Revista Matemática Iberoamericana
Vol. 10, N.° 2, 1994

Hilbert transforms and maximal functions along rough flat curves

Anthony Carbery and Sarah Ziesler

Introduction.

Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous curve of class $C^1(\mathbb{R}\backslash\{0\})$ with $\Gamma(0) = 0$. It is a question of considerable interest to find necessary and/or sufficient conditions on Γ so that the operators $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ defined by

$$\mathcal{H}_\Gamma f(x) = \text{p.v.} \int_{-\infty}^{+\infty} f(x - \Gamma(t)) \, \frac{dt}{t}$$

and

$$\mathcal{M}_\Gamma f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - \Gamma(t))| \, dt$$

are bounded on $L^p(\mathbb{R}^n)$ for certain $1 < p < \infty$. The case when Γ is well-curved at the origin (*i.e.* $\{\Gamma'(0), \dots, \Gamma^{(k)}(0)\}$ spans $\mathbb{R}^n$ for some k with $k \geq n$) is by now very well understood (see [SW]) and when $n = 2$ and Γ is flat and convex (or "biconvex") a great deal is known too, (see [CCC...], [CCVWW], [NVWW1], [NVWW2]). However, in higher dimensions, the case of flat curves is much less well-understood (for known results see [NVWW3], [CVWW], [Z2]) even to the extent that it is not clear which basic class(es) of curves one should be studying. In the three above-mentioned papers, the following substitute notion for convexity was proposed: the curve $\Gamma(t)$ should be of the form

$(t, \gamma_2(t), \ldots, \gamma_n(t))$ where each γ_j is of class $C^n(0, +\infty)$, and for each $j = 2, \ldots, n$ the determinant

$$D_j = \det \begin{pmatrix} 1 & \gamma_2' & \cdots & \gamma_j' \\ 0 & \gamma_2'' & \cdots & \gamma_j'' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j)} & \cdots & \gamma_j^{(j)} \end{pmatrix}$$

should be positive on $(0, +\infty)$. (A similar condition would be supposed on $(-\infty, 0)$.) In two dimensions this reduces to $\gamma_2'' > 0$, that is, convexity of $(t, \gamma(t))$. Associated with curves in this class there are auxiliary functions h_j, $(j = 1, \ldots, n)$ defined by

$$h_j(t) = \frac{1}{D_{j-1}(t)} \det \begin{pmatrix} t & \gamma_2(t) & \cdots & \gamma_j(t) \\ 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j-1)}(t) & \cdots & \gamma_j^{(j-1)}(t) \end{pmatrix}$$

where $D_0 \equiv 1$. In [NVWW3] it is shown that this convexity condition is equivalent to the positivity of h_j and h_j' for $1 \le j \le n$. (Note that when $n = 2$, $h_1(t) = t$ and $h_2(t) = t\gamma_2'(t) - \gamma_2(t)$.) It is also shown in [NVWW3] that a necessary and sufficient condition for *odd* curves of this class to have $\mathcal{H}_\Gamma$ bounded on L^2 is the bounded doubling of each of the functions h_j, that is the existence of a constant $C > 1$ so that $h_j(Ct) \ge h_j(t)$. In [Z2] it was shown that doubling of the h_j's is sufficient for L^2 boundedness of $\mathcal{M}_\Gamma$, while in [CVWW] it was shown that a slightly stronger condition (h_j's "infinitesimally doubling") is sufficient for L^p boundedness of $\mathcal{H}_\Gamma$ (for Γ odd) and $\mathcal{M}_\Gamma$ for $1 < p < \infty$. Thus a reasonably complete theory is available for curves of this class. Nevertheless the theorem of Nagel, Stein and Wainger [NSW] concerning differentiation in lacunary directions implies that if Γ is the polygonal curve obtained by joining points on (t, t^2, t^3) of the form $\pm 2^j$, $(j \in \mathbb{Z})$, by straight line segments, then the associated $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ are bounded on L^p, $1 < p < \infty$. These Γ do not fall under the scope of the theory of the curves considered in [CVWW]. The purpose of this paper is to provide a theory which includes these curves as a special case. It will turn out that while our theory does handle these curves, it does not handle curves which near the origin behave like $(t, e^{-1/|t|}\operatorname{sgn} t,\ e^{-1/|t|}\log(1/|t|)\operatorname{sgn} t)$ in which the ratio of derivatives of coordinates varies slowly. Such curves do fall under the scope of

