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Estimates on the solution of an elliptic equation related to Brownian motion with drift

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1. Introduction.

In this paper we are concerned with studying the Dirichlet problem for an elliptic equation on a domain in $\mathbb{R}^3$. For simplicity we shall assume that the domain is a ball Ω_R of radius R. Thus

$$\Omega_R = \{x \in \mathbb{R}^3 : \ |x| < R\} . \tag{1.1}$$

The equation we are concerned with is given by

$$(-\Delta - \mathbf{b}(x) \cdot \nabla)\, u(x) = f(x) , \qquad x \in \Omega_R , \tag{1.2}$$

with zero Dirichlet boundary conditions,

$$u(x) = 0 , \qquad x \in \partial\Omega_R . \tag{1.3}$$

Here we shall think of the functions $\mathbf{b}(x), f(x)$ as defined on all of $\mathbb{R}^3$. Thus we shall assume that

$$\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3 , \qquad f : \mathbb{R}^3 \to \mathbb{R} , \tag{1.4}$$

are Lebesgue measurable functions. It is well known [5], [11] that the solution of (1.2)-(1.3) has -at least in the case of smooth functions $\mathbf{b}, f$- a representation as an expectation value with respect to Brownian motion with drift $\mathbf{b}$. Thus

$$u(x) = E_x\left[\int_0^\tau f(X_{\mathbf{b}}(t))\,dt\right], \tag{1.5}$$

where E_x denotes the expectation is taken with respect to the drift process $X_{\mathbf{b}}(t)$ starting at $x \in \Omega_R$, and τ is the first hitting time on the boundary $\partial\Omega_R$.

Our main goal here is to prove existence and uniqueness of solutions to the boundary value problem (1.2)-(1.3) when the drift $\mathbf{b}$ is allowed to have singularities. To specify which kind of singularities $\mathbf{b}$ can have we define the Morrey spaces $M_p^q(\mathbb{R}^3)$ for $1 \le p \le q < \infty$. A measurable function $g : \mathbb{R}^3 \to \mathbb{C}$ is in $M_p^q(\mathbb{R}^3)$ if $|g|^p$ is locally integrable and there is a constant C such that

$$\int_Q |g|^p\,dx \le C^p\,|Q|^{1-p/q}, \tag{1.6}$$

for all cubes $Q \subset \mathbb{R}^3$. Here $|Q|$ denotes the volume of Q. The norm of g, $\|g\|_{q,p}$ is defined as

$$\|g\|_{q,p} = \inf\{C :\ (1.6) \text{ holds for } C \text{ and all cubes } Q\}. \tag{1.7}$$

It is easy to see that, with the definition (1.7) of norm, the space $M_p^q(\mathbb{R}^3)$ is a Banach space. Let $L^q(\mathbb{R}^3)$ be the standard L^q space on $\mathbb{R}^3$ with norm denoted by $\|\cdot\|_q$. Then one has the relationships for $1 \le r \le p \le q < \infty$,

$$\begin{aligned} L^q(\mathbb{R}^3) &= M_q^q(\mathbb{R}^3) \subset M_p^q(\mathbb{R}^3) \subset M_r^q(\mathbb{R}^3), \\ \|g\|_q &= \|g\|_{q,q} \ge \|g\|_{q,p} \ge \|g\|_{q,r}. \end{aligned} \tag{1.8}$$

Our first theorem is a perturbation theory result.

Theorem 1.1. *Suppose $1 < r < p \le q$ and $|\mathbf{b}| \in M_p^3$, $f \in M_r^q$ for some q, with $3/2 < q < 3$. Then there exists an $\varepsilon_0 > 0$ depending only on r, p, q such that if $\varepsilon \in \mathbb{C}$, $|\varepsilon| < \varepsilon_0/\|\mathbf{b}\|_{3,p}$, then the boundary value problem*

$$(-\Delta - \varepsilon\,\mathbf{b}(x)\cdot\nabla)\,u_\varepsilon(x) = f(x), \qquad x \in \Omega_R, \tag{1.9}$$

$$u_\varepsilon(x) = 0, \qquad x \in \partial\Omega_R, \tag{1.10}$$

has a unique solution u_ε *in the following sense:*

a) u_ε *is uniformly Hölder continuous on* Ω_R *and satisfies the boundary condition* (1.10),

b) The distributional Lapacian Δu_ε *of* u_ε *on* Ω_R *is in* M^q_r *and the equation* (1.9) *holds for almost every* $x \in \Omega_R$,

c) $u_\varepsilon(x)$ *is an analytic function of* ε *in the disk* $|\varepsilon| < \varepsilon_0$ *for any fixed* $x \in \Omega_R$,

d) The L^∞ *norm of* u_ε *is bounded by*

$$\|u_\varepsilon\|_\infty \le C\, R^{2-3/q}\, \|f\|_{q,r}\ , \tag{1.11}$$

where the constant C *depends only on* p, q, r.

REMARK. The restriction that f is L^q integrable for $q < 3$ is artificial since if $f \in L^{q_0}$ for some $q_0 \ge 3$ then $f \in L^q$ for all $q \le q_0$. The $q < 3$ restriction is related to $b)$ and the value of ε_0 .

Theorem 1.1 will be derived from a theorem on integral equations. Let T be an integral operator with measurable kernel $k_T : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$. Thus for measurable $f : \mathbb{R}^3 \to \mathbb{C}$ one defines Tf by

$$Tf(x) = \int_{\mathbb{R}^3} k_T(x,y)\, f(y)\, dy\, . \tag{1.12}$$

Theorem 1.2. *Suppose the kernel* k_T *of the integral operator* T *satisfies the inequality*

$$|k_T(x,y)| \le \frac{|\mathbf{b}(x)|}{|x-y|^2}\ , \qquad x, y \in \mathbb{R}^3\ , \tag{1.13}$$

where $|\mathbf{b}| \in M^3_p$, $1 < p \le 3$. *Then for any* r, q *which satisfy the inequalities*

$$1 < r < p\, , \qquad r \le q < 3\, , \tag{1.14}$$

the operator T *is a bounded operator on the space* M^q_r. *The norm of* T *satisfies the inequality*

$$\|T\| \le C\, \|\mathbf{b}\|_{3,p}\ , \tag{1.15}$$

where the constant C depends only on r, p, q.

Theorem 1.2 generalizes a result of Kerman and Sawyer [8] which proves the theorem in the case of L^q spaces, *i.e.* $r = q$. The more general Theorem 1.2 is necessary to prove Theorem 1.1 even if we assume $f \in L^q$. The Kerman-Sawyer theorem does apply to Theorem 1.1 if we assume $\mathbf{b} \in M_p^3$ with $p > 3/2$.

Next we turn to the non perturbative situation. It is easy to see -by considering the case of $|\mathbf{b}(x)| = \varepsilon/|x|$ with large ε- that (1.2) need not have a solution for $|\mathbf{b}| \in M_p^3$ if we make no restriction on the norm of $|\mathbf{b}|$. To obtain an appropriate non perturbative theorem we pursue an analogy with a problem which has already been studied in great detail. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a measurable potential and consider the problem of estimating the number of bound states $N(V)$ of the Schrödinger operator $-\Delta + V$. It was shown independently by Cwickel-Lieb-Rosenbljum [10] that $N(V)$ satisfies the inequality

$$N(V) \leq C \int_{\mathbb{R}^3} |V(x)|^{3/2}\, dx\,, \tag{1.16}$$

for some universal constant C. The best value for the constant C was obtained by Lieb [9] and is $C = .116$. This is to be contrasted with the lower bound on C, $C \geq .078$ obtained from semi-classical asymptotics. Hence the bound (1.16) with constant $C = .116$ is in some sense very sharp. However it may in fact be a bad estimate such as in the case $V(x) = -\varepsilon/|x|^2$ with ε small. In this situation the right hand side of (1.16) is infinity whereas in fact $N(V) = 0$.

In order to understand the cases where (1.16) gives bad estimates Fefferman and Phong [3] obtained new estimates on $N(V)$ which imply (1.16) and remain finite in the case $V(x) = -\varepsilon/|x|^2$ for small ε. The price one pays is that the constant C in (1.16) which follows from their estimates is far from optimal. The Fefferman-Phong estimate is as follows: Suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. Let $\varepsilon > 0$ be an arbitrary positive number. A cube Q is said to be minimal with respect to ε if

$$\begin{aligned} &\int_Q |V|^p\, dx \geq \varepsilon^p\, |Q|^{1-2p/3}\,, \\ &\int_{Q'} |V|^p\, dx < \varepsilon^p\, |Q'|^{1-2p/3}\,, \quad Q' \subset Q\,, \end{aligned} \tag{1.17}$$

for all dyadic subcubes $Q' \subset Q$. Here p is some fixed number, $1 < p \leq 3/2$. Let $N_\varepsilon(V)$ be the number of minimal cubes in the dyadic decomposition. Then the Fefferman-Phong inequality is given by

$$N(V) \leq C_\varepsilon N_\varepsilon(V), \tag{1.18}$$

where the constant C_ε is finite provided $\varepsilon > 0$ is sufficiently small. Since it is clear that

$$N_\varepsilon(V) \leq \varepsilon^{-3/2} \int_{\mathbb{R}^3} |V(x)|^{3/2}\, dx, \tag{1.19}$$

the inequality (1.16) follows from (1.18).

The analogy between the drift problem (1.2)-(1.3) and the bound state problem for the Schrödinger operator is roughly in making the identification $-|\mathbf{b}|^2 = V$. It has been shown in a previous paper [1] that one can directly estimate the solution of the drift problem with $V = -|\mathbf{b}|^2$. However these estimates are not sharp. In fact there are important differences between the drift problem and the potential problem. For example there is no semi-classical asymptotic limit in which the inequality analogous to (1.16) becomes an identity. That said, our analysis will be close in spirit to the Fefferman-Phong analysis of the potential problem.

We consider the drift problem with non perturbative drift $\mathbf{b}$. Let p be a fixed number $1 < p < 3$ and $\varepsilon > 0$ be arbitrary. Suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. A cube Q is minimal with respect to ε if

$$\begin{aligned} &\int_Q |\mathbf{b}|^p\, dx \geq \varepsilon^p\, |Q|^{1-p/3}, \\ &\int_{Q'} |\mathbf{b}|^p\, dx < \varepsilon^p\, |Q'|^{1-p/3}, \quad Q' \subset Q, \end{aligned} \tag{1.20}$$

for all dyadic subcubes $Q' \subset Q$. Let $N_\varepsilon(\mathbf{b})$ be the number of minimal cubes in the dyadic decomposition. Then we have the following theorem:

Theorem 1.3. *Suppose $f \in M_r^q$, $1 < r \leq q$, $r < p$, $p > 2$, $3/2 < q < 3$. Then there exists $\varepsilon > 0$ depending only on p, q, r such that if $N_\varepsilon(\mathbf{b}) < +\infty$ the boundary value problem (1.2)-(1.3) has a unique solution $u(x)$ in the following sense:*

a) u is uniformly Hölder continuous on Ω_R *and satisfies the boundary condition* (1.3),

b) The distributional Laplacian Δu *of* u *is in* M_r^q *and the equation* (1.2) *holds for almost every* $x \in \Omega_R$.

Our final theorem generalizes the estimate (1.11) on the L^∞-norm of the solution u of (1.2)-(1.3), to the nonperturbative situation. For $\mathbf{b} \in M_p^3$, $p > 1$, and n an integer define a function $a_n : \mathbb{R}^3 \to \mathbb{R}$ by

$$a_n(x) = \left(2^{n(3-p)} \int_{|x-y|<2^{-n}} |\mathbf{b}|^p \, dy \right)^{1/p} . \tag{1.21}$$

We then have the following

Theorem 1.4. *For* $f \in M_r^q, \mathbf{b} \in M_p^3$ *with* $N_\varepsilon(\mathbf{b}) < +\infty$, *let* $u(x)$, $x \in \Omega_R$, *be the solution of the Dirichlet problem* (1.2)-(1.3) *given by Theorem* 1.3. *Let* n_0 *be the integer which satisfies the inequality*

$$4R > 2^{-n_0} \geq 2R . \tag{1.22}$$

Then there exists γ, $0 < \gamma < 1$, *depending only on* $p > 2$ *such that* u *satisfies the* L^∞ *estimate*

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega_R} \exp\left(C_2 \sum_{j=0}^{m} a_{n_0+j}(x) \right) . \tag{1.23}$$

The constant C_1 *depends only on* p, q, r *and* C_2 *only on* $p > 2$.

Theorem 1.4 will be proved in Section 6. We shall also show there that Theorem 1.4 implies the bound

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \exp(C_2 N_\varepsilon(\mathbf{b})) , \tag{1.24}$$

provided ε is sufficiently small depending only on $p > 2$. Since $N_\varepsilon(\mathbf{b})$ satisfies the inequality

$$N_\varepsilon(\mathbf{b}) \leq \varepsilon^{-3} \|\mathbf{b}\|_3^3 , \tag{1.25}$$

the inequality (1.24) implies the bound

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \exp(C_2 \|\mathbf{b}\|_3^3) . \tag{1.26}$$

Inequality (1.26) with $q = r = 3$, is already known [6]. This is the Alexandrov-Pucci estimate, which has been proved here using different ideas.

Our method is based on studying how the drift process $X_{\mathbf{b}}(t)$ differs from Brownian motion $X(t)$. The technical tool we use for this is the Cameron-Martin formula [10] which expresses expectations with respect to the drift process as Brownian motion expectations. Our main idea is that if $p > 2$ then the sets on which $|\mathbf{b}|$ is large have dimension strictly less than 1. Hence, by the nonrecurrence property of Brownian motion in dimension strictly larger than 2, most paths do not often visit sets where $|\mathbf{b}|$ is large.

While there is an extensive recent literature on elliptic equations with nonsmooth coefficients [2], [4], [7], there appears to be little studying the singular drift problem. The most recent paper we could find on the subject was the 1980 paper of Trudinger [12]. See also the book by Friedlin [5] for the relation between functional integration and partial differential equations.

2. A Theorem in Integral Equations.

Our goal in this section is to prove Theorem 1.2. For $x \in \mathbb{R}^3$ and $r > 0$, let $B(x,r)$ be the ball of radius r centered at x,

$$B(x,r) = \{y \in \mathbb{R}^3 : \ |x-y| < r\}. \tag{2.1}$$

We define operators S_n on locally integrable functions u on $\mathbb{R}^3$ for any integer $n \in \mathbb{Z}$ by

$$S_n u(x) = 2^{-n} \, |B(x, 2^{-n})|^{-1} \int_{B(x,2^{-n})} |u(y)| \, dy. \tag{2.2}$$

It is evident then from (1.3) that the operator T satisfies the inequality

$$|Tu(x)| \le C \sum_{n=-\infty}^{\infty} |\mathbf{b}(x)| S_n u(x), \tag{2.3}$$

for some universal constant C.

Let Q_0 be the cube centered at the origin with side of length 2^{-n_0}. We define an operator T_0 by

$$\begin{aligned} T_0 u(x) &= Tu(x), \qquad x \notin Q_0, \\ T_0 u(x) &= 0, \qquad x \in Q_0. \end{aligned} \tag{2.4}$$

Lemma 2.1. *Suppose the support of u is contained in the ball $B(0,2^{-n_0-2})$. Then for $1 < r < p \le 3$, $r \le q \le 3$, there is a constant C depending only on p, q, r such that*

$$\|T_0 u\|_{q,r} \le C\, \|\mathbf{b}\|_{3,p}\, \|u\|_{q,r}\,. \tag{2.5}$$

PROOF. We need to show

$$\Big(\int_Q |T_0 u|^r\, dx\Big)^{1/r} \le C\, \|\mathbf{b}\|_{3,p}\, \|u\|_{q,r}\, |Q|^{1/r-1/q}\,, \tag{2.6}$$

for all cubes Q. First let us consider the case

$$|Q| \le 2^{-3n_0}\,. \tag{2.7}$$

We use the inequality

$$|T_0 u(x)| \le A\, |\mathbf{b}(x)|\, 2^{2n_0}\, \|u\|_1\,, \qquad x \in \mathbb{R}^3\,, \tag{2.8}$$

where the constant A is universal. Hence the left hand side of (2.6) is bounded by

$$A\, 2^{2n_0} \|u\|_1 \Big(\int_Q |\mathbf{b}|^r\, dx\Big)^{1/r} \le A\, 2^{2n_0} \|u\|_1\, \|\mathbf{b}\|_{3,p}\, |Q|^{1/r-1/3}\,, \tag{2.9}$$

on using Hölder's inequality. Next we use the fact that

$$\|u\|_1 \le \|u\|_{q,r}\, |Q_0|^{1-1/q}\,, \tag{2.10}$$

whence

$$\Big(\int_Q |T_0 u|^r\, dx\Big)^{1/r} \le A\, \|\mathbf{b}\|_{3,p}\, \|u\|_{q,r}\, |Q|^{1/r-1/q}\,, \tag{2.11}$$

in view of (2.7).

Next let us suppose that

$$2^{-3k} < |Q| \le 2^{-3(k-1)}\,, \qquad k \le n_0\,, \tag{2.12}$$

and the double of Q contains the origin. Then, by the property of the support of u, one has

$$\int_Q |T_0 u|^r\, dx \le A\, \|u\|_1^r \sum_{n=k-3}^{n_0} 2^{2nr} \int_{Q_n} |\mathbf{b}|^r\, dx\,, \tag{2.13}$$

where the cubes Q_n have side of length 2^{-n} and center at the origin. Using the fact that $\mathbf{b} \in M_p^3$, $p > r$, the inequality (2.13) yields

$$\begin{aligned}\int_Q |T_0 u|^r \, dx &\leq A \|u\|_1^r \sum_{n=k-3}^{n_0} 2^{2nr} \|\mathbf{b}\|_{3,p}^r |Q_n|^{1-r/3} \\ &= A \|u\|_1^r \|\mathbf{b}\|_{3,p}^r \sum_{n=k-3}^{n_0} 2^{3n(r-1)} \\ &\leq B \|u\|_1^r \|\mathbf{b}\|_{3,p}^r \, 2^{3n_0(r-1)} ,\end{aligned} \tag{2.14}$$

for some constant B depending on $r > 1$.

