

Formulas for approximate
solutions of the $\partial\bar{\partial}$ -equation
in a strictly
pseudoconvex domain

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Abstract. Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n . We construct approximative solution formulas for the equation $i\partial\bar{\partial}u = \theta$, θ being an exact $(1,1)$ -form in D . We show that our formulas give simple proofs of known estimates and indicate further applications.

Introduction.

Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n . The main result of this paper is a weighted approximate solution formula for the equation

$$(0.1) \quad i\partial\bar{\partial}u = \theta,$$

where θ is an exact $(1,1)$ -form (current) in D .

The equation (0.1) is of interest mainly because of its connection to divisors (zero sets) of holomorphic functions. Namely, to each such divisor there is associated a positive closed $(1,1)$ -current and the solutions to (0.1) are precisely functions of the form $u = \log |f|$ (disregarding for

the moment possible topological obstructions) where f is a holomorphic function that defines the divisor. Hence bounds on the solution u in (0.1) proves the existence of holomorphic functions in various function classes with given zero set θ . For results of this kind, see for instance [H], [S], [V2] and [B].

When $n = 1$, (0.1) is just Poisson's equation and can be solved by the Newton kernel $(1/2\pi) \log |\zeta - z|$ if θ has finite mass in D . However, if e.g. θ just satisfies the Blaschke condition

$$(0.2) \quad \int_D d(\zeta, \partial D) |\theta(\zeta)| < +\infty,$$

then one has to use a weighted solution kernel; the Green's functions in D , which is not explicitly given in general but nevertheless well understood, at least if ∂D has some regularity.

If $n > 1$, (0.1) is usually solved by following a two step method that goes back to Lelong. One first solves

$$(0.3) \quad i dw = \theta$$

so that $w_{0,1} = -\bar{w}_{1,0}$ (assuming θ real). For bidegree reasons $\bar{\partial}w_{0,1} = 0$ so one can solve

$$(0.4) \quad \bar{\partial}v = w_{0,1}.$$

Then $u = v + \bar{v}$ solves (0.1).

In 1975 it was proved independently by Henkin [H] and Skoda [S] that (0.1) admits a solution in $L^1(\partial D)$ if θ satisfies the Blaschke condition (0.2). This result had been conjectured for some time and the main problem for the solution was to get $L^1(\partial D)$ -estimates for (0.4). This was solved by Henkin and Skoda by introducing solution formulas with weights that gave the desired estimate.

However, in other situations it is (0.3) that offers the greatest difficulties, as for instance in Varopoulos' result, [V2], that there is a solution u in $BMO(\partial D)$ if θ satisfies a certain Carleson condition.

Explicit solution formulas for the L^2_α -minimal solutions to (0.1) in the ball in \mathbb{C}^n were obtained in [An1] and [An2] ($L^2_\alpha = L^2((1 - |\zeta|^2)^{\alpha-1} d\lambda)$). For appropriate choices of α these formulas admitted simple and natural proofs of the (known) estimates for (0.1) discussed above. Earlier Berndtsson [B] used an explicit formula for the ball in \mathbb{C}^2 to show that (0.1) has a negative solution if θ has finite mass. Recently this result has been generalized by Arlebrink [Ar] to the strictly pseudoconvex case.

Let ρ be a strictly plurisubharmonic C^4 defining function for D and put $L_\alpha^p = L^p((- \rho)^{\alpha-1} d\lambda)$ if $\alpha > 0$ and $L_0^p = L^p(\partial D)$. In this paper we use ideas from [An1], [An2] and [Ar] to construct operators M_α , acting on (1,1)-forms θ in D , and P_α and F_α , acting on functions, such that

$$(0.5) \quad u = M_\alpha(i\partial\bar{\partial}u) + P_\alpha u + F_\alpha u,$$

where $P_\alpha u$ is pluriharmonic and $F_\alpha u$ is a weakly singular integral operator and hence somewhat smoothing; roughly $F_\alpha^k M_\alpha \theta$ is nicer than $M_\alpha \theta$ and F_α^m maps L_α^1 into $C(\bar{D})$ if m is large enough. If θ is an exact (1,1) current and u_0 is an L_α^1 -solution of $i\partial\bar{\partial}u = \theta$, then by (0.5), $u_1 = M\theta + F_\alpha u_0$ also solves (0.1). Repeating this argument, we get a new solution u_m ,

$$(0.6) \quad u_m = M_\alpha \theta + F_\alpha M_\alpha \theta + F_\alpha^2 M_\alpha \theta + \dots + F_\alpha^{m-1} M_\alpha \theta + F_\alpha^m u_0.$$

Thus, given a starting solution $u_0 \in L_\alpha^1$, estimates of the solution u_m are reduced to estimates for the explicitly given $M_\alpha \theta$.

We also get a similar expansion of the L_α^2 -minimal solution of (0.1) in terms of $M_\alpha \theta$. As a by-product we get an expression for the L_α^2 -orthogonal pluriharmonic projection

$$\Pi_\alpha : L_\alpha^2 \cap \mathcal{H} \longrightarrow D,$$

(\mathcal{H} denotes pluriharmonic functions), such that

$$\Pi_\alpha u = P_\alpha u + R_\alpha^1 u + R_\alpha^2 \Pi_\alpha u,$$

where P_α and R_α^j are explicit and R_α^j are regularizing (compact). In particular, when $\alpha = 0$, $P_0 u$ only depends on the boundary values of u , so if $u \in L^2(\partial D)$ has pluriharmonic extension U to D , then

$$U = P_0 u + K_0 u,$$

where K_0 is a compact operator on $L^2(\partial D)$. When $n = 1$, $P_0 u$ is the classical double layer potential of u , which provides an approximate solution to Dirichlet's problem.

In order to clarify our argument for (0.5), we conclude this paragraph with a sketch of the proof in a simple nontrivial case, namely when D is the unit disc Δ in \mathbb{C} , $\alpha = 1$ and $z \in \partial\Delta$.

Claim. *If u is smooth on $\bar{\Delta}$, and $z \in \partial\Delta$, then*

$$(0.7) \quad u(z) = \int_{\Delta} \frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}z|^2} \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} d\lambda \\ + 2 \operatorname{Re} \frac{1}{\pi} \int_{\Delta} \frac{u(\zeta) d\lambda}{(1-\bar{\zeta}z)^2} - \frac{1}{\pi} \int_{\Delta} u(\zeta) d\lambda.$$

The following argument for (0.7) is possibly not the most simple, but it follows the general scheme in Section 3. First we write ($w = \partial u / \partial \bar{\zeta}$)

$$(0.8) \quad u(z) = \frac{1}{\pi} \int \frac{u(\zeta) d\lambda}{(1-\bar{\zeta}z)^2} + \frac{1}{\pi} \int \frac{(1-|\zeta|^2)\bar{z}w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} = Gu + Kw.$$

Then we rewrite Kw as

$$Kw = \frac{1}{\pi} \int \frac{(1-|\zeta|^2)(\bar{z}-\bar{\zeta})w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} + \frac{1}{\pi} \int \frac{(1-|\zeta|^2)\bar{\zeta}w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \\ = \frac{1}{\pi} \int \frac{(1-|\zeta|^2)\bar{z}w}{1-\zeta\bar{z}} + I.$$

An integration by parts shows that

$$(0.9) \quad I = -\frac{1}{2\pi} \int \frac{1}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \frac{\partial}{\partial\zeta} (1-|\zeta|^2)^2 w \\ = \frac{1}{2\pi} \int \frac{(1-|\zeta|^2)^2}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} + \frac{1}{2\pi} \int \frac{(1-|\zeta|^2)^2 \bar{z}w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})^2}.$$

Now we apply a trivial instance of the crucial Proposition 3.2, namely

$$(1-|\zeta|^2)w\bar{z} = (1-\bar{\zeta}z)w\bar{z} + (1-\zeta\bar{z})\bar{\zeta}w,$$

to the last term in (0.9) and get

$$I = \frac{1}{2\pi} \int \frac{1-|\zeta|^2}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} + \frac{1}{2\pi} \int \frac{(1-|\zeta|^2)\bar{z}w}{(1-\zeta\bar{z})^2} \\ + \frac{1}{2\pi} \int \frac{(1-|\zeta|^2)\bar{\zeta}w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})},$$

and since the last term here is $I/2$, we can solve for I and get

$$\begin{aligned} Kw &= \frac{1}{\pi} \int \frac{(1 - |\zeta|^2) \bar{z} w}{1 - \zeta \bar{z}} + \frac{1}{\pi} \int \frac{(1 - |\zeta|^2) \bar{z} w}{(1 - \zeta \bar{z})^2} \\ &\quad + \frac{1}{\pi} \int \frac{(1 - |\zeta|^2)^2}{(1 - \zeta z)(1 - \zeta \bar{z})} \frac{\partial^2 u}{\partial \zeta \partial \bar{\zeta}}. \end{aligned}$$

Finally one can integrate by parts in the two integrals involving w , and then summing up one arrives at (0.7).

To prove (0.5) in general we need suitable integral formulas to replace G and K in (0.8). We describe them in Section 1. In Section 2 we state our main results and also point out some applications, as *e.g.* $L^p(\partial D)$ -estimates for (0.1) and the BMO-estimate of Varopoulos. In Section 3 and Section 4 we construct our operators and show their relations, whereas some estimates are left to Section 5.

1. Some preliminaries.

Let $D = \{\rho < 0\}$ be a bounded strictly pseudoconvex domain in \mathbb{C}^n where ρ is a C^3 strictly plurisubharmonic defining function. Suppose that $\phi(\zeta, z) : \bar{D} \times \bar{D} \rightarrow \mathbb{C}$ is C^1 and satisfies

$$(1.1) \quad 2 \operatorname{Re} \phi \geq -\rho(\zeta) - \rho(z) + \delta |\zeta - z|^2$$

and

$$(1.2) \quad d_\zeta \phi|_{\zeta=z} = \partial \rho|_\zeta = d_\zeta \bar{\phi}|_{\zeta=z}, \quad \zeta = z \in D.$$

Then $|\phi(\zeta, p)| \sim |\phi(p, \zeta)|$ if $p \in \partial D$ and $\zeta \in \bar{D}$, and for $p \in \partial D$,

$$B_t(p) = \{\zeta \in \partial D : |\phi(\zeta, p)| < t\}$$

and

$$Q_t(p) = \{\zeta \in D : |\phi(\zeta, p)| < t\}$$

are the Koranyi balls in ∂D and in D around $p \in \partial D$, see for example the discussion in [AnC]; indeed $B_t(p)$ ($Q_t(p)$) is $\sim \sqrt{t}$ in the complex tangential directions at p and t in the last one (ones). In particular, $|B_t(p)| \sim t^n$ and $|Q_t(p)| \sim t^{n+1}$. Also, if $\check{\phi}$ is another function that

satisfies (1.1) and (1.2) then $|\tilde{\phi}| \sim |\phi|$. We recall the following well known estimates

$$(1.3) \quad \int_{\partial D} \frac{d\sigma(\zeta)}{|\phi(\zeta, z)|^{n+\alpha}} \lesssim \left(\frac{1}{-\rho(z)} \right)^\alpha, \quad \alpha > 0,$$

and

$$(1.4) \quad \int_D \frac{(-\rho(\zeta))^\beta d\lambda(\zeta)}{|\phi(\zeta, z)|^{n+1+\alpha+\beta}} \lesssim \left(\frac{1}{-\rho(z)} \right)^\alpha, \quad \alpha > 0, \quad \beta > -1.$$

