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Fourier coefficients of Jacobi forms over Cayley numbers

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Abstract. In this paper, we shall compute explicitly the Fourier coefficients of the Eisenstein series

$$E_{k,m}(z,w) = \frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}\sum_{t\in o}\exp\Big\{2\pi i m\Big(\frac{az+b}{cz+d}N(t) + \sigma\big(t,\frac{w}{cz+d}\big) - \frac{cN(w)}{cz+d}\Big)\Big\}$$

which is a Jacobi form of weight k and index m defined on $\mathcal{H}_1 \times \mathcal{C}_{\mathbb{C}}$, the product of the upper half-plane and Cayley numbers over the complex field $\mathbb{C}$. The coefficient of $e^{2\pi i(nz+\sigma(t,w))}$ with $nm > N(t)$, has the form

$$-\frac{2(k-4)}{B_{k-4}}\prod_p S_p\,.$$

Here S_p is an elementary factor which depends only on $\nu_p(m)$, $\nu_p(t)$, $\nu_p(n)$ and $\nu_p(nm-N(t))$. Also $S_p = 1$ for almost all p. Indeed, one has $S_p = 1$ if $\nu_p(m) = \nu_p(nm-N(t)) = 0$. An explicit formula for S_p will be given in details. In particular, these Fourier coefficients are rational numbers.

1. Notation and Introduction.

As usual $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, and the field of rational numbers, real numbers, and complex numbers, respectively. $\mathcal{C}_f$ is the Cayley numbers over the field f and $\mathbf{o}$ is the ring of integral Cayley numbers in $\mathcal{C}_{\mathbb{R}}$. $\mathcal{C}_f$ is an eight-dimensional vector space over f with a basis $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ which is characterized by the follwing rules for multiplication ([1]):

1) $xe_0 = e_0 x = x$, for all $x \in \mathcal{C}$,

2) $e_i^2 = -e_0, \quad i = 1, 2, \ldots, 7$,

3) $e_1e_2e_4 = e_2e_3e_5 = e_3e_4e_6 = e_4e_5e_7 = e_5e_6e_1 = e_7e_1e_3 = -e_0$.

Also $\mathbf{o}$ has a $\mathbb{Z}$-basis α_0, α_1, α_2, α_3, α_4, α_5, α_6, α_7 given by

$$\alpha_0 = e_0\,, \quad \alpha_1 = e_1\,, \quad \alpha_2 = e_2\,, \quad \alpha_3 = e_4\,,$$
$$\alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4)\,, \quad \alpha_5 = \frac{1}{2}(-e_0 - e_1 - e_4 + e_5)\,,$$
$$\alpha_6 = \frac{1}{2}(-e_0 + e_1 - e_2 + e_6)\,, \quad \alpha_7 = \frac{1}{2}(-e_0 + e_2 + e_4 + e_7)\,.$$

For $x = \sum_{j=0}^{7} x_j e_j\,,\ y = \sum_{j=0}^{7} y_j e_j\,;\ x_j, y_j \in f$, we define

$$N(x) = \sum_{j=0}^{7} x_j^2\,, \qquad \sigma(x,y) = 2\sum_{j=0}^{7} x_j y_j\,.$$

Let k, m be a pair of positive integers. A holomorphic function ψ on $\mathcal{H}_1 \times \mathcal{C}_{\mathbb{C}}$ is a Jacobi form of weight k and index m if it satisfies the following conditions

(J.1) $\psi(\frac{az+b}{cz+d}, \frac{w}{cz+d}) = (cz+d)^k \exp\{2\pi i m N(w) \frac{c}{cz+d}\}\, \psi(z,w)$,

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$.

(J.2) $\psi(z, w + \lambda z + \mu) = \exp\{-2\pi i m(zN(\lambda) + \sigma(\lambda, w))\}\, \psi(z,w)$,

for all $\lambda, \mu \in \mathbf{o}$.

(J.3) ψ possesses a Fourier expansion of the form

$$\psi(z,w)=\sum_{n\geq 0}\sum_{\substack{t\in\mathbf{o}\\ nm\geq N(t)}}\alpha_{\psi}(n,t)\,e^{2\pi i(nz+\sigma(t,w))}\,.$$

For positive integers k,m with k even and $k\geq 10$, we let

$$E_{k,m}(z,w)=\frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}\sum_{t\in\mathbf{o}}\exp\Big\{2\pi im\Big(\frac{az+b}{cz+d}N(t)+\sigma\big(t,\frac{w}{cz+d}\big)-\frac{c\,N(w)}{cz+d}\Big)\Big\}$$

with $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$. Then a direct verification shows that $E_{k,m}$ satisfies (J.1) and (J.2). For the proof, see [5].

In this paper, we shall show that the Fourier coefficient $e_{k,m}(n,t)$ with $nm>N(t)$ in the Fourier expansion

$$E_{k,m}(z,w)=\sum_{n=0}^{\infty}\sum_{\substack{nm\geq N(t)\\ t\in\mathbf{o}}}e_{k,m}(n,t)\,e^{2\pi i(nz+\sigma(t,w))}$$

of $E_{k,m}(z,w)$ is a rational number of the form

$$-\frac{2(k-4)}{B_{k-4}}\prod_p S_p\,.$$

Let ν_p be the standard discrete valuation in $\mathbb{Q}_p$ with $\nu_p(p^j)=j$ for all $j\in\mathbb{Z}$. For $t=\sum_{j=0}^{7}t_j\alpha_j\in\mathbf{o}$, we set $\nu_p(t)=\min_{0\leq j\leq 7}\nu_p(t_j)$. For our convenience, we set $\triangle=mn-N(t)$ and $\triangle'=\triangle/m=n-N(t)/m$.