We have then

$$\left(\int_Q |T_0 u|^r \, dx \right)^{1/r} \leq C \|u\|_1 \|\mathbf{b}\|_{3,p} |Q_0|^{1/r-1} . \tag{2.15}$$

Using (2.10) again we conclude that

$$\begin{aligned}\left(\int_Q |T_0 u|^r \, dx \right)^{1/r} &\leq C' \|\mathbf{b}\|_{3,p} \|u\|_{q,r} |Q_0|^{1/r-1/q} \\ &\leq C' \|\mathbf{b}\|_{3,p} \|u\|_{q,r} |Q|^{1/r-1/q} .\end{aligned} \tag{2.16}$$

Finally, if Q satisfies (2.12) and the double of Q does not contain the origin then the inequality (2.16) continues to hold.

Let K be an arbitrary cube in $\mathbb{R}^3$ with side of length 2^{-n_K} for some integer n_K. We associate with K an operator T_K on integrable functions $u : K \to \mathbb{C}$. To do this we decompose K into a dyadic decomposition of cubes Q_n with sides of length 2^{-n}, where $n \geq n_K$. For any cube $Q_n \subset K$ let u_{Q_n} be the average of $|u|$ on Q_n. Then for any $n \geq n_K$ we define the operator S_n by

$$S_n u(x) = 2^{-n} u_{Q_n} , \qquad x \in Q_n . \tag{2.17}$$

The operator T_K is then given by

$$T_K u(x) = \sum_{n=n_K}^{\infty} |\mathbf{b}(x)| \, S_n u(x) , \qquad x \in K . \tag{2.18}$$

We relate the operators T_K to the operator T by the following

Lemma 2.2. *For $z \in Q_0$ let $\tilde{Q}_0(z)$ be the cube centered at z with side of length 2^{2-n_0}. Let u be an arbitrary integrable function supported in the ball $B(0, 2^{-n_0-2})$ and Q an arbitrary cube. Then there is a universal constant C such that for any $r \geq 1$,*

$$\int_{Q\cap Q_0} |Tu(x)|^r \, dx \leq C \int_{Q_0} \frac{dz}{|Q_0|} \int_{Q\cap Q_0} |T_{\tilde{Q}_0(z)} u(x)|^r \, dx \,. \tag{2.19}$$

PROOF. This is a consequence of Jensen's inequality. In fact Jensen implies that

$$\begin{aligned} &\int_{Q_0} \frac{dz}{|Q_0|} \int_{Q\cap Q_0} |T_{\tilde{Q}_0(z)} u(x)|^r \, dx \\ &\qquad \geq \int_{Q\cap Q_0} \Big(\int_{Q_0} \frac{dz}{|Q_0|} T_{\tilde{Q}_0(z)} u(x) \Big)^r dx \,. \end{aligned} \tag{2.20}$$

Now one merely has to note that, because of the restriction on the support of u, one has

$$\int_{Q_0} \frac{dz}{|Q_0|} T_{\tilde{Q}_0(z)} u(x) \geq C\, |Tu(x)| \,, \qquad x \in Q_0 \,, \tag{2.21}$$

for some universal constant C.

The main work in this section will be concerned with bounding the operators T_K.

Theorem 2.3. *Suppose $1 < r < p \leq 3$, $1 < r \leq q < 3$. Then there is a constant C depending only on p, q, r such that*

$$\Big(\int_Q |T_K u|^r \, dx \Big)^{1/r} \leq C \, \|\mathbf{b}\|_{3,p} \, \|u\|_{q,r} \, |Q|^{1/r-1/q} \,, \tag{2.22}$$

where Q is any dyadic subcube of the cube K.

PROOF OF THEOREM 1.2. We can assume without loss of generality that u has compact support where the support of u is contained in a ball $B(0, 2^{-n_0-2})$ for some integer n_0. Let Q be an arbitrary cube. Then by Lemma 2.1 one has

$$\begin{aligned} \Big(\int_Q |Tu|^r \, dx \Big)^{1/r} &\leq \Big(\int_{Q\cap Q_0} |Tu|^r \, dx \Big)^{1/r} + \Big(\int_{Q\setminus Q_0} |Tu|^r \, dx \Big)^{1/r} \\ &\leq \Big(\int_{Q\cap Q_0} |Tu|^r \, dx \Big)^{1/r} + C \, \|\mathbf{b}\|_{3,p} \, \|u\|_{q,r} \, |Q|^{1/r-1/q} \,. \end{aligned} \tag{2.23}$$

Let K be one of the cubes $\tilde{Q}_0(z)$ from Lemma 2.2. Then it is clear that the set $Q \cap Q_0$ is contained in the union of at most eight dyadic subcubes Q' of $\tilde{Q}_0(z)$ with $|Q'| \le |Q|$. Hence one has

$$\begin{aligned}\left(\int_{Q\cap Q_0} |T_K u|^r\,dx\right)^{1/r} &\le \sum_{Q'} \left(\int_{Q'} |T_K u|^r\,dx\right)^{1/r} \\ &\le 8\,C\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,|Q|^{1/r-1/q}\,,\end{aligned} \tag{2.24}$$

by Theorem 2.1.

Now Lemma 2.2 implies

$$\left(\int_{Q\cap Q_0} |Tu|^r\,dx\right)^{1/r} \le C'\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,|Q|^{1/r-1/q}\,, \tag{2.25}$$

for some constant C' depending only on p, q, r. Theorem 1.2 follows now from (2.23), (2.25).

We begin the proof of Theorem 2.3. We shall assume without loss of generality that

$$\|\mathbf{b}\|_{3,p} \le 1\,. \tag{2.26}$$

Lemma 2.4. *Suppose $u : K \to \mathbb{C}$ is an integrable function and $Q' \subset K$ is a dyadic subcube of K such that for all dyadic $Q \subset Q'$ there is the inequality*

$$|Q|^{1/3+\varepsilon} u_Q \le |Q'|^{1/3+\varepsilon} u_{Q'}\,, \tag{2.27}$$

for sufficiently small $\varepsilon > 0$ depending only on r, p. Then the inequality (2.22) holds on Q'.

PROOF. Let N be the integer such that the length of Q' is $2^{-N}, N \ge n_K$. Then one has

$$\begin{aligned}\left(\int_{Q'} |Tu_K|^r\,dx\right)^{1/r} &\le \left(\int_{Q'} \left(|\mathbf{b}(x)| \sum_{n=n_K}^{N-1} S_n u(x)\right)^r dx\right)^{1/r} \\ &\quad + \left(\int_{Q'} \left(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x)\right)^r dx\right)^{1/r}.\end{aligned} \tag{2.28}$$

We estimate the first term on the right in (2.28). Since $q < 3$ one has

$$\sum_{n=n_K}^{N-1} S_n u(x) \le C \, \|u\|_{q,r} \, |Q'|^{1/3-1/q} \,. \tag{2.29}$$

Thus the first term is bounded by

$$\begin{aligned} C \, \|u\|_{q,r} \, |Q'|^{1/3-1/q} \Big(\int_{Q'} |\mathbf{b}(x)|^r \, dx \Big)^{1/r} \\ \le C \, \|u\|_{q,r} \, |Q'|^{1/3-1/q} \, |Q'|^{1/r-1/3} \end{aligned} \tag{2.30}$$

in view of (2.26). Hence the first term is bounded by

$$C \, \|u\|_{q,r} |Q'|^{1/r-1/q} \,, \tag{2.31}$$

which has the form of the right side of (2.22).

To bound the second term on the right in (2.28) we need to decompose $|\mathbf{b}|$. For m an integer let E_m be the set

$$E_m = \{ x \in \mathbb{R}^3 : \; 2^{m-1} < |\mathbf{b}(x)| \le 2^m \} \,. \tag{2.32}$$

We write the sum of $S_n u(x)$ over n as

$$\begin{aligned} \Big(\sum_{n=N}^{\infty} S_n u(x) \Big)^r \\ &= S_N u(x)^r + \sum_{k=N}^{\infty} \Big(\Big(\sum_{n=N}^{k+1} S_n u(x) \Big)^r - \Big(\sum_{n=N}^{k} S_n u(x) \Big)^r \Big) \\ &= S_N u(x)^r \\ &\quad + \sum_{k=N}^{\infty} r \int_0^1 \Big(\sum_{n=N}^{k} S_n u(x) + t \, S_{k+1} u(x) \Big)^{r-1} S_{k+1} u(x) \, dt \,. \end{aligned} \tag{2.33}$$

Now we use (2.27) to obtain the bound

$$\sum_{n=N}^{k} S_n u(x) + t \, S_{k+1} \, u(x) \le (k+2-N) \, 2^{3\varepsilon(k+1-N)} \, |Q'|^{1/3} \, u_{Q'} \,. \tag{2.34}$$

We have then the estimate

$$\begin{aligned}(2.35)\qquad &\Big(\sum_{n=N}^{\infty} S_n u(x)\Big)^r \le \big(|Q'|^{1/3}\, u_{Q'}\big)^{r-1} \\ &\cdot \Big(S_N u(x) + \sum_{k=N+1}^{\infty} r\,(k+1-N)^{r-1}\, 2^{3\varepsilon(r-1)(k-N)}\, S_k u(x)\Big)\,.\end{aligned}$$

For m, k integers with $k \ge N$ let

$$(2.36)\qquad a_{m,k} = \sum_{Q_k \subset Q'} |E_m \cap Q_k|\, u_{Q_k}\,,$$

where the Q_k are dyadic subcubes of Q' with side of length 2^{-k}. Then one has

$$\begin{aligned}&\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x)\Big)^r dx \\ (2.37)\qquad &\le \Big(|Q'|^{1/3}\, u_{Q'}\Big)^{r-1} \sum_{m=-\infty}^{\infty} 2^{mr} \Big(2^{-N}\, a_{m,N} \\ &\quad + \sum_{k=N+1}^{\infty} r\,(k+1-N)^{r-1}\, 2^{3\varepsilon(r-1)(k-N)}\, 2^{-k}\, a_{m,k}\Big)\,.\end{aligned}$$

There are two estimates on $a_{m,k}$ which we use. The first follows from (2.27). Thus

$$(2.38)\qquad a_{m,k} \le |E_m \cap Q'|\, 2^{(1+3\varepsilon)(k-N)}\, u_{Q'}\,.$$

The second is obtained by observing that $|E_m \cap Q_k| \le |Q_k|$, whence

$$(2.39)\qquad a_{m,k} \le |Q'|\, u_{Q'}\,.$$

It follows that for any $\alpha, 0 < \alpha < 1$, the right side of (2.37) is bounded by

$$\begin{aligned}&\Big(|Q'|^{1/3} u_{Q'}\Big)^{r-1} \sum_{m=-\infty}^{\infty} 2^{mr} \Big(2^{-N} \Big(|Q'|\, u_{Q'}\Big)^{\alpha} \Big(|E_m \cap Q'|\, u_{Q'}\Big)^{1-\alpha} \\ &+ \sum_{k=N+1}^{\infty} r(k+1-N)^{r-1} 2^{3\varepsilon(r-1)(k-N)} 2^{-k} \Big(|Q'|\, u_{Q'}\Big)^{\alpha}\end{aligned}$$

$$
\begin{aligned}
&\cdot \Big(|E_m \cap Q'|\, 2^{(1+3\varepsilon)(k-N)}\, u_{Q'} \Big)^{1-\alpha} \Big) \\
&\qquad \le C_\alpha\, |Q'|^{r/3+\alpha}\, u_{Q'}^r \sum_{m=-\infty}^{\infty} 2^{mr}\, |E_m \cap Q'|^{1-\alpha}\,,
\end{aligned} \tag{2.40}
$$

where the constant C_α depends only on $\alpha > 0$.

We bound the sum with respect to m on the right in (2.40) by writing

$$
\begin{aligned}
&\sum_{m=-\infty}^{\infty} 2^{mr} |E_m \cap Q'|^{1-\alpha} \\
&\qquad \le |Q'|^{1-\alpha} \sum_{m=-\infty}^{N-1} 2^{mr} + \sum_{m=N}^{\infty} 2^{mr} |E_m \cap Q'|^{1-\alpha} \\
&\qquad \le C\, |Q'|^{1-\alpha-r/3} + \sum_{m=N}^{\infty} 2^{mr} |E_m \cap Q'|^{1-\alpha}\,.
\end{aligned} \tag{2.41}
$$

In view of (2.26) one has

$$
2^{mp} |E_m \cap Q'| \le 2^p |Q'|^{1-p/3}\,, \tag{2.42}
$$

and consequently it follows because $r < p$ that

$$
\sum_{m=N}^{\infty} 2^{mr} |E_m \cap Q'|^{1-\alpha} \le C_\alpha\, |Q'|^{1-\alpha-r/3}\,, \tag{2.43}
$$

provided $\alpha > 0$ is sufficiently small. We conclude then that there is a constant C depending only on r, p, q such that

$$
\begin{aligned}
\Big(\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x) \Big)^r dx \Big)^{1/r} &\le C\, |Q'|^{1/r} u_{Q'} \\
&\le C\, \|u\|_{q,r}\, |Q'|^{1/r-1/q}\,.
\end{aligned} \tag{2.44}
$$

The inequality (2.44) combined with (2.31) proves the result.

Next we need to remove the restriction (2.27) on the growth of the averages of u on dyadic cubes. To do this we define a Calderón-Zygmund decomposition of Q'. We can assume without loss of generality that $u \in L^\infty(Q')$. Define a function N_1 on Q',

$$
N_1 : Q' \to \{ k \in \mathbb{Z} \cup \{\infty\} :\ k \ge N \}\,, \tag{2.45}
$$

by

a) $N_1(x) = \infty$ if $|Q|^{1/3+\varepsilon} u_Q \le |Q'|^{1/3+\varepsilon} u_{Q'}$ for all dyadic subcubes Q of Q' such that $x \in Q$,

b) Otherwise $2^{-N_1(x)}$ is the length of the side of the largest dyadic cube $Q, x \in Q \subset Q'$, such that $|Q|^{1/3+\varepsilon} u_Q > |Q'|^{1/3+\varepsilon} u_{Q'}$.

We define the set G_1 to be

$$G_1 = \{x \in Q' : \ N_1(x) = \infty\}. \tag{2.46}$$

Since $u \in L^\infty(Q')$ there is a unique finite family $\mathcal{F}_1$ of disjoint dyadic subcubes of Q' such that

$$\bigcup_{Q \in \mathcal{F}_1} Q = Q' \backslash G_1 . \tag{2.47}$$

If $\mathcal{F}_1$ is nonempty then we define a function N_2 on Q' which is analogous to N_1. Thus

a) $N_2(x) = \infty$ if $x \in G_1$,

b) $N_2(x) = \infty$ if $x \in Q' \backslash G_1$ and $|Q|^{1/3+\varepsilon} u_Q \le |\overline{Q}|^{1/3+\varepsilon} u_{\overline{Q}}$ for all dyadic subcubes with $x \in Q \subset \overline{Q} \in \mathcal{F}_1$,

c) Otherwise $2^{-N_2(x)}$ is the length of the side of the largest dyadic cube Q, $x \in Q \subset \overline{Q} \in \mathcal{F}_1$, such that $|Q|^{1/3+\varepsilon} u_Q > |\overline{Q}|^{1/3+\varepsilon} u_{\overline{Q}}$.

Observe that $N_2(x)$ is defined uniquely for x not on the boundary of any cube $\overline{Q} \in \mathcal{F}_1$. Thus it is defined up to a set of measure 0. Furthermore, one has

$$N_2(x) \ge N_1(x) + 1 , \qquad \text{a.e. } x \in Q' . \tag{2.48}$$

Now define G_2 to be the set

$$G_2 = \{x \in Q' \backslash G_1 : \ N_2(x) = \infty\}. \tag{2.49}$$

Then, as with N_1, there is a unique finite family $\mathcal{F}_2$ of disjoint dyadic subcubes of Q' with

$$\bigcup_{Q \in \mathcal{F}_2} Q = Q' \backslash G_1 \backslash G_2 . \tag{2.50}$$

One can continue this procedure inductively to construct a sequence of functions $N_j, j \geq 1$, on Q', a sequence of disjoint subsets $G_j, j \geq 1$, of Q', and a sequence of families $\mathcal{F}_j$ with the properties:

$$a) \quad \bigcup_{j=1}^{\infty} G_j = Q' ,$$

b) $\mathcal{F}_j$ is a finite collection of disjoint dyadic subcubes of Q' such that

$$\bigcup_{Q \in \mathcal{F}_k} Q = Q' \setminus \bigcup_{j=1}^{k} G_j ,$$

c) For any $Q \in \mathcal{F}_k$ let $\overline{Q} \in \mathcal{F}_{k-1}$ be the unique subcube containing Q. Then

$$|Q|^{1/3+\varepsilon} u_Q > |\overline{Q}|^{1/3+\varepsilon} u_{\overline{Q}} ,$$

$$d) \quad N_k(x) = \infty \text{ for } x \in \bigcup_{j=1}^{k} G_j .$$

Otherwise $N_k(x)$ is defined by $2^{-3N_k(x)} = |Q|$ where Q is the unique cube in $\mathcal{F}_k$ with $x \in Q$.