There is a C^1 -function $\phi(\zeta, z) : \bar{D} \times \bar{D} \rightarrow \mathbf{C}^n$ which is holomorphic in z for fixed $\zeta \in \bar{D}$ and such that

$$\phi(\zeta, z) = \langle q(\zeta, z), z - \zeta \rangle - \rho(\zeta)$$

satisfies (1.1) and (1.2), see [F]. If we put $s(\zeta, z) = -q(z, \zeta)$ and make the identifications $s \sim \Sigma s_j d\zeta_j$ and similarly $q \sim \Sigma q_j d\zeta_j$, we can define, for $\alpha > 0$,

$$(1.5) \quad H_\alpha u(z) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha) n!} \left(\frac{i}{2\pi} \right)^n \int_D \frac{(-\rho)^{\alpha-1} u [-\rho \bar{\partial} q - n q \wedge \bar{\partial} \rho] \wedge (\bar{\partial} q)^{n-1}}{\phi^{n+\alpha}},$$

for functions u , and

$$(1.6) \quad Q_\alpha w(z) = \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-1)!} \left(\frac{i}{2\pi} \right)^n \int_D \frac{(-\rho)^{\alpha-1} s \wedge w \wedge [-\rho \bar{\partial} q - (n-1)q \wedge \bar{\partial} \rho] \wedge (\bar{\partial} q)^{n-2}}{\phi(\zeta, z)^{n+\alpha-1} \phi(z, \zeta)},$$

for (0,1)-forms w and $z \in \partial D$. Then, see [AnB, Example 1], $Q_\alpha w(z)$ is the boundary values of a function $Q_\alpha w(z)$ on D , such that

$$(1.7) \quad Q_\alpha \bar{\partial} u = u - H_\alpha u$$

and

$$(1.8) \quad \bar{\partial} Q_\alpha w = w \quad \text{if } \bar{\partial} w = 0.$$

REMARK 1. If we let α tend to 0 in (1.5) and (1.6) we get

$$(1.5') \quad H_0 u(z) = \left(\frac{i}{2\pi}\right)^n \int_{\partial D} \frac{u q \wedge (\bar{\partial} q)^{n-1}}{\phi^n}$$

and

$$(1.6') \quad Q_0 w(z) = \left(\frac{i}{2\pi}\right)^n \int_{\partial D} \frac{w \wedge s \wedge q \wedge (\bar{\partial} q)^{n-2}}{\phi(\zeta, z)^{n-1} \phi(z, \zeta)}$$

and (1.7) and (1.8) still hold. We also notice that

$$H_\alpha : L_\alpha^2 \rightarrow L_\alpha^2 \cap O(D)$$

boundedly. For $\alpha > 0$ this follows by (1.4) and Shur's lemma, whereas for $\alpha = 0$, H_0 is a singular integral operator and the argument is more involved and uses Cotlar's lemma, see [KS].

We will use the solution operator Q_α above later on, but for our primary purposes we need analogues of (1.5)-(1.8) with a modification v of ϕ that we now construct.

To begin with we let $\eta_j = z_j - \zeta_j$ and put

$$(1.9) \quad -v(\zeta, z) = \rho + \sum \rho_j \eta_j + \frac{1}{2} \sum \rho_{jk} \eta_j \eta_k,$$

where $\rho = \rho(\zeta)$, $\rho_j = \partial\rho/\partial\zeta_j$ and so on. Clearly v satisfies (1.2) and since ρ is strictly plurisubharmonic it also satisfies (1.1) near the diagonal $\Delta \subset \bar{D} \times \bar{D}$. Let $\chi = \chi(|z - \zeta|)$ be a smooth function supported and identically 1 near Δ and put

$$-q_j(\zeta, z) = \chi \left(\rho_j + \frac{1}{2} \sum_k \rho_{jk} \eta_k \right) - (1 - \chi) \bar{\eta}_j.$$

Then we define v globally by

$$(1.10) \quad -v(\zeta, z) = \langle q(\zeta, z), z - \zeta \rangle + \rho(\zeta).$$

This v coincides with v in (1.9) near Δ and (1.1) holds globally (with v instead of ϕ) on $\bar{D} \times \bar{D}$. The main reason for requiring that v be as in (1.9) near Δ is that

$$(1.11) \quad v(\zeta, z) = \overline{v(z, \zeta)} + O(|z - \zeta|^3),$$

i.e. $v(\zeta, z)$ is almost conjugate symmetric. For further discussion of $v(\zeta, z)$, see Propositions 3.1 and 3.2 in Section 3. We also put $s(\zeta, z) = -q(z, \zeta)$ so that

$$-(s(\zeta, z), z - \zeta) + \rho(z) = v(z, \zeta) = \overline{v(\zeta, z)} + O(|\eta|^3).$$

Again identifying s and q by (1,0)-forms, we can build the operators H_α and Q_α , *cf.* (1.5) and (1.6). Then (1.7) still holds, but since $H_\alpha u$ no longer has a holomorphic kernel, (1.8) cannot hold in general. However, since $q(\zeta, z)$ is holomorphic in z near Δ , we have, *cf.* Example 1 and the proof of Theorem 1 in [AnB],

$$(1.8') \quad \bar{\partial}Q_\alpha w = w + \mathcal{X}_\alpha w, \quad \text{if } \bar{\partial}w = 0,$$

where

$$\mathcal{X}_\alpha \bar{\partial}u = \int_D O((- \rho)^\alpha) \wedge \bar{\partial}u = \int_D O((- \rho)^{\alpha-1})u$$

and $O((- \rho)^\alpha)$ denotes a smooth kernel which is $O((- \rho)^\alpha)$ and has bidegree (0,1) in z (for $\alpha = 0$ the last integral is over the boundary). Since clearly $\mathcal{X}_\alpha w$ is $\bar{\partial}$ -closed if w is, we can apply any reasonable solution operator for $\bar{\partial}$, *e.g.* Q_α from (1.6), to $\mathcal{X}_\alpha w$ and then obtain new operators $\tilde{\mathcal{L}}_\alpha$, such that

$$\bar{\partial}\tilde{\mathcal{L}}_\alpha w = \bar{\partial}Q_\alpha w - w,$$

and \mathcal{L}_α such that

$$\bar{\partial}\mathcal{L}_\alpha u = \bar{\partial}Q_\alpha \bar{\partial}u - \bar{\partial}u.$$

Moreover, it follows from *e.g.* Section 5 that $\tilde{\mathcal{L}}_\alpha$ and \mathcal{L}_α have smooth kernels that are $O((- \rho)^\alpha)$ and $O((- \rho)^{\alpha-1})$, respectively. Finally, we put

$$(1.12) \quad K_\alpha w = Q_\alpha w - \tilde{\mathcal{L}}_\alpha w, \quad G_\alpha u = H_\alpha u + \mathcal{L}_\alpha u.$$

Then

$$(1.13) \quad K_\alpha \bar{\partial}u = u - G_\alpha u$$

and

$$(1.14) \quad \bar{\partial}K_\alpha w = w \quad \text{if } \bar{\partial}w = 0.$$

Moreover, G_α has a holomorphic kernel and maps $L_\alpha^2 \rightarrow L_\alpha^2 \cap O(D)$ boundedly, since H_α does. An important consequence of (1.11) (and (1.12)) that we need later on is that (letting small letters denote the corresponding kernels)

$$(1.15) \quad \begin{aligned} g_\alpha(\zeta, z) - \overline{g_\alpha(z, \zeta)} &\sim h_\alpha(\zeta, z) - \overline{h_\alpha(z, \zeta)} \\ &= (-\rho)^{\alpha-1} O\left(\frac{|\eta|^3}{|v|^{n+\alpha+1}}\right), \end{aligned}$$

which makes it a weakly singular kernel and hence represents a (some-what) smoothing operator (and a compact operator on L_α^2).

2. Main results.

With the notation introduced in Section 1 we can describe the boundary values of our solutions for $\partial\bar{\partial}$. However to describe them for $z \in D$ we also need the following notation. If $\alpha > -1$ and $(\zeta, z) \in \bar{D} \times \bar{D}$ we let

$$(2.1) \quad h_{\alpha,j,k}(\zeta, z) = \frac{\alpha+1}{\pi} \int_{|\tau|<1} \frac{(1-|\tau|^2)^\alpha d\lambda(\tau)}{(1-a\bar{\tau})^j (1-\bar{a}\tau)^k},$$

and if $\alpha = -1$,

$$(2.1') \quad h_{-1,j,k}(\zeta, z) = \frac{1}{\pi} \int_{|\tau|=1} \frac{d\sigma(\tau)}{(1-a\bar{\tau})^j (1-\bar{a}\tau)^k},$$

where

$$a(\zeta, z) = \frac{\sqrt{\rho(\zeta)\rho(z)}}{v(\zeta, z)}.$$

Because of (1.1), $|a| \leq 1$ with equality if and only if $\zeta = z$. Also $h_{\alpha,j,k} \equiv 1$ if z or ζ is on ∂D . Moreover, one easily verifies that $h_{\alpha,j,0} = h_{\alpha,0,k} \equiv 1$, $h_{\alpha,j,k} \sim (|v|^2/\sigma)^{j+k-\alpha-2}$ if $j+k > \alpha+2$, $h_{\alpha,j,k} \sim 1 + \log|v|^2/\sigma$ if $j+k = \alpha+2$ and $h_{\alpha,j,k} \sim 1$ if $j+k < \alpha+2$, where $\sigma = (1-|a|^2)|v|^2 = |v|^2 - \rho(\zeta)\rho(z)$, so that $\sigma \sim |z-\zeta|^2$ if $(\zeta, z) \in K \subset\subset D \times D$. More precisely, $h_{\alpha,j,k}$ can be expressed in terms of hypergeometric functions, see [An2], but we are only interested in its asymptotic behaviour. We also put $h_{\alpha,\ell} = h_{\alpha,\ell/2,\ell/2}$, so that $|h_{\alpha,j,k}| \leq h_{\alpha,j+k}$ and $h_{\alpha+1,\ell+1} \sim h_{\alpha,\ell}$ if $l > \alpha+2$. Now we can state our main result (recall (1.12) for the definition of G_α).