Theorem. *For positive integers m, k with k even and $k\geq 10$, the Fourier coefficient $e_{k,m}(n,t)=0$ if $nm<N(t)$. If $mn=N(t)$, then*

$$e_{k,m}(n,t)=\begin{cases}1\,, & \text{if } t=mt' \text{ and } n=mN(t') \text{ for some } t'\in\mathbf{o}\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

If $mn>N(t)$, then

$$e_{k,m}(n,t)=-\frac{2(k-4)}{B_{k-4}}\prod_p S_p\,,$$

where S_p is given by

1) *If* $\nu_p(m)=0$, *then* $S_p=\sum_{j=0}^{\nu_p(\triangle)} p^{j(k-5)}$.

2) *If* $\nu_p(m)>\nu_p(t)$, *then*

$$S_p=p^{l_1}\,\frac{1-p^{8-k}}{1-p^{4-k}}\sum_{j=0}^{\alpha} p^{j(9-k)}+\begin{cases} p^{l_2}(1-p^{4-k})^{-1}, & \text{if } \nu_p(t)\le\nu_p(n),\\ 0, & \text{if } \nu_p(t)>\nu_p(n).\end{cases}$$

3) *If* $\nu_p(m)\le\nu_p(t)$, *then*

$$S_p=p^{l_1}\,\frac{1-p^{8-k}}{1-p^{4-k}}\sum_{j=0}^{\beta} p^{j(9-k)}+\begin{cases} p^{l_3}(1-p^{4-k})^{-1}, & \text{if } \nu_p(m)\le\nu_p(n),\\ 0, & \text{if } \nu_p(m)>\nu_p(n).\end{cases}$$

$$+\sum_{j=\gamma+1}^{\nu_p(\triangle')} p^{(k-5)(\nu_p(\triangle')-j)}-\begin{cases} p^{l_4}(1-p^{4-k})^{-1}, & \text{if } \nu_p(m)\le\nu_p(\triangle'),\\ 0, & \text{if } \nu_p(m)>\nu_p(\triangle').\end{cases}$$

In 2) *and* 3), *one has*

$$l_1=-(k-1)\,\nu_p(m)+(k-5)\,\nu_p(\triangle),$$
$$l_2=-(k-1)\,\nu_p(m)+(k-5)\,\nu_p(\triangle)+(9-k)\,\nu_p(t)+8-k,$$
$$l_3=(10-2k)\,\nu_p(m)+(k-5)\,\nu_p(\triangle)+8-k,$$
$$l_4=(10-2k)\,\nu_p(m)+(k-5)\,\nu_p(\triangle)+4-k.$$

Also

$$\alpha=\min\{\nu_p(t),\nu_p(n)\},\qquad \beta=\min\{\nu_p(m),\nu_p(n)\}$$

and

$$\gamma=\min\{\nu_p(m),\nu(\triangle')\}.$$

REMARK. Note that in the above, (1) is a special case of (3) and S_p in (1) can be obtained from S_p given in (3) by setting $\nu_p(m)=0$.

In particular, we have

$$e_{k,1}(n,t)=\begin{cases} 1, & \text{if } n=N(t),\\ -\dfrac{2\,(k-4)}{B_{k-4}}\,\sigma_{k-5}(n-N(t)), & \text{if } n>N(t).\end{cases}$$

From this, we conclude that $E_{k,1}(z,w)$ is a product of $E_{k-4}(z)$, the normalized Eisenstein series of weight $k-4$, and

$$\theta(z,w)=\sum_{t\in\mathfrak{o}} e^{2\pi i\,(zN(t)+\sigma(t,w))}$$

which is a Jacobi form of weight 4 and index 1.

The Fourier coefficient $\tilde{e}_{k,m}(n,r)$ of the Eisenstein series

$$\tilde{E}_{k,m}(z,w)=\frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}\sum_{\lambda\in\mathbb{Z}}\exp\left\{2\pi i m\Big(\frac{az+b}{cz+d}\lambda^2+2\lambda\frac{w}{cz+d}-\frac{cw^2}{cz+d}\Big)\right\}$$

was given in [4] in terms of Cohen's function $H(s,N)$; *i.e.*

$$\tilde{e}_{k,1}(n,r)=\frac{H(k-1,4n-r^2)}{\zeta(3-2k)}\,.$$

If m is square free, then

$$\tilde{e}_{k,m}(n,r)=\frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)}\sum_{d|(n,m,r)}d^{k-1}H(k-1,\frac{4nm-r^2}{d^2})$$

by the relation $\tilde{E}_{k,1}\,|_{T(m)}\,(z,w)=\sigma_{k-1}(m)\,\tilde{E}_{k,m}(z,w)$. Here $T(m)$ is the Hecke operator on the space of Jacobi forms of weight k and index 1, $J_{k,1}$, defined by

$$\psi|_{T(m)}(z,w)=m^{k-1}\sum_{ad=m}\sum_{0\le b<d}d^{-k}\,\psi(\frac{az+b}{d},\frac{mw}{d})\,.$$

However, we do not see any relation such as

$$E_{k,1}\,|_{T(m)}\,(z,w)=\sigma_{k-1}(m)E_{k,m}(z,w)$$

in the cases for Cayley numbers even if m is a prime number.