We have constructed families $\mathcal{F}_j, j \geq 1$, of dyadic subcubes of Q'. Let $\mathcal{F}_0 = \{Q'\}$. Then we have

Lemma 2.5. *Suppose $u \in L^\infty(Q')$. Then there is a constant C depending only on r, p, q such that*

$$\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x) \Big)^r dx \leq C \sum_{j=0}^{\infty} \sum_{Q \in \mathcal{F}_j} |Q| \, u_Q^r . \tag{2.51}$$

PROOF. Define a sequence a_j, $j \geq 0$, by

$$\begin{aligned} a_0 = & \int_{Q'} |\mathbf{b}(x)|^r \big(S_N u(x)\big)^r dx \\ & + \int_{Q'} |\mathbf{b}(x)|^r \sum_{k=N}^{N_1(x)-2} r \int_0^1 \Big(\sum_{n=N}^{k} S_n u(x) + t\, S_{k+1} u(x) \Big)^{r-1} \\ & \qquad \cdot S_{k+1} u(x) \, dt \, dx , \end{aligned}$$

(2.52)
$$a_j = \int_{Q'} |\mathbf{b}(x)|^r \sum_{k=N_j(x)-1}^{N_{j+1}(x)-2} r \int_0^1 \Big(\sum_{n=N}^{k} S_n u(x) + t\, S_{k+1}u(x)\Big)^{r-1} \cdot S_{k+1}u(x)\, dt\, dx\,, \qquad j \geq 1\,.$$

In view of (2.33) the left hand side of (2.51) is given by

$$\sum_{j=0}^{\infty} a_j\,. \tag{2.53}$$

It follows directly from the proof of Lemma 2.4 that there is a constant C such that

$$a_0 \leq C\, |Q'|\, u_{Q'}^r\,. \tag{2.54}$$

We wish to show that for $j \geq 1$, one has

$$a_j \leq C \sum_{Q \in \mathcal{F}_j} |Q|\, u_Q^r\,. \tag{2.55}$$

Evidently one has

$$a_j = \int_{N_j(x)<\infty} (\,\cdot\,)\, dx = \sum_{Q\in\mathcal{F}_j} \int_Q (\,\cdot\,)\, dx\,, \tag{2.56}$$

where $(\,\cdot\,)$ denotes the integrand in the formula for a_j. Let us fix a particular $\overline{Q} \in \mathcal{F}_j$ with side of length 2^{-M}, $M > N$, whence $N_j(x) = M$, $x \in \overline{Q}$. By definition of the families $\mathcal{F}_j$ it follows that

$$\begin{aligned}\sum_{n=N}^{M} S_n u(x) &\leq \sum_{n=N}^{M} 2^{-3\varepsilon(M-n)} |\overline{Q}|^{1/3}\, u_{\overline{Q}} \\ &\leq C\, |\overline{Q}|^{1/3} u_{\overline{Q}}\,, \qquad x \in \overline{Q}\,.\end{aligned} \tag{2.57}$$

On the other hand, for $M < k+1 \leq N_{j+1}(x) - 1$, one has

$$\begin{aligned}&\sum_{n=M+1}^{k} S_n u(x) + t\, S_{k+1}u(x) \\ &\qquad \leq (k+1-M)\, 2^{3\varepsilon(k+1-M)} |\overline{Q}|^{1/3}\, u_{\overline{Q}}\,,\end{aligned} \tag{2.58}$$

which is analogous to (2.34). Now one can proceed just as in Lemma 2.4 to obtain (2.55). The result follows then from (2.53) and (2.55).

Lemma 2.6. *There is a constant C depending only on r, p, q such that*

$$\sum_{j=0}^{\infty} \sum_{Q \in \mathcal{F}_j} |Q|\, u_Q^r \le C \int_{Q'} |u|^r \, dx \,. \tag{2.59}$$

PROOF. Since we can assume $\|u\|_\infty < \infty$ there exists an integer $t \ge 1$ such that $\mathcal{F}_t$ is empty. Thus

$$Q' = \bigcup_{j=1}^{t} G_j \,. \tag{2.60}$$

Let us consider a particular $Q \in \mathcal{F}_j$, $0 \le j \le t-1$. It is evident that

$$Q \subset \bigcup_{m=j+1}^{t} G_m \,. \tag{2.61}$$

We wish to estimate $|Q \cap G_m|$ for $m \ge j+1$. We have now

$$\begin{aligned} |Q|\, u_Q &= \int_Q |u| \, dx \\ &\ge \sum_{i=m}^{t} \int_{Q \cap G_i} |u| \, dx \\ &= \sum_{\overline{Q} \in \mathcal{F}_{m-1}, \overline{Q} \subset Q} |\overline{Q}|\, u_{\overline{Q}} \\ &\ge \sum_{\overline{Q} \in \mathcal{F}_{m-1}, \overline{Q} \subset Q} |\overline{Q}| \left(\frac{|Q|}{|\overline{Q}|} \right)^{1/3} u_Q \\ &\ge 2^{(m-j-1)}\, u_Q \sum_{\overline{Q} \in \mathcal{F}_{m-1}, \overline{Q} \subset Q} |\overline{Q}| \\ &= 2^{(m-j-1)}\, u_Q \, |Q \cap \bigcup_{i=m}^{t} G_i| \,. \end{aligned} \tag{2.62}$$

We conclude therefore that

$$\frac{|Q \cap G_m|}{|Q|} \leq 2^{-(m-j-1)} . \tag{2.63}$$

Next we consider

$$\begin{aligned} |Q|\, u_Q^r &= \frac{1}{|Q|^{r-1}} \Big(\int_Q |u|\, dx \Big)^r \\ &= \frac{1}{|Q|^{r-1}} \Big(\sum_{m=j+1}^{t} \int_{Q \cap G_m} |u|\, dx \Big)^r \\ &\leq \frac{1}{|Q|^{r-1}} \Big(\sum_{m=j+1}^{t} a_m^{r'} \Big)^{r/r'} \sum_{m=j+1}^{t} a_m^{-r} \Big(\int_{Q \cap G_m} |u|\, dx \Big)^r , \end{aligned} \tag{2.64}$$

by Hölder's inequality, where a_m is an arbitrary positive sequence and $1/r + 1/r' = 1$. We choose a_m to be given by

$$a_m = \left(\left(\frac{3}{2}\right)^{m-j-1} \frac{|Q \cap G_m|}{|Q|} \right)^{1/r'} \tag{2.65}$$

In view of (2.63) the inequality (2.64) yields

$$\begin{aligned} |Q|\, u_Q^r &\leq C \sum_{m=j+1}^{t} \left(\frac{2}{3}\right)^{(m-j-1)(r-1)} \frac{1}{|Q \cap G_m|^{r-1}} \\ &\qquad \cdot \Big(\int_{Q \cap G_m} |u|\, dx \Big)^r \\ &= C \sum_{m=j+1}^{t} \left(\frac{2}{3}\right)^{(m-j-1)(r-1)} |Q \cap G_m|\, u_{Q \cap G_m} . \end{aligned} \tag{2.66}$$

We conclude then that

$$\sum_{Q \in \mathcal{F}_j} |Q|\, u_Q^r \leq C \sum_{m=j+1}^{t} \left(\frac{2}{3}\right)^{(m-j-1)(r-1)} \int_{G_m} |u|^r\, dx \tag{2.67}$$

by Jensen's inequality. Now if we sum (2.67) with respect to j and use the fact that the sets G_j are disjoint we obtain the inequality (2.59).

Theorem 2.3 now follows immediately from the previous two lemmas and the estimate (2.31) in Lemma 2.4 on the first term on the right hand side of (2.28).

3. Perturbative existence and uniqueness.

We turn to the proof of Theorem 1.1. We first consider the problem of uniqueness of the solution to (1.9), (1.10). Let us write $g(x) = -\Delta u_\varepsilon(x), x \in \Omega_R$, the distributional Laplacian which is assumed to exist by *b*) of Theorem 1.1. Since $g \in M_r^q$ and *a*) of the same theorem it follows by Weyl's lemma that u_ε is given by the formula

$$u_\varepsilon(x) = \int_{\Omega_R} G_D(x,y)\, g(y)\, dy\,, \tag{3.1}$$

where G_D is the Green's function for the Dirichlet Laplacian on Ω_R. Thus

$$G_D(x,y) = \frac{1}{4\pi\,|x-y|} - \frac{1}{4\pi}\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\,, \tag{3.2}$$

where $\overline{y}$ is the conjugate of y in the sphere $\partial\Omega_R$. It follows easily from the representation (3.1) that the distributional gradient ∇u_ε exists as an integrable function on Ω_R and is given by the formula

$$\nabla u_\varepsilon(x) = \int_{\Omega_R} \nabla_x G_D(x,y)\, g(y)\, dy\,, \qquad x \in \Omega_R\,. \tag{3.3}$$

Now let T be the integral operator with kernel k_T given by

$$k_T(x,y) = \begin{cases} \mathbf{b}(x)\cdot\nabla_x G_D(x,y)\,, & x,y \in \Omega_R\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{3.4}$$

It is clear from (3.2) that $\nabla_x G_D$ satisfies the inequality

$$|\nabla_x G_D(x,y)| \le \frac{1}{2\pi\,|x-y|^2}\,. \tag{3.5}$$

Hence Theorem 1.2 applies to the operator T. In view of (3.1), (3.3) equation (1.9) is the same as

$$(I - \varepsilon T)\, g = f\,, \tag{3.6}$$

provided we extend f, g by zero outside Ω_R. By Theorem 1.2 there is an appropriate $\varepsilon_0 > 0$ such that $|\varepsilon| < \varepsilon_0/\|\mathbf{b}\|_{3,p}$ implies that εT as an operator on M_r^q has norm strictly less than 1. Since f, g are assumed to be in M_r^q equation (3.5) implies that g is given by

$$g = (I - \varepsilon T)^{-1} f .$$

Hence g is uniquely determined by f. Since (3.1) shows that u_ε is uniquely determined by g uniqueness of the solution follows.

To prove existence we define g by (3.7) and u_ε by (3.1). Thus $g \in M_r^q$ and $\|g\|_{q,r} \le C \|f\|_{q,r}$ for some constant C depending on ε_0. We shall show that the estimate (1.11) holds. In fact from (3.1) we have

$$\begin{aligned} |u_\varepsilon(x)| &\le \frac{1}{4\pi} \int_{\Omega_R} \frac{|g(y)|}{|x-y|}\, dy \\ &\le \frac{1}{2\pi} \sum_{n=n_0}^{\infty} 2^n \int_{Q_n} |g(y)|\, dy , \end{aligned} \tag{3.8}$$

where Q_n is the cube centered at x with side of length 2^{-n}, and n_0 is the unique integer satisfying

$$4R \le 2^{-n_0} < 8R . \tag{3.9}$$

Using the fact that $g \in M_r^q$ it follows that

$$\begin{aligned} |u_\varepsilon(x)| &\le \sum_{n=n_0}^{\infty} 2^n\, |Q_n|^{1-1/q}\, \|g\|_{q,r} \\ &\le C\, 2^{-2n_0+3n_0/q}\, \|f\|_{q,r} , \end{aligned} \tag{3.10}$$

since $q > 3/2$. The inequality (1.11) follows from (3.9), (3.10). We can generalize the above argument to show that u_ε is Hölder continuous. It is also clear from (3.2) that u_ε satisfies the boundary condition (1.10). Hence $a)$ of Theorem 1.1 holds. To prove $b)$ we use the fact that the distributional gradient of u_ε must be given by (3.3) and the distributional Laplacian of u_ε satisfies $-\Delta u_\varepsilon = g$. Thus we have

$$-\Delta u_\varepsilon(x) - \varepsilon\, \mathbf{b}(x) \cdot \nabla u_\varepsilon(x) = (I - \varepsilon T)\, g(x) , \tag{3.11}$$

for almost every $x \in \Omega_R$.

It follows now from the definition (3.7) of g that the right hand side of (3.11) is just the function $f(x)$. This concludes the proof of $b)$.

Part c) follows by expanding (3.7) out in a Taylor series in ε. This completes the proof of Theorem 1.1.

Next we prove some results which are perturbative in nature but which will be needed to understand the nonperturbative problem. Let $g : \partial\Omega_R \to \mathbb{R}$ be a continuous function and $u(x)$, $|x| < R$ be the solution of the Dirichlet problem

$$\begin{cases} -\Delta u(x) = 0\,, & |x| < R\,, \\ u(x) = g(x)\,, & x \in \partial\Omega_R\,. \end{cases} \tag{3.12}$$

Then u is given by the Poisson formula

$$u(x) = Pg(x) = \frac{1}{4\pi R} \int\limits_{|z|=R} \frac{R^2 - |x|^2}{|x-z|^3}\, g(z)\, dz\,. \tag{3.13}$$

Now the solution of the Dirichlet problem

$$\begin{cases} (-\Delta - \mathbf{b}(x) \cdot \nabla)\, u(x) = 0\,, & |x| < R\,, \\ u(x) = g(x)\,, & x \in \partial\Omega_R\,, \end{cases} \tag{3.14}$$

is given formally by the expression

$$u = Pg + (-\Delta_D)^{-1}(I - T)^{-1}\mathbf{b} \cdot \nabla Pg\,, \tag{3.15}$$

where $(-\Delta_D)^{-1}$ is the inverse of the Dirichlet Laplacian and has kernel (3.2).

The formula (3.15) is not appropriate for drifts $\mathbf{b} \in M_p^3$. The reason is that even if g is Hölder continuous on $\partial\Omega_R$ the function $\nabla Pg(x)$ is not in general an L^∞ function for $|x| < R$. To get around this difficulty we average over the radius of the ball on which we solve the Poisson problem. Thus let us suppose we have a Hölder continuous $g \in C^\alpha(\Omega_R)$ for some α, $0 < \alpha \le 1$. We define $Kg(x)$ formally for $x \in \Omega_{R/2}$ by

$$Kg(x) = \frac{2}{R} \int_{R/2}^{R} u_\lambda(x)\, d\lambda\,, \tag{3.16}$$

where u_λ denotes the solution (3.15) of the Poisson problem on the ball of radius λ. More precisely let P_λ, T_λ, $(-\Delta_{D,\lambda})^{-1}$ be the operators in

(3.15) acting on the ball of radius λ. Denote the kernel of $(I-T_\lambda)^{-1}$ by $H_\lambda(x,y)$ and $(-\Delta_{D,\lambda})^{-1}$ by $G_{D,\lambda}(x,y)$. We can think of H_λ and $G_{D,\lambda}$ as being defined on $\mathbb{R}^3\times\mathbb{R}^3$ by simply extending the functions by zero outside $\Omega_\lambda\times\Omega_\lambda$. It is clear that for $|x|<\lambda$ one has

$$\nabla P_\lambda g(x) = \frac{1}{4\pi\lambda}\int\limits_{|z|=\lambda} \nabla_x\Big(\frac{|z|^2-|x|^2}{|x-z|^3}\Big)\,(g(z)-g(x))\,dz \tag{3.17}$$

since the left hand side of (3.13) is constant for g a constant. It follows therefore that the formal definition (3.16) of K corresponds to

$$\begin{aligned} Kg(x) &= \frac{2}{R}\int\limits_{R/2<|z|<R} \frac{1}{4\pi|z|}\,\frac{|z|^2-|x|^2}{|x-z|^3}\,g(z)\,dz \\ &\quad + \frac{2}{R}\int_{\Omega_R} dw \int_{\Omega_R} dy \int\limits_{R/2<|z|<R} dz\; G_{D,|z|}(x,w)\,H_{|z|}(w,y) \\ &\quad \cdot \mathbf{b}(y)\cdot\Big(\nabla_y \frac{|z|^2-|y|^2}{4\pi|z|\,|y-z|^3}\Big)\,(g(z)-g(y))\,. \end{aligned} \tag{3.18}$$

Proposition 3.1. *There exists a constant $\varepsilon_0>0$ depending only on $p>1$ such that if $\|\mathbf{b}\|_{3,p}<\varepsilon_0$ then $u(x)=Kg(x)$ defined by* (3.18) *on $\Omega_{R/2}$ exists and is Hölder continuous. Further, the distributional Laplacian Δu is in M_r^q for any $r<p$, $q<3$ and*

$$-\Delta u(x) - \mathbf{b}(x)\cdot\nabla u(x) = 0\,, \qquad \textit{for almost every } x\in\Omega_{R/2}\,. \tag{3.19}$$

PROOF. Evidently the first integral on the right in (3.18) yields a Hölder continuous function. We shall show first that the second integral is uniformly bounded. Let T be the integral operator with kernel $|\mathbf{b}(x)|/2\pi|x-y|^2$. Then if ε_0 is sufficiently small the operator $(I-T)^{-1}$ exists in the sense of Theorem 1.2 with kernel $H(x,y)$ say. It is easy to see now that

$$|H_{|z|}(w,y)| \le H(w,y)\,, \qquad w,y\in\mathbb{R}^3\,. \tag{3.20}$$

We also have

$$|G_{D,|z|}(x,w)| \le \frac{1}{4\pi\,|x-w|}\,, \qquad x,w\in\mathbb{R}^3\,.$$

Consequently the second integral is bounded in absolute value by

$$
\begin{aligned}
&\frac{2}{R}\int_{\Omega_R} dw \int_{\Omega_R} dy \int\limits_{R/2<|z|<R} dz \frac{1}{4\pi\,|x-w|} \\
(3.22) \qquad &\cdot H(w,y)\,|\mathbf{b}(y)| \left|\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|z-y|^3}\right| |g(z)-g(y)| \\
&\leq C\int_{\Omega_R} dw \int_{\Omega_R} dy\, \frac{1}{|x-w|} H(w,y)\,|\mathbf{b}(y)|\,,
\end{aligned}
$$

using the fact that g is Hölder continuous. Since $|\mathbf{b}| \in M_r^q$ for any $r < p, q < 3$, Theorem 1.2 implies that the last integral is uniformly bounded in x. The Hölder continuity of u follows similarly.