Theorem 1. *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n and ρ a strictly plurisubharmonic C^4 defining function. For each integer $\alpha \geq 1$ we have operators P_α , F_α and M_α such that for smooth functions u ,*

$$(2.2) \quad u = M_\alpha(i\partial\bar{\partial}u) + P_\alpha u + F_\alpha u.$$

Here

$$P_\alpha u = G_\alpha u + \bar{G}_\alpha u$$

is pluriharmonic,

$$(2.3) \quad |F_\alpha u| \lesssim \int_D \frac{(-\rho)^{\alpha-1}|u|}{|v|^{n+\alpha-1/2}} h_{\alpha-2, n+\alpha-1/2} d\lambda,$$

$$(2.4) \quad \begin{aligned} M_\alpha \theta = & \sum_{\substack{j+k=n+\alpha \\ j, k \geq 1}} c_{jk} \int_D \frac{(-\rho)^{\alpha+1} \theta \wedge (i\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k} h_{\alpha, j, k} \\ & - \sum_{\substack{j+k=n+\alpha+1 \\ j, k \geq 1}} c'_{jk} \\ & \int_D \frac{(-\rho)^{\alpha+1} \theta \wedge (-\rho(z)i\partial\bar{\partial}\rho + i(n-1)\partial\bar{v} \wedge \bar{\partial}v) \wedge (i\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k} h_{\alpha, j, k} \\ & + \sum_{\substack{j+k=n+\alpha \\ j, k \geq 1}} c''_{jk} \\ & \int_D \frac{(-\rho)^{\alpha+2} \theta \wedge (-\rho(z)i\partial\bar{\partial}\rho + i(n-1)\partial\bar{v} \wedge \bar{\partial}v) \wedge (i\partial\bar{\partial}\rho)^{n-2}}{v^{j+1} \bar{v}^{k+1}} \\ & \cdot h_{\alpha+1, j+1, k+1} + R_\alpha \theta, \end{aligned}$$

where

$$(2.5) \quad \begin{aligned} |R_\alpha \theta| \\ \lesssim \int_D \frac{(-\rho)^\alpha (-\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|))}{|v|^{n+\alpha-1/2}} h_{\alpha-1, n+\alpha-1/2}. \end{aligned}$$

In particular,

$$(2.6) \quad |M_\alpha \theta| \lesssim \int_D \frac{(-\rho)^\alpha}{|v|^{n+\alpha}} \cdot \left(-\rho|\theta| + \sqrt{-\rho} (|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho| \right) h_{\alpha-1, n+\alpha}.$$

The exact values of the constants are

$$c_{j,k} = \frac{1}{(2\pi)^n} \frac{(n+\alpha)(n+\alpha-j-1)!(n+\alpha-k-1)!}{(n-1)!(\alpha+1)!(n+\alpha-j-k)!},$$

$$c'_{j,k} = \frac{1}{(2\pi)^n} \frac{(n+\alpha-j)!(n+\alpha-k)!}{(n-1)!(\alpha+1)!(n+\alpha-j-k+1)!}$$

and

$$c''_{j,k} = \frac{1}{(2\pi)^n} \frac{j k (n+\alpha-j-1)!(n+\alpha-k-1)!}{(n-1)!(\alpha+2)!(n+\alpha-j-k)!}.$$

Notice that the kernel for M_α is $\sim |h_{\alpha-1, n+\alpha}| \sim |\zeta - z|^{-(2n-2)}$ if $n > 1$ (and $\sim \log |\zeta - z|$ for $n = 1$) when (ζ, z) is in the interior of $D \times D$.

REMARK 1. If D is the ball, then M_α is the solution operator M_α from [An1] and [An2], G_α is the L_α^2 -Bergman projection, $F_\alpha u = -G_\alpha u(0)$ so that $P_\alpha + F_\alpha$ is the L_α^2 -orthogonal pluriharmonic projection.

REMARK 2. For $\alpha = 0$, we have the same result as in Theorem 1 for $z \in \partial D$; *i.e.* everything holds for $\alpha = 0$ if $z \in \partial D$, and

$$|F_0 u(z)| \lesssim \int_{\partial D} \frac{|u(\zeta)|}{|v|^{n-1/2}} + \int_D \frac{|u(\zeta)|}{|v|^{n+1/2}}, \quad z \in \partial D.$$

Our main application of Theorem 1 is to estimate solutions to the $\partial\bar{\partial}$ -equation, and to this end we need the following technical result:

Proposition 2.1. *Under the conditions of Theorem 2.1,*

$$(2.7) \quad |F_\alpha^k u(z)| \lesssim \int_D \frac{(-\rho)^{\alpha-1} |u|}{|v|^{n+\alpha-k/2}} h_{\alpha-2, n+\alpha-k/2} d\lambda,$$

if $k < 2(n + \alpha) - 1$,

$$(2.8) \quad |F_\alpha^k M_\alpha \theta(z)| \lesssim \int_D \frac{(-\rho)^\alpha}{|v|^{n+\alpha-k/2}} \left(-\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) \right. \\ \left. + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho| \right) h_{\alpha-1, n+\alpha-k/2} d\lambda,$$

if $k < 2(n + \alpha - 1)$, and if k is large enough F_α^k maps L_α^1 into $C(\bar{D})$ boundedly.

This proposition is proved in Section 5.

Now, suppose u_0 is a solution to $i\partial\bar{\partial}u = \theta$ in L_α^1 . Then $u_1 = M_\alpha\theta + F_\alpha u_0$ is another solution, and by iteration so is

$$u = M_\alpha\theta + F_\alpha M_\alpha\theta + F_\alpha^2 M_\alpha\theta + \cdots + F_\alpha^m M_\alpha\theta + F_\alpha^m u_0.$$

From Proposition 2.1 we then get

Theorem 2. *If $u_0 \in L_\alpha^1$ solves $i\partial\bar{\partial}u_0 = \theta$ so does*

$$u = M_\alpha\theta + \mathcal{R}_\alpha\theta + T_\alpha u_0,$$

where $M_\alpha\theta$ is as in Theorem 1, $T_\alpha u_0 \in C(\bar{D})$ and

$$|\mathcal{R}_\alpha\theta| \lesssim \int_D \frac{(-\rho)^\alpha}{|v|^{n+\alpha-1/2}} \\ \cdot \left(-\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho| \right) \\ \cdot h_{\alpha-1, n+\alpha-1/2}.$$

In order to apply Theorem 2 to get various estimates of solutions to $i\partial\bar{\partial}u = \theta$ we need an a priori solution u_0 in some $L_\alpha^1(D)$. This is provided by

Theorem 3. *Assume that θ is a d -exact $(1, 1)$ -form (current) such that*

$$(2.9) \quad (-\rho)^\alpha \left(-\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho| \right)$$

is a finite measure in D , $\alpha \geq 0$. Then there is a solution $u \in L^1_\alpha(D)$ to $i\partial\bar{\partial}u = \theta$.

If $\alpha = 0$ this is the Henkin-Skoda theorem, [H] and [S], and the case $\alpha > 0$ is due to Dautov and Henkin, [DH]. This theorem was the first outcome of solution formulas for the $\bar{\partial}$ -equation with weights. With the modern technique the proof is rather simple and for the readers convenience we sketch it when α is an integer.

SKETCH OF PROOF. Assume first that $\alpha = 0$ and that θ is a d -exact real (1,1)-current such that

$$\|\theta\| = -\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho|$$

is a finite measure (if θ is positive this is equivalent to $\int_D -\rho \text{trace } \theta < +\infty$). First we look for a solution to $id\omega = \theta$ such that

$$(2.10) \quad \int_D |\omega| + \frac{1}{\sqrt{-\rho}} (|\partial\rho \wedge \omega| + |\bar{\partial}\rho \wedge \omega|) < +\infty.$$

If D is convex, one can use the simple homotopy for d obtained by contracting D to a point, i.e. $\omega = i \int_0^1 h^* \theta dt$, where $h(t, z) = tz$ (assuming $0 \in D$). For a general strictly pseudoconvex domain D one can piece together such local solutions to a global one by a cohomology argument. It is at this point the d -exactness of θ comes into play. If $\omega_{0,1}$ is the (0,1)-part of ω , then for bidegree reasons, $\bar{\partial}\omega_{0,1} = 0$. We may assume that ω is chosen so that $\bar{\omega}_{0,1} = -\omega_{1,0}$. We now apply the solution operator Q_α from (1.6) and put $v = Q_1\omega_{0,1}$. Then (1.3), (2.10) and Fubini's theorem immediately yield that

$$\int_{\partial D} |v| d\sigma \lesssim \int_D |\omega_{0,1}| + \frac{1}{\sqrt{-\rho}} |\omega_{0,1} \wedge \bar{\partial}\rho| < \infty.$$

Putting $u = 2 \text{Re } v = v + \bar{v}$, we get a solution $u \in L^1(\partial D)$ to $i\partial\bar{\partial}u = \theta$.

It follows from the \mathbb{R}^{2n} -Riesz decomposition that also $u \in L^1(D)$; however we show this with another argument that also covers the case $\alpha \in \mathbb{Z}_+$. Let $\tilde{D} \subset \mathbb{C}^{n+\alpha}$ be the strictly pseudoconvex domain defined by $\tilde{D} = \{(z, z') \in \mathbb{C}^{n+\alpha} : \tilde{\rho}(z, z') = \rho(z) + |z'|^2 < 0\}$. Then (2.9) implies, cf. Section 5,

$$\int_{\tilde{D}} -\tilde{\rho}|\theta| + \sqrt{-\tilde{\rho}}(|\theta \wedge \partial\tilde{\rho}| + |\theta \wedge \bar{\partial}\tilde{\rho}|) + |\theta \wedge \partial\tilde{\rho} \wedge \bar{\partial}\tilde{\rho}| < +\infty.$$

Hence, by the case $\alpha = 0$ applied to $\theta(z, z') = \theta(z)$ in \tilde{D} , we obtain a solution $u \in L^1(\partial\tilde{D})$. We may assume that u is independent of z' and then, by Lemma 5.1, $\int_D (-\rho)^\alpha |u| \sim \int_{\partial\tilde{D}} |u| d\tilde{\sigma} < +\infty$.

Recall that a measure μ in D is a Carleson measure if $\mu(Q_t(p)) \lesssim t^n$ where $Q_t(p)$ are the Koranyi balls in D . As an application of Theorem 2 we can prove Varopoulos' theorem:

Theorem 4. *Assume that θ is d -exact and that $\|\|\theta\|\|$ is a Carleson measure. Then there is a solution $u \in \text{BMO}(\partial D)$ to $i\partial\bar{\partial}u = \theta$.*

By Theorem 3, there is a solution $u_0 \in L^1(D)$ and hence $T_\alpha u_0 \in C(\bar{D})$. It is easy to see that $\mathcal{R}_\alpha \theta$ is bounded on ∂D , and a standard estimation of the integral defining $M_\alpha \theta$ shows that $M_\alpha \theta \in \text{BMO}(\partial D)$. For the details of this argument see Section 6. Thus we have obtained a relatively simple proof of Varopoulos' theorem that avoids the delicate task of solving the Poincaré equation $id\omega = \theta$ with Carleson estimates, cf. Section 0.

By interpolation, Theorems 3 and 4 imply that there is a solution in $L^p(\partial D)$, $1 < p < \infty$, if $\|\|\theta\|\| \in W^{1-1/p}$. Here W^α are the interpolation spaces between the finite measures W^0 and the Carleson measures W^1 in D . This can also be seen by simple estimates of the integrals, using the following characterization of W^α , see [AmB],

$$\mu \in W^\alpha \quad \text{if and only if} \quad \mu = k d\tau,$$

where

$$\tau \in W^1 \quad \text{and} \quad k \in L^{1/(1-\alpha)}(d\tau).$$

EXAMPLE 1. If ∂D has enough regularity then the operator T_α in Theorem 2 will map L_α^1 into $C^k(\bar{D})$, and so our technique can be used also to study $C^{n+\alpha}$ -regularity for the solution. However, we do not pursue these things in more detail in this paper.

We will go one step further and show that in fact $M_\alpha \theta$ is the principal term of the L_α^2 -minimal solution $N_\alpha \theta$ to $i\partial\bar{\partial}u = \theta$, but to this end we first have to study the L_α^2 -orthogonal projection

$$\Pi_\alpha : L_\alpha^2 \rightarrow L_\alpha^2 \cap \mathcal{H},$$

where \mathcal{H} denotes the space of pluriharmonic functions in D . First we note that, cf. (2.2),

$$(2.11) \quad u = (P_\alpha + F_\alpha)u$$

if u is pluriharmonic, $\alpha \geq 1$.