2. Fourier Coefficients of $E_{k,m}$.

From the formula for $E_{k,m}(z,w)$, we separate the sum over c and d into two sums $E^0_{k,m}(z,w)$ and $E^1_{k,m}(z,w)$ according to c is zero or not. If $c=0$, then $d=1$ or -1. We choose $a=d=1$ or -1, and $b=0$, so that

$$E^0_{k,m}(z,w)=\sum_{t\in\mathfrak{o}}e^{2\pi i m(zN(t)+\sigma(t,w))}\,. \tag{1}$$

Obviously, $E^0_{k,m}$ is a linear combination of $e^{2\pi i(nz+\sigma(t,w))}$ with coefficient 1 or 0 according to $nm = N(t)$ with $t = mt'$, $n = mN(t')$ for some $t' \in \mathbf{o}$ or not. For those terms with $c \neq 0$, we can rewrite the sum as

$$E^1_{k,m}(z,w) = \frac{1}{2}\sum_{(c,d)=1} c^{-k} \sum_{t\in\mathbf{o}} \Big(z+\frac{d}{c}\Big)^{-k}$$

$$\cdot \exp\Big\{2\pi i m\Big(-\frac{N(w-t/c)}{z+d/c}+\frac{aN(t)}{c}\Big)\Big\}. \tag{2}$$

Note that the substitutions $d \mapsto d+cp$ and $t \mapsto t+c\lambda$ correspond to $z \mapsto z+p$ and $w \mapsto w+\lambda$ in $E^1_{k,m}$, respectively. Here p is an integer and λ is an integral Cayley number. Hence

$$E^1_{k,m}(z,w) = \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{(c,d)=1\\ d(\bmod\ c)}} \sum_{t\in\mathbf{o}/c\mathbf{o}} \exp\Big(\frac{2\pi i m N(t)}{cd}\Big)$$

$$\cdot F_{k,m}\big(z+\frac{d}{c}, w-\frac{t}{c}\big) \tag{3}$$

with

$$F_{k,m}(z,w) = \sum_{p\in\mathbb{Z}}\sum_{\lambda\in\mathbf{o}} (z+p)^{-k}\exp\Big(-2\pi i m\,\frac{N(w+\lambda)}{z+p}\Big). \tag{4}$$

The function $F_{k,m}(z,w)$ is a periodic function in z and w, so it has Fourier expansion of the form

$$F_{k,m}(z,w) = \sum_{n\in\mathbb{Z}}\sum_{t\in\mathbf{o}} \gamma(n,t)\, e^{2\pi i(nz+\sigma(t,w))}.$$

In order to compute the Fourier coefficient $\gamma(n,t)$ of $F_{k,m}$, we need the following lemma which follows from the well known Poisson summation formula.

Lemma 1. *For any $h > 0$, we have*

$$\sum_{\lambda\in\mathbf{o}} \exp\{-2\pi h N(w+\lambda)\} = \frac{1}{h^4}\sum_{t\in\mathbf{o}} \exp\Big\{-2\pi\Big(\frac{N(t)}{h}+i\,\sigma(t,w)\Big)\Big\}.$$

Proposition 1. *Notation as above, then one has for* $k \geq 10$,

$$(5) \quad F_{k,m}(z,w) = \frac{\alpha_k}{m^{k-1}} \sum_{n\in\mathbb{N}} \sum_{\substack{t\in\mathfrak{o} \\ nm>N(t)}} (nm - N(t))^{k-5} e^{2\pi i(nz+\sigma(t,w))}$$

with

$$\alpha_k = \frac{(-2\pi i)^{k-4}}{(k-5)!} .$$

PROOF. By Lemma 1 and a standard argument (see [2, p. 226]), we get

$$\sum_{\lambda\in\mathfrak{o}} \exp\Big(-2\pi m \frac{N(w+\lambda)}{z+p}\Big) = \frac{(z+p)^4}{m^4} \sum_{t\in\mathfrak{o}} \exp\Big(2\pi i \frac{N(t)\,(z+p)}{m+2\pi i\,\sigma(t,w)}\Big),$$

for any $z \in \mathcal{H}_1$ and $p \in \mathbb{Z}$. It follows

$$F_{k,m}(z,w) = \frac{1}{m^4} \sum_{p\in\mathbb{Z}} \sum_{t\in\mathfrak{o}} (z+p)^{-k+4} \exp\Big(2\pi i \frac{N(t)\,(z+p)}{m}\Big)\, e^{2\pi i\,\sigma(t,w)} .$$

Note that the series

$$\sum_{p\in\mathbb{Z}} (z+p)^{-k+4} \exp\Big(2\pi i \frac{N(t)\,(z+p)}{m}\Big)$$

is a periodic function in $z = x + iy$. Let

$$\sum_{p\in\mathbb{Z}} (x+iy+p)^{-k+4} \exp\Big(2\pi i \frac{N(t)\,(x+iy+p)}{m}\Big) = \sum_{p\in\mathbb{Z}} c_n(y)\, e^{2\pi i n x} .$$