Let us define $h(x)$ for $x \in \Omega_R$ by

$$
\begin{aligned}
h(x) = \frac{2}{R}&\int_{\Omega_R} dy \int\limits_{R/2<|z|<R} dz\; H_{|z|}(x,y) \\
(3.23) \qquad &\cdot \mathbf{b}(y)\cdot\left(\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|y-z|^3}\right)(g(z)-g(y))\,.
\end{aligned}
$$

By our previous argument it follows that $h \in M_r^q$ for any $r < p$, $q < 3$. We wish to show that the distributional Laplacian Δu on Ω_R is given by

$$
(3.24) \qquad -\Delta u(x) = h(x)\,, \qquad x \in \Omega_R\,.
$$

We have

$$
(3.25) \qquad h(x) = \int\limits_{R/2<|z|<R} h(x,z)\,dz
$$

and

$$
\begin{aligned}
u(x) &= \frac{2}{R}\int_{R/2}^{R} d\lambda\; P_\lambda g(x) \\
(3.26) \qquad &+ \int_{\Omega_R} dw \int\limits_{R/2<|z|<R} dz\; G_{D,|z|}(x,w)\,h(w,z)\,.
\end{aligned}
$$

Let φ be a C^∞ function with compact support in $\Omega_{R/2}$. Then it follows from (3.26) that

$$\begin{aligned}
&\int -\Delta\varphi(x)\,u(x)\,dx \\
&= \int_{\Omega_R} dw \int_{R/2<|z|<R} dz \left(\int -\Delta\varphi(x)\,G_{D,|z|}(x,w)\,dx\right) h(w,z) \\
&= \int dw\,dz\;\varphi(w)\,h(w,z) \\
&= \int dw\;\varphi(w)\,h(w)\,,
\end{aligned} \tag{3.27}$$

on application of Fubini's theorem. Hence we have (3.24). Similarly one has that the distributional gradient ∇u is given by

$$\begin{aligned}
\nabla u(x) = \frac{2}{R}\int \frac{1}{4\pi\,|z|}\nabla_x\Big(\frac{|z|^2-|x|^2}{|x-z|^3}\Big)\,g(z)\,dz \\
+\frac{2}{R}\int_{\Omega_R} dw \int_{\Omega_R} dy \int dz\;\nabla_x G_{D,|z|}(x,w)\,H_{|z|}(w,y) \\
\cdot\,\mathbf{b}(y)\cdot\Big(\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|y-z|^3}\Big)\,(g(z)-g(y))\,.
\end{aligned} \tag{3.28}$$

Let us define for $\delta>0$, $y\in\Omega_R$, $z\in\Omega_R\backslash\Omega_{R/2}$, $f_{\delta,|z|}(y)$ by

$$f_{\delta,|z|}(y) = \mathbf{b}(y)\cdot\Big(\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|y-z|^3+\delta}\Big)\,(g(z)-g(y))\,. \tag{3.29}$$

Then we have for almost every $x\in\Omega_{R/2}$, the identity

$$\begin{aligned}
\int_{\Omega_R} dw \int_{\Omega_R} dy\;\mathbf{b}(x)\cdot\nabla_x G_{D,|z|}(x,w)\,H_{|z|}(w,y)\,f_{\delta,|z|}(y) \\
= -f_{\delta,|z|}(x) + \int_{\Omega_R} dy\;H_{|z|}(x,y)\,f_{\delta,|z|}(y)\,.
\end{aligned} \tag{3.30}$$

This follows since the left hand side of (3.30) is the operator $T_{|z|}(I-T_{|z|})^{-1}$ applied to the function $f_{\delta,|z|}\in M^q_r$ for any $r<p$, $q<3$, and the right hand side is the operator $-I+(I-T_{|z|})^{-1}$ applied to the same function. It is clear then by dominated convergence that

$$\lim_{\delta\to 0}\frac{2}{R}\int_{\Omega_R} dy \int_{R/2<|z|<R} dz\;H_{|z|}(x,y)\,f_{\delta,|z|}(y) = h(x)\,, \tag{3.31}$$

for almost every $x \in \Omega_{R/2}$. It follows trivially that

$$\begin{aligned}\lim_{\delta\to 0}\frac{2}{R}\int\limits_{R/2<|z|<R} dz\, f_{\delta,|z|}(x)\\ = \mathbf{b}(x)\cdot\frac{2}{R}\int\limits_{R/2<|z|<R}\frac{1}{4\pi|z|}\nabla_x\Big(\frac{|z|^2-|x|^2}{|x-z|^3}\Big)\, g(z)\, dz\,,\end{aligned}\tag{3.32}$$

for any $x \in \Omega_{R/2}$. Again dominated convergence and (3.28) implies that

$$\begin{aligned}&\lim_{\delta\to 0}\frac{2}{R}\int_{\Omega_R} dw\int_{\Omega_R} dy\\ &\qquad\int\limits_{R/2<|z|<R} dz\; \mathbf{b}(x)\cdot\nabla_x G_{D,|z|}(x,w)\, H_{|z|}(w,y)\, f_{\delta,|z|}(y)\\ &= \mathbf{b}(x)\cdot\nabla u(x)\\ &\quad - \mathbf{b}(x)\cdot\frac{2}{R}\int\limits_{R/2<|z|<R}\frac{1}{4\pi|z|}\nabla_x\Big(\frac{|z|^2-|x|^2}{|x-z|^3}\Big)\, g(z)\, dz\,.\end{aligned}\tag{3.33}$$

It follows then from (3.30) to (3.33) that

$$\mathbf{b}(x)\cdot\nabla u(x) = h(x)\,,\qquad \text{for almost every } x\in\Omega_{R/2}\,.\tag{3.34}$$

Hence (3.24) and (3.34) implies (3.19).

Proposition 3.2. *Suppose u is a Hölder continuous function on the closure of Ω_R, the distributional Laplacian Δu is in M^q_r for some r, q, $1 < r < p$, $3/2 < q < 3$ and u satisfies the equation*

$$-\Delta u(x) - \mathbf{b}(x)\cdot\nabla u(x) = 0\,,\qquad \textit{for almost every } x\in\Omega_{R/2}\,.\tag{3.35}$$

Then

$$u(x) = Ku(x)\,,\qquad \textit{for all } x\in\Omega_{R/2}\,,\tag{3.36}$$

provided $\|\mathbf{b}\|_{3,p} < \varepsilon_0$ where ε_0 depends only on r, p, q.

PROOF. Let us consider the problem

$$(3.37)\qquad \begin{cases} -\Delta u(x) - \mathbf{b}(x)\cdot\nabla u(x) = f(x)\,, & |x| < \lambda\,, \\ u(x) = g(x)\,, & |x| = \lambda\,. \end{cases}$$

We shall assume that $f \in M_r^q$ and $\nabla P_\lambda g(x)$ is an L^∞ function for $|x| < \lambda$. Then it is easy to see that (3.37) has a unique solution u given by

$$(3.38)\qquad \begin{aligned} u = P_\lambda g + (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1}\mathbf{b}\cdot\nabla P_\lambda g \\ + (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1} f\,. \end{aligned}$$

In fact if we put

$$(3.39)\qquad v = u - P_\lambda g\,,$$

then

$$(3.40)\qquad \begin{cases} -\Delta v(x) - \mathbf{b}(x)\cdot\nabla v(x) = \mathbf{b}(x)\cdot\nabla P_\lambda g(x) + f(x)\,, & |x| < \lambda\,, \\ v(x) = 0\,, & |x| = \lambda\,. \end{cases}$$

Since we are assuming $\nabla P_\lambda g$ is L^∞, Theorem 1.1 implies (3.38).

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ function with support in the unit ball centered at the origin and with integral 1. Then for $\delta > 0$ the functions φ_δ,

$$(3.41)\qquad \varphi_\delta(x) = \delta^{-3}\,\varphi(x/\delta)\,, \qquad x \in \mathbb{R}^3\,,$$

are approximate Dirac δ functions. We consider functions $u_\delta = \varphi_\delta * u$ where u is the solution of (3.35) given in the statement of Proposition 3.2. Then u_δ is a C^∞ function in the ball $|x| < R - \delta$, and

$$(3.42)\qquad -\Delta u_\delta(x) - \mathbf{b}(x)\cdot\nabla u_\delta(x) = f_\delta(x)\,, \qquad |x| < R - \delta\,,$$

where

$$(3.43)\qquad f_\delta(x) = \varphi_\delta * (\mathbf{b}\cdot\nabla u)(x) - \mathbf{b}(x)\cdot\nabla u_\delta(x)\,.$$

Since $f \in M_r^q$ for some $r < p$, $q > 3/2$, it follows that if $\lambda < R - \delta$, then

$$(3.44)\qquad \begin{aligned} u_\delta = P_\lambda u_\delta + (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1}\mathbf{b}\cdot\nabla P_\lambda u_\delta \\ + (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1} f_\delta\,. \end{aligned}$$

Let η satisfy $0 < \eta < R/2$ and consider $\delta < \eta$. Then if we integrate (3.44) with respect to λ over the interval $R/2 < \lambda < R-\eta$, we have for any x with $|x| < R/2$,

$$
\begin{aligned}
u_\delta(x) = & \frac{2}{R-2\eta} \int\limits_{R/2<|z|<R-\eta} \frac{1}{4\pi|z|} \frac{|z|^2-|x|^2}{|x-z|^3} u_\delta(z)\, dz \\
& + \frac{2}{R-2\eta} \int_{\Omega_R} dw \int_{\Omega_R} dy \int\limits_{R/2<|z|<R-\eta} dz\; G_{D,|z|}(x,w)\, H_{|z|}(w,y) \\
& \qquad \cdot \mathbf{b}(y) \cdot \Big(\nabla_y \frac{|z|^2-|y|^2}{4\pi|z|\,|y-z|^3}\Big)\, (u_\delta(z) - u_\delta(y)) \\
& + \frac{2}{R-2\eta} \int_{\Omega_R} dw \int_{\Omega_R} dy \\
& \qquad \int\limits_{R/2<|z|<R-\eta} dz\; G_{D,|z|}(x,w)\, H_{|z|}(w,y)\, f_\delta(y)\,.
\end{aligned} \tag{3.45}
$$

Since $\delta < \eta$ it follows that $f_\delta \in M_r^q(\Omega_{R-\eta})$.

We shall show that

$$\lim_{\delta\to 0} \|f_\delta\|_{q,r} = 0\,. \tag{3.46}$$

Let us put $h = -\Delta u$. Then by (3.35) we have that

$$\lim_{\delta\to 0} \|\varphi_\delta * (\mathbf{b}\cdot\nabla u) - h\|_{q,r} = 0\,. \tag{3.47}$$

Next we consider the limit of $\mathbf{b}\cdot\nabla u_\delta$. By Weyl's lemma we have that

$$u(x) = \int_{\Omega_R} G_{D,R}(x,y)\, h(y)\, dy + P_R u(x)\,, \tag{3.48}$$

for all $x \in \Omega_R$. Hence the distributional gradient of u is given by the formula

$$\nabla u(x) = \int_{\Omega_R} \nabla_x G_{D,R}(x,y)\, h(y)\, dy + \nabla_x P_R u(x)\,, \quad |x| < R\,. \tag{3.49}$$

Thus

$$\nabla u(x) = \int_{\Omega_R} \frac{1}{4\pi} \nabla_x \Big(\frac{1}{|x-y|}\Big)\, h(y)\, dy + \mathbf{w}(x)\,, \tag{3.50}$$

where $\mathbf{w}$ is a C^∞ function in $|x| < R$. Hence we have

$$(3.51) \qquad \nabla u_\delta(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi} \nabla_x \Big(\frac{1}{|x-y|}\Big) \varphi_\delta * h(y)\, dy + \varphi_\delta * \mathbf{w}(x)\,,$$

in $|x| < R-\delta$. Here we have extended h to all of $\mathbb{R}^3$ by setting h to zero outside Ω_R. Evidently $\varphi_\delta * \mathbf{w}(x)$ converges uniformly in $|x| < R - \eta$ as $\delta \to 0$ to $\mathbf{w}(x)$. Also $\varphi_\delta * h$ converges to h as $\delta \to 0$ in the space M_r^q. It follows then from the fact that the operator with kernel

$$(3.52) \qquad \mathbf{b}(x) \cdot \frac{1}{4\pi} \nabla_x \Big(\frac{1}{|x-y|}\Big)$$

is bounded on the space M_r^q that

$$\lim_{\delta \to 0} \|\mathbf{b} \cdot \nabla u_\delta - \mathbf{b} \cdot \nabla u\|_{q,r} = 0\,.$$

The identity (3.46) follows from (3.47), (3.53).

Next we take the limit as $\delta \to 0$ in (3.45). In view of (3.46) the final integral on the right hand side vanishes in the limit. Since u is Hölder continuous the first 2 integrals converge to identical integrals with u_δ replaced by u. Now if we let $\eta \to 0$ we obtain the formula (3.36).

4. Nonperturbative uniqueness.

We shall prove the uniqueness part of Theorem 1.3 in this section. Throughout the section we shall assume that $\mathbf{b} \in M_p^3$ with $2 < p < 3$.

Assume for the moment that $\mathbf{b}$ is a C^∞ function. Then for any $\lambda > 0$, the solution of the Poisson problem

$$(4.1) \qquad \begin{cases} -\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = 0\,, & |x| < \lambda\,, \\ u(x) = g(x)\,, & |x| = \lambda\,, \end{cases}$$

is given as an integral

$$(4.2) \qquad u(x) = \int_{|y|=\lambda} \rho(x,y)\, g(y)\, dy\,.$$

The function $\rho(x, y)$ in (4.2) is defined and continuous on the set $\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| < |y|\}$. The solution (4.2) can be represented as an expectation value with respect to Brownian motion with drift $\mathbf{b}$. One has

$$u(x) = E_x[g(X_{\mathbf{b}}(\tau_\lambda)], \tag{4.3}$$

where τ_λ is the first hitting time of the drift process started at x on the sphere $|y| = \lambda$. Let $R < \lambda$. Then if we condition on the hitting distribution of the process on the sphere $|y| = R$, we have from (4.3),

$$\begin{aligned} u(0) &= E_0\left[E_{X_{\mathbf{b}}(\tau_R)}[g(X_{\mathbf{b}}(\tau_\lambda))]\right] \\ &= E_0[u(X_{\mathbf{b}}(\tau_R))] \\ &= \int\limits_{|z|=R} \rho(0, z)\, u(z)\, dz\,. \end{aligned} \tag{4.4}$$

We conclude then that

$$\rho(0, y) = \int\limits_{|z|=R} \rho(0, z)\, \rho(z, y)\, dz\,, \tag{4.5}$$

for any y with $|y| > R$.

The Cameron-Martin formula [10] enables one to write the probability measure for the drift process $X_{\mathbf{b}}(t)$ in terms of the Wiener measure for Brownian motion $X(t)$. In particular, the drift expectation (4.3) becomes a Brownian motion expectation given by

$$\begin{aligned} u(x) = E_x\Big[\exp\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t)) \cdot dX(t) \\ - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\, dt\Big)\, g(X(\tau_\lambda))\Big]\,. \end{aligned} \tag{4.6}$$

Since $g \equiv 1$ implies $u \equiv 1$ it follows from (4.6) that for any $\theta \in \mathbb{R}$ one has the identity

$$E_x\Big[\exp\Big(\frac{\theta}{2}\int_0^\tau \mathbf{b}(X(t)) \cdot dX(t) - \frac{\theta^2}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\, dt\Big)\Big] \equiv 1\,. \tag{4.7}$$

For any integer $n \in \mathbb{Z}$ let A_n be the spherical shell,

$$A_n = \{x \in \mathbb{R}^3 : \ 2^{-n} < |x| < 2^{-n+1/2}\}. \tag{4.8}$$

We define a measure μ on $\cup_{n=-\infty}^{\infty} A_n$ by

$$d\mu(x) = \frac{dx}{4\pi\,|x|^2\,(2^{-n+1/2} - 2^{-n})}\,, \qquad x \in A_n,\ n \in \mathbb{Z}. \tag{4.9}$$

Hence $\mu(A_n) = 1$ so A_n is a probability space with respect to μ. We define an integral operator T_n from functions on A_n to functions on A_{n-1} by

$$T_n f(x) = 4\pi\,|x|^2 \int_{A_n} \rho(y,x)\, f(y)\, \mu(y)\,, \qquad x \in A_{n-1}\,. \tag{4.10}$$

Let $\rho_n : A_n \to \mathbb{R}$ be given by

$$\rho_n(x) = 4\pi\,|x|^2\,\rho(0,x)\,, \qquad x \in A_n\,. \tag{4.11}$$

Then (4.5) imples that

$$T_n \rho_n = \rho_{n-1}\,, \qquad n \in \mathbb{Z}. \tag{4.12}$$

We write

$$T_n = P_n + Q_n\,, \tag{4.13}$$

where P_n is the same operator as T_n for the case $\mathbf{b} \equiv 0$. Hence the kernel $\rho(y,x)$ for P_n is just the Poisson kernel. It follows easily that

$$P_n(1) = 1\,, \tag{4.14}$$

where 1 denotes the function identically equal to 1.

Lemma 4.1. *Suppose $f \in L^r_\mu(A_n)$ for some r, $1 \le r \le \infty$ and satisfying*

$$\int_{A_n} f\, d\mu = 0\,. \tag{4.15}$$

Then there exists a universal constant $\gamma, 0 < \gamma < 1$, *such that* $P_n f \in L^r_\mu(A_{n-1})$ *and there is the inequality*

$$\|P_n f\|_r \le \gamma \, \|f\|_r \,. \tag{4.16}$$

PROOF. First we prove (4.16) with $\gamma = 1$. We have

$$P_n f(x) = 4\pi \, |x|^2 \int_{A_n} \rho(y,x) \, f(y) \, d\mu(y) \,, \tag{4.17}$$

where ρ is the Poisson kernel. It follows then from (4.14) and Jensen's inequality that

$$|P_n f(x)|^r \le 4\pi \, |x|^2 \int_{A_n} \rho(y,x) \, |f(y)|^r \, d\mu(y) \,. \tag{4.18}$$

Now (4.16) with $\gamma = 1$ follows on integrating (4.18) and using the fact that

$$\int_{A_{n-1}} 4\pi \, |x|^2 \, \rho(y,x) \, d\mu(x) = 1 \,. \tag{4.19}$$

To obtain $\gamma < 1$ we use (4.15). Observe that (4.15) implies

$$\int_{A_{n-1}} P_n f(x) \, d\mu(x) = 0 \,. \tag{4.20}$$

Since $P_n f(x)$ is a continuous function there exists $x_0 \in A_{n-1}$ with $P_n f(x_0) = 0$. Now let us write f as a sum of its positive and negative parts,

$$f = f_+ - f_- \,, \qquad P_n f = P_n f_+ - P_n f_- \,. \tag{4.21}$$

By the properties of the Poisson kernel there exist universal constants c_1, c_2 such that for $x \in A_{n-1}$,

$$P_n f_-(x) \ge c_1 \, P_n f_-(x_0) = \frac{c_1}{2} \, P_n |f|(x_0) \ge c_2 \, P_n |f|(x) \,, \tag{4.22}$$

where $0 < c_2 < 1$. Hence

$$P_n f(x) = P_n f^+(x) - P_n f^-(x) \le (1 - c_2) \, P_n |f|(x) \,. \tag{4.23}$$

Since we can obtain a similar lower bound on $P_n f(x)$ we conclude that

$$\|P_n f\|_r \le (1-c_2)\, \|P_n |f|\,\|_r \le (1-c_2)\,\|f\|_r\,. \tag{4.24}$$

Thus we can take $\gamma = 1 - c_2 < 1$.

Lemma 4.2. *Let $1 < r < \infty$. Then for any $\delta > 0$ there exists $\varepsilon > 0$ depending only on r, p, δ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies the inequality*

$$\|Q_n f\|_r \le \delta\,\|f\|_r\,, \qquad f \in L^r_\mu(A_n)\,,\ n \in \mathbb{Z}\,. \tag{4.25}$$

PROOF. Let r' be the conjugate to r, $1/r + 1/r' = 1$. We consider the adjoint Q_n^* of Q_n. We shall show that Q_n^* is a bounded operator from $L^{r'}_\mu(A_{n-1})$ to $L^{r'}_\mu(A_n)$ and satisfies

$$\|Q_n^* f\|_{r'} \le \delta\,\|f\|_{r'}\,. \tag{4.26}$$

This will imply (4.25).