Since Π_α is pluriharmonic, $P_\alpha \Pi_\alpha = \Pi_\alpha - F_\alpha \Pi_\alpha$ and since $P_\alpha : L_\alpha^2 \rightarrow L_\alpha^2 \cap \mathcal{H}$, cf. Section 1, $\Pi_\alpha P_\alpha = P_\alpha$. Taking adjoints we get $P_\alpha^* = P_\alpha^* \Pi_\alpha^*$, and after subtracting $\Pi_\alpha - F_\alpha \Pi_\alpha - P_\alpha^* = (P_\alpha - P_\alpha^*) \Pi_\alpha$ and thus

$$(2.12) \quad \Pi_\alpha = P_\alpha + F_\alpha + A_\alpha(I - \Pi_\alpha),$$

if $A_\alpha = P_\alpha^* - P_\alpha - F_\alpha$. Note that, since $P_\alpha = G_\alpha + \bar{G}_\alpha$, (1.15) and (2.3) imply that A_α is weakly singular. By iteration we get

Theorem 5. *The L_α^2 -orthogonal projection $\Pi_\alpha : L_\alpha^2 \rightarrow L_\alpha^2 \cap \mathcal{H}$ can be written*

$$(2.13) \quad \begin{aligned} \Pi_\alpha &= P_\alpha + F_\alpha + A_\alpha(I - P_\alpha - F_\alpha) - A_\alpha^2(I - P_\alpha - F_\alpha) \\ &+ \cdots + (-1)^m A_\alpha^{m-1}(I - P_\alpha - F_\alpha) + (-1)^{m+1} A_\alpha^m(I - \Pi_\alpha). \end{aligned}$$

Since A_α^m maps L_α^2 into $C(\bar{D})$ (or even into $C^k(\bar{D})$ if ∂D is sufficiently smooth) if m is large enough, cf. Proposition 2.1, $\Pi_\alpha u$ has the same regularity as $P_\alpha u$; e.g. if ∂D is C^∞ then P_α maps $C^\infty(\bar{D})$ into $C^\infty(\bar{D}) \cap \mathcal{H}$ and hence also Π_α does.

EXAMPLE 2. Formula (2.11) also holds for $\alpha = 0$, cf. Remark 2, for $z \in \partial D$. Then $P_0 u(z)$, $z \in \partial D$, has to be interpreted as the boundary values of the pluriharmonic function $P_0 u(z)$. Since $P_0 u$ only depends on the boundary values of u , we can take any operator V , e.g., the \mathbb{R}^{2n} -Poisson integral, which represents a pluriharmonic function in terms of its boundary values, and then

$$(2.14) \quad U = P_0 u + F_0 V u$$

is a representation of the pluriharmonic function U in terms of its boundary values u , $P_0 u$ is pluriharmonic and one can show that $F_0 V : L^2(\partial D) \rightarrow L^2(\partial D)$ is compact. When $n = 1$,

$$P_0 u(z) = 2 \operatorname{Re} \frac{1}{2\pi i} \int_{\partial D} \frac{u(\zeta) d\zeta}{\zeta - z}$$

is the classical double layer potential of u , which provides an approximative solution of Dirichlet's problem.

Finally we state our result about the L^2_α -minimal solution $N_\alpha\theta$ of $i\partial\bar{\partial}u = \theta$.

Theorem 6. *If θ is exact in D , then the L^2_α -minimal solution $N_\alpha\theta$ of $i\partial\bar{\partial}u = \theta$ is given by*

$$(2.15) \quad \begin{aligned} N_\alpha\theta &= M_\alpha\theta - A_\alpha M_\alpha\theta + A_\alpha^2 M_\alpha\theta \\ &+ \cdots + (-1)^m A_\alpha^{m-1} M_\alpha\theta + (-1)^{m+1} A_\alpha^m N_\alpha\theta. \end{aligned}$$

PROOF. This follows immediately from (2.2) and (2.13) once one has noticed that if u is any solution, then

$$N_\alpha\theta = u - \Pi_\alpha u.$$

3. Proof of Theorem 1 when $z \in \partial D$.

We first assume that $z \in \partial D$. The general case will then be obtained by applying the first result to a certain set $\tilde{D} \subset \mathbb{C}^{n+1}$ where $D = \mathbb{C}^n \cap \tilde{D}$. Our starting point is the relation ($w = \bar{\partial}u$)

$$(3.1) \quad u = G_\alpha u + K_\alpha w,$$

see (1.13). For convenience we recall that

$$(3.2) \quad \begin{aligned} G_\alpha u(z) &= \frac{\Gamma(n+\alpha)}{n! \Gamma(\alpha)} \left(\frac{i}{2\pi}\right)^n \\ &\cdot \int_D \frac{(-\rho)^{\alpha-1} u [-\rho \bar{\partial}q - nq \wedge \bar{\partial}\rho] \wedge (\bar{\partial}q)^{n-1}}{v^{n+\alpha}(\zeta, z)} \\ &+ \text{nice operator} \end{aligned}$$

and, for $z \in \partial D$,

$$(3.3) \quad \begin{aligned} K_\alpha w(z) &= \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-1)!} \left(\frac{i}{2\pi}\right)^n \\ &\cdot \int_D \frac{(-\rho)^{\alpha-1} s \wedge w \wedge [-\rho \bar{\partial}q - (n-1)q \wedge \bar{\partial}\rho] \wedge (\bar{\partial}q)^{n-2}}{v(\zeta, z)^{n+\alpha-1} v(z, \zeta)} \\ &+ \text{nice operator.} \end{aligned}$$

The objective now is to generalize the argument given at the end of Section 0, *i.e.* rewrite $K_\alpha w$ in an appropriate manner to obtain (2.2), and to this end we need

Proposition 3.1. *If ρ is C^3 and v, s, q are defined as in Section 1, then*

$$(3.4) \quad \partial\bar{v} = -s + O(|\eta|) = q + O(|\eta|) = -\partial\rho + O(|\eta|)$$

$$(3.5) \quad s \wedge q = O(|\eta|), \quad \partial\rho \wedge \partial\bar{v} = O(|\eta|),$$

$$(3.6) \quad \bar{\partial}q = \partial\bar{\partial}\rho + O(|\eta|),$$

$$(3.7) \quad \partial\rho \wedge \partial\bar{v} = s \wedge q + O(|\eta|^2),$$

$$(3.8) \quad v(z, \zeta) = \overline{v(\zeta, z)} + O(|\eta|^3),$$

and

$$(3.9) \quad \partial v(\zeta, z) = O(|\eta|^2).$$

and

Proposition 3.2. *If ρ is C^3 , v is defined by (1.9) and $z \in \partial D$, then*

$$\begin{aligned} & (n-1)w \wedge d\bar{v} \wedge \bar{\partial}\rho \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &= \left(-(n-1)w \wedge d\bar{v} \wedge dv \wedge \partial\rho - \rho w \wedge d\bar{v} \wedge \partial\bar{\partial}\rho \right. \\ & \quad \left. - vw \wedge d\bar{v} \wedge \partial\bar{\partial}\rho + \bar{v}w \wedge \partial\rho \wedge \partial\bar{\partial}\rho \right) \wedge (\partial\bar{\partial}\rho)^{n-2} + |w| O(|\eta|^3) \end{aligned}$$

and O is in $C^1(\bar{D} \times \partial D)$.

All differentials are with respect to ζ . Note that (3.8) means that $v(\zeta, z)$ is nearly conjugate symmetric (self-adjoint) and (3.9) means that it is nearly antiholomorphic in ζ .

The equations (3.4) to (3.6) follow quite easily from the definitions. Clearly (3.9) follows from (3.8), which is wellknown and used *e.g.* in [KS]. It can be verified by a straightforward computation but it also follows immediately from (3.11) below, which we anyway need in the proof of the much harder Proposition 3.2.

The equation (3.7) was first used in [Ar] and can be verified as follows:

PROOF OF (3.7). Since $\sum q_k \eta_k = -v(\zeta, z) - \rho(\zeta)$ and, by (3.8), $\sum s_k \eta_k = -v(\zeta, z) - \rho(z) + O(|\eta|^3)$, we have for small η , $\sum \eta_k \partial q_k - q = -\partial \rho$ and $\sum \eta_k \partial s_k - s = -\partial \bar{v} + O(|\eta|^2)$. Hence, since by (3.4) $s = -q + O(|\eta|)$,

$$\partial s \wedge \partial \bar{v} = q \wedge s + \sum \eta_k \partial(q_k + s_k) + O(|\eta|^2).$$

Now

$$q_k = -\rho_k(\zeta) - \frac{1}{2} \sum_j \rho_{jk}(\zeta) \eta_j$$

and

$$s_k = \rho(z) - \frac{1}{2} \sum_j \rho_{jk}(z) \eta_j$$

so that $\partial(q_k + s_k) = O(|\eta|)$ and thus (3.7) follows.

PROOF OF PROPOSITION 3.2. By (3.9) and for bidegree reasons it is enough to prove that $A \wedge (\partial \bar{\partial} \rho)^{n-2} = O(|\eta|^3)$, where

$$A = (n-1)w \wedge \partial \bar{v} \wedge \bar{\partial}(\rho+v) \wedge \partial \rho + (\rho+v)w \wedge \partial \bar{v} \wedge \partial \bar{\partial} \rho - \bar{v}w \wedge \partial \rho \wedge \partial \bar{\partial} \rho.$$

To simplify the computation we want to choose suitable holomorphic coordinates. The definition (1.9) depends on the choice of coordinates but if v' is defined by (1.9) with respect to new holomorphic coordinates z' , we claim that

$$(3.10) \quad v = v' + O(|\eta|^3) \quad \text{and} \quad \bar{\partial}_\zeta v = \bar{\partial}_\zeta v' + O(|\eta|^3).$$

Let us assume (3.10) for a moment and complete the proof of Proposition 3.2. Let $\zeta \in D$ be fixed. By (3.10) we may assume that v is defined with respect to holomorphic coordinates such that

$$\partial \bar{\partial} \rho(z) = \sum d\zeta_i \wedge d\bar{\zeta}_i + O(|\eta|^2)$$

and hence also $\rho_{i\bar{j}\bar{k}}(\zeta) = \rho_{i\bar{j}k}(\zeta) = 0$. By linearity we may assume that $w = d\bar{\zeta}_1$. Now $(\partial \bar{\partial} \rho)^{n-2}$ is a sum of terms $\wedge_i (d\zeta_i \wedge d\bar{\zeta}_i)$ where i assumes all but two of the numbers $1, \dots, n$, and only the differentials in A with respect to these two variables make any contribution to such a term.

Thus if we let $\gamma = d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_k \wedge d\bar{\zeta}_k \wedge (\partial\bar{\partial}\rho)^{n-2}$ (γ is independent of k), we have

$$\begin{aligned} & w \wedge \partial\bar{v} \wedge \bar{\partial}(\rho + v) \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &= \sum_{k=2}^n d\bar{\zeta}_1 \wedge (\partial_{1k}\rho + \bar{\eta}_1 d\zeta_1 + \bar{\eta}_k d\zeta_k) \wedge \eta_k d\bar{\zeta}_k \wedge \partial_{1k}\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &= \sum_{k=2}^n (-\rho_1 \eta_k \bar{\eta}_k + \rho_k \bar{\eta}_1 \eta_k) \gamma, \end{aligned}$$

$$\begin{aligned} & (\rho + v)w \wedge \partial\bar{v} \wedge \partial\bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &= \sum_{k=2}^n (\rho + v) d\zeta_1 \wedge \partial_1 \bar{v} \wedge d\zeta_k \wedge d\bar{\zeta}_k \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &= (n-1)(\rho + v)(\rho_1 + \bar{\eta}_1) \gamma \end{aligned}$$

and

$$\begin{aligned} \bar{v} \wedge \partial\rho \wedge \partial\bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-2} &= \sum_{k=2}^n \bar{v} d\bar{\zeta}_1 \wedge \partial_1 \rho \wedge d\zeta_k \wedge d\bar{\zeta}_k \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &= -(n-1)\bar{v} \rho_1 \gamma. \end{aligned}$$

Hence,

$$\begin{aligned} A \wedge (\partial\bar{\partial}\rho)^{n-2} &= (n-1) \left(\rho_1(\rho - v - \bar{v} - \sum_{k=2}^n \eta_k \bar{\eta}_k) \right. \\ &\quad \left. + (\rho + v)\bar{\eta}_1 + \sum_{k=2}^n \rho_k \bar{\eta}_1 \eta_k \right) \gamma + O(|\eta|^3) \\ &= (n-1)\rho_1(-\rho(z) + O(|\eta|^3)) + O(|\eta|^3) \\ &= O(|\eta|^3) \end{aligned}$$

as $\rho(z) = 0$, and the proof is complete.