Then

$$\begin{aligned} c_n(y) &= \int_0^1 \sum_{p\in\mathbb{Z}} (x+iy+p)^{-k+4} \exp\Big\{2\pi i\Big(\frac{N(t)\,(x+iy+p)}{m} - nx\Big)\Big\}\, dx \\ &= \exp(-s\pi N(t)\, y/m) \int_{-\infty}^{+\infty} \frac{e^{2\pi i x(-n+N(t)/m)}}{(x+iy)^{k-4}}\, dx \\ &= \begin{cases} \alpha_k(\Delta')^{k-5}\, e^{-2\pi n y}, & \text{if } nm > N(t), \\ 0, & \text{if } nm \leq N(t). \end{cases} \end{aligned}$$

Here $\triangle' = n - N(t)/m$. For detail of calculations, see [4, p. 19]. Hence

$$\sum_{p\in\mathbb{Z}} (z+p)^{-k+4} \exp(2\pi i N(t)\,(z+p)/m)$$
$$= \frac{\alpha_k}{m^{k-1}} \sum_{\substack{n\in\mathbb{N} \\ nm>N(t)}} (nm - N(t))^{k-5}\, e^{2\pi i n z}$$

and our assertion for $F_{k,m}(z,w)$ follows.

In our next proposition, we shall express $e_{k,m}(n,t)$ as a Dirichlet series with an Euler product.

Proposition 2. *For $nm > N(t)$ and k even, $k \geq 10$, one has*

$$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}} (nm - N(t))^{k-5} \frac{1}{\zeta(k-8)} \sum_{a=1}^{\infty} T_a(Q)\, a^{-(k-1)}$$

with

$$T_a(Q) = \#\{\lambda \in \mathfrak{o}/a\mathfrak{o} : \; m\,N(\lambda) - \sigma(t,\lambda) + n \equiv 0 \,(\mathrm{mod}\;\, a)\}\,.$$

PROOF. We substitute $F_{k,m}(z,w)$ in Proposition 1 into (3), and get

(6) $$E^1_{k,m}(z,w) = \sum_{n\in\mathbb{N}} \sum_{\substack{t\in\mathfrak{o} \\ nm>N(t)}} e_{k,m}(n,t)\, e^{2\pi i(nz+\sigma(t,w))}$$

with

(7) $$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}} (nm - N(t))^{k-5} \cdot \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{(c,d)=1 \\ d\leq c}} \sum_{t\in\mathfrak{o}/c\,\mathfrak{o}} \exp\left\{2\pi i \left(\frac{m\,N(\lambda)}{cd} + \sigma(t, -\frac{\lambda}{c}) + \frac{nd}{c}\right)\right\}.$$

Since $(c,d) = 1$, we can replace λ by $d\lambda$ in the third summation of $e_{k,m}(n,t)$. Hence

(8) $$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}} (nm - N(t))^{k-5} \cdot \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{(c,d)=1 \\ d(\mathrm{mod}\ c)}} \sum_{\lambda\in\mathfrak{o}/c\,\mathfrak{o}} \exp\left(2\pi i \frac{d}{c} (m\,N(\lambda) + \sigma(t,-\lambda) + n)\right).$$

Let $Q(\lambda) = m\,N(\lambda) - \sigma(t,\lambda) + n$. Use the well known formula

$$\sum_{\substack{(c,d)=1 \\ d(\mathrm{mod}\ c)}} e^{2\pi i dN/c} = \sum_{a|(c,N)} \mu(\frac{c}{a})\,a \tag{9}$$

with $\mu(a)$ the Möbius function. Hence

$$\begin{aligned} e_{k,m}(n,t) &= \frac{\alpha_k}{m^{k-1}}\,(nm - N(t))^{k-5} \sum_{c=1}^{\infty} c^{-k} \sum_{a|(c,Q(\lambda))} \mu(\frac{c}{a})\,a \sum_{\substack{\lambda\in\mathbf{o}/c\,\mathbf{o} \\ a|Q(\lambda)}} 1 \\ &= \frac{\alpha_k}{m^{k-1}}\,(nm - N(t))^{k-5} \sum_{c=1}^{\infty} c^{-k} \sum_{a|(c,Q(\lambda))} \mu(\frac{c}{a})\,a\,\big(\frac{c}{a}\big)^8 \sum_{\substack{\lambda\in\mathbf{o}/c\,\mathbf{o} \\ a|Q(\lambda)}} 1\,. \end{aligned}$$

Let $c = ab$ and use the formula

$$\sum_{b=1}^{\infty} \mu(b)\,b^{-s} = \frac{1}{\zeta(s)}\,, \qquad \text{for } \operatorname{Re} s > 1\,,$$

to get

$$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}}\,(nm - N(t))^{k-5}\,\frac{1}{\zeta(k-8)} \sum_{a=1}^{\infty} T_a(Q)\,a^{-(k-1)}\,. \tag{10}$$

Here

$$T_a(Q) = \#\{\lambda \in \mathbf{o}/a\mathbf{o}:\ Q(\lambda) \equiv 0\,(\mathrm{mod}\ a)\}\,.$$

To obtain the explicit formula for $e_{k,m}(n,t)$ when $nm > N(t)$, we have to find the value of the Dirichlet series

$$\sum_{a=1}^{\infty} T_a(Q)\,a^{-s}$$

at $s = k-1$. Here

$$T_a(Q) = \#\{\lambda \in \mathbf{o}/a\mathbf{o}:\ Q(\lambda) = m\,N(\lambda) - \sigma(\lambda,t) + n \equiv 0\,(\mathrm{mod}\ a)\}\,.$$

By the multiplicativity of $T_a(Q)$, it suffices to consider the case $a = p^{\nu}$ ($\nu \in \mathbb{Z},\ \nu \geq 0$).

