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Hiperbolic singular integral operators

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Abstract. We define a class of integral operators which are singular relative to the hyperbolic metric on simply connected domains of the plane. We study the necessary and sufficient conditions for such operators to be bounded on L^2 of the upper half plane relative to the hyperbolic metric.

Introduction.

Let Ω be an open simply connected subset of $\mathbb{R}^2$. Let us denote by $\partial\Omega$ the boundary of Ω and by $\delta(x)$ the euclidean distance from x to $\partial\Omega$. Let $\rho(x,y)$ be the hyperbolic distance between x and y in Ω. And let $m(x,y) = \inf\{\mathrm{vol}_h(B) : B \text{ is a ball containing } x \text{ and } y\}$, where by $\mathrm{vol}_h(B)$ we denote the hyperbolic volume of B and B is the ball defined relative to the hyperbolic metric (when $\Omega = \mathbb{R}^2_+$ and B has hyperbolic radius r, $\mathrm{vol}_h(B)$ is like $\sinh^2(r/2)$).

We consider a class of operators given by kernels satisfying standard estimates -like the usual Calderón-Zygmund operators- but with respect to the hyperbolic metric. We study the necessary and sufficient conditions for such "hyperbolic singular integral operators" T to extend to a bounded operators on $L^2(\Omega, dx/\delta(x)^2)$.

In hyperbolic spaces, the volume of a ball grows exponentially as a function of its radius. We can not then have a doubling measure. Therefore, hyperbolic spaces are not examples of spaces of homogeneous type and we cannot view them within the same framework of general Calderón-Zygmund theory developed for these spaces (*cf.* for example [CW], [DJS])

The motivation for considering operators given by this kind of kernels arises when looking at the Green's function of the upper half plane in two-dimensions:

$$G(x,y) = \log \frac{|x-\bar{y}|}{|x-y|}\,, \qquad \text{for } x, y \in \mathbb{R}^2_+ \,.$$

It is well known that the Green's operator is not bounded on $L^2(\mathbb{R}^2_+\,, dx)$, where dx is Lebesgue measure. But if we consider

$$\tilde{G}(x,y) = \log \frac{|x-\bar{y}|}{|x-y|}\, \chi_{\{\rho(x,y)>1\}}\,,$$

then the operator associated to $\tilde{G}(x,y)$ is bounded on $L^2(\mathbb{R}^2_+\,, dh)$, $dh = dx/\delta(x)^2 = dx/x_2^2$. This is a consequence of $|G(x,y)| \leq c/m(x,y)$; that is the Green's function decays like the inverse of the volume of the smallest ball -relative to the hyperbolic metric- containing x and y.

The philosophy to deal with such "hyperbolic singular integral operators" would be the following. If the hyperbolic distance between x and y is larger than one, then the kernel would decrease "exponentially" and we have enough decay to handle the L^2-boundedness via Schur's Lemma. If the distance between x and y is less than one then these points lie "in the same" Whitney cube where euclidean distance and hyperbolic distance are comparable. We are then reduced to the euclidean case and the $T1$-Theorem of David and Journé applies (*cf.* [DJ]).

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1. Definitions, examples and statement of results.

Definition 1.1. *A hyperbolic standard kernel is a continuous function* $K : \Omega \times \Omega \setminus \Delta \longrightarrow \mathbb{C}$ *for which there exists a constant* $c > 0$ *such that*

1) $|K(x,y)| \leq \dfrac{c}{m(x,y)}$, *for all* $(x,y) \in \Omega \times \Omega \setminus \Delta$.

2) *If* $(x,y) \in \Omega \times \Omega \setminus \Delta$ *are such that* $\rho(x,y) < 1$, *then*

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \le c \, \frac{\delta(x)\, \delta(y)}{|x-y|^3}$$

($\Delta = \{(x,y) : x = y\}$ *and the gradients are taken in the distributional sense and assumed to be functions*).

Definition 1.2. *A hyperbolic singular integral operator* T *is an operator taking* $C_0^\eta(\Omega)$ *into* $L^1_{\text{loc}}(\Omega)$ *and associated to a hyperbolic standard kernel* K*: for every* $f \in C_0^\eta(\Omega)$,

$$Tf(x) = \int K(x,y)\, f(y)\, \frac{dy}{\delta(y)^2} \,,$$

for $x \notin \operatorname{supp} f$.