PROOF OF (3.10). We first assume that ρ is real analytic and let $u(\zeta, z)$ be the unique function that equals $-\rho$ for $\zeta = z$, is holomorphic in z and antiholomorphic in ζ . Then $\overline{u(\zeta, z)} = u(z, \zeta)$. Since v (and v') is nothing but the Taylor expansion of u up to second order, we have

$$(3.11) \quad v = u + O(|\eta|^3).$$

Note also that $\bar{\partial}_\zeta v(\zeta, z) = \bar{\partial}_\zeta u(\zeta, z) + O(|\eta|^3)$. The same formulas hold for v' and since u and $\bar{\partial}u$ are invariantly defined, (3.10) now follows if ρ is real analytic. The general case can be obtained by approximation.

We now replace $v(z, \zeta) = \bar{v}(\zeta, z) + O(|\eta|^3)$ by \bar{v} in (3.3), $\bar{\partial}q$ by $\partial\bar{\partial}\rho$ and $s \wedge w$ by $\partial\rho \wedge \partial\bar{v}$. Then we get

$$(3.12) \quad K_\alpha w = \frac{\Gamma(n + \alpha - 1)}{\Gamma(\alpha)(n - 1)!} \left(\frac{i}{2\pi}\right)^n \left(-\int_D \frac{(-\rho)^\alpha d\bar{v} \wedge w \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^{n+\alpha-1}\bar{v}}\right. \\ \left.+ (n - 1) \int_D \frac{(-\rho)^{\alpha-1} \partial\rho \wedge w \wedge d\bar{v} \wedge \bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^{n+\alpha-1}\bar{v}}\right) + \tilde{F}_\alpha w,$$

where

$$\tilde{F}_\alpha w = \sum \int \frac{(-\rho)^{\alpha-1+\ell} w \wedge O(|\eta|^{2m+1})}{v^j \bar{v}^k} \\ + \int \frac{(-\rho)^{\alpha-1+\ell} w \wedge \bar{\partial}\rho \wedge O(|\eta|^{2m})}{v^j \bar{v}^k},$$

if $\ell, j, k, m \geq 0$, and $\ell + m - (j + k) \geq 1 - n - \alpha$, and $O(|\eta|^\beta)$ is in $C^1(\bar{D} \times \bar{D})$. To obtain (3.12) we have used Proposition 3.1 and that

$$\frac{1}{\bar{v} + O(|\eta|^3)} - \frac{1}{\bar{v}} = O\left(\frac{|\eta|^3}{|\bar{v}|^3}\right).$$

We need the following auxiliary notation:

$$A_{\alpha,j,k} = \left(\frac{i}{2\pi}\right)^n \int_D \frac{(-\rho)^\alpha w \wedge d\bar{v} \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k}, \\ B_{\alpha,j,k} = \left(\frac{i}{2\pi}\right)^n \int_D \frac{(-\rho)^\alpha w \wedge d\bar{v} \wedge \bar{\partial}\rho \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}, \\ C_{\alpha,j,k} = \left(\frac{i}{2\pi}\right)^n \int_D \frac{(-\rho)^\alpha \partial w \wedge \partial\bar{v} \wedge \bar{\partial}v \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}$$

and

$$D_{\alpha,j,k} = \left(\frac{i}{2\pi}\right)^n \int_D \frac{(-\rho)^\alpha \partial w \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k}.$$

Hence (3.12) can be written

$$K_\alpha w = \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-1)!} A_{\alpha,n+\alpha-1,1} - \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-2)!} B_{\alpha,1,n+\alpha-1} + \tilde{F}_\alpha w.$$

Let $R_\alpha = R_\alpha \theta$ denote any term that satisfies (2.5) (since $z \in \partial D$, $h_{\alpha,n+\alpha-1/2} = 1$). Then we have

Lemma 3.3. *If $j+k = n+\beta+1$ and $\beta \geq \alpha-1$ then*

$$(3.13) \quad \begin{aligned} B_{\beta,j,k} &= -\frac{1}{\beta+1} A_{\beta+1,j,k} \\ &+ \frac{j}{(\beta+1)(\beta+2)} C_{\beta+2,j+1,k} + \tilde{F}_\alpha + R_\alpha. \end{aligned}$$

Thus

$$(3.14) \quad \begin{aligned} K_\alpha w &= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)(n-1)!} A_{\alpha,n+\alpha-1,1} \\ &+ \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+2)(n-2)!} C_{\alpha+1,n+\alpha,1} + \tilde{F}_\alpha + R_\alpha. \end{aligned}$$

By repeated use of

Lemma 3.4. *If $j+k = n+\alpha$ then*

$$\begin{aligned} A_{\alpha,j,k} &= \frac{k}{n+\alpha-k} A_{\alpha,j-1,k+1} + \frac{n+\alpha}{(n+\alpha-k)(\alpha+1)} D_{\alpha+1,j,k} \\ &+ \frac{jk(n-1)}{(\alpha+1)(\alpha+2)(n+\alpha-k)} C_{\alpha+2,j+1,k+1} \\ &- \frac{k(n-1)}{(\alpha+1)(n+\alpha-k)} C_{\alpha+1,j,k+1} + \tilde{F}_\alpha + R_\alpha, \end{aligned}$$

and recalling that $\theta = i \partial w$ we obtain from (3.14) that ($z \in \partial D$)

$$(3.15) \quad K_\alpha w = M_\alpha + A_{\alpha,0,n+\alpha} + \tilde{F}_\alpha$$

and then by (3.1), (2.2) is proved since we have

Lemma 3.5.

$$(3.16) \quad A_{\alpha,0,n+\alpha} = \bar{G}_\alpha u + F_\alpha u$$

and

$$(3.17) \quad \tilde{F}_\alpha w = F_\alpha u.$$

As $-\rho \lesssim |v|$ and $|\eta| \lesssim \sqrt{|v|}$, (2.6) follows from (2.4) by (3.4), and Theorem 1 is completely proved for $z \in \partial D$.

The rest of this paragraph is devoted to the proofs of Lemmas 3.3 to 3.5. Lemmas 3.3 and 3.5 are obtained only by some elementary integrations by parts whereas the proof of Lemma 3.4 is somewhat involved and depends on the crucial Proposition 3.2.

PROOF OF LEMMA 3.3. Note that

$$B_{\beta,j,k} = \int_D \frac{d(-\rho)^{\beta+1} \wedge w \wedge d\bar{v} \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}.$$

Thus an integration by parts yields

$$(3.18) \quad \begin{aligned} B_{\beta,j,k} &= A_{\beta+1,j,k} + \int_D \frac{(-\rho)^{\beta+1} \wedge w \wedge d\bar{v} \wedge dv \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^{j+1} \bar{v}^k} \\ &\quad + \int_D \frac{(-\rho)^{\beta+1} \partial w \wedge d\bar{v} \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}. \end{aligned}$$

To handle the first integral, notice that

$$\begin{aligned} (-\rho)^{\beta+1} w \wedge d\bar{v} \wedge dv \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\ = \frac{1}{\beta+2} d(-\rho)^{\beta+2} \wedge w \wedge d\bar{v} \wedge dv \wedge (\partial\bar{\partial}\rho)^{n-2} \\ + (-\rho)^{\beta+1} w \wedge \bar{\partial}\rho \wedge O(|\eta|^2), \end{aligned}$$

and so after an integration by parts, the integral becomes

$$-\frac{1}{\beta+2} \int \frac{(-\rho)^{\beta+2} \wedge \partial w \wedge d\bar{v} \wedge dv \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^{j+1} \bar{v}^k} + \tilde{F}_\alpha.$$

However, for bidegree reasons and (3.9),

$$\partial w \wedge d\bar{v} \wedge dv \wedge (\partial\bar{\partial}\rho)^{n-2} = \partial w \wedge \partial\bar{v} \wedge \bar{\partial}v \wedge (\partial\bar{\partial}\rho)^{n-2} + \partial w \wedge O(|\eta|^4)$$

and hence the first integral in (3.18) is

$$C_{\beta+2,j+1,k} + \tilde{F}_\alpha + R_\alpha.$$

Now consider the last integral in (3.18). Again, for bidegree reasons, $d\bar{v} \wedge \partial\rho$ can be replaced by $\bar{\partial}v \wedge d\rho$ and then an integration by parts yields

$$\frac{1}{\beta+2} \int_D (-\rho)^{\beta+2} \partial w \wedge d \frac{\partial\bar{v}}{v^j \bar{v}^k} \wedge (\partial\bar{\partial}\rho)^{n-2}$$

which is an R_α if $\beta \geq \alpha - 1$ and $j + k = n + \beta + 1$.

PROOF OF LEMMA 3.4. In this proof \equiv means equality modulo terms R_α and \tilde{F}_α . By Lemma 3.3,

$$(3.19) \quad A_{\alpha+1,j,k+1} \equiv -(\alpha+1) B_{\alpha,j,k+1} + \frac{j}{\alpha+2} C_{\alpha+2,j+1,k+1}.$$

Now we apply Proposition 3.2 to the B -term in (3.19) and get, using the same argument as when handling the first integral in (3.18) in the proof of Lemma 3.3,

$$\begin{aligned} A_{\alpha+1,j,k+1} &\equiv -C_{\alpha+1,j,k+1} + \frac{j}{\alpha+2} C_{\alpha+2,j+1,k+1} \\ &\quad - \frac{\alpha+1}{n-1} A_{\alpha+1,j,k+1} + \frac{\alpha+1}{n-1} A_{\alpha,j-1,k+1} + \frac{\alpha+1}{n-1} A_{\alpha,j,k}. \end{aligned}$$

Solving for $A_{\alpha+1,j,k+1}$ yields

$$(3.20) \quad \begin{aligned} A_{\alpha+1,j,k+1} &\equiv -\frac{n-1}{n+\alpha} C_{\alpha+1,j,k+1} + \frac{j(n-1)}{(n+\alpha)(\alpha+2)} C_{\alpha+2,j+1,k+1} \\ &\quad + \frac{\alpha+1}{n+\alpha} A_{\alpha,j-1,k+1} + \frac{\alpha+1}{n+\alpha} A_{\alpha,j,k}. \end{aligned}$$

Note that

$$\begin{aligned} A_{\alpha,j,k} &\equiv - \int_D \frac{(-\rho)^\alpha w \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k} \\ &\equiv - \frac{1}{\alpha+1} \int_D \frac{d(-\rho)^{\alpha+1} \wedge w \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k}, \end{aligned}$$

and hence an integration by parts yields

$$(3.21) \quad A_{\alpha,j,k} \equiv \frac{k}{\alpha+1} A_{\alpha+1,j,k+1} + \frac{1}{\alpha+1} D_{\alpha+1,j,k}.$$

Combining (3.20) with (3.21) and solving for $A_{\alpha,j,k}$ finally one gets Lemma 3.4.