In the following consideration, we set $T_\nu(Q) = T_{p^\nu}(Q)$, $\omega_\nu = e^{2\pi i/p^\nu}$. We also set

$$Z(s) = \sum_{\nu=0}^{\infty} T_\nu(Q)\, p^{-\nu s}.$$

Proposition 3. *For any positive integer ν, we have for*

$$\lambda = \sum_{j=0}^{7} \lambda_j \alpha_j\,, \qquad t = \sum_{j=0}^{7} t_j \alpha_j\,,$$

that

$$(11)\quad T_\nu(Q) = p^{3\nu} \sum_{\tau=0}^{\nu} p^{4\tau} \sum_{\alpha'}{}^{\tau} \omega_{\nu-\tau}^{\alpha' n} \prod_{j=0}^{3} \Big(\sum_{m\lambda_j \equiv t_j \,(\mathrm{mod}\ p^{\nu-\tau})} \omega_{\nu-\tau}^{\alpha' t_{j+4}\lambda_j} \Big).$$

where α' ranges over all positive integers between 1 and $p^{\nu-\tau}$ with $(\alpha', p) = 1$ in the summation $\sum^{\tau}$.

PROOF. By the p-adic version of Siegel's Babylonian reduction process, we can express $T_\nu(Q)$ as a Gaussian sum given by

$$T_\nu(Q) = p^{-\nu} \sum_{\alpha=1}^{p^\nu} \sum_{\lambda \in \mathfrak{o}/p^\nu \mathfrak{o}} \omega_\nu^{\alpha(mN(\lambda) - \sigma(\lambda,t) + n)}.$$

Over the p-adic integers, the quadratic form N is equivalent to the quadratic form with matrix $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, where E is the 4×4 identity matrix. Thus

$$\sum_{\lambda \in \mathfrak{o}/p^\nu \mathfrak{o}} \omega_\nu^{\alpha(mN(\lambda) - \sigma(\lambda,t) + n)}$$

$$= \omega_\nu^{\alpha n} \prod_{j=0}^{3} \Big(\sum_{\lambda_j=1}^{p^\nu} \sum_{\lambda_{j+4}=1}^{p^\nu} \omega_\nu^{\alpha(m\lambda_j \lambda_{j+4} - t_j \lambda_{j+4} - t_{j+4}\lambda_j)} \Big)$$

$$= \omega_\nu^{\alpha n} \prod_{j=0}^{3} \Big(\sum_{\lambda_j=1}^{p^\nu} \omega_\nu^{-a t_{j+4}\lambda_j} \sum_{\lambda_{j+4}=1}^{p^\nu} \omega_\nu^{\alpha(m\lambda_j - t_j)\lambda_{j+4}} \Big).$$

Note that

$$\sum_{\lambda_{j+4}=1}^{p^\nu} \omega_\nu^{\alpha(m\lambda_j - t_j)\lambda_{j+4}} = \begin{cases} p^\nu, & \text{if } \alpha(m\lambda_j - t_j) \equiv 0\,(\mathrm{mod}\ p^\nu), \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, we let $\alpha = \alpha' p^\tau$ with $(\alpha', p) = 1$ and get our assertion by an elementary calculation.

REMARK. For fixed $\nu \geq 1$ and $0 \leq \tau \leq \nu$, the product

$$\prod_{j=0}^{3} \Big(\sum_{\lambda_j+1}^{p^\nu} \sum_{\lambda_{j+4}=1}^{p^\nu} \omega_{\nu-\tau}^{\alpha'(m\lambda_j \lambda_{j+4} - t_j \lambda_{j+4} - t_{j+4}\lambda_j)} \Big) \tag{12}$$

is zero unless the congruences

$$m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad j = 0, 1, 2, 3$$

have a solution. By the symmetry of t_j and $t_{j+4'}$ we conclude that the product in (12) is zero unless the congruences

$$m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad \text{for } j = 0, 1, 2, 3, 4, 5, 6, 7$$

have at least a solution.

3. Cases with $\nu_p(m) = 0$.

From Proposition 3 and its remark, we note that the evaluation of $T_\nu(Q)$ depends on solving the congruences

$$m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad j = 0, 1, 2, \ldots, 7. \tag{13}$$

Obviously, the solvability of the congruences is wholy determined by $\nu_p(m)$, $\nu_p(t)$ and $\nu - \tau$.

In this Section, we shall investigate those cases with $(m, p) = 1$. Under such assumption, the congruences in (13) have always a unique solution.