Notice that when $\rho(x,y) < 1$ we have that $m(x,y)$ is comparable to $\delta(x)\,\delta(y)/|x-y|^2$ and that $1/4 \le \delta(x)/\delta(y) \le 4$; therefore our hyperbolic singular integral operator coincides with a usual singular integral operator in euclidean geometry.

We refer the reader to [B] and [BP] for precise definitions and properties about hyperbolic geometry.

EXAMPLES. i) Let $\Omega = \mathbb{R}^2_+$, $(x_1, x_2) = x$, $y = (y_1, y_2)$ and consider the Riesz transforms

$$\Big(x_2 \, \frac{\partial}{\partial x_1}\Big)^2 G(x,y) \, ; \qquad G(x,y) = \log \frac{|x - \bar{y}|}{|x-y|} \, .$$

where the derivatives are taken in the distributional sense.

Then $(x_2\, \partial/\partial x_1)^2 G(x,y)$ equals

$$K(x,y) = -x_2^2 \Big(\frac{(x_1-y_1)^2 - (x_2-y_2)^2}{|x-y|^4} - \frac{(x_1-y_1)^2 - (x_2+y_2)^2}{|x-\bar{y}|^4} \Big) \, .$$

$K(x,y)$ is a C^1-function away from the diagonal and it is easy to see that it satisfies 1) and 2).

ii) Take $\Omega = \mathbb{R}^2_+$ and for $0 < r < 1$, let $B(i,r)$ be the ball of hyperbolic radius r centered at i. Given $x, y \in \mathbb{R}^2_+$ there exists a

Möbius transformation γ such that $\rho(x,y)=\rho(\gamma x,\gamma y)=\rho(i,pi)$. Since $\rho(i,pi)=|\log\ p|$ we have that,

$$\mathrm{vol}_h B(x,\rho(x,y))=\mathrm{vol}_h B(i,|\log\ p|)\,.$$

Let k be a smooth function so that $|k(r)|\le c/r^2$ and $|k'(r)|\le c/r^3$ and define $k(r)=k(\rho(x,y))=k(|\log\ p|)$. Then let $K(x,y)=k(\rho(x,y))$, if $\rho(x,y)<1$ and $K(x,y)=0$ otherwise. Clearly K satisfies 1) and 2). And we have that,

$$Tf(x)=Tf(\gamma i)=\int_G k(\rho(\gamma i,\mu i))\,f(\mu i)\,dh(\mu)=k*_G f$$

if $\gamma i=x$ and $\mu i=y$; $\gamma,\mu\in G$, the group of Möbius transformations (*cf.* [CW2, Chapter 10]).

iii) Let $k(r)=(1/\sinh^2 r)^{1+i}$ and let $K(x,y)=k(\rho(x,y))$. Then it is clear that K satisfies 1) and 2).

Let $\Omega=\mathbb{R}^2_+$. In Section 2 we prove,

Theorem 2.1. *Let T be a hyperbolic singular integral operator. Then, T extends to a bounded operator in $L^2(\mathbb{R}^2_+,dx/x_2^2)$ if and only if, for any $0<\varepsilon<1$,*

1) $T(w)\in w\,\mathrm{BMO}(dh)$; $w(x)=x_2^{\varepsilon}$,

2) $T^*(w)\in w\,\mathrm{BMO}(dh)$; $w(x)=x_2^{\varepsilon}$,

3) *T satisfies the "local weak boundedness property"* (LWBP):

Let $\{Q_j\}$ be the Whitney decomposition of $\mathbb{R}^2_+$. Fix Q_j and let $d\omega_j$ be $dx/|Q_j|$. Then $\omega_j(Q)=|Q|/|Q_j|$ for $Q\subseteq Q_j$. Let $f,g\in C_0^\eta$ such that $\mathrm{supp}\,f,\mathrm{supp}\,g\subseteq Q$, $Q\subseteq Q_j$ and $|f(x)-f(y)|\le c\,|x-y|^\eta\,\omega_j(Q)^{-\eta/2}$; same condition also for g. Then,

$$|\langle Tf,g\rangle|=\left|\int Tf(x)\,g(x)\,d\omega_j\right|\le c\,\omega_j(Q)\,\|f\|_\infty\,\|g\|_\infty\,,$$

where c is a constant independent of j.