PROOF OF LEMMA 3.5. For bidegree reasons, $w \wedge d\bar{v} = \bar{\partial}u \wedge d\bar{v}$ can be replaced by $du \wedge \partial\bar{v}$ in the definition of $A_{\alpha,0,n+\alpha}$ and hence

$$\begin{aligned} A_{\alpha,0,n+\alpha} &= \left(\frac{i}{2\pi}\right)^n \int_D \frac{(-\rho)^\alpha du \wedge \partial\bar{v} \wedge (\partial\bar{\partial}\rho)^{n-1}}{\bar{v}^{n+\alpha}} \\ &= \left(\frac{i}{2\pi}\right)^n \alpha \int_D \frac{(-\rho)^{\alpha-1} u \wedge \partial\bar{v} \wedge \bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-1}}{\bar{v}^{n+\alpha}} + F_\alpha u \\ &= \bar{G}_\alpha u + F_\alpha u, \end{aligned}$$

cf. (3.2) and Proposition 3.1. This proves (3.16), and (3.17) is obtained in the same manner.

A FINAL REMARK ABOUT THE CASE $\alpha = 0$. Anything we have done above works equally well for $\alpha = 0$ if only integrals as $\alpha \int_D (-\rho)^{\alpha-1} d\rho \wedge \gamma$ are interpreted as $\int_{\partial D} \gamma$. In particular,

$$|F_0 u| \lesssim \int_{\partial D} |u| O\left(\frac{1}{|v|^{n-1/2}}\right) + \int_D |u| O\left(\frac{1}{|v|^{n+1/2}}\right).$$

4. Proof of Theorem 1 for $z \in D$.

Let ρ and D be as before and consider the strictly pseudoconvex domain

$$\tilde{D} = \{(\zeta, \zeta_{n+1}) \in \mathbb{C}^{n+1} : \rho(\zeta) + |\zeta_{n+1}|^2 < 0\}$$

and put $\tilde{\rho}(\tilde{\zeta}) = \rho(\zeta) + |\zeta_{n+1}|^2$, where $\tilde{\zeta} = (\zeta, \zeta_{n+1})$. Then $D = \tilde{D} \cap \{\zeta_{n+1} = 0\}$ so a function u in D can be considered as a function in \tilde{D} , not depending on the last variable. By Theorem 1 we now have operators $\tilde{P}_{\alpha-1}u$, $\tilde{F}_{\alpha-1}u$, $\tilde{M}_{\alpha-1}\partial\bar{\partial}u$ and so on so that (2.2) holds for $\tilde{z} \in \partial\tilde{D}$, $\alpha \geq 1$.

Proposition 4.1. *With the notation above, $\tilde{P}_{\alpha-1}u(z, z_{n+1})$ does not depend on the last variable and $P_\alpha u(z) = \tilde{P}_{\alpha-1}u(z, z_{n+1})$. For $z \in D$ we thus have*

$$(4.1) \quad u(z) = P_\alpha u(z) + \tilde{F}_{\alpha-1}u(z, \sqrt{-\rho(z)}) + \tilde{M}_{\alpha-1}(i\partial\bar{\partial}u)(z, \sqrt{-\rho(z)}).$$

It is therefore natural to *define*

$$M_\alpha(i\partial\bar{\partial}u)(z) = \tilde{M}_{\alpha-1}(i\partial\bar{\partial}u)(z, \sqrt{-\rho(z)})$$

for $z \in D$, and $F_\alpha u(z)$ similarly, and then it remains to check that (2.3)-(2.6) hold.

PROOF OF PROPOSITION 4.1. To be precise, we first construct $\tilde{P}_{\alpha-1}$ in the following way: We let $\tilde{v}(\tilde{\zeta}, \tilde{z}) = v(\zeta, z) - \bar{\zeta}_{n+1}z_{n+1}$ and form the corresponding weighted formula $\tilde{H}_{\alpha-1}$, cf. (1.5), in \tilde{D} . Then $\tilde{H}_{\alpha-1}u(z, z_{n+1})$ will not depend on z_{n+1} if $u = u(z)$ does not. Then we modify it by a smooth kernel $\tilde{\mathcal{L}}_{\alpha-1}u$ as in (1.12). Since $\tilde{H}_{\alpha-1}$ is already holomorphic in the last variable, this can be done in such a way that $\tilde{G}_{\alpha-1}u$ and hence $\tilde{P}_{\alpha-1} = \tilde{G}'_{\alpha-1} + \tilde{G}_{\alpha-1}$ is independent of z_{n+1} if u is. Thus the proposition follows from

$$(4.2) \quad \tilde{H}_{\alpha-1}u(z, z_{n+1}) = H_\alpha u(z).$$

To see (4.2), we first note that (with obvious notation)

$$\tilde{q}(\tilde{\zeta}, \tilde{z}) = q(\zeta, z) + \bar{\zeta}_{n+1}d\zeta_{n+1}$$

and

$$(4.3) \quad \tilde{H}_{\alpha-1}u = \int_{\tilde{D}} \frac{(-\tilde{\rho})^{\alpha-2}u(\zeta)(-\tilde{\rho}\partial\tilde{q} - (n+1)\tilde{q} \wedge \bar{\partial}\tilde{q}) \wedge (\bar{\partial}\tilde{q})^n}{(v(\zeta, z) - \bar{\zeta}_{n+1}z_{n+1})^{n+\alpha}}.$$

Now,

$$\begin{aligned} & (-\tilde{\rho}\partial\tilde{q} - (n+1)\tilde{q} \wedge \bar{\partial}\tilde{\rho}) \wedge (\bar{\partial}\tilde{q})^n \\ &= \left((-\rho + |\zeta_{n+1}|^2)\bar{\partial}q - (n+1)q \wedge \bar{\partial}\rho \right) \wedge n(\bar{\partial}q)^{n-1} \\ & \quad + (-\rho + |\zeta_{n+1}|^2 - (n+1)|\zeta_{n+1}|^2)(\bar{\partial}q)^n \wedge d\zeta_{n+1} \wedge d\bar{\zeta}_{n+1} \\ &= (n+1)(-\rho\bar{\partial}q - nq \wedge \bar{\partial}\rho) \wedge (\bar{\partial}q)^{n-1} \wedge d\zeta_{n+1} \wedge d\bar{\zeta}_{n+1}. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{H}_{\alpha-1} u(z) &= (n+1) \int_D (-\rho \bar{\partial} q - nq \wedge \bar{\partial} \rho) \wedge (\bar{\partial} q)^{n-1} u \\ &\quad \int_{|\zeta_{n+1}| < \sqrt{-\rho(\zeta)}} \frac{(-\rho(\zeta) - |\zeta_{n+1}|^2)^{\alpha-2} d\zeta_{n+1} \wedge d\bar{\zeta}_{n+1}}{(v - \zeta_{n+1} z_{n+1})^{n+\alpha}} \end{aligned}$$

and if we make the change of variables $\sqrt{-\rho(\zeta)} \tau = \zeta_{n+1}$ in the inner integral we get

$$\tilde{H}_{\alpha-1} u(z) = \int_D \frac{(-\rho)^{\alpha-1} u(-\rho \bar{\partial} q - nq \wedge \bar{\partial} \rho) \wedge (\bar{\partial} q)^{n-1}}{v^{n+\alpha}} h_{\alpha-2, n+\alpha, 0},$$

cf. (2.1), with $a = \sqrt{-\rho(\zeta)} \sqrt{-\rho(z)} / v(\zeta, z)$ and hence (4.2) follows, since $h_{\alpha-2, n+\alpha, 0} \equiv 1$.

Next we compute $\tilde{M}_{\alpha-1} \theta(z, \sqrt{-\rho(z)})$ for $\theta = \theta(\zeta)$. For simplicity we just consider a typical term, namely

$$I = \int_{\tilde{D}} \frac{(-\tilde{\rho})^\alpha \theta \wedge \partial \tilde{v} \wedge \bar{\partial} \tilde{v} \wedge (\partial \bar{\partial} \tilde{\rho})^{n-1}}{\tilde{v}^j \bar{v}^k}$$

with $\tilde{z} = (z, \sqrt{-\rho(z)})$. We first notice that $\partial \tilde{v} = \partial v - \bar{z}_{n+1} d\zeta_{n+1}$ and $\bar{\partial} \tilde{v} = \bar{\partial} v - z_{n+1} d\bar{\zeta}_{n+1}$ so that

$$\begin{aligned} &\theta \wedge \partial \tilde{v} \wedge \bar{\partial} \tilde{v} \wedge (\partial \bar{\partial} \tilde{\rho})^{n-1} \\ &= \theta \wedge (|z_{n+1}|^2 \partial \bar{\partial} \rho + (n-1) \partial \bar{v} \wedge \bar{\partial} v) \wedge (\partial \bar{\partial} \rho)^{n-2} \wedge d\zeta_{n+1} \wedge d\bar{\zeta}_{n+1}. \end{aligned}$$

Noting that $|z_{n+1}|^2 = -\rho(z)$ and proceeding as in the proof of Proposition 4.1 above we get

$$I = \int_D \frac{(-\rho)^{\alpha+1} \theta \wedge (-\rho(z) \partial \bar{\partial} \rho + (n-1) \partial \bar{v} \wedge \bar{\partial} v) \wedge (\partial \bar{\partial} \rho)^{n-2}}{v^j \bar{v}^k} h_{\alpha, j, k}$$

with $a = \sqrt{-\rho(\zeta)} \sqrt{-\rho(z)} / v(\zeta, z)$. In the same way,

$$F_\alpha u(z) = \tilde{F}_{\alpha-1} u(z, \sqrt{-\rho(z)})$$

so that

$$(4.4) \quad |F_\alpha u(z)| \lesssim \int_{\tilde{D}} \frac{(-\tilde{\rho})^{\alpha-2} |u|}{|\tilde{v}|^{n+\alpha-1/2}}$$

(if $\alpha = 1$,

$$F_1 u(z) \lesssim \int_{\partial\tilde{D}} |u| \frac{d\sigma(\tilde{\zeta})}{|\tilde{v}|^{n+1/2}} + \int_{\tilde{D}} |u| \frac{d\lambda(\tilde{\zeta})}{|\tilde{v}|^{n+3/2}})$$

and proceeding as before we get

$$|F_\alpha u(z)| \lesssim \int_D \frac{(-\rho)^{\alpha-1} |u|}{|v|^{n+\alpha-1/2}} h_{\alpha-2, n+\alpha-1/2}.$$

This completes the proof of Theorem 1.

5. Proof of Proposition 2.1.

We recall from the last paragraph that if u is defined on D , then $F_\alpha u(z) = \tilde{F}_{\alpha-1} u(\tilde{z})$ and $M_\alpha \theta(z) = \tilde{M}_{\alpha-1} \theta(\tilde{z})$ where $\tilde{z} = (z, \sqrt{-\rho(z)}) \in \partial\tilde{D}$. By writing the operators this way we avoided the factors $h_{\alpha,j,k}$ in their integral representation. However, since $\tilde{F}_{\alpha-1}$ was defined as an integral over \tilde{D} , to compute compositions such as $\tilde{F}_{\alpha-1} \circ \tilde{F}_{\alpha-1}$, etc., we need to know $F_{\alpha-1} u(\tilde{z})$ also for $\tilde{z} \in \tilde{D}$. We will avoid this difficulty by rewriting $\tilde{F}_{\alpha-1} u$ as an integral over the boundary of a domain $D_\alpha \subset \mathbb{C}^{n+\alpha}$.