Proposition 4. *If $(m, p) = 1$ and $\delta = \nu_p(n - N(t)/m)$, then one has for* $\operatorname{Re} s > 8$,

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = \frac{1 - p^{3-s}}{1 - p^{7-s}} \sum_{j=0}^{\delta} p^{-(s-4)j}.$$

PROOF. Denote the solution of the congruences

$$m\lambda_j \equiv t_j \,(\mathrm{mod}\ p^{\nu-\tau}), \qquad j = 0,1,2,3$$

by $\lambda_j = t_j/m$, $j = 0,1,2,3$. Hence by Proposition 3, we have

$$T_\nu(Q) = p^{3\nu} \sum_{\tau=0}^{\nu} p^{4\tau} {\sum_{\alpha'}}^{\tau} \omega_{\nu-\tau}^{\alpha'(n-N(t)/m)} .$$

Apply (9) to the second summation; we get

$$T_\nu(Q) = p^{3\nu} \sum_{\tau=0}^{\nu} p^{4\tau} \sum_{j=0}^{\min\{\delta,\nu-\tau\}} \mu(p^{\nu-\tau-j}) p^j .$$

Note that $\mu(1) = 1$, $\mu(p) = -1$ and $\mu(p^l) = 0$ for $l \geq 2$. It follows

$$\begin{aligned} T_\nu(Q) &= p^{3\nu}\Big(\sum_{0 \leq \nu-\tau \leq \delta} p^{4\tau} p^{\nu-\tau} - \sum_{0 \leq \nu-\tau-1 \leq \delta} p^{4\tau} p^{\nu-\tau-1}\Big) \\ &= p^{4\nu}\Big(\sum_{0 \leq \nu-\tau \leq \delta} p^{3\tau} - \sum_{0 \leq \nu-\tau-1 \leq \delta} p^{3\tau-1}\Big) \end{aligned}$$

Now we shall prove by induction on δ that our assertion is true. In order to distinguish the cases for different δ, we let

$$Z_q(s) = \sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s}, \qquad \delta = q .$$

When $\delta = 0$, then $\tau = \nu$ in the first summation and $\tau = \nu - 1$ in the second summation. Hence

$$Z_0(s) = \sum_{\nu=0}^{\infty} p^{7\nu} p^{-\nu s} - \sum_{\nu=1}^{\infty} p^{7\nu-4} p^{-\nu s} = \frac{1-p^{3-s}}{1-p^{7-s}} .$$

Suppose that for $\delta = q$ the assertion is true. Now

$$\begin{aligned} Z_{q+1}(s) - Z_q(s) &= \sum_{\nu=q+1}^{\infty} p^{3(\nu-q-1)} p^{-\nu(s-4)} - \sum_{\nu=q+2}^{\infty} p^{3(\nu-q-2)-1} p^{-\nu(s-4)} \\ &= p^{-(q+1)(s-4)} \frac{1-p^{3-s}}{1-p^{7-s}} . \end{aligned}$$

Thus the formula is also true for $\delta = q + 1$ and our proof is complete.

Corollary. *If* $n > N(t)$, *then*

$$e_{k,1}(n,t) = -\frac{2(k-4)}{B_{k-4}}\,\sigma_{k-5}(n - N(t))\,.$$

PROOF. From (10) and Proposition 4 we have

$$\begin{aligned} e_{k,1}(n,t) &= \alpha_k (n - N(t))^{k-5} \frac{1}{\zeta(k-8)} \frac{\zeta(k-8)}{\zeta(k-4)} \sum_{d \mid [n-N(t)]} d^{-(k-5)} \\ &= \frac{\alpha_k}{\zeta(k-4)}\,\sigma_{k-5}(n - N(T))\,. \end{aligned}$$

But

$$\frac{\alpha_k}{\zeta(k-4)} = \frac{(-2\pi i)^{k-4}}{(k-5)!\,\zeta(k-4)} = -\frac{2(k-4)}{B_{k-4}}\,,$$

hence our assertion follows.

Corollary. $E_{k,1}(z,w) = E_{k-4}(z)\,\theta(z,w)$ *with*

$$\theta(z,w) = \sum_{t \in \mathbf{o}} e^{2\pi i (N(t) z + \sigma(t,w))}\,.$$

PROOF. Note that $e_{k,1}(n,t) = 0$ unless $n \geq N(t)$. Also $e_{k,1}(N(t),t) = 1$ by an observation. Then we have

$$\begin{aligned} E_{k,1}(z,w) &= \sum_{t \in \mathbf{o}} e^{2\pi i (N(t) z + \sigma(t,w))} \\ &\quad - \frac{2(k-4)}{B_{k-4}} \sum_{n > N(t)} \sigma_{k-5}(n - N(t))\, e^{2\pi i (nz + \sigma(t,w))} \\ &= \sum_{t \in \mathbf{o}} e^{2\pi i (N(t) z + \sigma(t,w))} \\ &\quad - \frac{2(k-4)}{B_{k-4}} \sum_{n=1}^{\infty} \sum_{t \in \mathbf{o}} \sigma_{k-5}(n)\, e^{2\pi i (n + N(t)) z + \sigma(t,w)} \\ &= \Big(1 - \frac{2(k-4)}{B_{k-4}} \sum_{n=1}^{\infty} \sigma_{k-5}(n)\, e^{2\pi i n z}\Big)\,\theta(z,w) \\ &= E_{k-4}(z)\,\theta(z,w)\,. \end{aligned}$$

Corollary.

$$\theta(z,w)=\sum_{t\in\mathfrak{o}} e^{2\pi i(N(t)z+\sigma(t,w))},$$

is a Jacobi form of weight 4 *and index* 1.