By $\mathrm{BMO}(dh)$ we mean, modulo constants the space of functions f such that

$$\sup_{\{Q:\mathrm{vol}_h(Q)\le 1\}}\frac{1}{\mathrm{vol}_h(Q)}\int_Q|f-(mh)_Q f|\,dh(x)\,,$$

where Q is a cube in $\mathbb{R}^2_+$ with sides parallel to the coordinate axis, $dh(x) = dx/x_2^2$ and

$$(mh)_Q f = \frac{1}{\mathrm{vol}_h(Q)} \int f \, dh \, .$$

For other domains different than $\mathbb{R}^2_+$ we can get a partial result for Ω a simply connected domain in $\mathbb{R}^2$ bounded by a Jordan curve that is a K-quasicircle with $K = 1 + \varepsilon$, $\varepsilon > 0$ very small and R an operator associated to a kernel $R(z, w)$ that is closely related to a hyperbolic standard kernel when $\rho^*(z, w) \geq 1$, ρ^* is the hyperbolic distance in Ω.

In Section 3 we prove,

Theorem 3.1. *Let Ω be a simply connected domain in $\mathbb{R}^2$ bounded by a Jordan curve that is a K-quasicircle with $K = 1 + \varepsilon$ and $\varepsilon > 0$ very small. Let ρ^* be the hyperbolic distance function in Ω and $\delta(z')$ the euclidean distance* $\operatorname{dist}\{z', \partial\Omega\}$. *Let $R(z', w')$ be a kernel defined on $\Omega \times \Omega$ such that $R(z', w') = 0$, if $\rho^*(z', w') < 1$ and $|R(z', w')| \leq c\, e^{-\rho^*(z',w')/K^2}$, if $\rho^*(z', w') \geq 1$.*

Then, if R is the operator associated to $R(z', w')$, we have that there exists $\eta = \eta(\varepsilon) > 0$ such that

$$R(\delta^\eta)(z') \leq c\,\delta^\eta(z') \, , \qquad dh \textit{ almost everywhere,}$$

where $dh = dz'/\delta(z')^2$ is equivalent to the hyperbolic measure on Ω.

By Schur's Lemma it is an immediate consequence of Theorem 3.1 that R defines a bounded operator on $L^2(\Omega, dz'/\delta(z')^2)$.

Actually, if $\tilde{G}(z', w')$ is the Green's function on any simply connected domain of $\mathbb{R}^2$ (with non trivial boundary), then $|\tilde{G}(z', w')| \leq c\, e^{-\rho^*(z',w')}$ if $\rho^*(z', w') \geq 1$.

REMARK. We also have that

$$|\tilde{G}(z', w')| \leq c \left(1 + \log \frac{1}{\rho^*(z', w')^2}\right), \qquad \text{for } \rho^*(z', w') < 1 \, .$$

This estimate is enough to prove that $\tilde{G}$, the operator associated to the kernel $\tilde{G}(z', w')\, \chi_{\rho^*(z',w')<1}$, satisfies $\tilde{G}(\delta^\eta)(z') \leq \delta^\eta(z')$, dh almost everywhere, for $0 < \eta < 1$, Ω simply connected in $\mathbb{R}^2$.

The Green's operator on Δ, the unit disk, defines a bounded operator on $L^2(\Delta, dh)$. Therefore the Green's operator on Ω, any simply connected domain in the plane, defines a bounded operator on $L^2(\Omega, dh)$.

REMARK. Kernels $K(z', w')$ defined on $\Omega \times \Omega$, $\partial\Omega$ a K-quasicircle, $K = 1 + \varepsilon$, satisfying: $K(z', w') = 0$ if $\rho^*(z', w') < 1$ and

$$|K(z', w')| \leq c\, \frac{\min\{\delta(z')^2, \delta(w')^2\}}{|z' - w'|^2}\,, \qquad \text{if } \rho^*(z', w') > 1\,,$$

are also kernels of the kind described in Theorem 3.1:

Being $\partial\Omega$ a K-quasicircle, there exists $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $F(\infty) = \infty$, $f = F|_\Delta : \Delta \longrightarrow \Omega$ is univalent and F is a K^2-global quasiconformal map (recall that F^{-1} is also K^2-quasiconformal).