REMARK. When $\alpha = 1$, $\tilde{F}_{\alpha-1} u$ consists of both a boundary integral and an integral over \tilde{D} . So in this case the argument is slightly different as we only need to rewrite this last integral as an integral over ∂D_2 . We omit those details and assume that $\alpha > 1$ in the sequel.

Let $\zeta^\alpha = (\zeta_1, \dots, \zeta_n, \zeta_{n+1}, \dots, \zeta_{n+\alpha}) = (\zeta, \zeta') \in \mathbb{C}^{n+\alpha}$ where $\zeta = (\zeta_1, \dots, \zeta_n)$. We define the strictly pseudoconvex domain D_α by $D_\alpha = \{\zeta^\alpha \in \mathbb{C}^{n+\alpha} : \rho^\alpha(\zeta^\alpha) < 0\}$ where $\rho^\alpha(\zeta^\alpha) = \rho(\zeta) + \sum_{i=1}^{\alpha} |\zeta_{n+i}|^2$. We then have $v_\alpha(\zeta^\alpha, z^\alpha) = v(\zeta, z) + \zeta_{n+1} z_{n+1} + \dots + \zeta_{n+\alpha} z_{n+\alpha}$. When $\alpha = 1$, we write \tilde{D} for D_1 and $\tilde{\zeta}$ for ζ^1 (as in Section 4). Note that D_α is obtained from D by applying the map $D \mapsto \tilde{D}$, α times.

Lemma 5.1. *If u is defined on D , then*

$$a) \quad \int_D (-\rho(\zeta))^\alpha u(\zeta) d\lambda(\zeta) = \frac{\alpha}{\pi} \int_{\tilde{D}} (-\tilde{\rho}(\tilde{\zeta}))^{\alpha-1} u(\zeta) d\lambda(\tilde{\zeta})$$

and

$$b) \quad \int_D u(\zeta) d\lambda(\zeta) \sim \int_{\partial\tilde{D}} u(\zeta) d\sigma(\tilde{\zeta}).$$

PROOF. We have

$$\begin{aligned} \int_{\tilde{D}} (-\tilde{\rho})^{\alpha-1} u \, d\lambda &= \int_D u(\zeta) \, d\lambda(\zeta) \int_{|\zeta_{n+1}| < \sqrt{-\rho(\zeta)}} (-\rho(\zeta) - |\zeta_{n+1}|^2)^{\alpha-1} \, d\lambda(\zeta_{n+1}) \\ &= \frac{\pi}{\alpha} \int_D (-\rho(\zeta))^\alpha u(\zeta) \, d\lambda(\zeta). \end{aligned}$$

We obtain *b*) from *a*) as

$$\int_{\partial\tilde{D}} \frac{u(\zeta) \, d\sigma(\tilde{\zeta})}{|d\tilde{\rho}|} = \lim_{\alpha \rightarrow 0} \alpha \int_{\tilde{D}} (-\tilde{\rho})^{\alpha-1} u(\zeta) \, d\lambda(\tilde{\zeta}).$$

We also need

Lemma 5.2. *If D is a strictly pseudoconvex domain in \mathbb{C}^n , then for $z \in \tilde{D}$, $w \in \partial D$*

$$\int_{\partial D} \frac{d\sigma(\zeta)}{|v(z, \zeta)|^a |v(\zeta, w)|^b} \lesssim \frac{1}{|v(z, w)|^{a+b-n}}$$

if $0 < a, b < n$ and $a + b > n$. If $a + b < n$ the integral is bounded.

PROOF. We first observe that

$$(5.1) \quad \text{I} = \int_{d(\zeta, z) < \delta} \frac{d\sigma(\zeta)}{d^a(\zeta, z)} \lesssim \delta^{n-a}, \quad \text{if } 0 < a < n,$$

and

$$(5.2) \quad \text{II} = \int_{d(\zeta, z) > \delta} \frac{d\sigma(\zeta)}{d^a(\zeta, z)} \lesssim \delta^{n-a}, \quad \text{if } a > n.$$

Here d is the pseudometric that defines the Koranyi balls in D , see Section 1 and [AnC]. Then $d(z, w) \sim |v(z, w)|$ if w and/or z is on the boundary.

We first prove (5.1) and (5.2) for $z \in \partial D$. Then

$$\begin{aligned} \text{I} &\sim \int_{d(\zeta, z) < \delta} d\sigma(\zeta) \int_{t > d(\zeta, z)} \frac{dt}{t^{a+1}} = \int_0^{+\infty} \frac{dt}{t^{a+1}} \int_{d(\zeta, z) < \min\{t, \delta\}} d\sigma(\zeta) \\ &\lesssim \int_{-\infty}^{\delta} \frac{t^n}{t^{a+1}} dt + \delta^n \int_{\delta}^{+\infty} \frac{dt}{t^{a+1}} \lesssim \delta^{n-a}. \end{aligned}$$

Similarly,

$$\text{II} \sim \int_{\delta}^{+\infty} \frac{dt}{t^{a+1}} \int_{\delta < d(\zeta, z) < t} d\sigma(\zeta) \lesssim \int_{\delta}^{+\infty} \frac{t^n}{t^{a+1}} \sim \delta^{n-a}.$$

If $z \in D$, let $z_0 \in \partial D$ satisfy $d(z) = d(z, z_0) = d(z, \partial D)$. Then $d(\zeta, z) \gtrsim d(\zeta, z_0)$ and (5.1) follows if we apply it to z_0 . To prove (5.2) we consider two cases. If $d(z) \lesssim C\delta$, then if $d(\zeta, z) > \delta$, we have $d(\zeta, z) \sim d(\zeta, z_0)$, and we are done by the case $z_0 \in \partial D$. If $d(z) > C\delta$, then (1.3) (it is proved in the same way as (5.2) by observing that $d(\zeta, z) \sim -\rho(z) + d(\zeta, z_0)$) implies that $\text{II} \lesssim (-\rho(z))^{n-a} \sim d(z)^{n-a} \lesssim \delta^{n-a}$.

Now choose c small enough (so that $cC \leq 1/2$, where C is the constant in the triangle inequality for d), let $\delta = cd(z, w)$ and put $B^c = \partial D \setminus (B_{\delta}(z) \cup B_{\delta}(w))$. Then

$$\begin{aligned} \int_{\partial D} \frac{d\sigma(\zeta)}{|v(z, \zeta)|^a |v(\zeta, w)|^b} &\sim \int_{B_{\delta}(z)} + \int_{B_{\delta}(w)} + \int_{B^c} \frac{d\sigma(\zeta)}{d^a(z, \zeta) d^b(\zeta, w)} \\ &= A + B + C. \end{aligned}$$

The integrals A and B are estimated in the same way. Observe that if $\zeta \in B_{\delta}(z)$, then $d(z, w) \leq C(d(z, \zeta) + d(\zeta, w)) \leq d(z, w)/2 + Cd(\zeta, w)$ and hence $d(\zeta, w) \gtrsim d(z, w)$. Thus by (5.1),

$$A \lesssim \frac{1}{d^b(z, w)} \int_{d(\zeta, w) < \delta} \frac{d\sigma(\zeta)}{d^a(\zeta, z)} \lesssim \frac{1}{d(z, w)^{a+b-n}}.$$

To estimate C , we note that if $\zeta \in B^c$ then $d(z, \zeta) \geq cd(z, w)$. Hence $d(\zeta, w) \lesssim d(\zeta, z) + d(z, w) \lesssim d(\zeta, z)$ and by symmetry we have $d(\zeta, w) \sim d(\zeta, z)$. This implies by (5.2)

$$C \leq \int_{d(\zeta, z) > \delta} \frac{d\sigma(\zeta)}{d^{a+b}(\zeta, z)} \leq \frac{1}{d^{a+b-n}(z, w)}$$

if $a + b > n$. If $a + b < n$ the integral is bounded.

PROOF OF PROPOSITION 2.1. First we claim that

$$\begin{aligned} (5.3) \quad \int_D \frac{(-\rho)^{\alpha-1} |u(\zeta)|}{|v(\zeta, z)|^l} h_{\alpha-2, l} d\lambda(\zeta) &= \int_{\tilde{D}} \frac{(-\tilde{\rho})^{\alpha-2} |u(\zeta)|}{|\tilde{v}(\tilde{\zeta}, \tilde{z})|^l} d\lambda(\tilde{\zeta}) \\ &= \int_{\partial D_{\alpha}} \frac{|u(\zeta)|}{|v_{\alpha}(\zeta^{\alpha}, z^{\alpha})|^l} d\sigma(\zeta^{\alpha}), \end{aligned}$$

if $|(z_{n+1}, \dots, z_{n+\alpha})| = \sqrt{-\rho(z)}$. In fact, for the first equality cf. the proof of Proposition 4.1; the last equality is obtained for $z^\alpha = (z, \sqrt{-\rho(z)}, 0, \dots, 0)$ by repeated use of Lemma 5.1, and then the general case follows since the integral is rotation invariant with respect to $(\zeta_{n+1}, \dots, \zeta_{n+\alpha})$.

By (5.3), (2.7) is equivalent to

$$(5.4) \quad |F_\alpha^k u(z)| \lesssim \int_{\partial D_\alpha} \frac{|u(\zeta)|}{|v_\alpha(z^\alpha, \zeta^\alpha)|^{n+\alpha-k/2}} d\sigma(\zeta^\alpha),$$

$1 \leq k < 2(n + \alpha)$. We will prove (5.4) by induction. It is true for $k = 1$ since then (2.7) is nothing but (2.3). Assuming (5.4) for k we have

$$\begin{aligned} |F_\alpha^{k+1} u(z)| &\lesssim \int_{\partial D_\alpha} \frac{|F_\alpha^k u(\zeta)| d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha, \zeta^\alpha)|^{n+\alpha-1/2}} \\ &\lesssim \int_{\partial D_\alpha} \frac{d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha, \zeta^\alpha)|^{n+\alpha-k/2}} \int_{\partial D_\alpha} \frac{|F_\alpha^k u(w)| d\sigma(w^\alpha)}{|v_\alpha(\zeta^\alpha, w^\alpha)|^{n+\alpha-1/2}}. \end{aligned}$$

By Fubini's theorem and Lemma 5.2 we now get (5.4) for $k + 1$.

We also see that if $k > 2(n + \alpha) - 1$, $F_\alpha^k u$ has a bounded kernel (when $k = 2(n + \alpha) - 1$ it has a logarithmic singularity), and since the kernel of F_α is continuous (and much more) off the diagonal and this is preserved under composition, we have $F_\alpha^k u \in C(\bar{D})$.