[CVWW]. It should be pointed out that in 2-dimensions a result of Ziesler [Z1] includes both results. It turns out that the analysis in the present paper is much simpler than that in [CVWW]. Probably the reason is that for curves like $(t, e^{-1/|t|}\operatorname{sgn} t,\ e^{-1/|t|}\log(1/|t|)\operatorname{sgn} t)$ the intrinsic geometry just is more complicated.

Let us say that a curve $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ is of class (K) if Γ is continuous, $\Gamma \in C^1(\mathbb{R}\backslash\{0\})$, $\Gamma(0) = 0$, and if $\Gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, then

$$|\gamma_1'|\,,\quad \left|\frac{\gamma_{j+1}'}{\gamma_j'}\right|, \qquad (j = 1, \ldots, n-1)$$

are increasing for $t > 0$ and decreasing for $t < 0$. We say that Γ is *balanced* if for some $C \geq 1$, all $1 \leq j \leq n$, and all $t > 0$ we have

$$\left|\frac{\gamma_j(-t)}{\gamma_j(C\,t)}\right| \leq 1\,, \qquad \left|\frac{\gamma_j(t)}{\gamma_j(-C\,t)}\right| \leq 1\,.$$

Theorem. *Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a balanced curve of class* (K). *If Γ also satisfies*

$$\text{(D)} \qquad \left|\frac{\gamma_{j+1}'(\lambda t)}{\gamma_j'(\lambda\,t)}\right| \geq 2\left|\frac{\gamma_{j+1}'(t)}{\gamma_j'(t)}\right|$$

for $1 \leq j \leq n-1$, some $\lambda > 1$ and all $t \neq 0$, then $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

REMARKS.

1. When $n = 2$ and $\gamma_1(t) = t$, this theorem is in [CCC...]; see also [CóRdeF]; note that class (K) in this case reduces to monotonicity and single-signedness of γ_2' on $(-\infty, 0)$ and on $(0, +\infty)$. Thus class (K) gives an alternative variant of convexity in higher dimensions.

2. The behaviour of Γ for $t < 0$ is irrelevant for $\mathcal{M}_\Gamma$. So in (K) and (D) no assumptions are necessary for $t < 0$, and balance is dropped when considering $\mathcal{M}_\Gamma$.

3. The C^1 assumption in (K) can be relaxed to allow, for example, the piecewise linear curves discussed above. That the theory applies to flat curves is clear since we require no more than one derivative to exist.

4. Even when $n = 2$ there are curves satisfying the hypotheses of our theorem for which it is *not* true that the map $t \mapsto \xi\,\Gamma'(t)$ has

boundedly many changes of monotonicity in any dyadic interval. For example, let $\Gamma(t) = (t^2/2, t^3/3 + \int_0^t s^7 \sin(s^{-5}/6)\,ds)$. We thank Jim Wright for pointing this out to us.

5. As in [CóRdeF], balance is necessary for a curve γ in class (K) to have $\mathcal{H}_\Gamma$ bounded on any L^p. Indeed, if $\mathcal{H}_\Gamma$ is bounded on $L^p(\mathbb{R}^n)$ and π is a projection onto any subspace of $\mathbb{R}^n$, then $\mathcal{H}_{\pi\Gamma}$ is bounded on $L^p(\pi\mathbb{R}^n)$ and hence on $L^2(\pi\mathbb{R}^n)$. In particular,

$$\Big| \int_{-\infty}^{+\infty} e^{i\xi_k\, \gamma_k(t)}\, \frac{dt}{t} \Big| \le C\,, \qquad (1 \le k \le n)\,.$$