We have from (4.10) that T_n^* is given by the formula

$$T_n^* f(x) = \frac{2^{n+1}}{\sqrt{2}-1}\int_{A_{n-1}} \rho(x,y)\,f(y)\,dy\,. \tag{4.27}$$

To obtain Q_n^* we need to subtract off from (4.27) the operator corresponding to $\mathbf{b} = 0$. This can easily be done from the formula (4.6). Comparing (4.2), (4.6), (4.27) we have

$$\begin{aligned} Q_n^* f(x) = \frac{2^{n+1}}{\sqrt{2}-1}&\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \int_0^1 ds \\ &\cdot E_x\Big[\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\ &\cdot \exp\Big(\frac{s}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\ &\qquad - \frac{s}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\,f(X(\tau_\lambda))\Big]\,. \end{aligned} \tag{4.28}$$

Equation (4.28) can be written as

$$Q_n^* f(x) = \frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \int_0^1 ds$$

$$\begin{aligned}(4.29)\qquad & \cdot E_x\Big[\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\ & \cdot \exp\Big(\frac{s}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{rs}{2}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\ & \cdot \exp\Big(s\,(\frac{r}{2}-\frac{1}{4})\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\, f(X(\tau_\lambda))\Big]\,.\end{aligned}$$

Now we apply the generalized Hölder inequality to (4.29). Let m be an integer satisfying $m > r$. Then, observing that

$$(4.30)\qquad \frac{1}{2m}+\frac{1}{2r}+\Big(\frac{1}{2r}-\frac{1}{2m}\Big)+\frac{1}{r'}=1\,,$$

we have

$$\begin{aligned} & |Q_n^* f(x)| \\ & \le \int_0^1 ds\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\ & \qquad \cdot E_x\Big[\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big]\Big)^{1/2m} \\ & \quad \cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\ & \qquad \cdot E_x\Big[\exp\Big(rs\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - r^2 s\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{1/2r} \\ (4.31) & \\ & \quad \cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\ & \qquad \cdot E_x\Big[\exp\Big(\frac{mrs(2r-1)}{2(m-r)}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{(1/2r-1/2m)} \\ & \quad \cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda\, E_x\big[|f(X(\tau_\lambda))|^{r'}\big]\Big)^{1/r'}\,. \end{aligned}$$

If we use (4.7) with $\theta = 2\,r$, we can conclude from (4.31) that for $x \in A_n$,

$$\begin{aligned} & |Q_n^* f(x)|^{r'} \\ & \le C\,\|f\|_{r'}^{r'}\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \end{aligned}$$

(4,32)

$$\cdot E_x\Big[\Big(\frac{1}{2}\int_0^T \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big]\Big)^{r'/2m}$$

$$\cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda$$

$$\cdot E_x\Big[\exp\Big(\frac{m\,r(2r-1)}{2(m-r)}\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{r'(1/2r-1/2m)}$$

where C is a universal constant. Observe now that

$$\begin{aligned} & E_x\Big[\Big(\frac{1}{2}\int_0^T \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big] \\ (4.33) \qquad & \qquad \le E_x\Big[\Big(\int_0^T \mathbf{b}(X(t))\cdot dX(t)\Big)^{2m}\Big] \\ & \qquad\quad + \frac{1}{4^m}E_x\Big[\Big(\int_0^T |\mathbf{b}|^2(X(t))dt\Big)^{2m}\Big]. \end{aligned}$$

It follows from (4.7) that there exists a constant C_m depending only on m such that

$$\begin{aligned} & E_x\Big[\Big(\int_0^T \mathbf{b}(X(t))\cdot dX(t)\Big)^{2m}\Big] \\ (4.34) \qquad & \qquad \le C_m\,E_x\Big[\Big(\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^{m}\Big]. \end{aligned}$$

We can assume without loss of generality that $2m > r'$. Hence if we integrate (4.32) with respect to x over A_n and apply Hölder with exponents $2m/r'$ and $2m/(2m-r')$ we obtain the inequality

$$\|Q_n^* f\|_{r'}^{r'} \le C'\,\|f\|_{r'}^{r'}$$

$$\cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda$$

$$\cdot\int_{A_n} d\mu(x)\,E_x\Big[\Big(\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^{m}$$

$$+\Big(\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big]\Big)^{r'/2m}$$

$$\cdot\Big(\int_{A_n} d\mu(x)\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda$$

$$(4.35)\qquad \cdot E_x\Big[\exp\Big(\frac{m\,r(2r-1)}{2(m-r)} \cdot \int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{\frac{2m}{2m-r'}\frac{r'(m-r)}{2rm}}\Big)^{\frac{2m-r'}{2m}},$$

for some constant C' depending only on m. It follows from Jensen's inequality that

$$(4.36)\qquad \begin{aligned}&\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\ &\cdot E_x\Big[\exp\Big(\frac{m\,r(2r-1)}{2(m-r)}\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{\frac{2m}{2m-r'}\frac{r'(m-r)}{2rm}} \\ &\le \frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda\, E_x\Big[\exp\Big(\alpha\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\end{aligned}$$

where

$$(4.37)\qquad \alpha = \frac{m\,r(2r-1)}{2(m-r)}\max\Big\{1, \frac{2m}{2m-r'}\,\frac{r'(m-r)}{2rm}\Big\}.$$

It follows from Theorem 1.1.b) of [1] that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and ε is sufficiently small depending only on α, then

$$(4.38)\qquad \begin{aligned}&\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\ &\cdot \int_{A_n} d\mu(x)\, E_x\Big[\exp\Big(\alpha\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)\Big] \le 2.\end{aligned}$$

On the other hand by the same argument one has

$$(4.39)\qquad \begin{aligned}&\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\ &\cdot \int_{A_n} d\mu(x)\, E_x\Big[\Big(\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^m \\ &\qquad + \Big(\int_0^T |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big] \le C\,\varepsilon^{2m},\end{aligned}$$

where C depends only on m. We conclude therefore from (4.35), (4.38), (4.39) that

$$\|Q_n^* f\|_{r'} \leq \delta \, \|f\|_{r'} \,, \tag{4.40}$$

provided ε is sufficiently small. The inequality (4.25) follows directly from this.

Lemma 4.3. *Let ρ_n be the density* (4.11), *and $1 < r < \infty$. Then for any $\delta > 0$ there exists $\varepsilon > 0$ depending only on r, p, δ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies the inequality*

$$\|\rho_n - 1\|_r < \delta \,, \qquad \textit{for all} \;\; n \in \mathbb{Z} \,. \tag{4.41}$$

PROOF. From (4.12) we have

$$\begin{aligned} \rho_{n-1} - 1 &= T_n \rho_n - 1 \\ &= P_n \rho_n + Q_n \rho_n - 1 \\ &= P_n(\rho_n - 1) + Q_n(\rho_n - 1) + Q_n 1 \,. \end{aligned} \tag{4.42}$$

Hence by Lemmas 4.1 and 4.2 we have

$$\|\rho_{n-1} - 1\|_r \leq \gamma \, \|\rho_n - 1\|_r + \delta' \, \|\rho_n - 1\|_r + \delta' \,, \tag{4.43}$$

where δ' can be chosen arbitrarily small depending on ε. Since $\gamma < 1$ we can therefore have $\gamma + \delta' < 1$. It follows then by induction from (4.43) that for any $M \geq n$ one has the inequality

$$\|\rho_n - 1\|_r \leq (\gamma + \delta')^{M-n} \, \|\rho_M - 1\|_r + \frac{\delta'}{1 - \gamma - \delta'} \,. \tag{4.44}$$

Letting $M \to \infty$ and choosing δ' such that $\delta'/(1 - \gamma - \delta') < \delta$ yields the inequality (4.41).

Now let us return to the operator K on functions $g \in C^\alpha(\Omega_R)$ defined by (3.18).

Lemma 4.4. *Suppose $x_0 \in \Omega_R \backslash \Omega_{R/2}$ and $g(x) \geq g_0 > 0$ for $|x - x_0| < r_0$, $g(x) \geq 0, x \in \Omega_R \backslash \Omega_{R/2}$. Then there exists a positive constant $c(r_0/R)$ depending only on r_0/R such that*

$$K \, g(0) \geq c(r_0/R) \, g_0 \,, \tag{4.45}$$

provided $\|\mathbf{b}\|_{3,p}$ *is sufficiently small.*

PROOF. With $\rho(x,y)$ defined as in (4.2) we have the identity

$$\begin{aligned} K\,g(0) &= \frac{2}{R}\int_{R/2}^{R} d\lambda \int_{|z|=\lambda} \rho(0,z)\,g(z)\,dz \\ &= \frac{2}{R}\int_{\Omega_R\setminus\Omega_{R/2}} \rho(0,z)\,g(z)\,dz\,. \end{aligned} \tag{4.46}$$

Since $g(z) \geq 0$ for $z \in \Omega_R\setminus\Omega_{R/2}$ it follows that

$$K\,g(0) \geq \frac{2}{R}\,g_0 \int_{\{z\in\Omega_R\setminus\Omega_{R/2}:|z-x_0|<r_0\}} \rho(0,z)\,dz\,. \tag{4.47}$$

For $x, y \in \mathbb{R}^3$ with $x \neq y$ we define a function $\xi(x,y)$ as follows: For $\lambda > 0$ let O_λ be an arbitrary open subset of the sphere $\{z : |z| = \lambda\}$. Then

$$\int_{y-x\in O_\lambda} \xi(x,y)\,dy = \text{probability that the drift process started at } x \text{ exits the sphere } |y-x| = \lambda \text{ through the set } x + O_\lambda\,. \tag{4.48}$$

It is clear that ξ and the previously defined function ρ are related by the equation

$$\rho(0,y) = \xi(0,y)\,, \qquad y \in \mathbb{R}^3\setminus\{0\}\,. \tag{4.49}$$

Let N be the integer $N = [4|x_0|/r_0]$, where $[\cdot]$ denotes integer part. For $j = 1,\ldots,N-1$, let $z_j \in \mathbb{R}^3$ be given by

$$z_j = j\,\frac{r_0}{4}\,\frac{x_0}{|x_0|}\,. \tag{4.50}$$

It is clear then from the definition of N that

$$\frac{r_0}{4} \leq |z_{N-1} - x_0| \leq \frac{r_0}{2}\,, \qquad |z_{N-1}| \leq |x_0| - \frac{r_0}{4}\,. \tag{4.51}$$

Next we define z_N by

$$z_N = x_0 + \frac{r_0}{4}\frac{x_0}{|x_0|}\,. \tag{4.52}$$

For $j = 1,\ldots,N$, $\delta > 0$, let $B_{j,\delta}$ be the ball of radius $\delta\, r_0$ centered at z_j. We first choose $\delta < 1/8$. This ensures that the spheres $B_{j,\delta}$, $j = 1,\ldots,N$ are disjoint. Let $K(\delta)$ be given by

$$K(\delta) = \sup\{|z-w|/r_0 :\ z \in B_{N-1,\delta}\,,\ w \in B_{N,\delta}\}\,. \tag{4.53}$$

It is clear that

$$K(\delta) \le \frac{3}{4} + 2\,\delta\,. \tag{4.54}$$

Now for arbitrary $z \in B_{N-1,\delta}$ let y satisfy

$$|y-z| = K(\delta)\, r_0\,, \qquad |x_0| - \frac{r_0}{8} \le |y| \le |x_0| + \frac{r_0}{8}\,. \tag{4.55}$$

Then we need to choose δ sufficiently small such that if y satisfies (4.55) then $|y-x_0| < r_0$. This is clearly possible provided δ is chosen to depend on the ratio $r_0/R < 1$. We then have the inequality

$$\int_{\{z\in\Omega_R\backslash\Omega_{R/2}:|z-x_0|<r_0\}} \rho(0,z)\,dz \ge \Big(\frac{1}{4r_0\delta}\Big)^N \frac{r_0}{8}\,\Big(\prod_{j=1}^{N}\int_{B_{j,\delta}} dy_j\Big) \cdot \xi(0,y_1)\,\xi(y_1,y_2)\cdots\xi(y_{N-1},y_N)\,. \tag{4.56}$$

The inequality (4.56) can be explained as follows: First constrain the integration on the left hand side to the surface of the sphere $|z| = |x_0| + \varepsilon$, where $-r_0/8 < \varepsilon < r_0/8$. Second, constrain the variables $y_j, j = 1,\ldots,N$ to lie on surfaces $|y_1| = \varepsilon_1$, $|y_j - y_{j-1}| = \varepsilon_j$, $j = 2,\ldots,N$, where

$$\begin{gathered} r_0\,(1/4-\delta) < \varepsilon_1 < r_0\,(1/4+\delta)\,,\\ r_0\,(1/4-2\,\delta) < \varepsilon_j < r_0\,(1/4+2\,\delta)\,, \quad j = 2,\ldots,N-1,\\ (K(\delta) - 4\,\delta)\, r_0 < \varepsilon_N < K(\delta)\, r_0\,. \end{gathered} \tag{4.57}$$

Then we have the inequality

$$\int \rho(0,z)\,dz \ge \Big(\prod_{j=1}^{N}\int dy_j\Big)\xi(0,y_1)\,\xi(y_1,y_2)\cdots\xi(y_{N-1},y_N)\,. \tag{4.58}$$

This is true because the left hand side is the probability of the drift process starting at 0 exiting the sphere $|z| = |x_0| + \varepsilon$ where it intersects the ball $|z - x_0| < r_0$. The right hand side gives the probability of a set of paths which accomplish this. The second condition on δ following (4.55) guarantees that any path included on the right hand side exits through the intersection with the ball $|z - x_0| < r_0$. The inequality (4.56) is obtained from (4.58) by doing the radial integrations and observing the constraints (4.57) on the ε_j, $j = 1, \dots, N$.

The inequality (4.45) will follow if we can show that

$$\frac{1}{4r_0\delta} \int_{B_{j,\delta}} \xi(y_{j-1}, y_j)\, dy_j \geq \gamma > 0\,, \tag{4.59}$$

where γ depends only on $\|\mathbf{b}\|_{3,p}$. However, this is an immediate consequence of Lemma 4.3.

The previous lemmas enable us to prove a maximum principle for the solutions of the elliptic equation (1.2). This will then imply uniqueness of the solution as given in Theorem 1.3.

Theorem 4.5. *Suppose* $\mathbf{b}$ *satisfies the conditions of Theorem* 1.3, u *is a Hölder continuous function on* Ω_R *with distributional Laplacian* Δu *in* M_r^q *for some* r, q, $1 < r < p$, $3/2 < q < 3$ *and* u *satisfies the equation*

$$-\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = 0\,, \qquad \text{for almost every } x \in \Omega_R\,. \tag{4.60}$$

Then if u *has a maximum interior to* Ω_R *the function* u *is a constant.*

PROOF. Suppose u has a maximum at an interior point $x_0 \in \Omega_R$. By the conditions on $\mathbf{b}$ there exists an open ball $B(x_0, \delta)$ centered at x_0 with radius δ such that the closure is contained in Ω_R and $\|\mathbf{b}\|_{3,p} < \varepsilon$ when $\mathbf{b}$ is restricted to $B(x_0, \delta)$. We can therefore apply Proposition 3.2 to u on $B(x_0, \delta)$ to conclude that

$$u(x_0) = K_\delta u(x_0)\,, \tag{4.61}$$

where K_δ is the operator (3.18) for the ball $B(x_0, \delta)$. It follows from (4.61) that

$$0 = K_\delta g(x_0)\,, \tag{4.62}$$

where $g(x) = u(x_0) - u(x)$. Since $g(x) \geq 0, \quad x \in B(x_0, \delta)$ it follows from Lemma 4.4 that $g(x) = 0$ for all $x, \delta/2 < |x - x_0| < \delta$. One can further deduce that $g(x) = 0$ for all $x \in B(x_0, \delta)$. The result then is a consequence of the connectedness of Ω_R.

5. Nonperturbative existence.

We shall complete the proof of Theorem 1.3 in this section. The basic input is that the boundary value problem (1.2)-(1.3) has a C^∞ solution $u(x)$ provided $\mathbf{b}$ and f are C^∞ functions. This is a well known result [6]. We then prove existence of the solution to (1.2)-(1.3) for nonsmooth $\mathbf{b}$ and f by smoothing $\mathbf{b}$ and f with approximate Dirac δ functions and taking limits.

Let Q_0 be the smallest cube concentric with Ω_R and containing it which has side of length $2^{-n_0}, n_0$ an integer. Suppose now $\mathbf{b} \in M_p^3, \varepsilon > 0$ and $N_\varepsilon(\mathbf{b}) < +\infty$. Then there exists a unique minimal integer $m_\varepsilon(\mathbf{b}) \geq n_0$ such that every dyadic subcube $Q \subset Q_0$ with side of length 2^{-m_ε} has the property

$$\int_{Q'} |\mathbf{b}|^p \, dx < \varepsilon^p \, |Q'|^{1-p/3} , \tag{5.1}$$

for all dyadic subcubes $Q' \subset Q$.

Our main theorem in this section is the following

Theorem 5.1. *Suppose* $\mathbf{b}$ *and* f *are* C^∞ *functions and* u *is the solution of the boundary value problem* (1.2)-(1.3). *Then there exists* $\varepsilon > 0$ *depending only on* p, q, r *such that*

$$\|u\|_\infty \leq C_1 \, R^{2-3/q} \, \|f\|_{q,r} \, \exp(C_2 \, m_\varepsilon(\mathbf{b})/n_0) , \tag{5.2}$$

for some constants C_1, C_2 *depending only on* p, q, r.