To see (2.8) first note that, by (2.6) and the argument for (5.3),

$$|M_\alpha \theta(\zeta)| \lesssim \int_{D_\alpha} \frac{|||\theta|||_D d\lambda(w^\alpha)}{|v_\alpha(\zeta^\alpha, w^\alpha)|^{n+\alpha}}$$

if $|\zeta_{n+1}, \dots, \zeta_{n+\alpha}| = \sqrt{-\rho(\zeta)}$ (recall that $|||\theta||| = -\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho|$). Hence, by (5.4),

$$\begin{aligned} |F_\alpha^k M_\alpha \theta(z)| &\lesssim \int_{\partial D_\alpha} \frac{|M_\alpha \theta(\zeta)| d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha, \zeta^\alpha)|^{n+\alpha-k/2}} \\ &\lesssim \int_{\partial D_\alpha} \frac{d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha, \zeta^\alpha)|^{n+\alpha-k/2}} \int_{D_\alpha} \frac{|||\theta|||_D d\lambda(w^\alpha)}{|v_\alpha(\zeta^\alpha, w^\alpha)|^{n+\alpha}} \\ &\sim \int_{D_\alpha} |||\theta|||_D d\lambda(w^\alpha) \\ &\quad \cdot \int_{\partial D_\alpha} \frac{d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha, \zeta^\alpha)|^{n+\alpha-k/2} |v_\alpha(\zeta^\alpha, w^\alpha)|^{n+\alpha}} \end{aligned}$$

$$\begin{aligned} &\lesssim \int_{D_\alpha} \frac{|||\theta|||_D d\lambda(w^\alpha)}{|v_\alpha(\zeta^\alpha, w^\alpha)|^{n+\alpha-k/2}} \\ &\sim \int_D (-\rho)^\alpha \frac{|||\theta||| d\lambda(w)}{|v(z, w)|^{n+\alpha-k/2}} h_{\alpha-1, n+\alpha-k/2}, \end{aligned}$$

and the proof of Proposition 2.1 is complete.

6. Proof of Theorem 4.

In this section we assume that $\mu = -\rho|\theta| + \sqrt{-\rho}(|\theta \wedge \partial\rho| + |\theta \wedge \bar{\partial}\rho|) + |\theta \wedge \partial\rho \wedge \bar{\partial}\rho|$ is a Carleson measure and prove that $M_\alpha\theta \in \text{BMO}(\partial D)$. It is easy to see that $R_\alpha\theta \in L^\infty$; in fact,

$$\begin{aligned} |R_\alpha\theta(z)| &\lesssim \int_D \frac{(-\rho)^\alpha}{|v(\zeta, z)|^{n+\alpha-1/2}} d\mu(\zeta) \\ &\sim \int_D d\mu(\zeta) \int_{d(\zeta, z)}^{+\infty} \frac{dt}{t^{n+1/2}} \\ &= \int_0^{+\infty} \frac{dt}{t^{n+1/2}} \int_{d(\zeta, z) < t} d\mu(\zeta) = \int_0^{+\infty} \frac{\mu(Q_t(\zeta))}{t^{n+1/2}} dt < +\infty, \end{aligned}$$

as $\mu(Q_t(\zeta)) \lesssim t^n$. The other terms in $M_\alpha\theta$ are estimated in the same fashion, so instead of giving detailed arguments for each of them we concentrate on a typical one. Our choice is

$$f(z) = \int_D \frac{(-\rho)^{\alpha+1} \theta \wedge \partial\bar{v}(\zeta, z) \wedge \bar{\partial}v(\zeta, z) \wedge (\partial\bar{\partial}\rho)^{n-2}}{v(\zeta, z)^{n+1} \bar{v}(\zeta, z)^\alpha}.$$

We want to estimate

$$M_h(p) = \frac{1}{|B_h(p)|} \int_{B_h(p)} |f(z) - f_h| d\sigma(z),$$

where f_h is the mean value of f over $B_h(p)$. To this end we need

Lemma 6.1. *If μ is a Carleson measure then*

$$J = \int_{-\rho(\zeta) > h} \frac{d\mu(\zeta)}{d^{n+\alpha}(\zeta, p)} \lesssim \frac{1}{h^\alpha}.$$

PROOF. Let $\zeta = (y, x)$ where $y = -\rho(\zeta)$ and $x \in \partial D$. Put

$$E_{k,m} = \{\zeta : 2^{k-1}h \leq y \leq 2^k h, m 2^k h \leq d(x, p) \leq (m+1) 2^k h\}$$

if $k \geq 1$. When $k = 0$, we replace the lower bound for y by 0. Then $\{-\rho(\zeta) > h\} = \cup_{k \geq 1} \cup_{m \geq 0} E_{k,m}$. Since $|E_{k,m}| \approx (2^k h)^{n+1} m^{n-1}$, $E_{k,m}$ can be covered by $\lesssim m^{n-1}$ Koranyi balls $Q_{2^k h}(q_i)$, and hence

$$\mu(E_{k,m}) \leq \sum \mu(Q_{2^k h}(q_i)) \lesssim m^{n-1} (2^k h)^n.$$

From this we obtain

$$\begin{aligned} J &\lesssim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2^k h (1+m))^{n+\alpha}} \mu(E_{k,m}) \\ &\lesssim \frac{1}{h^\alpha} \sum_{k=1}^{\infty} 2^{-k\alpha} \sum_{m=0}^{\infty} \frac{m^{n-1}}{(1+m)^{n+\alpha}} \lesssim \frac{1}{h^\alpha}, \end{aligned}$$

as desired.

We also need

$$(6.1) \quad |v(z, \zeta) - v(w, \zeta)| \lesssim (h d(\zeta, p))^{1/2},$$

if $z, w \in B_h(p)$, $\zeta \notin B_{Ch}(p)$. This follows immediately if we write

$$\begin{aligned} v(z, \zeta) - v(w, \zeta) &= v(z, w) + \langle q(z, w) - q(z, \zeta), z - w \rangle \\ &\quad + \langle q(z, \zeta) - q(w, \zeta), \zeta - w \rangle. \end{aligned}$$

Let us return to the estimate of $M_h(p)$. We have

$$M_h(p) \lesssim \frac{1}{|B_h(p)|^2} \int_{z \in B_h(p)} \int_{w \in B_h(p)} |f(z) - f(w)| d\sigma(z) d\sigma(w),$$

and

$$\begin{aligned} f(z) - f(w) &= \left(\int_{y > Ch} + \int_{y \leq Ch} \right) (-\rho)^{\alpha+1} \\ &\quad \cdot \left(\frac{\partial \bar{v}(z) \wedge \bar{\partial} v(z)}{v^{n+1}(z) \bar{v}^\alpha(z)} - \frac{\partial \bar{v}(w) \wedge \bar{\partial} v(w)}{v^{n+1}(w) \bar{v}^\alpha(w)} \right) \wedge \theta \wedge (\partial \bar{\partial} \rho)^{n-2} \\ &= m_\infty(z, w) + m_0(z, w), \end{aligned}$$

where $y = -\rho(\zeta)$ and $v(z) = v(z, \zeta)$ for short. Consider first the part where $y > Ch$. Then

$$\begin{aligned} & \frac{\partial\bar{v}(z) \wedge \bar{\partial}v(z)}{v^{n+1}(z)\bar{v}^\alpha(z)} - \frac{\partial\bar{v}(w) \wedge \bar{\partial}v(w)}{v^{n+1}(w)\bar{v}^\alpha(w)} \\ &= \partial\bar{v}(z) \wedge \bar{\partial}v(z) \left(\frac{1}{v^{n+1}(z)\bar{v}^\alpha(z)} - \frac{1}{v^{n+1}(w)\bar{v}^\alpha(w)} \right) \\ & \quad + \frac{1}{v^{n+1}(z)\bar{v}^\alpha(z)} (\partial\bar{v}(z) \wedge \bar{\partial}v(z) - \partial\bar{v}(w) \wedge \bar{\partial}v(w)). \end{aligned}$$

By (6.1),

$$\begin{aligned} \left| \frac{1}{v^{n+1}(z)\bar{v}^\alpha(z)} - \frac{1}{v^{n+1}(w)\bar{v}^\alpha(w)} \right| &\lesssim \frac{|v(z) - v(w)|}{d^{n+\alpha+2}(\zeta, p)} \\ &\leq \frac{\sqrt{h}}{d^{n+\alpha+3/2}(\zeta, p)}. \end{aligned}$$

Furthermore by (3.4),

$$\partial\bar{v}(z) \wedge \bar{\partial}v(z) = \partial\rho \wedge \bar{\partial}\rho + O(|\eta|)(\partial\rho + \bar{\partial}\rho) + O(|\eta|^2)$$

and

$$\begin{aligned} \partial\bar{v}(z) \wedge \bar{\partial}v(z) - \partial\bar{v}(w) \wedge \bar{\partial}v(w) &= \partial\bar{v}(z) \wedge (\bar{\partial}v(z) - \bar{\partial}v(w)) \\ & \quad + (\partial\bar{v}(z) - \partial\bar{v}(w)) \wedge \bar{\partial}v(w) \\ &= O(\sqrt{h})(\partial\rho + \bar{\partial}\rho + O(|\eta|)). \end{aligned}$$

Hence the integrand in m_∞ is bounded by $\sqrt{h} d(\zeta, p)^{-(n+1/2)} d\mu$. By Lemma 6.1 this implies

$$\begin{aligned} & \frac{1}{|B_h(p)|^2} \int_{B_h(p)} \int_{B_h(p)} m_\infty(z, w) d\sigma(z) d\sigma(w) \\ (6.2) \quad & \lesssim \frac{1}{|B_h(p)|^2} \int_{B_h(p)} d\sigma(z) \int_{B_h(p)} d\sigma(w) \int_{y>Ch} \frac{\sqrt{h}}{d(\zeta, p)^{n+1/2}} d\mu \\ & \lesssim \frac{\sqrt{h}}{\sqrt{h}} = 1. \end{aligned}$$

The contribution from the part where $y < Ch$, is dominated by two terms of the form

$$\begin{aligned} & \int_{y \leq Ch} \frac{1}{|B_h(p)|} \int_{z \in B_h(p)} (-\rho)^{\alpha+1} \frac{|\partial\bar{v} \wedge \bar{\partial}v \wedge \theta \wedge (\partial\bar{\partial}\rho)^{n-2}|}{|v^{n+\alpha+1}(\zeta, z)|} d\sigma(z) \\ & \lesssim \sum_{m=0}^{\infty} \int_{E_{0,m}} \frac{1}{h^n} \int_{z \in B_h(p)} (-\rho) \frac{|\partial\bar{v} \wedge \bar{\partial}v \wedge \theta \wedge (\partial\bar{\partial}\rho)^{n-2}|}{d(\zeta, z)^{n+1}}. \end{aligned}$$

Again we use $\partial\bar{v}(z) \wedge \bar{\partial}v(z) = \partial\rho \wedge \bar{\partial}\rho + O(|\eta|)(\partial\rho + \bar{\partial}\rho) + O(|\eta|^2)$ to obtain

$$\begin{aligned} \text{I} &= \frac{1}{|B_h(p)|^2} \int \int m_0(z, w) d\sigma(z) d\sigma(w) \\ &\lesssim \sum_{m=0}^{\infty} \frac{1}{h^n} \int_{E_{0,m}} d\mu \int_{z \in B_h(p)} \frac{-\rho}{d(\zeta, z)^{n+1}} d\sigma(z) \\ &\lesssim \sum_{m=0}^{\infty} \frac{1}{h^n} \int_{E_{0,m}} d\mu(y, x) \int_{z \in B_h(p)} \frac{y}{(y + d(x, z))^{n+1}} d\sigma. \end{aligned}$$

But if $\zeta = (y, x) \in E_{0,m}$, then

$$\int_{z \in B_h(p)} \frac{y}{(y + d(x, z))^{n+1}} d\sigma \lesssim \frac{1}{(1+m)^{n+1}},$$

and we obtain

$$(6.3) \quad \text{I} \lesssim \sum_{m=0}^{\infty} \frac{1}{(1+m)^{n+1}} \mu(E_{0,m}) \lesssim \sum_{m=0}^{\infty} \frac{m^{n-1}}{(1+m)^{n+1}} \lesssim 1.$$

By (6.2) and (6.3) we get $M_h(p) \lesssim 1$, and the proof is complete.

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