But if $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, satisfies $\rho(0) = 0$ and $|\rho'|$ is increasing for $t > 0$ and decreasing for $t < 0$, and if ρ also satisfies

$$\Big| \int_{-\infty}^{+\infty} e^{i\xi\, \rho(t)}\, \frac{dt}{t} \Big| \le C\,,$$

then for any $s, t > 0$ such that $|\rho(t)/\rho(-s)| = 1$, we must have $|\log(s/t)| \le 8 + C$; see [CóRdeF] (where $\rho'(0) = 0$ was assumed but not used). An additional feature of this present paper is the clarification of the role of balance as being the condition necessary to ensure compatibility of the two Calderón-Zygmund theories naturally associated to the two halves of the curve corresponding to $t > 0$ and $t < 0$.

6. If $\gamma_1, \gamma_3, \ldots$ are odd and $\gamma_2, \gamma_4, \ldots$ are even, then (D) is necessary for a curve of class (K) with $\gamma'_{j+1}(0)/\gamma'_j(0) = 0$, $j = 1, \ldots, n-1$ to have $\mathcal{H}_\Gamma$ bounded on any L^p. Indeed it is enough to see that if $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$ in $\mathbb{R}^2$ with γ_1 odd and γ_2 even, $\gamma'_2(0)/\gamma'_1(0) = 0$, γ'_1, γ'_2/γ'_1 increasing on $(0, +\infty)$, then doubling of γ'_2/γ'_1 is necessary for $\mathcal{H}_\Gamma$ to be bounded on $L^p(\mathbb{R}^2)$. When $\gamma_1(t) = t$, this was done in [NVWW1]; see also [CCC...]. The argument of the latter paper easily adapts in the present situation.

7. If $\Gamma : [0, +\infty) \longrightarrow \mathbb{R}^n$ is convex in the sense that its D_j's are positive and also Γ satisfies certain normalisation conditions at the origin, it is not difficult to see (*cf.* [Z1, Lemmas 3.3 and 3.4]) that after applying an appropriate lower triangular matrix with ones on the diagonal to Γ, (not affecting its convexity) we may assume that γ'_j is positive and increasing for $j = 1, \ldots, n$ and that γ'_{j+1}/γ'_j is increasing for $j = 1, \ldots, n-1$. So (K) is satisfied by this modified curve. If this modified curve is extended to be odd, doubling of the γ'_{j+1}/γ'_j ($j = 1, \ldots, n-1$) now implies doubling of the h_j's for the modified (and

hence the original) curve since this latter condition is equivalent to L^2 boundedness of $\mathcal{H}_\Gamma$ for such curves.

2. Proof of Theorem.

We define the measures σ_k and μ_k by

$$\int f\,d\mu_k = \frac{1}{2^k}\int_{2^k}^{2^{k+1}} f(\Gamma(t))\,dt\,,$$

$$\int f\,d\sigma_k = \int\limits_{2^k\le|t|\le 2^{k+1}} f(\Gamma(t))\,\frac{dt}{t}\,.$$

We decompose $\mathcal{H}_\gamma f$ as

$$\mathcal{H}_\Gamma f = \sum_{k\in\mathbb{Z}} \sigma_k * f$$

and majorize $\mathcal{M}_\Gamma f$ by

$$\mathcal{M}_\Gamma f \le C \sup_k \mu_k * |f|\,.$$

Notice that

$$\hat{\mu}_k(\xi) = \frac{1}{2^k}\int_{2^k}^{2^{k+1}} e^{2\pi i\xi\cdot\Gamma(t)}\,dt$$

and

$$\hat{\sigma}_k(\xi) = \int\limits_{2^k\le|t|\le 2^{k+1}} e^{2\pi i\xi\cdot\Gamma(t)}\,\frac{dt}{t}\,.$$

In keeping with [CCVWW] and [CVWW] we shall introduce to the problem a family of dilation matrices which allows us to normalize the measures μ_k and σ_k so that the Fourier transforms of the normalized measures have uniform decay estimates (save for certain exceptional sets of directions which we shall handle separately, as in [CCC...], [CóRdeF], [NVWW2], [Z2].) Indeed, a combination of the ideas of [CCVWW], [CCC...], [DRdeF], and [Ch] yields the following proposition, which may be found essentially in [Z0, Theorem 2.4.1, p. 22].