Next we consider a possibly singular $\mathbf{b} \in M_p^3$ with $N_\varepsilon(\mathbf{b}) < +\infty$ for some $\varepsilon > 0$. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative C^∞ function such that

$$\int_{\mathbb{R}^3} \varphi(x) dx = 1 , \qquad \operatorname{supp} \varphi \subset \{x : \ |x| \leq 1\} . \tag{5.3}$$

For $\delta > 0$ let $\varphi_\delta(x) = \delta^{-3}\varphi(x/\delta)$, $x \in \mathbb{R}^3$, and put $\mathbf{b}_\delta = \varphi_\delta * \mathbf{b}$. Since $\mathbf{b} \in M_p^3$ it is clear that $\mathbf{b}_\delta$ is a C^∞ function. We choose δ to satisfy the inequality

$$\delta < \delta_0 = 2^{-m_\varepsilon - 1}\,. \tag{5.4}$$

Let $y \varepsilon \mathbb{R}^3$ be an arbitrary vector satisfying $|y| < \delta$ and $Q \subset Q_0$ be a dyadic subcube with side of length 2^{-m_ε}. Then for all dyadic subcubes Q' of Q we have

$$\int_{Q'} |\mathbf{b}(x+y)|^p \, dx \le 8\,\varepsilon^p \, |Q'|^{1-p/3} < (8\,\varepsilon)^p \, |Q'|^{1-p/3}\,, \tag{5.5}$$

since the translate of Q' by y intersects at most 8 dyadic cubes with side of length $2\delta_0$. We see from Jensen's inequality that

$$\int_{Q'} |\mathbf{b}_\delta|^p \, dx < (8\,\varepsilon)^p \, |Q'|^{1-p/3}\,, \tag{5.6}$$

for all dyadic subcubes Q' of the cube Q. It follows in particular that

$$m_{8\varepsilon}(\mathbf{b}_\delta) \le m_\varepsilon(\mathbf{b})\,, \tag{5.7}$$

provided δ satisfies (5.4). We shall need the following

Lemma 5.2. *Let $R > 0$, $g : \Omega_R \backslash \Omega_{R/2} \to \mathbb{R}$ an L^∞ function and $\mathbf{b}$ be a C^∞ drift. Let τ_λ be the first hitting time for the drift process started at $x, |x| < \lambda$, on the sphere $\{y : |y| = \lambda\}$. Define $v(x)$ for $|x| < R/4$ by*

$$v(x) = \frac{2}{R} \int_{R/2}^{R} E_x[g(X_{\mathbf{b}}(\tau_\lambda))]\, d\lambda\,. \tag{5.8}$$

Then there exists $\varepsilon > 0$ depending only on $p > 2$ such that, if $\|\mathbf{b}\|_{3,p} < \varepsilon$, the function $v(x)$ is Hölder continuous for $|x| \le R/4$. In particular $v(x)$ satisfies the inequalities

$$\|v\|_\infty \le \|g\|_\infty\,, \qquad |v(x) - v(y)| \le C\,\|g\|_\infty \left(\frac{|x-y|}{R}\right)^\alpha, \tag{5.9}$$

where C and $\alpha > 0$ depend only on p, ε.

PROOF. The first estimate in (5.9) is immediate from the definition (5.8). To obtain the second estimate we use the method employed in Section 4 to prove Lemmas 4.1 and 4.2. For x, y satisfying $|x|, |y| < R/4$ we choose $(x+y)/2$ as our origin and define regions A_n as in (4.8). Let n_1 be the smallest integer such that $|x-y| \geq 2^{-n_1-1}$ and n_0 be the smallest integer such that $R/8 \geq 2^{-n_0-1}$. If $n_1 \leq n_0$ then the second inequality of (5.9) follows from the first inequality. Hence we shall assume $n_1 \geq n_0+1$. For $n \leq n_1$ let $\rho_{x,n}$ be the density corresponding to (4.11) on the set A_n for the drift process starting at x. This can be constructed exactly as in Lemmas 4.1 and 4.2 by using spherical shells centered at x up to radius 2^{-n_1-2} and then making the next transformation to the spherical shell A_{n_1} centered at $(x+y)/2$. We conclude that for ε sufficiently small there is an inequality $\|\rho_{x,n_1}\|_r \leq C_r$ where the constant C_r depends only on $r > 1$. Since there is a similar inequality for ρ_{y,n_1} we conclude that

$$\|\rho_{x,n_1} - \rho_{y,n_1}\|_r \leq C_r \ , \tag{5.10}$$

for some suitable universal constant depending only on $r > 1$. Now by Lemmas 4.1 and 4.2 one has the inequality

$$\|\rho_{x,n} - \rho_{y,n}\|_r \leq \gamma^{n_1-n}\, C_r \ , \qquad n \geq n_1 \ , \tag{5.11}$$

where γ is a constant depending only on ε, r, $0 < \gamma < 1$. In particular (5.11) holds for $n = n_0$. Next we can use the method of Lemmas 4.1 and 4.2 to estimate the densities of the drift process starting at x and y on $\Omega_R \backslash \Omega_{R/2}$. If we denote these by ρ_x, ρ_y it easily follows from (5.11) that

$$\|\rho_x - \rho_y\|_r \leq \gamma^{n_1-n_0}\, C'_r \ . \tag{5.12}$$

Since

$$v(x) = \int_{\Omega_R \backslash \Omega_{R/2}} \rho_x(z)\, g(z)\, d\mu(z) \,, \tag{5.13}$$

where

$$d\mu(z) = \frac{2}{R}\, \frac{dz}{4\pi\, |z|^2} \ , \tag{5.14}$$

it easily follows from (5.12), (5.13) that the second inequality of (5.9) holds with α defined by

$$2^{-\alpha} = \gamma\,, \qquad \text{where } 1/2 < \gamma < 1\,. \tag{5.15}$$

This completes the proof.

PROOF OF THEOREM 1.3: EXISTENCE. We shall use Theorem 5.1 and Lemma 5.2 to construct a solution of the boundary value problem. Let ε_0 be chosen so that Lemma 5.2 and the perturbation Theorem 1.1 holds for $\|\mathbf{b}\|_{3,p} < \varepsilon_0$, while Theorem 5.1 holds for $\varepsilon = \varepsilon_0$. We restrict ε so that $\varepsilon < \varepsilon_0/64$. Now for δ satisfying (5.4) let u_δ be the solution of the boundary value problem (1.2)-(1.3) with drift $\mathbf{b}_\delta$ and observable $f_\delta = \varphi_\delta * f$, $f \in M^q_r$, $q > 3/2$. In view of (5.7) and Theorem 5.1 we have the inequality

$$\|u_\delta\|_\infty \le C_1\, R^{2-3/q}\, \|f\|_{q,r} \exp(C_2\, m_\varepsilon(\mathbf{b})/n_0)\,, \tag{5.16}$$

since $\|f_\delta\|_{q,r} \le \|f\|_{q,r}$.

Now let x_0 be an arbitrary point in Ω_R and consider $\mathbf{b}_\delta$ restricted to the ball centered at x_0 with radius δ_0, $B(x_0, \delta_0)$. It follows from (5.6) that $\|\mathbf{b}_\delta\|_{3,p} < \varepsilon_0$. We consider x in the ball $B(x_0, \delta_0/4)$ and for $\delta_0/2 < \lambda < \delta_0$ let τ_λ be the hitting time for the drift process started at x on the boundary of the ball $B(x_0, \lambda)$. Then if τ is the hitting time on the sphere $\partial\Omega_R$ we have

$$\begin{aligned} u_\delta(x) &= E_x\Big[\int_0^\tau f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] \\ &= E_x\Big[\int_0^{\tau_\lambda} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] + E_x\Big[\int_{\tau_\lambda}^{\tau} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] \\ &= E_x\Big[\int_0^{\tau_\lambda} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] + E_x\Big[u_\delta(X_{\mathbf{b}_\delta}(\tau_\lambda))\Big]\,. \end{aligned} \tag{5.17}$$

Integrating with respect to λ we have then for $|x - x_0| < \delta_0/4$ the representation

$$\begin{aligned} u_\delta(x) = \frac{2}{\delta_0} \int_{\delta_0/2}^{\delta_0} d\lambda\; E_x\Big[\int_0^{\tau_\lambda} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] \\ + \frac{2}{\delta_0} \int_{\delta_0/2}^{\delta_0} d\lambda\; E_x\Big[u_\delta(X_{\mathbf{b}_\delta}(\tau_\lambda))\Big] \\ = w_\delta(x) + v_\delta(x)\,. \end{aligned} \tag{5.18}$$

In view of Lemma 5.2 and (5.16) we have v_δ is Hölder continuous and

$$
\begin{aligned}
&|v_\delta(x) - v_\delta(y)| \\
&\qquad \le C\left(\frac{|x-y|}{\delta_0}\right)^{\alpha} R^{2-3/q}\|f\|_{q,r} \exp(C_2\, m_\varepsilon(\mathbf{b})/n_0)\,.
\end{aligned}
\tag{5.19}
$$

It follows now from Theorem 1.1 that w_δ is Hölder continuous and

$$
|w_\delta(x) - w_\delta(y)| \le C\left(\frac{|x-y|}{\delta_0}\right)^{\beta} \delta_0^{2-3/q}\,\|f\|_{q,r}\,, \tag{5.20}
$$

where the exponent β depends on $q > 3/2$. Hence the functions $u_\delta, \delta < \delta_0$, form an equicontinuous family, which by (5.16) is uniformly bounded. The Ascoli-Arzelà theorem implies then that there exists a sequence δ_n, $n \ge 1$, with $\lim_{n\to\infty} \delta_n = 0$ such that the u_{δ_n} converge uniformly to a limiting function u. The function u must necessarily be Hölder continuous in view of the uniform Hölder continuity of the functions u_δ.

We shall show that u is the solution to the boundary value problem (1.2)-(1.3) in the sense of Theorem 1.3. Evidently *a*) of Theorem 1.3 follows immediately from our preceding work. To prove *b*) we consider equation (5.18) again. Letting K_δ be the operator K of (3.18) adapted to the ball $B(x_0, \delta_0)$ with drift $\mathbf{b}_\delta$ and $T_{\delta,\lambda}$ be the integral operator on M_r^q with kernel (3.4) corresponding to the drift $\mathbf{b}_\delta$ and ball $B(x_0, \lambda)$, we can write (5.18) as

$$
u_\delta(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\,(-\Delta_{D,\lambda})^{-1}(I - T_{\delta,\lambda})^{-1} f_\delta(x) + K_\delta(u_\delta)\,. \tag{5.21}
$$

Now we take $\delta = \delta_n, n \ge 1$, in (5.21) and let $\delta \to 0$. Since $\mathbf{b}_\delta$ converges to $\mathbf{b}$ in M_p^3 and f_δ to f in M_r^q and u_δ is uniformly Hölder continuous as $\delta \to 0$ it follows that

$$
u(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\,(-\Delta_{D,\lambda})^{-1}(I - T_{0,\lambda})^{-1} f(x) + K_0(u)\,, \tag{5.22}
$$

where K_0 and $T_{0,\lambda}$ are the operators which correspond to the drift $\mathbf{b}$. It follows easily from (5.22) that the distributional Laplacian $\Delta u(x)$ for $|x - x_0| < \delta_0/4$ is given by

$$
-\Delta u(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\,(I - T_{0,\lambda})^{-1} f(x) - \Delta K_0(u)\,. \tag{5.23}
$$

Proposition 3.1 and Theorem 1.2 then imply that $\Delta u \in M_r^q$. Finally Proposition 3.1 and the perturbative existence argument at the beginning of Section 3 imply from (5.23) that

$$-\Delta u(x) - \mathbf{b}(x)\cdot\nabla u(x) = f(x)\,, \qquad |x - x_0| < \delta_0/4\,, \tag{5.24}$$

where ∇u is the distributional gradient of u. The proof of Theorem 1.3 is complete.

We turn to the proof of Theorem 5.1. We shall pursue the same method we used in Section 4 to prove uniqueness. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ observable and $\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3$ a C^∞ drift. For any $\eta > 0$ we define a function $\rho_{f,\eta}(x,y)$ on the set $\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| < |y|\}$ as follows: Let $\lambda > 0$ be arbitrary and τ_λ be the first hitting time for the drift process started at a point $x, |x| < \lambda$, on the sphere $|y| = \lambda$. Then for any continuous function g on the sphere $|y| = \lambda$, one has

$$\begin{aligned}\int\limits_{|y|=\lambda} &\rho_{f,\eta}(x,y)\, g(y)\, dy \\ &= E_x\Big[g(X_{\mathbf{b}}(\tau_\lambda)\, \chi\Big(\eta\, \lambda^{2-3/q} \|f\|_{q,r} - \int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\, dt\Big)\Big]\,,\end{aligned} \tag{5.25}$$

where χ is the Heaviside function, $\chi(s) = 1$, for $s \ge 0$, $\chi(s) = 0$, for $s < 0$. It is clear from our definition that $\rho_{f,\eta}$ is an increasing function of η and

$$\lim_{\eta\to\infty} \rho_{f,\eta}(x,y) = \rho(x,y)\,, \tag{5.26}$$

where $\rho(x,y)$ is defined by (4.2). Let A_n be the region (4.8). For any integer $n \in \mathbb{Z}$ we can define $\rho_{f,\eta,n} : A_n \to \mathbb{R}$ in analogy to $\rho_n : A_n \to \mathbb{R}$ given by (4.11). Thus we define $\rho_{f,\eta,n}$ by

$$\rho_{f,\eta,n}(x) = 4\pi\, |x|^2\, \rho_{f,\eta}(0,x)\,, \qquad x \in A_n\,. \tag{5.27}$$

Lemma 5.3. *There exists $\varepsilon > 0, C > 0$ depending only on r, p, q such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies the inequality*

$$\|\rho_n - \rho_{f,\eta,n}\|_1 < \frac{C}{\eta}\,. \tag{5.28}$$

PROOF. Since $\|\rho_n\|_1 = 1$, $\|\rho_{f,\eta,n}\|_1 \le 1$, the inequality (5.28) holds for small η. Therefore we may assume that η is large. Now we have

$$\|\rho_n - \rho_{f,\eta,n}\|_1 = \frac{2^n}{\sqrt{2}-1} \cdot \int_{2^{-n}}^{2^{-n+1/2}} d\lambda \, E_0\Big[\chi\Big(\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt - \eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big)\Big]. \tag{5.29}$$

From Theorem 1.1 and (1.11) we have that if ε is sufficiently small then

$$E_0\Big[\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt\Big] \le C_1\,\lambda^{2-3/q}\,\|f\|_{q,r} \tag{5.30}$$

for some constant C_1 depending only on r, p, q. Hence (5.29), (5.30) and Chebyshev's inequality implies that

$$\|\rho_n - \rho_{f,\eta,n}\|_1 \le \frac{2^n}{\sqrt{2}-1}\int_{2^{-n}}^{2^{-n+1/2}} d\lambda\,\frac{C_1}{\eta} = \frac{C_1}{\eta}\,. \tag{5.31}$$

The proof is complete.

To complete the proof of Theorem 5.1 we follow the argument of Lemma 4.4. Thus for $x, y \in \mathbb{R}^3$ we define a function $\xi_{f,\eta}(x,y)$ in analogy to the function $\xi(x,y)$ of Lemma 4.4. For $\lambda > 0$ and O_λ an arbitrary open subset of the sphere $\{z : |z| = \lambda\}$ we define

$$\int_{y-x\in O_\lambda} \xi_{f,\eta}(x,y)\,dy = \begin{array}{l}\text{probability that the drift}\\ \text{process started at } x\\ \text{exits the sphere } |y-x| = \lambda\\ \text{through the set } x + 0_\lambda, \text{ and}\end{array} \tag{5.32}$$
$$\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt < \eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\,.$$

Now let x_0 be an arbitrary point in Ω_R and τ be the time for the drift process starting at x_0 to hit $\partial 0\Omega_R$. We define points z_j, $j = 0, 1, 2, \ldots$ by

$$z_j = x_0 + j\,2^{-m_\varepsilon(\mathbf{b})}\mathbf{k}\,, \tag{5.33}$$

where $\mathbf{k} = (0,0,1)$ is the unit vector in $\mathbb{R}^3$ in the positive z direction, $m_\varepsilon(\mathbf{b})$ is as given in the statement of Theorem 5.1. Let $B_{j,\delta}$ be the ball of radius $\delta\, 2^{-m_\varepsilon(\mathbf{b})}$ centered at $z_j, j = 1,2,\ldots$ We choose $\delta < 1/2$ so that the balls $B_{j,\delta}$ do not intersect. Then, in analogy to the inequality (4.56) we have

$$
\begin{aligned}
&P_{x_0}\Big(\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt < N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) \\
&\qquad \geq \Big(\frac{1}{2\lambda\delta}\Big)^N \Big(\prod_{j=1}^N \int_{B_{j,\delta}} dy_j\Big) \\
&\qquad\quad \cdot\, \xi_{f,\eta}(x_0,y_1)\,\xi_{f,\eta}(y_1,y_2)\cdots\xi_{f,\eta}(y_{N-1},y_N)\,,
\end{aligned} \tag{5.34}
$$

where

$$
N = \frac{m_\varepsilon(\mathbf{b})}{n_0} + 1\,, \qquad \lambda = 2^{-m_\varepsilon(\mathbf{b})+1}\,. \tag{5.35}
$$

Lemma 4.3 and Lemma 5.3 imply that

$$
\frac{1}{2\lambda\delta}\int_{B_{j,\delta}} \xi_{f,\eta}(y_{j-1},y_j)\,dy_j \geq \gamma > 0\,. \tag{5.36}
$$

where γ depends only on p,q,r,ε, provided η is sufficiently large. We conclude then from (5.34) that

$$
P_{x_0}\Big(\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt < N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) \geq \gamma^N\,, \tag{5.37}
$$

whence it follows that

$$
\sup_{x\in\Omega_R} P_x\Big(\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt > N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) < 1-\gamma^N\,, \tag{5.38}
$$

where $0 < \gamma < 1$. The estimate (5.2) follows from (5.38) and the Markov property. In fact

$$
\begin{aligned}
|u(x)| &\leq E_x\Big[\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt\Big] \\
&\leq N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\sum_{k=0}^\infty (1-\gamma^N)^k \\
&= N\,\eta\,\gamma^{-N}\,\lambda^{2-3/q}\,\|f\|_{q,r}\,,
\end{aligned} \tag{5.39}
$$

since (5.38) implies by the Markov property that

$$\sup_{x\in\Omega_R} P_x\Big(\int_0^T |f|(X_{\mathbf{b}}(t))\,dt > k\,N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) < (1-\gamma^N)^k\,, \tag{5.40}$$

for $k = 1, 2, \ldots$ It is finally easy to see that

$$N\,\gamma^{-N}\,\lambda^{2-3/q} \le C_1\,R^{2-3/q}\,\exp(C_2 m_\varepsilon(\mathbf{b})/n_0)\,, \tag{5.41}$$

for some constants C_1, C_2 depending only on p, q, r.

6. L^∞-bounds.

We shall prove Theorem 1.4 here by refining the estimates already proved in [1]. It is clear we may assume $\mathbf{b}$ and f are C^∞ functions on Ω_R. Hence the drift process $X_{\mathbf{b}}(t)$ is defined and also the expectations of f we shall be considering.