4. Cases with $0 \le \nu_p(t) < \nu_p(m)$.

For fixed $\nu \ge 1$ and $0 \le \tau \le \nu$. If $0 \le \nu_p(t) < \nu_p(m)$, then the congruences

$$m\lambda_j \equiv t_j \,(\mathrm{mod}\ p^{\nu-\tau}), \qquad j=0,1,2,3$$

have solutions only if $\nu-\tau \le \nu_p(t)$. Moreover the number of solutions is $p^{4(\nu-\tau)}$.

Proposition 5. *Under the condition* $0 \le \nu_p(t) < \nu_p(m)$, *then one has for* $\mathrm{Re}\, s > 8$,

$$\sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} = \sum_{j=0}^{\alpha} p^{(8-s)j} + \begin{cases} p^{(8-s)\nu_p(t)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(t) \le \nu_p(n),\\ 0, & \text{if } \nu_p(t) > \nu_p(n),\end{cases}$$

where $\alpha = \min\{\nu_p(n), \nu_p(t)\}$.

PROOF. Begin with (11) of Proposition 3 and the observation above, and get

$$\sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} = 1+\sum_{\nu=1}^{\infty} p^{-\nu s} \sum_{\nu-\tau\le\nu_p(t)} p^{3\nu+4\tau}p^{4\nu-4\tau}\sum_{\alpha'}^{\tau} \omega_{\nu-\tau}^{\alpha' n}.$$

Apply (9) to the third summation, we get

$$\sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} = 1+\sum_{\nu=1}^{\infty} p^{-\nu s} \sum_{\nu-\tau\le\nu_p(t)} \sum_{j=0}^{\min\{\nu-\tau,\nu_p(n)\}} \mu(p^{\nu-\tau-j})p^j.$$

Denote the coefficient of $p^{(7-s)\nu}$ by A_ν. According to $\nu_p(t) \le \nu_p(n)$ or $\nu_p(t) > \nu_p(n)$, we have the following two cases.

Case I. $\nu_p(t) \le \nu_p(n)$. Then $\min\{\nu-\tau, \nu_p(n)\} = \nu-\tau$ since $\nu-\tau \le \nu_p(t) \le \nu_p(n)$. Therefore

$$A_\nu = \sum_{\nu-\tau\le\nu_p(t)} \sum_{0\le j\le\nu-\tau} \mu(p^{\nu-\tau-j})p^j = \begin{cases} p^\nu, & \text{if } \nu \le \nu_p(t), \\ p^{\nu_p(t)}, & \text{if } \nu > \nu_p(t). \end{cases}$$

Hence

$$\begin{aligned}\sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} &= 1 + \sum_{\nu=1}^{\infty} p^{(7-s)\nu} A_\nu \\ &= \sum_{j=0}^{\nu_p(t)} p^{j(8-s)} + p^{(8-s)\nu_p(t)+7-s}(1-p^{7-s})^{-1}.\end{aligned}$$

Case II. $\nu_p(t) > \nu_p(n)$. Then

$$A_\nu = \sum_{\nu-\tau\le\nu_p(n)} \sum_{0\le j\le\nu-\tau} \mu(p^{\nu-\tau-j})p^j + \sum_{\nu-\tau>\nu_p(n)}^{\nu_p(t)} \sum_{0\le j\le\nu_p(n)} \mu(p^{\nu-\tau-j})p^j.$$

Note that the first sum in A_ν can be computed as in the case I. The second sum in A_ν is zero unless $\nu \ge \nu_p(n)+1$, $\nu-\tau = \nu_p(n)+1$ and $j = \nu_p(n)$. For such exceptional cases, the sum is $-p^{\nu_p(n)}$. Consequently, we have

$$A_\nu = \begin{cases} p^\nu, & \text{if } \nu \le \nu_p(t), \\ 0, & \text{if } \nu > \nu_p(t). \end{cases}$$

It follows

$$\sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} = 1 + \sum_{\nu=1}^{\infty} p^{(7-s)\nu} A_\nu = \sum_{j=0}^{\nu_p(n)} p^{j(8-s)}.$$

This proves our assertions.

5. Cases with $\nu_p(m) \le \nu_p(t)$.

For fixed $\nu \ge 1$ and $0 \le \tau \le \nu$. If $\nu_p(m) \le \nu_p(t)$, then the congruences

$$m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad j = 0,1,2,3$$

always have solutions. The number of solutions is $p^{4\nu_p(m)}$ if $\nu_p(m) < \nu - \tau$, and if $\nu_p(m) \geq \nu - \tau$, the number of solutions is $p^{4(\nu-\tau)}$.