Proposition. *Let $n \geq 2$. Suppose that $\{A_k\}_{k\in\mathbb{Z}}$ is a family of matrices in $GL(n,\mathbb{R})$ satisfying*

$$\|A_{k+1}^{-1} A_k\| \leq \alpha < 1 . \tag{1}$$

Suppose $\{\nu_k\}_{k\in\mathbb{Z}}$ is a family of measures such that

$$A_{k+1}^{-1} \operatorname{supp} \nu_k \subseteq B \tag{2}$$

(where B is some fixed ball). Suppose that

$$\int d\nu_k = 0 \tag{3}$$

and that

$$|\hat{\nu}_k(\xi)| \leq C\,|A_k^* \xi|^{-1}, \quad \text{except when } \xi \text{ belongs to a cone } C_k . \tag{4}$$

Letting $\widehat{(T_k f)}(\xi) = \chi_{C_k}(\xi)\,\hat{f}(\xi)$, we further suppose that

$$\Big\| \Big(\sum_k |T_k f|^2 \Big)^{1/2} \Big\|_p \leq C_p \, \|f\|_p , \qquad (1 < p < \infty) . \tag{5}$$

Then $f \mapsto \sum \nu_k * f$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The differences between this proposition and Theorem 2.4.1 of [Z0] are that in [Z0], the conclusion was stated directly in terms of $\mathcal{H}_\Gamma$ (for Γ odd) and $\mathcal{M}_\Gamma$, and that an auxiliary Littlewood-Paley inequality

$$\Big\| \sum_k T_k f_k \Big\|_p \leq C \Big\| \Big(\sum_k |T_k f_k|^2 \Big)^{1/2} \Big\|_p$$

was required there also. However, when T_k corresponds to the characteristic function of a cone, this inequality follows from (5) as may be seen by taking $S_k = T_k$ in the following standard lemma whose proof is omitted.

Lemma 1. *Assume $\Big\| \Big(\sum_k |T_k f|^2 \Big)^{1/2} \Big\|_{p'} \leq C\,\|f\|_{p'}$ for all $f \in L^{p'}$. Then*

$$\Big\| \sum_k T_k S_k f_k \Big\|_p \leq C \Big\| \Big(\sum_k |S_k f_k|^2 \Big)^{1/2} \Big\|_p .$$

Corollary. *If*

$$\Big\| \Big(\sum_k |T_k f|^2 \Big)^{1/2} \Big\|_p \le C_p \, \|f\|_p$$

and

$$\Big\| \Big(\sum_k |S_k f|^2 \Big)^{1/2} \Big\|_p \le C \, \|f\|_p$$

for all $f \in L^p$, $1 < p < \infty$, *then*

$$\Big\| \Big(\sum_k |T_k S_k f|^2 \Big)^{1/2} \Big\|_p \le C'_p \, \|f\|_p \,, \qquad 1 < p < \infty \,.$$

PROOF. Take $f_k = \pm f$ in the lemma and average over the choice of $\pm$ in the usual way.

Notice that if Φ is a fixed C^∞ function of compact support in $\mathbb{R}^n$ then the measures $\Phi_k(x)\,dx = (\det A_k)^{-1}\Phi(A_k^{-1}x)\,dx$ satisfy (2) and (4) (with no exceptional set of directions).

To handle the maximal function we set $A_k = A(2^k)$ where, for $t > 0$,

$$A(t) = \begin{pmatrix} \gamma_1(t) & 0 & \dots & 0 \\ 0 & \gamma_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_n(t) \end{pmatrix}$$

(in keeping with the diagonally invariant nature of the problem), and $\nu_k = \pm[\mu_k - \Phi_k \, dx]$ where Φ is normalized so that $\int d\nu_k = 0$ and so (3) holds. Then (1) and (2) follow from the fact that each $|\gamma_j'|$ is increasing.

