal{N}}$ satisfy (I.B.10) and (I.B.11). Moreover, the size of $\widehat{\mathcal{N}}$ is strictly less than that of \mathcal{N} , and the size of $\check{\mathcal{N}}$ is strictly less than that of $\widetilde{\mathcal{N}}$. (For, one node at level $\bar{\ell}$, and possible additional nodes, are deleted from \mathcal{N} , $\widetilde{\mathcal{N}}$ to make $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$). Hence, by (A.1) and (A.2), the uniqueness Theorem I.B.9 applies to $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$. Therefore, $\widehat{\mathcal{N}}$ is isomorphic to $\check{\mathcal{N}}$. Also, by definition of $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$ and by (1), (2), (3), the nets $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$

are identical below their output level, *i.e.* $\widehat{\omega}_{jk}^\ell = \check{\omega}_{jk}^\ell$, $\widehat{\theta}_j^\ell = \check{\theta}_j^\ell$ for $\ell \leq \bar{\ell}$. Also, for fixed ℓ , k , we have $\widehat{\omega}_{jk}^\ell \neq 0$ and $|\widehat{\omega}_{jk}^\ell| \neq |\widehat{\omega}_{j'k}^\ell|$ for $j \neq j'$, by (32) and hypothesis (I.B.10). Therefore, Lemma I.C.1 applies, and shows that the nets $\widehat{\mathcal{N}}, \check{\mathcal{N}}$ are identical. In particular,

$$(37) \quad \omega_{j_0}^{\bar{\ell}+1} = \varepsilon_{j_0} \widetilde{\omega}_{j_0 k}^{\bar{\ell}+1} \quad \text{for } 1 \leq k \leq D_{\bar{\ell}} - 1, \quad \text{and} \quad \theta_{j_0}^{\bar{\ell}+1} = \varepsilon_{j_0} \widetilde{\theta}_{j_0}^{\bar{\ell}+1}.$$

Since $j_0, 1 \leq j_0 \leq D_{\bar{\ell}+1}$, was arbitrary, Lemma 16 follows from (9), (15) and (37).

To finish the proof of Theorem I.B.9, we distinguish two cases.

CASE 1: $L > \bar{\ell} + 1$. Then by subjecting $\check{\mathcal{N}}$ to an isomorphism that changes the signs of the nodes at level $\bar{\ell} + 1$, we can achieve (5) and (6). (That is obvious from (1), (2), (3), (16).) We already proved (4), so we have completed the inductive step in the proof of Theorem I.B.9 in Case 1.

CASE 2: $L = \bar{\ell} + 1$. Then (1), (2), (3), (16) show that $D_\ell = \widetilde{D}_\ell$, $0 \leq \ell \leq L$, and that

$$(38) \quad \omega_{jk}^\ell = \widetilde{\omega}_{jk}^\ell, \quad \theta_j^\ell = \widetilde{\theta}_j^\ell, \quad \text{if } \ell < L,$$

$$(39) \quad \omega_{1k}^L = \varepsilon \widetilde{\omega}_{1k}^L, \quad \theta_1^L = \varepsilon \widetilde{\theta}_1^L, \quad \text{with } \varepsilon = \pm 1.$$

It follows at once that $x_1^L(t, \mathcal{N}) = \varepsilon x_1^L(t, \widetilde{\mathcal{N}})$ for all $t \in \mathbb{R}$. However, from the hypothesis of Theorem I.B.9, we know that $x_1^L(t, \mathcal{N}) = x_1^L(t, \widetilde{\mathcal{N}})$ since $L = \widetilde{L}$. Since also $x_1^L(t, \mathcal{N})$ is a non-constant function (see (III.D.15)), it follows that $\varepsilon = +1$, hence (38), (39) show that the nets $\mathcal{N}, \widetilde{\mathcal{N}}$ are identical. In particular, we have achieved (4), (5), (6), completing the inductive step in the proof of theorem (I.B.9) in *Case 2*.

The proof of Theorem I.B.9 is complete.

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Recibido: 18 de junio de 1.993

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* This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under Contract F49620-92-C-0072. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding and copyright notation hereon. This work was also supported by the National Science Foundation.