Proposition 6. *Under the condition* $\nu_p(m) \leq \nu_p(t)$, *one has for* $\operatorname{Re} s > 8$

$$\begin{aligned}\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = & \sum_{j=0}^{\beta} p^{(8-s)j} \\ & + \begin{cases} p^{(8-s)\nu_p(m)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \leq \nu_p(n), \\ 0, & \text{if } \nu_p(m) > \nu_p(n), \end{cases} \\ & + p^{4\nu_p(m)} \frac{1-p^{3-s}}{1-p^{7-s}} \sum_{j=\gamma+1}^{\nu_p(\triangle')} p^{(4-s)j} \\ & - \begin{cases} p^{(8-s)\nu_p(m)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \leq \nu_p(\triangle'), \\ 0, & \text{if } \nu_p(m) > \nu_p(\triangle'). \end{cases}\end{aligned}$$

Here $\beta = \min\{\nu_p(m), \nu_p(n)\}$ *and* $\gamma = \min\{\nu_p(m), \nu_p(\triangle')\}$.

PROOF. We begin with (11) of Proposition 3, and separate the series into two subseries according to $\nu - \tau > \nu_p(m)$ or $\nu_p(m) \leq \nu - \tau$. Hence

$$\begin{aligned}\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = 1 + & \sum_{\nu=1}^{\infty} p^{(3-s)\nu} \sum_{\nu-\tau>\nu_p(m)} p^{4\tau+4\nu_p(m)} \sum_{\alpha'}{}^{\tau} \omega_{\nu-\tau}^{\alpha'\triangle'} \\ & + \sum_{\nu=1}^{\infty} \sum_{\nu-\tau\leq\nu_p(m)} p^{(7-s)\nu} \sum_{\alpha'}{}^{\tau} \omega_{\nu-\tau}^{\alpha' n},\end{aligned}$$

where α' ranges over all positive integers between 1 and $p^{\nu-\tau}$ with $(\alpha', p) = 1$ in the summation $\sum^{\tau}$.

Let $Z_1(s)$ be the subseries corresponding to the summation $\nu - \tau > \nu_p(m)$ and $Z_2(s)$ be the remaining sum. By the computations in Proposition 5, we have

$$\begin{aligned}Z_2(s) = & \sum_{j=0}^{\beta} p^{(8-s)j} \\ & + \begin{cases} p^{(8-s)v_p(m)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \leq \nu_p(n), \\ 0, & \text{if } \nu_p(m) > \nu_p(n)', \end{cases}\end{aligned}$$

where $\beta = \min\{\nu_p(m), \nu_p(n)\}$.

Also we have

$$\begin{aligned}
Z_1(s) &= p^{4\bar{\nu}_p(m)}\Big(\sum_{\nu=0}^{\infty} p^{(3-s)\nu} \sum_{0\le\tau\le\nu} p^{4\tau} \sum_{j=0}^{\min\{\nu-\tau,\nu_p(\triangle')\}} \mu(p^{\nu-\tau-j})p^j \\
&\quad - \sum_{\nu=0}^{\infty} p^{(3-s)\nu} \sum_{\nu-\tau\le\nu_p(m)} \sum_{j=0}^{\min\{\nu-\tau,\nu_p(\triangle')\}} \mu(p^{\nu-\tau-j})p^j\Big) \\
&= p^{4\nu_p(m)}\Big(\sum_{\nu=0}^{\infty} p^{(4-s)\nu}\Big(\sum_{0\le\tau\le\nu(\triangle')} p^{3\tau} - \sum_{0\le\tau-1\le\nu(\triangle')} p^{3\tau-1}\Big)\Big) \\
&\quad - p^{4\nu_p(m)} \sum_{\nu=0}^{\infty} p^{(4-s)\nu}\Big(\sum_{0\le\tau\le\gamma} p^{3\tau} - \sum_{0\le\tau-1\le\gamma} p^{3\tau-1}\Big) \\
&\quad - \begin{cases} p^{5\nu_p(m)} \displaystyle\sum_{\nu=\nu_p(m)+1}^{\infty} p^{(3-s)\nu} p^{4(\nu-\nu_p(m)-1)}, & \text{if } \nu_p(m) \le \nu_p(\triangle'), \\ 0, & \text{if } \nu_p(m) > \nu_p(\triangle'). \end{cases}
\end{aligned}$$

Here $\gamma = \min\{\nu_p(m), \nu_p(\triangle')\}$. Now by the computations of Proposition 4, we conclude that

$$\begin{aligned}
Z_1(s) &= p^{4\nu_p(m)} \frac{1-p^{3-s}}{1-p^{7-s}} \sum_{j=\gamma+1}^{\nu(\triangle')} p^{-j(s-4)} \\
&\quad - \begin{cases} p^{(8-s)\nu_p(m)+3-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \le \nu_p(\triangle'), \\ 0, & \text{if } \nu_p(m) > \nu_p(\triangle'). \end{cases}
\end{aligned}$$

Combine Proposition 2 and Propositions 4, 5, 6 with $s = k-1$ together. Also using the well known result

$$\frac{\alpha_k}{\zeta(k-4)} = \frac{(-2\pi i)^{k-4}}{(k-5)!\,\zeta(k-4)} = -\frac{2(k-4)}{B_{k-4}},$$

we get

$$e_{k,m}(n,t) = -\frac{2(k-4)}{B_{k-4}} \prod_p S_p,$$

where S_p is as we claimed in the Theorem.

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