Let $M > 1$ be large and fixed and let

$$C_k = \bigcup_{1 \le i < j \le n} C_k^{ij}$$

where

$$C_k^{ij} = \left\{ \xi : \ \frac{1}{M} \left| \frac{\gamma_j'(2^k)}{\gamma_i'(2^k)} \right| \le \left| \frac{\xi_i}{\xi_j} \right| \le M \left| \frac{\gamma_j'(2^{k+1})}{\gamma_i'(2^{k+1})} \right| \right\}.$$

We shall show that if $\xi \notin C_k$, then

$$|\hat{\mu}_k(\xi)| \le \frac{C}{|A_k^* \xi|} \,,$$

(*i.e.* (4) holds for μ_k, see Lemma 3 below), and that (5) holds for $\{C_k\}$.

Assuming we have done this we have then

$$\begin{aligned}\mathcal{M}_\Gamma f &\le \sup_k \mu_k * |f| \\ &\le \Big(\sum_k |\mu_k * f - \Phi_k * f|^2\Big)^{1/2} + \sup_k |\Phi_k * f|.\end{aligned}$$

the first term is controlled on L^p by averaging the conclusion of the Proposition over all choices of arbitrary $\pm$ in the standard way, while the second is controlled by the Hardy-Littlewood maximal function (defined with respect to the dilations $A(t)$) as in [NVWW2] or [CCVWW]. Thus $\mathcal{M}_\Gamma$ is bounded on L^p.

Before proceeding to prove (4) and (5), let us see what differences are involved when we instead consider $\mathcal{H}_\Gamma$. We repeat the above arguments, but now with $\sigma_k^{\pm}$ in place of μ_k where

$$\int f \, d\sigma_k^+ = \int_{2^k \le t \le 2^{k+1}} f(\Gamma(t)) \, \frac{dt}{t}$$

and $\sigma_k^- = \sigma_k^+ - \sigma_k$. For σ_k^+ we have matrices A_k^+ and for σ_k^- we have A_k^-; the same arguments as for $\mathcal{M}_\Gamma$ allow us to conclude that both

$$\sum(\sigma_k^+ * f - \Phi_k^+ * f)$$

and

$$\sum(\sigma_k^- * f - \Phi_k^- * f)$$

are bounded on L^p, $1 < p < \infty$, once the estimates corresponding to (4) and (5) have been established. Hence

$$\sum \sigma_k * f = \sum(\sigma_k^+ - \sigma_k^-) * f$$

differs from an L^p bounded operator $(1 < p < \infty)$ by

$$\sum(\Phi_k^+ - \Phi_k^-) * f.$$

This latter operator is easily seen to be a Calderón-Zygmund operator (with respect to the dilations $A(t)$, as in [CCVWW]) and so is L^p

bounded ($1 < p < \infty$) and of weak type 1 if and only if it is L^2 bounded; this occurs if and only if

$$\Big|\sum_k \hat{\Phi}_k^+(\xi) - \hat{\Phi}_k^-(\xi)\Big| = \Big|\sum_k \hat{\Phi}(A_k^{+*}\xi) - \hat{\Phi}(A_k^{-*}\xi)\Big|$$

defines a bounded function on $\mathbb{R}^n$.

Lemma 2. *If there is an $r \geq 0$ so that $j - k \geq r$ implies that*

$$\|A_j^{+^{-1}} A_k^-\| \leq 1 \qquad \textit{and} \qquad \|A_j^{-^{-1}} A_k^+\| \leq 1$$

for all j, then

$$\sup_{\xi\in\mathbb{R}^n} \Big|\sum_k \hat{\Phi}(A_k^{+*}\xi) - \hat{\Phi}(A_k^{-*}\xi)\Big|$$

is finite.