Let Q_0 be a cube concentric with Ω_R having side of length 2^{-n_0}, where n_0 is defined by (1.22). We have the following

Lemma 6.1. *Suppose for some integer $m \ge 0$, the drift $\mathbf{b}$ satisfies the inequality*

$$\int_Q |\mathbf{b}|^p\,dx < \varepsilon^p\,|Q|^{1-p/3}\,, \tag{6.1}$$

on all dyadic subcubes $Q \subset Q_0$ with side of length 2^{-n}, $n \ge m + n_0$. Let u be the solution of the Dirichlet problem (1.2)-(1.3). Then if ε is sufficiently small, depending only on $p > 2$, there exist constants C_1 depending only on p, q, r, and C_2 only on $p > 2$, such that

$$\|u\|_\infty \le C_1\,R^{2-3/q}\,\|f\|_{q,r}\,\sup_{x\in\Omega_r}\exp\Big(C_2\sum_{j=0}^{m} a_{n_0+j}(x)\Big)\,. \tag{6.2}$$

PROOF. If $m = 0$, the perturbative Theorem 1.1 applies and the estimate (6.2) is just the same as (1.11). Therefore we may assume m is large. In that case we modify the proof of Theorem 1.4 of [1]. Consider the function $\xi(x)$, $x \in \Omega_R$, given by

$$\xi(x) = E_x\Big[\exp\Big(-\frac{1}{\mu}\int_0^T |f|(X_{\mathbf{b}}(t))\,dt\Big)\Big]\,, \tag{6.3}$$

where τ is the time for the drift process starting at x to hit ∂Q_0. The parameter μ is given by the formula

$$\mu = C\, R^{2-3/q}\, \|f\|_{q,r}\,, \tag{6.4}$$

where the constant C is to be chosen large, depending only on p, q, r. Let U be the set

$$U = \{y:\ 2^{-m-n_0-1} \le |x-y| \le 2^{-m-n_0}\}\,. \tag{6.5}$$

We define a density $\rho : U \to \mathbb{R}$ by the relation

$$\begin{aligned} 2^{m+n_0+1}\int_{2^{-m-n_0-1}}^{2^{-m-n_0}} d\lambda\, E_x\Big[g\,(X(\tau_\lambda))\,\exp\Big(-\frac{1}{\mu}\int_0^{\tau_\lambda} |f|\,(X_{\mathbf{b}}(t))\,dt\Big)\Big] \\ = \int_U \rho(y)\,g(y)\,dy\,, \end{aligned} \tag{6.6}$$

for all continuous functions $g : U \to \mathbb{R}$. Here τ_λ denotes the hitting time for the drift process started at x on the sphere $\{y : |x-y| = \lambda\}$. From Sections 2 and 3 it is clear that

$$\int_{\rho(y)<2^{3(m+n_0)}} \rho(y)\,dy > \frac{1}{2}\,, \tag{6.7}$$

provided ε is sufficiently small depending on $p > 2$, and C in (6.4) is chosen sufficiently large depending on p, q, r.

It follows from (6.3) and (6.6) that

$$\xi(x) = \int_U dy\ \rho(y)\,E_y\Big[\exp\Big(-\frac{1}{\mu}\int_0^{\tau} |f|(X_{\mathbf{b}}(t))\,dt\Big)\Big]\,. \tag{6.8}$$

Now we apply the same argument as in Section 5 of [1] to conclude that

$$\xi(x) \ge \eta(x)^2\,, \tag{6.9}$$

where

$$\begin{aligned} \eta(x) = \int_U dy\ \rho(y)\,E_y\Big[\exp\Big(-\frac{1}{2\mu}\int_0^{\tau} |f|(X(t))\,dt \\ -\frac{1}{4}\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\,, \end{aligned} \tag{6.10}$$

and $X(t)$ is standard Brownian motion. Applying the same argument as in the proof of Theorem 1.1.*a*) of [1] to (6.10) we obtain the inequality

$$\eta(x) \geq \frac{1}{2} \exp\Big(- C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big) , \tag{6.11}$$

where $C_2 > 0$ is universal provided C in (6.4) is chosen sufficiently large depending on p, q, r.

Evidently (6.11) implies a lower bound on $\xi(x)$. The inequality (6.2) follows from this bound and Lemma 5.1 of [1].

Next we consider the probability of hitting a dyadic subcube Q_n of Q_0 with side of length 2^{-n}, $n > n_0$, before exiting Ω_R.

Lemma 6.2. *For $n \in \mathbb{Z}$, let Ω_n be the region*

$$\Omega_n = \{x \in \mathbb{R}^3 : \ 2^{-n-1} < |x| < 2^{-n+1}\} . \tag{6.12}$$

For $x \in \Omega_n$ let P_x be the probability that the drift process started at x exits Ω_n through the sphere $\{y : |y| = 2^{-n+1}\}$. Let δ be a number satisfying $0 < \delta < 2/3$. Then if $|x| = 2^{-n}$ there is a constant C depending only on $\delta < 2/3$ and $p > 2$ such that

$$P_x \geq \delta \exp(-C\, a_{n-1}(0)) . \tag{6.13}$$

PROOF. Let χ be the function defined on the boundary $\partial\Omega_n$ of Ω_n by

$$\chi(z) = \begin{cases} 1, & \text{if } |z| = 2^{-n+1} , \\ 0, & \text{otherwise.} \end{cases} \tag{6.14}$$

We define a function $\xi(x)$ analogous to (6.3) by

$$\xi(x) = E_x[\chi(X_{\mathbf{b}}(\tau))] , \tag{6.15}$$

where τ is the first hitting time on $\partial\Omega_n$ for the drift process started at $x \in \Omega_n$. Hence $\xi(x)$ is the probability of exiting Ω_n though the outer sphere. We wish to generalize the inequality (6.9). Let $K > 0$ be some arbitrary constant to be specified later and put

$$\eta(x) = E_x\Big[\chi(X(\tau)) \exp\Big(- \frac{K}{4} \int_0^{\tau} |\mathbf{b}|^2(X(t))\, dt\Big)\Big] , \tag{6.16}$$

where $X(t)$ denotes Brownian motion in (6.16) and τ is its first exit time from Ω_n. Then for any $s > 1, 1/s + 1/s' = 1$, we have by Hölder's inequality and the Cameron-Martin formula.

$$
\begin{aligned}
\eta(x) = E_x\Big[\chi(X(\tau) \exp\Big(\frac{1}{2s}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\
- \frac{1}{4s}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\
\cdot \exp\Big(-\frac{1}{2s}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\
- \frac{1}{4}\Big(K - \frac{1}{s}\Big)\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big] \\
\leq \xi(x)^{1/s} E_x\Big[\exp\Big(\frac{-s'}{2s}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\
- \frac{s'}{4}\Big(K - \frac{1}{s}\Big)\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]^{1/s'}.
\end{aligned}
\tag{6.17}
$$

In view of (4.7) we have the inequality

$$\eta(x) \leq \xi(x)^{1/s}, \tag{6.18}$$

provided

$$s'\Big(K - \frac{1}{s}\Big)\Big(\frac{s}{s'}\Big)^2 \geq 1. \tag{6.19}$$

Hence if we choose K by the formula

$$K = \frac{1}{s-1}, \tag{6.20}$$

then the inequality (6.18) holds. Note that K diverges if we let s approach 1.

We can estimate $\eta(x)$ from below in exactly the same way Theorem 1.1.*a*) of [1] was proved. In fact we have the inequality

$$\eta(x) \geq E_x[\chi(X(\tau))] \exp\Big(-C\,K^{1/2}\sum_{m=n-2}^{\infty} a_m(x)\Big), \tag{6.21}$$

where the constant C is universal, provided x lies in the region

$$2^{-n-1/2} < |x| < 2^{-n+1/2}. \tag{6.22}$$

It is a simple matter to compute $E_x[\chi(X(\tau))]$. In fact we have

$$E_x[\chi(X(\tau))] = \frac{4}{3}\left(1 - \frac{2^{-n-1}}{|x|}\right). \tag{6.23}$$

Now consider an arbitrary point $x_0 \in \Omega_n$ with $|x_0| = 2^{-n}$. Choose $s > 1$ sufficiently small so that $\delta^{1/s} < 2/3$, and let B be the ball

$$B = \left\{x : |x - x_0| < \min\left\{2^{-n}\left(\frac{2 - 3\,\delta^{1/s}}{4 - 3\,\delta^{1/s}}\right), 2^{-n}(1 - 2^{-1/2})\right\}\right\}. \tag{6.24}$$

It is clear from (6.23) that B is contained in the region (6.22) and

$$E_x[\chi(X(\tau))] \geq \delta^{1/s}, \qquad x \in B. \tag{6.25}$$

Let $X(t)$ be an arbitrary continuous path with $X(0) = x_0$, $X(t) \in B$, $t < \tau$, and $X(\tau) \in \partial B$. We claim that there exists an $x = X(t)$ for some t, $0 \leq t \leq \tau$, such that

$$\sum_{m=n-2}^{\infty} a_m(x) \leq C_1\, a_{n-1}(0), \tag{6.26}$$

where the constant C_1 depends only on $\delta^{1/s} < 2/3$ and $p > 2$. Here we are taking $\mathbf{b}(x) = 0$ for $x \notin \Omega_n$ in our definition of $a_m(x)$. The inequality (6.13) clearly follows from (6.18), (6.21), (6.25), (6.26).

We are left to prove (6.26). Let $C_1 > 0$, β be constants with $0 < \beta < 1$ and consider the sets

$$S_m = \{x \in B :\ a_m(x) > C_1\, \beta^{m-n}\, a_{n-1}(0)\}, \qquad m \geq n-2. \tag{6.27}$$

We shall assume that

$$\{X(t) :\ 0 \leq t \leq \tau\} \subset \bigcup_{m=n-2}^{\infty} S_m. \tag{6.28}$$

Otherwise there exists a point x on the path $X(t)$ in the complement of all the sets S_m, $m \geq n-2$, in which case (6.26) clearly holds since $\beta < 1$. For each $x \in S_m$ let D_x be the open ball centered at x with radius 2^{-m}. From (6.28) it follows that the sets $\{D_x : x \in S_m,\ m \geq n-2\}$ form an open cover of the path $X(t)$, $0 \leq t \leq \tau$. By compactness of the path there exists a finite subcover $\Gamma = \{D_j : 1 \leq j \leq N\}$ for some integer

N. For each integer $m \geq n-2$, let Γ_m be the subset of Γ consisting of balls with radius 2^{-m}. Let D be an arbitrary ball and $\tilde{D}$ the ball concentric with D but with three times the radius. Then there exists a subset $\tilde{\Gamma}_m \subset \Gamma_m$ of disjoint balls such that

$$\bigcup_{D \in \Gamma_m} D \subset \bigcup_{D \in \tilde{\Gamma}_m} \tilde{D}. \tag{6.29}$$

Since the balls in $\tilde{\Gamma}_m$ are disjoint it follows from the definition (6.27) of S_m that the cardinality $|\tilde{\Gamma}_m|$ of the set $\tilde{\Gamma}_m$ satisfies the inequality

$$\begin{aligned} |\tilde{\Gamma}_m| \, (C_1 \, \beta^{m-n} \, a_{n-1}(0))^p &\leq 2^{m(3-p)} \int_{\Omega_n} |\mathbf{b}|^p \, dy \\ &\leq 2^{(3-p)(m+1-n)} \, a_{n-1}(0)^p \,, \end{aligned} \tag{6.30}$$

which implies the bound,

$$|\tilde{\Gamma}_m| \leq \frac{2^{3-p}}{C_1^p} \left(\frac{2^{3-p}}{\beta^p} \right)^{m-n}. \tag{6.31}$$

We choose β so that

$$\frac{2^{3-p}}{\beta^p} < 2. \tag{6.32}$$

This is possible since $2 < p < 3$. It is clear that for any point x on the path $X(t)$, $0 \leq t \leq \tau$, one must have the inequality

$$|x - x_0| \leq \sum_{m=n-2}^{\infty} 6 \cdot 2^{-m} \, |\tilde{\Gamma}_m| \leq A \, \frac{2^{-n}}{C_1^p} \,, \tag{6.33}$$

where A depends only on β satisfying (6.32). Since $X(\tau)$ lies on the boundary of the ball B in (6.24) the inequality (6.33) is violated for $x = X(\tau)$ provided C_1 is chosen sufficiently large. Hence we have a contradiction to our assumption (6.28). The proof is complete.

Lemma 6.3. *Let $S_0, S_1, \ldots, S_N$ be a set of concentric spheres with radii $r_0, r_1, \ldots, r_N$ satisfying $r_0 < r_1 < r_2 < \cdots < r_N$. For $j = 1, \ldots, N-1$ let $q_j(x,y)$ be non negative functions of $x \in S_j$, $y \in S_{j-1}$ satisfying*

$$0 < \int_{S_{j-1}} q_j(x,y) \, dy \leq q_j < 1 \,, \qquad x \in S_j \,, \tag{6.34}$$

for some positive numbers $q_1, \dots, q_{N-1}$.

Suppose now the $q_j(x,y)$ *are probability density functions for a stochastic process* $Y(t)$ *with continuous paths in the following sense: For any open set* $0 \subset S_{j-1}$,

$$\text{Prob}\,\{Y \textit{ started at } x \in S_j \textit{ exits the region between } S_{j-1} \textit{ and } S_{j+1} \textit{ through } 0\} = \int_0 q_j(x,y)\,dy\,. \tag{6.35}$$

Let $x \in S_{N-m}$ *for some* m, $1 \le m \le N-1$, *and* P_x *be the probability that* Y *started at* x *exits the region between* S_0 *and* S_N *through* S_0. *Then there is the inequality*

$$P_x \cdot \le \frac{1 + \dfrac{p_{N-1}}{q_{N-1}} + \dfrac{p_{N-1}}{q_{N-1}}\dfrac{p_{N-2}}{q_{N-2}} + \cdots + \displaystyle\prod_{j=1}^{m-1} \dfrac{p_{N-j}}{q_{N-j}}}{1 + \dfrac{p_{N-1}}{q_{N-1}} + \dfrac{p_{N-1}}{q_{N-1}}\dfrac{p_{N-2}}{q_{N-2}} + \cdots + \displaystyle\prod_{j=1}^{N-1} \dfrac{p_{N-j}}{q_{N-j}}}\,, \tag{6.36}$$

where the p_j *are defined by* $p_j = 1 - q_j$, $j = 1, \dots, N-1$.

PROOF. Observe that the right hand side of (6.36) is just u_{N-m} where u_n, $n = 0, 1, \dots, N$ is the solution of the finite difference equation

$$\begin{aligned} &u_n = p_n\, u_{n+1} + q_n\, u_{n-1}\,, \qquad 1 \le n \le N-1\,, \\ &u_0 = 1\,, \qquad u_N = 0\,. \end{aligned} \tag{6.37}$$

Hence the lemma merely states that P_x is bounded by the probability for a random walk on the spheres S_j, $j = 0, \dots, N$, with transition probabilities determined by the upper bound q_j in (6.34), $j = 1, \dots, N-1$.

To prove the lemma we first consider the case when $N = 3$. For $x \in S_1 \cup S_2$ let $u(x)$ be the probability that Y started at x hits S_0 before hitting S_3. Then $u(x)$ must satisfy the equations

$$\begin{aligned} u(x) &= \int_{S_1} q_2(x,y)\,u(y)\,dy, \qquad x \in S_2\,, \\ u(x) &= \int_{S_2} p_1(x,y)\,u(y)\,dy + \int_{S_0} q_1(x,y)\,dy\,, \qquad x \in S_1\,, \end{aligned} \tag{6.38}$$

where $p_1(x,y)$ is the probability density for the process started at $x \in S_1$ of hitting S_2 before S_0. It follows from (6.38) that

$$u(x) = \int_{S_1}\int_{S_2} q_2(x,y)\,p_1(y,z)\,u(z)\,dz\,dy + \int_{S_1}\int_{S_0} q_2(x,y)\,q_1(y,z)\,dz\,dy\,, \quad x \in S_2. \tag{6.39}$$

Putting $u_2 = \sup_{x\in S_2} u(x)$ it follows from (6.39) that

$$u_2 \le \sup_{x\in S_2} A(x)\,, \tag{6.40}$$

where

$$A(x) = \frac{\int_{S_1}\int_{S_0} q_2(x,y)\,q_1(y,z)\,dz\,dy}{1 - \int_{S_1}\int_{S_2} q_2(x,y)\,p_1(y,z)\,dz\,dy}\,. \tag{6.41}$$

Using the fact that for any $y \in S_1$,

$$\int_{S_2} p_1(y,z)\,dz = 1 - \int_{S_0} q_1(y,z)\,dz\,, \tag{6.42}$$

we have that

$$\frac{1}{A(x)} = \frac{1 + \int_{S_1} q_2(x,y)\,dy\left(\left(\int_{S_1} q_2(x,y)\,dy\right)^{-1} - 1\right)}{\int_{S_1}\int_{S_0} q_2(x,y)\,q_1(y,z)\,dz\,dy}\,. \tag{6.43}$$

Using (6.34) for $j = 1$, we have from (6.43) that $A(x)$ satisfies the inequality

$$\frac{1}{A(x)} \ge 1 + \frac{1}{q_1}\left(\left(\int_{S_1} q_2(x,y)\,dy\right)^{-1} - 1\right). \tag{6.44}$$

Next, applying (6.34) for $j = 2$ to the right side of (6.44) yields

$$\frac{1}{A(x)} \ge 1 + \frac{1}{q_1}\left(\frac{1}{q_2} - 1\right) = 1 + \frac{p_2}{q_2} + \frac{p_2}{q_2}\frac{p_1}{q_1}\,. \tag{6.45}$$

Inequality (6.45) together with (6.40) implies (6.36) for $N = 3$, $m = 1$.