Also F satisfies: if $z, w, u \in \mathbb{R}^2$ then (*cf.* [A])

$$\min\left\{\frac{|z-w|}{|w-u|}^{1/K^2}, \frac{|z-w|}{|w-u|}^{K^2}\right\} \leq \left|\frac{F(z) - F(w)}{F(w) - F(u)}\right| \leq \max\left\{\frac{|z-w|}{|w-u|}^{1/K^2}, \frac{|z-w|}{|w-u|}^{K^2}\right\}.$$

Therefore,

$$\frac{\min\{\delta(z')^2, \delta(w')^2\}}{|z' - w'|^2} \leq e^{-\rho(h(z'), h(w'))/K^2}\,,$$

where $h = f^{-1}$ and ρ is the hyperbolic distance function in Δ.

Before proving Theorem 2.1 we wish to recall,

Schur's Lemma. *If $K(x, y)$ is a nonnegative kernel, if p and q are strictly positive measurable functions on X and Y respectively, and if α and β are positive numbers such that*

$$\int K(x, y)\, q(y)\, d\mu(y) \leq \alpha\, p(x)\,, \qquad \textit{for almost every } x\text{-}d\mu'\,,$$

$$\int K(x, y)\, p(x)\, d\mu'(x) \leq \beta\, q(y)\,, \qquad \textit{for almost every } y\text{-}d\mu\,,$$

then $K(x, y)$ is a bounded kernel and $\|K\|^2 \leq \alpha\,\beta$. That is, the operator associated to $K(x, y)$ maps $L^2(X, d\mu') \to L^2(Y, d\mu)$ continuously. $(X, d\mu')$ and $(Y d\mu)$ are measure spaces, μ' and μ are positive measures (*cf.* [HS]).

2. Proof of Theorem 2.1.

Write,

$$T(f) = \int_{\{\rho(x,y)<1\}} K(x,y)\, f(y)\, \frac{dy}{\delta(y)^2} + \int_{\{\rho(x,y)\geq 1\}} K(x,y)\, f(y)\, \frac{dy}{\delta(y)^2}$$
$$= T_1(f) + T_2(f)\,,$$

for $f \in C_0^\eta$ and $x \notin \operatorname{supp} f$. Then, the theorem will follow from:

A) $T_2 : x_2^\varepsilon\, L^\infty \longrightarrow x_2^\varepsilon\, L^\infty$. Recall that on $\mathbb{R}^2_+$, $\delta(x) = x_2$ if $x = (x_1, x_2)$.

B) If $T : L^2(dh) \longrightarrow L^2(dh)$, then $T_1 : x_2^\varepsilon\, L^\infty \longrightarrow x_2^\varepsilon\, \mathrm{BMO}(dh)$.

C) If T_1 has the LWBP on each Q_j and $T_1(1) \in \mathrm{BMO}(dh)$, $T_1^*(1) \in \mathrm{BMO}(dh)$, then T_1 is bounded on $L^2(dh)$.

PROOF OF A). We need to show that there exists a constant $c = c(\varepsilon) > 0$ such that $\|T_2(f)(\cdot)/\delta(\cdot)\|_\infty \leq c\,\|F\|_\infty$; $f = x_2^\varepsilon\, F$, $F \in L^\infty$.

$$|T_2(f)(x)| \leq \int_{\{\rho(x,y)>1\}} |K(x,y)|\, y_2^\varepsilon\, F(y)\, \frac{dy}{y_2^2}$$
$$\leq c\,\|F\|_\infty \int_{\{\rho(x,y)>1\}} \frac{1}{m(x,y)}\, y_2^\varepsilon\, \frac{dy}{y_2^2}$$
$$\leq c''\,\|F\|_\infty \int_{\{\rho(x,y)>1\}} \frac{x_2 y_2}{|x-\bar{y}|^2}\, y_2^\varepsilon\, \frac{dy}{y_2^2}\,,$$

since for $r = \rho(x,y) > 1$, $\sinh^2(r/2) \sim e^r$, and then,

$$\frac{1}{m(x,y)} \leq c' \frac{x_2\, y_2}{|x-\bar{y}|^2}\,.$$

By means of a (Möbius) transformation we get,

$$\leq c\,\|F\|_\infty\, x_2^\varepsilon \int \frac{x_2^2}{|x - \overline{x_2 w}|^2}\, w_2^{\varepsilon-1}\, dw$$
$$\leq c\,\|F\|_\infty\, x_2^\varepsilon \int_0^{+\infty} \frac{x_2^2}{w_2^{1-\varepsilon}(x_2 + x_2 w_2)^2} \int_{-\infty}^{+\infty} \frac{1}{1 + \left(\dfrac{x_1 - x_2 w_1}{x_2 + x_2 w_2}\right)^2}\, dw_1\, dw_2$$

$$\leq c\,\pi\,\|F\|_\infty\,x_2^\varepsilon \int_0^{+\infty} \frac{1}{w_2^{1-\varepsilon}(1+w_2)}\,dw_2$$

$$\leq c(\varepsilon)\,\pi\,\|F\|_\infty\,x_2^\varepsilon\,,$$

where

$$c(\varepsilon) = c\left(\frac{1}{\varepsilon} + \frac{1}{1-\varepsilon}\right).$$