The easy proof is left to the reader. (Note that this lemma provides a compatibility condition between the two halves of the curve so that the two Calderón-Zygmund theories generated should be consistent.) In our present case, the hypothesis of this lemma reduces to the condition that Γ should be balanced.

So, the proof of the theorem will be finished once we have established the estimates (4) and (5).

Lemma 3. *If $\xi \notin C_k$ then*

$$|\hat{\mu}_k(\xi)| \leq \frac{C}{|A_k^*\xi|}\,.$$

PROOF. We first observe that $\xi \notin C_k$ implies the existence of an m, $1 \leq m \leq n$, such that for $2^k \leq t \leq 2^{k+1}$ and $j \neq m$, we have

$$\left|\frac{\gamma_j'(t)}{\gamma_m'(t)}\right| \leq \frac{1}{M}\left|\frac{\xi_m}{\xi_j}\right|. \tag{6}$$

Indeed, $\xi \notin C_k$ implies that if $1 \leq i < j \leq n$, then either

$$\left|\frac{\gamma_j'(t)}{\gamma_i'(t)}\right| \geq \left|\frac{\gamma_j'(2^k)}{\gamma_i'(2^k)}\right| \geq M\left|\frac{\xi_i}{\xi_j}\right|,$$

for all t such that $2^k \leq t \leq 2^{k+1}$, or

$$\left|\frac{\gamma_j'(t)}{\gamma_i'(t)}\right| \leq \left|\frac{\gamma_j'(2^{k+1})}{\gamma_i'(2^{k+1})}\right| \leq \frac{1}{M}\left|\frac{\xi_i}{\xi_j}\right|,$$

for all t such that $2^k \leq t \leq 2^{k+1}$.

Thus of the functions $t \mapsto |\xi_j||\gamma_j'(t)|$ $(1 \leq j \leq n)$, one exceeds all the others by a factor of at least M uniformly on $[2^k, 2^{k+1}]$. If this is the function $|\xi_m|\,|\gamma_m'(t)|$, (6) follows. Now, with $\xi \notin C_k$ and m fixed and satisfying (6) we can write

$$\begin{aligned}\hat{\mu}_k(\xi) &= \frac{1}{2^k}\int_{2^k}^{2^{k+1}} e^{2\pi i\xi\cdot\Gamma(t)}\,dt \\ &= \frac{1}{2^k}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})} e^{2\pi i\xi\cdot\Gamma\circ\gamma_m^{-1}(s)}\,\frac{ds}{\gamma_m'\circ\gamma_m^{-1}(s)}\,.\end{aligned}$$

Letting

$$\phi(s) = 2\pi\xi\cdot\Gamma\circ\gamma_m^{-1}(s)$$

we see that

$$\phi'(s) = 2\pi\sum_{j=1}^{n}\xi_j\,\frac{\gamma_j'(\gamma_m^{-1}(s))}{\gamma_m'(\gamma_m^{-1}(s))}\,,$$

which, by virtue of (6) satisfies

$$|\phi'(s)| \geq 2\pi\,|\xi_m|\left(1-\frac{n-1}{M}\right) \geq C\,|\xi_m|$$

(if M is sufficiently large), for all $\gamma_m(2^k) \leq s \leq \gamma_m(2^{k+1})$. We now set

$$w(s) = \frac{e^{i\phi(s)}}{\phi'(s)\,\gamma_m'\circ\gamma_m^{-1}(s)}\,,$$

so that

$$\begin{aligned}w'(s) = {}& \frac{ie^{i\phi(s)}}{\gamma_m'\circ\gamma_m^{-1}(s)} + \frac{e^{i\phi(s)}}{\phi'(s)}\left(\frac{1}{\gamma_m'\circ\gamma_m^{-1}}\right)'(s) \\ & - \frac{e^{i\phi(s)}\,\phi''(s)}{(\phi'(s))^2\,\gamma_m'\circ\gamma_m^{-1}(s)}\end{aligned}$$