Next we consider the case $N = 3$, $m = 2$. From (6.38) we have

$$u(x) = \int_{S_2}\int_{S_1} p_1(x,y)\, q_2(y,z)\, u(z)\, dz\, dy + \int_{S_0} q_1(x,y)\, dy\,, \qquad x \in S_1\,. \tag{6.46}$$

Setting $u_1 = \sup_{x\in S_1} u(x)$ we obtain from (6.46) the inequality

$$u_1 \le \sup_{x\in S_1} B(x)\,, \tag{6.47}$$

where

$$B(x) = \frac{\displaystyle\int_{S_0} q_1(x,y)\, dy}{1 - \displaystyle\int_{S_2}\int_{S_1} p_1(x,y)\, q_2(y,z)\, dz\, dy}\,. \tag{6.48}$$

Using (6.34) with $j = 2$ it follows that

$$B(x) \le \frac{\displaystyle\int_{S_0} q_1(x,y)\, dy}{1 - q_2 \displaystyle\int_{S_2} p_1(x,y)\, dy} = \frac{\displaystyle\int_{S_0} q_1(x,y)\, dy}{1 - q_2 + q_2 \displaystyle\int_{S_0} q_1(x,y)\, dy}\,. \tag{6.49}$$

Next, applying (6.34) with $j = 1$, yields the inequality

$$B(x) \le \frac{q_1}{1 - q_2 + q_2\, q_1} = \frac{1 + \dfrac{p_2}{q_2}}{1 + \dfrac{p_2}{q_2} + \dfrac{p_2}{q_2}\dfrac{p_1}{q_1}}\,. \tag{6.50}$$

Hence (6.36) for $N = 3$, $m = 2$ follows from (6.50) and (6.47).

The situation for $N \ge 4$ can be derived from the $N = 3$ case by induction. Suppose we already know that (6.36) holds for any sequence of less than $N + 1$ spheres. We consider the case of $N + 1$ spheres $S_0, S_1, \dots, S_N$. Let k be an integer satisfying $2 \le k \le N - 1$, and consider the case of the four spheres S_0, S_{k-1}, S_k, S_N. Let Q_1 be an upper bound on the probability of Y starting at $x \in S_{k-1}$ of hitting S_0 before S_k. Similarly let Q_2 be an upper bound on the probability of Y starting at $x \in S_k$ of hitting S_{k-1} before S_N. Then by our result already obtained for the 4 sphere case, we have

$$P_x \leq \left(1 + \frac{1}{Q_1}\left(\frac{1}{Q_2} - 1\right)\right)^{-1}, \quad x \in S_k . \tag{6.51}$$

By our inductive assumptions we have bounds on Q_1, Q_2, namely

$$\frac{1}{Q_1} \geq 1 + \frac{p_{k-1}}{q_{k-1}} + \frac{p_{k-1}}{q_{k-1}}\frac{p_{k-2}}{q_{k-2}} + \cdots + \prod_{j=1}^{k-1} \frac{p_{k-j}}{q_{k-j}} ,$$

$$\frac{1}{Q_2} - 1 \geq \frac{\displaystyle\prod_{j=1}^{N-k} \frac{p_{N-j}}{q_{N-j}}}{1 + \dfrac{p_{N-1}}{q_{N-1}} + \cdots + \displaystyle\prod_{j=1}^{N-k-1} \frac{p_{N-j}}{q_{N-j}}} . \tag{6.52}$$

Substituting the right hand side of (6.52) into (6.51) clearly implies the bound (6.36).

Finally we must deal with the case $k = 1$. Here the four spheres are S_0, S_1, S_2, S_N. Let Q_2 be an upper bound on the probability of Y started at $x \in S_2$ of hitting S_1 before S_N. Then from (6.47), (6.50) we have

$$P_x \leq \frac{q_1}{1 - Q_2 + Q_2\, q_1} , \quad x \in S_1 . \tag{6.53}$$

By our induction assumption we have the bound

$$Q_2 \leq \frac{1 + \dfrac{p_{N-1}}{q_{N-1}} + \cdots + \displaystyle\prod_{j=1}^{N-3} \frac{p_{N-j}}{q_{N-j}}}{1 + \dfrac{p_{N-1}}{q_{N-1}} + \cdots + \displaystyle\prod_{j=1}^{N-2} \frac{p_{N-j}}{q_{N-j}}} . \tag{6.54}$$

Hence we have

$$\frac{1}{q_1}(1 - Q_2) + Q_2 = 1 - Q_2 + \frac{p_1}{q_1}(1 - Q_2) + Q_2$$

$$\geq \frac{1 + \dfrac{p_{N-1}}{q_{N-1}} + \cdots + \displaystyle\prod_{j=1}^{N-1} \frac{p_{N-j}}{q_{N-j}}}{1 + \dfrac{p_{N-1}}{q_{N-1}} + \cdots + \displaystyle\prod_{j=1}^{N-2} \frac{p_{N-j}}{q_{N-j}}} . \tag{6.55}$$

Substituting (6.55) into (6.53) yields the estimate (6.36) for $x \in S_1$.

Lemma 6.4. *For any integer $m \geq 0$ and arbitrary $\varepsilon > 0$ let $U_m \subset Q_0$ be the union of all dyadic subcubes Q of Q_0 with side of length 2^{-n_0-m} such that*

$$\int_Q |\mathbf{b}|^p \, dx \geq \varepsilon^p \, |Q|^{1-p/3} . \tag{6.56}$$

For $x \in \Omega_R$ let $X_{\mathbf{b}}(t)$ be the process with drift $\mathbf{b}$ starting at x, where we set $\mathbf{b}$ to be identically zero outside Q_0. For $\lambda > 0$ let τ_λ be the first hitting time on the sphere of radius λ centered at x and $P_m(x)$ be the probability

$$\begin{aligned} &P_m(x) \\ &= \operatorname{Prob}\{X_{\mathbf{b}}(t) \text{ hits } U_m \text{ in the time interval } \tau_{R/2} < t < \tau_R\} . \end{aligned} \tag{6.57}$$

Then there exists a constant $\gamma, 0 < \gamma < 1$, and constants C_1, C_2 depending only on $p > 2$ such that

$$P_m(x) \leq C_1 \, \varepsilon^{-p} \, \gamma^m \sup_{y \in Q_{n_0}} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(y)\Big) . \tag{6.58}$$

PROOF. We can assume m large since the right hand side will be larger than 1 if m is small. Let $Q \subset Q_0$ be a dyadic cube with side of length 2^{-n_0-m} and center x_0. Then for any path $X_{\mathbf{b}}(t)$ there exists a time t_0 with $0 \leq t_0 < \tau_{R/2}$ such that

$$|X_{\mathbf{b}}(t_0) - x_0| \geq \frac{R}{4} . \tag{6.59}$$

For $j = 0, 1, 2, \ldots$ let S_j be the sphere centered at x_0 with radius 2^{-n_0-m+j}. Then from (1.22) we see that $X_{\mathbf{b}}(t_0)$ lies in the region outside the sphere S_{m-4}. Also the ball of radius R centered at x lies inside the sphere S_{m+1}. Hence the probability P_Q of $X_{\mathbf{b}}(t)$ hitting Q in the time interval $\tau_{R/2} < t < \tau_R$ is less than the supremum of the probabilities of the drift process started at $x \in S_{m-4}$ of hitting S_0 before S_{m+1}.

Now we can use Lemmas 6.2 and 6.3 to estimate this last probability. From Lemma 6.2 we have that

$$q_j \leq 1 - \delta \exp\Big(- C \, a_{n_0+m-j-1}(x_0)\Big) . \tag{6.60}$$

Hence the inequality (6.36) yields

$$P_Q \le A\left(\frac{1}{\delta}-1\right)^m \exp\left(\frac{C}{1-\delta}\sum_{j=0}^{m} a_{n_0+j}(x_0)\right), \tag{6.61}$$

for some universal constant A. Here we have used the fact that $\mathbf{b} \equiv 0$ outside Q_{n_0} and that

$$\delta^{-1} e^{\xi} - 1 \le (\delta^{-1}-1)\exp\frac{\xi}{1-\delta}, \qquad \xi \ge 0. \tag{6.62}$$

Finally we estimate the number N_m of cubes $Q \subset U_m$. From (6.56) we have, if 0 is the center of Q_0, that

$$\varepsilon^p\, 2^{-(n_0+m)(3-p)}\, N_m \le 2^{-n_0(3-p)}\, a_{n_0}(0)^p, \tag{6.63}$$

whence

$$N_m \le \varepsilon^{-p}\, 2^{m(3-p)}\, a_{n_0}(0)^p. \tag{6.64}$$

Hence $P_m(x)$ is bounded by the product of the right side of (6.64) and the supremum over $x_0 \in Q_0$ of the right side of (6.61). Now using the fact that, $p > 2$ and δ can be chosen as close as we please to $2/3$, yields (6.58).

Lemma 6.5. *Let $f \in M^q_r$, $q > 3/2$, $1 < r \le q$. Then there exists γ, $0 < \gamma < 1$, depending only on $p > 2$ such that*

$$\begin{aligned} &\sup_{x\in\Omega_R} E_x\Big[\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big] \\ &\quad \le C_1\, R^{2-3/q}\, \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big). \end{aligned} \tag{6.65}$$

The constant C_1 depends only on p, q, r and C_2 only on $p > 2$.

PROOF. We write

$$\begin{aligned} &E_x\Big[\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big] \\ &\quad = \sum_{m=0}^{\infty} E_x\Big[\chi_m(X_{\mathbf{b}}) \int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big], \end{aligned} \tag{6.66}$$

where χ_m is the characteristic function of the set of paths which visit U_m between times $\tau_{R/2}$ and τ_R but do not visit any U_n with $n > m$. We use Schwarz's inequality to obtain

$$
\begin{aligned}
&E_x\Big[\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big] \\
&\le \sum_{m=0}^{\infty} E_x\Big[\chi_m(X_{\mathbf{b}})\Big(\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big)^2\Big]^{1/2} P_m(x)^{1/2}.
\end{aligned}
\tag{6.67}
$$

We have now that

$$
\begin{aligned}
&E_x\Big[\chi_m(X_{\mathbf{b}})\Big(\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big)^2\Big] \\
&\qquad \le E_x\Big[\Big(\int_0^{\tau_R} |f|(X_{\mathbf{b}_m}(t))\,dt\Big)^2\Big],
\end{aligned}
\tag{6.68}
$$

where $\mathbf{b}_m$ is equal to $\mathbf{b}$ on $Q_0\setminus \cup_{j=m+1}^{\infty} U_j$ but zero otherwise. This is true because the characteristic function χ_m restricts to paths which do not visit $\cup_{j=m+1}^{\infty} U_j$. The drift $\mathbf{b}_m$ satisfies the conditions for Lemma 6.1 and hence if ε is sufficiently small there is the inequality

$$
\begin{aligned}
&\sup_{x\in\Omega_R} E_x\Big[\int_0^{\tau_R} |f|(X_{\mathbf{b}_m}(t))\,dt\Big] \\
&\qquad \le C_1\, R^{2-3/q}\, \|f\|_{q,r} \sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big).
\end{aligned}
\tag{6.69}
$$

It follows from (6.69) using the Chebyshev inequality and the Markov property that

$$
\begin{aligned}
&\sup_{x\in\Omega_R} E_x\Big[\Big(\int_0^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big)^2\Big]^{1/2} \\
&\qquad \le 40\, C_1\, R^{2-3/q}\, \|f\|_{q,r} \sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big).
\end{aligned}
\tag{6.70}
$$

The inequality (6.65) follows now from (6.70), (6.67) and Lemma 6.4.

PROOF OF THEOREM 1.4. We bound the solution $u(x)$ of (1.2)-(1.3) by

$$
|u(x)| \le \sum_{k=1}^{\infty} E_x\Big[\int_{\tau_{R/2^k}}^{\tau_{R/2^{k-1}}} |f|(X_{\mathbf{b}}(t))\,dt\Big].
\tag{6.71}
$$

From Lemma 6.5 we have the inequality

$$
\begin{aligned}
(6.72)\qquad & E_x\Big[\int_{\tau_{R/2^k}}^{\tau_{R/2^{k-1}}} |f|(X_{\mathbf{b}}(t))\,dt\Big] \\
& \le C_1\, 2^{-(2-3/q)(k-1)}\, R^{2-3/q}\, \|f\|_{q,r} \\
& \quad \cdot \sum_{m=k-1}^{\infty} \gamma^{m-(k-1)} \sup_{y\in\Omega_R} \exp\Big(C_2 \sum_{j=k-1}^{m} a_{n_0+j}(y)\Big) \\
& \le C_1\, 2^{-(2-3/q)(k-1)/2}\, R^{2-3/q} \\
& \quad \cdot \sum_{m=0}^{\infty} \gamma_1^m \sup_{y\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(y)\Big),
\end{aligned}
$$

where $\gamma_1 = \max\{\gamma, 2^{-(2-3/q)/2}\}$.

Summing the right side of the last inequality with respect to k proves the theorem.

Our last result shows the inequality (1.24) follows from Theorem 1.4.

Proposition 6.6. *There exist universal constants C, c such that for any $\varepsilon > 0$, $x \in \mathbb{R}^3$, there is the inequality*

$$
(6.73)\qquad \sum_{n=-\infty}^{\infty} a_n(x)\, H(a_n(x) - \varepsilon) \le C\, N_{c\varepsilon}(\mathbf{b}),
$$

where $H(t)$ is the Heaviside function, $H(t) = 1$, if $t > 0$ and $H(t) = 0$, if $t \le 0$.

PROOF. For $n \in \mathbb{Z}$ let $c_n(x)$ be defined by

$$
(6.74)\qquad c_n(x) = \Big(2^{n(3-p)} \int_{2^{-n-1}<|x-y|<2^{-n}} |\mathbf{b}|^p\, dy\Big)^{1/p}.
$$

It is clear from (6.74), (1.21) that there is the identity

$$
(6.75)\qquad 2^{-(3-p)}\, a_{n+1}(x)^p + c_n(x)^p = a_n(x)^p.
$$

Let α be an arbitrary positive number, $0 < \alpha < 1$, such that

$$
(6.76)\qquad \delta^{-p} = (1-\alpha^p)\, 2^{3-p} > 1.
$$

If $c_n(x) > \alpha\, a_n(x)$ it is clear that

$$a_n(x)\, H(a_n(x) - \varepsilon) \leq \alpha^{-1}\, c_n(x)\, H(c_n(x) - \alpha\,\varepsilon)\,. \tag{6.77}$$

On the other hand if $c_n(x) < \alpha a_n(x)$, then (6.75) implies that $a_n(x) < \delta\, a_{n+1}(x)$, and hence

$$a_n(x)\, H(a_n(x) - \varepsilon) \leq \delta\, a_{n+1}(x)\, H(a_{n+1}(x) - \varepsilon)\,. \tag{6.78}$$

Putting (6.77), (6.78) together we conclude that for all values of $a_n(x)$ there is the inequality

$$\begin{aligned} a_n(x)\, H(a_n(x) - \varepsilon) \leq \delta\, a_{n+1}(x)\, H(a_{n+1}(x) - \varepsilon) \\ + \alpha^{-1}\, c_n(x)\, H(c_n(x) - \alpha\,\varepsilon)\,. \end{aligned} \tag{6.79}$$

If we sum (6.79) over $n \in \mathbb{Z}$ we obtain the inequality

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} a_n(x)\, H(a_n(x) - \varepsilon) \\ &\qquad \leq \frac{1}{\alpha(1-\delta)} \sum_{n=-\infty}^{\infty} c_n(x)\, H(c_n(x) - \alpha\,\varepsilon)\,. \end{aligned} \tag{6.80}$$

Now let us suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. For any $n \in \mathbb{Z}$, let S_n be the set,

$$\begin{aligned} S_n = \{Q : Q \cap \{y : 2^{-n-1} < |x-y| < 2^{-n}\} \\ \text{is not empty and } |Q| \leq 2^{-3(n+3)}\}\,. \end{aligned} \tag{6.81}$$

For any $\varepsilon > 0$ let $N_{\varepsilon,n}(\mathbf{b})$ be the number of minimal cubes for $\mathbf{b}$ which are in S_n. It is clear from the definition (6.81) of S_n that

$$\sum_{n=-\infty}^{\infty} N_{\varepsilon,n}(\mathbf{b}) \leq 2\, N_\varepsilon(\mathbf{b})\,. \tag{6.82}$$

Next let us suppose $c_n(x) > \alpha\,\varepsilon$. Then we have

$$(\alpha\,\varepsilon)^p < 2^{n(3-p)} \sum_{\substack{Q \in S_n \\ |Q| = 2^{-3(n+3)}}} \int_Q |\mathbf{b}|^p\, dy\,. \tag{6.83}$$

Since there are at most 2^{12} cubes $Q \in S_n$ with side of length 2^{-n-3} it follows from (6.83) that one of them must satisfy the inequality

$$\int_Q |\mathbf{b}|^p \, dy \geq (\alpha \, \varepsilon \, 2^{-12/p} \, 2^{3(3/p-1)})^p \, |Q|^{1-p/3} \,. \tag{6.84}$$

Hence if $c_n(x) > \alpha\varepsilon$ and c satisfies the inequality

$$c < \alpha \, 2^{-12/p} \, 2^{3(3/p-1)} \,, \tag{6.85}$$

then from (1.20) one must have $N_{c\,\varepsilon,n}(\mathbf{b}) \geq 1$.

Finally we use a result of Fefferman [3]. Let $\varepsilon > 0$ be arbitrary. Then there exist disjoint sets $E_1, E_2, \ldots, E_M$ with the properties:

a) $\displaystyle\bigcup_{Q \in S_n} Q = \bigcup_{j=1}^{M} E_j \,.$

b) Each E_j is a subset of a cube $Q_j \in S_n$.

c) $\displaystyle\int_{E_j} |\mathbf{b}|^p \, dy \leq C_1 \, \varepsilon^p \, |Q_j|^{1-p/3}$, for some universal constant C_1, $j = 1, \ldots, M$.

d) $M \leq C_2 \, (N_{\varepsilon,n}(\mathbf{b}) + 1)$, for some universal constant C_2.

We can bound $c_n(x)$ by using the Fefferman deconposition. Thus for any $c > 0$, we have

$$\begin{aligned} c_n(x)^p &\leq 2^{n(3-p)} \sum_{j=1}^{M} \int_{E_j} |\mathbf{b}|^p \, dy \\ &\leq 2^{n(3-p)} \sum_{j=1}^{M} C_1 \, (c\,\varepsilon)^p \, |Q_j|^{1-p/3} \\ &\leq C_1 \, (c\,\varepsilon)^p \, M \\ &\leq C_1 \, (c\,\varepsilon)^p \, C_2 \, (N_{c\,\varepsilon,n}(\mathbf{b}) + 1) \,. \end{aligned} \tag{6.86}$$

Now if we choose c to satisfy the inequality (6.85) we have that

$$c_n(x)^p \leq 2 \, C_1 \, (c\,\varepsilon)^p \, C_2 \, N_{c\,\varepsilon,n}(\mathbf{b}) \,, \tag{6.87}$$

provided $c_n(x) > \alpha\,\varepsilon$. The inequality (6.73) follows from (6.80), (6.82) and (6.87).

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