PROOF OF B). Assume $T : L^2(dh) \longrightarrow L^2(dh)$. By A) and Schur's Lemma we have that $T_2 : L^2(dh) \longrightarrow L^2(dh)$ continuously, therefore we know that $T_1 : L^2(dh) \longrightarrow L^2(dh)$. We define the action of T_1 on $x_2^\varepsilon L^\infty$:

Let Q be a cube such that $\mathrm{vol}_h(Q) < 1$, then everything is like in the euclidean case ([DJ], [DJS]).

Let $f = f\,\chi_{\bar{Q}} + f(1-\chi_{\bar{Q}}) = f_1 + f_2$. $\bar{Q} = 2\,Q$ is the cube with the same center as Q and sidelength $2\,\ell(Q)$.

$$(T_1 f)_Q = \frac{T_1(f_1)(x)}{x_2^\varepsilon} + \left|\frac{T_1(f_2)(x)}{x_2^\varepsilon} - c_Q\right|,$$

where

$$c_Q = \int\limits_{\{y:\,\rho(x_Q,y)<1,\,y\notin\bar{Q}\}} \frac{K(x_0,y)}{x_2^\varepsilon}\,f_2(y)\,\frac{dy}{y_2^2}\,,$$

where x_Q is the center of Q.

$$(T_1 f)_Q(x) = \frac{T_1(f_1)(x)}{x_2^\varepsilon} + \int\limits_{\{\rho(x,y)<1\}\cap\bar{Q}^c} \left(\frac{K(x,y)}{x_2^\varepsilon} - \frac{K(x_Q,y)}{x_2^\varepsilon}\right) f_2(y)\,\frac{dy}{y_2^2}\,.$$

$(T_1 f)_Q$ is well defined up to constants (depending on Q).

Now, denote by $d_Q = d(Q,\mathbb{R})$, and let $f_1(x) = x_2^\varepsilon F(x)\chi_{\bar{Q}}(x) \in L^2(\mathbb{R}_+^2, dh)$. By the boundedness of T_1 and Jensen's inequality we have,

$$\frac{1}{\mathrm{vol}_h(Q)}\int_Q \frac{|T_1 f_1(x)|}{x_2^\varepsilon}\,\frac{dx}{x_2^2} \leq c\,\frac{d_Q^{-\varepsilon}}{\mathrm{vol}_h(Q)}\int_Q |T_1 f_1(x)|\,\frac{dx}{x_2^2}$$

$$\leq d_Q^{-\varepsilon}\left(\frac{1}{\mathrm{vol}_h(Q)}\int_Q |T_1 f_1(x)|^2\,\frac{dx}{x_2^2}\right)^{1/2}$$

$$\leq d_Q^{-\varepsilon} \left(\frac{1}{\mathrm{vol}_h(Q)} \int_Q |f_1(x)|^2 \frac{dx}{x_2^2} \right)^{1/2} .$$

But $|f_1(x)| \leq \|F\|_\infty d_Q^\varepsilon$, therefore the last expression is less or equal than $c\,\|F\|_\infty$.

On the other hand,

$$\begin{aligned}
&\Big| \int\limits_{\{\rho(x,y)<1\}\cap \bar{Q}^c} \left(\frac{K(x,y)}{x_2^\varepsilon} - \frac{K(x_Q,y)}{x_2^\varepsilon} \right) f_2(y) \frac{dy}{y_2^2} \Big| \\
&\qquad \leq c\,\|F\|_\infty \int\limits_{\{\rho(x,y)<1\}\cap \bar{Q}^c} |K(x,y) - K(x_Q,y)| \frac{dy}{y_2^2} \\
&\qquad \leq c\,|Q|^{1/2}\,\|F\|_\infty \int\limits_{\{y:\rho(x,y)<1\}\cap \bar{Q}^c} \frac{1}{|x_Q - y|^3}\, dy \\
&\qquad \leq c'\,|Q|^{1/2}\,\|F\|_\infty \int_{|Q|^{1/2}}^{+\infty} \frac{1}{r^2}\, dr \\
&\qquad \leq c\,\|F\|_\infty .
\end{aligned}$$