and

$$\begin{aligned}
|\hat{\mu}_k(\xi)| &\leq \frac{1}{2^k}\left|w(\gamma_m(2^{k+1})) - w(\gamma_m(2^k))\right| \\
&\quad + \frac{C}{2^k}\frac{1}{|\xi_m|}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})}\left|\Big(\frac{1}{\gamma_m'\circ\gamma_m^{-1}}\Big)'\right| ds \\
&\quad + \frac{C}{2^k}\frac{1}{|\xi_m|^2}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})}\frac{|\phi''(s)|}{|\gamma_m'\circ\gamma_m^{-1}(s)|}\, ds \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

by the Fundamental Theorem of Calculus and the fact that $|\phi'(s)| \geq C|\xi_m|$.

Now

$$\begin{aligned}
\mathrm{I} &\leq \frac{C}{2^k|\xi_m|}\frac{1}{|\gamma_m'(2^k)|} \\
&\leq \frac{C}{2^k\sum_{j=1}^n|\xi_j|\,|\gamma_j'(2^k)|} \qquad \text{(by (6) with } t = 2^k) \\
&\leq \frac{C}{\sum_{j=1}^n|\xi_j|\,|\gamma_j(2^k)|} \leq \frac{C}{|A_k^*\xi|}
\end{aligned}$$

(since each $|\gamma_j'|$ is monotonic).

Similarly

$$\mathrm{II} \leq \frac{C}{2^k|\xi_m|}\frac{1}{|\gamma_m'(2^k)|} \leq \frac{C}{|A_k^*\xi|}$$

since $\gamma_m'\circ\gamma_m^{-1}$ is monotonic on $[2^k, 2^{k+1}]$.

Finally, for III, it is enough to show that for each $j \neq m$,

$$\frac{|\xi_j|}{|\xi_m|}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})}\left|\Big(\frac{\gamma_j'\circ\gamma_m^{-1}}{\gamma_m'\circ\gamma_m^{-1}}\Big)'(s)\right|\frac{ds}{|\gamma_m'\circ\gamma_m^{-1}(s)|} \leq \frac{C}{|\gamma_m'(2^k)|}$$

and then argue as for I. But since $|\gamma_m'\circ\gamma_m^{-1}(s)| \geq |\gamma_m'(2^k)|$ and $(\gamma_j'/\gamma_m')\circ\gamma_m^{-1}$ is monotonic, this estimate reduces to

$$\frac{|\xi_j|}{|\xi_m|}\left|\frac{\gamma_j'}{\gamma_m'}(2^{k+1}) - \frac{\gamma_j'}{\gamma_m'}(2^k)\right| \leq C,$$

which follows from (6). This completes the proof of the Lemma.

Finally, to establish (5), we first note that

$$\chi_{C_k} = \sum_{i<j} \chi_{C_k^{ij}} + \sum_{\substack{i<j \\ \ell<m \\ (i,j)\neq(l,m)}} \chi_{C_k^{ij}}\, \chi_{C_k^{lm}} + \ldots \pm \prod_{i<j} \chi_{C_k^{ij}} \,.$$

Setting $\widehat{(T_k^{ij} f)}(\xi) = \chi_{C_k^{ij}}(\xi)\, \hat{f}(\xi)$, we see that, using the Corollary to Lemma 1, it suffices to prove

$$\Big\| \Big(\sum_k |T_k^{ij} f|^2 \Big)^{1/2} \Big\|_p \le C_p \, \|f\|_p \,, \qquad 1 < p < \infty \,.$$

But this is now a two-dimensional inequality, and as such follows from the following lemma and assumption (D).

Lemma 4. *If $M > 1$, $\rho > 1$ and $\lambda_{k+1} \ge \lambda_k \, \rho$ ($k \in \mathbb{Z}$), and we define for $\xi \in \mathbb{R}^2$*

$$\widehat{(R_k f)}(\xi) = \begin{cases} \hat{f}(\xi)\,, & \text{if } \dfrac{\lambda_k}{M} \le \dfrac{|\xi_1|}{|\xi_2|} \le M \, \lambda_{k+1} \\ 0\,, & \text{otherwise,} \end{cases}$$

then

$$\Big\| \Big(\sum_k |R_k f|^2 \Big)^{1/2} \Big\|_p \le C_p \, \|f\|_p \,, \qquad 1 < p < \infty \,.$$

PROOF. It is a standard exercise, using the Marcinkiewicz multiplier theorem and the theorem of Nagel, Stein and Wainger on Differentiation in Lacunary Directions, see [NSW]. The proof of our main theorem is complete.

3. Concluding Remarks.

1. By projection on the first co-ordinate, the following assertion is contained in the theorem: if $\gamma(t)$ satisfies $|\gamma'(t)|$ increasing for $t > 0$ and

decreasing for $t < 0$, $\gamma(0) = 0$, and $|\gamma(-t)/\gamma(Ct)|$, $|\gamma(t)/\gamma(-Ct)| \leq 1$ (all $t > 0$, some C) then

$$f \mapsto \int_{-\infty}^{+\infty} f(x - \gamma(t)) \frac{dt}{t}$$

is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$. Indeed more is true: this operator is of weak type 1. (The corresponding fact for maximal functions, and the L^p boundedness of this operator are well-known -see [W] or [CóRdeF] and [W] respectively.) Since we already know that the operator is bounded on L^2 it is enough to see that the convolution kernel satisfies an appropriate variant of the Hörmander condition. As in the proof of the main theorem, the balance condition allows us to reduce this to verification of a Hörmander condition for $t > 0$. Now, formally

$$\int_0^{+\infty} f(x - \gamma(t)) \frac{dt}{t} = K * f(x),$$

where

$$K(x) = \frac{1}{\gamma'(\gamma^{-1}(x))\, \gamma^{-1}(x)},$$

and we may clearly assume without loss of generality that γ' is positive and increasing on $(0, +\infty)$ so that K is decreasing. Although we have no estimates on the derivative of K, monotonicity is enough to allow us to apply the Hörmander criterion in the form

$$\sup_{j \in \mathbb{Z}} \sup_{0 < y \leq 3\gamma(2^j)} \int_{x \geq 5\gamma(2^j)} |K(x) - K(x-y)|\, dy \leq C$$

(see [CCVWW, Theorem 2.3]). Indeed, for j fixed and $y \leq 3\gamma(2^j)$,

$$\begin{aligned} \int_{x \geq 5\gamma(2^j)} |K(x) - K(x-y)|\, dy &= \int_{5\gamma(2^j)-y}^{+\infty} K(x)\, dx - \int_{5\gamma(2^j)}^{+\infty} K(x)\, dx \\ &\leq \int_{2\gamma(2^j)}^{5\gamma(2^j)} K(x)\, dx \\ &= \log \frac{t_1}{t_0}, \end{aligned}$$

where $\gamma(t_0) = 2\gamma(2^j)$, and $\gamma(t_1) = 5\gamma(2^j)$. But γ' increasing implies γ doubles, so $\log(t_1/t_0)$ is bounded.

2. Finally, we ask the following question: is there a condition involving at most one derivative of an odd curve Γ which gives L^2 boundedness of $\mathcal{H}_\Gamma$ but not L^p boundedness for all $1 < p < \infty$? (In $\mathbb{R}^2$, γ' doubling is strictly stronger than h doubling for convex curves.)

Acknowledgement: Both authors would like to thank Dr. Jim Wright for valuable conversations on the contents of this paper and in particular on Lemma 3. We would also like to thank Professor S. Wainger and the anonymous referee for their useful comments.

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Recibido: 25 de febrero de 1.993

A. Carbery
Centre for Mathematical Analysis and its Applications
University of Sussex
Falmer, Brighton BN1 9QH
UNITED KINGDOM

and

S. Ziesler*
Mathematics Department
University of Wisconsin
Madison, WI 53706
U.S.A.

* Supported by SERC