Therefore, $T_1 : x_2^\varepsilon L^\infty \longrightarrow x_2^\varepsilon \,\mathrm{BMO}(dh)$, which proves B). Similarly, we have that if $T^*\colon L^2(dh) \longrightarrow L^2(dh)$ then, $T^*\colon x_2^\varepsilon L^\infty \longrightarrow x_2^\varepsilon\,\mathrm{BMO}(dh)$.

It is easy, and left to the reader, to check that if $T\colon L^2(dh) \longrightarrow L^2(dh)$ then T has the LWBP on each Whitney Q_j.

Now we should prove the converse. Once more we remark that A) has already established, by Schur's Lemma, the $L^2(dh)$-boundedness of T_2.

PROOF OF C). If T_1 has the LWBP on each Q_j and $T_1(1) \in \mathrm{BMO}(dh)$, $T_1^*(1) \in \mathrm{BMO}(dh)$, then T_1 is bounded on $L^2(dh)$.

To see this we take the Whitney decomposition $\{Q_j\}$ for the upper half plane and we divide it into 9 subfamilies $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots, \mathcal{F}_9$ so that if f is a function supported in Q_{j_i} and g a function supported in Q_{k_i}, and $Q_{k_i} \in \mathcal{F}_i$, $Q_{j_i} \in \mathcal{F}_i$, $k_i \neq j_i$. Then, $\operatorname{supp} T_1(f)$ and $\operatorname{supp} T_1(g)$ have disjoint interiors.

For Q_k a Whitney cube denote by $N_k = Q_k \cup \{8 \text{ neighbors}\}$. A neighbor of Q_k is a Whitney cube having one side or one vertex in common with Q_k.

Now let f be in $C_0^\eta(\mathbb{R}^2_+)$ and let $\{Q_j\}$ be the Whitney decomposition for $\mathbb{R}^2_+$ and write

$$f = \sum_j f\,\chi_{Q_j} = \sum_{i=1}^{9}\sum_{j_i} f\,\chi_{Q_{j_i}} = \sum_{i=1}^{9}\sum_{j_i} f_{j_i}\,.$$

Then,

$$\|T_1(f)\|_2^2 \leq c(9)\sum_{i=1}^{9}\sum_{j_i}\|T_1(f_{j_i})\|_2^2\,.$$

We wish to conclude that $\|T_1(f_{j_i})\|_2^2 \leq c_0\,\|f_{j_i}\|_2^2$ where c_0 is independent of Q_j, any Whitney cube. Observe that $\|f\|_2^2 = \sum_{i=1}^{9}\sum_{j_i}\|f_{j_i}\|_2^2\,.$

To see this, let us write T_1 in the following way: for h such that $\operatorname{supp} h \subseteq Q_j$

$$\begin{aligned} T_1(h) &= L_j(h) + E_j(h)\,,\\ L_j(h)(x) &= \int_{\{\rho(x,y)<1\}} \chi_{Q_j}(x)\,K(x,y)\,h(y)\,\frac{dy}{y_2^2}\,,\\ E_j(h)(x) &= \int_{\{\rho(x,y)<1\}} \chi_{N_j\setminus Q_j}(x)\,K(x,y)\,h(y)\,\frac{dy}{y_2^2}\,. \end{aligned}$$

Then $L_j : L^2(Q_j, d\omega_j) \longrightarrow L^2(Q_j, d\omega_j)$ continuously. Indeed, $T_2(w) \in w\,\mathrm{BMO}$ and $T(w) \in w\,\mathrm{BMO}$, $w(x) = x_2^\varepsilon$. Then $T_1(w) \in w\,\mathrm{BMO}$, and

$$\frac{T_1(w)}{w} \in \mathrm{BMO}(dh) \qquad \text{implies} \qquad \frac{T_1(w)}{w} \in \mathrm{BMO}(Q_j, d\omega_j)\,.$$

Now, if $x \in Q_j$ then

$$w(x) = x_2^\varepsilon = d_j^\varepsilon\, b(x)^\varepsilon\,,$$

where $1/4 \leq b(x) \leq 4$ is independent of j and $d_j = d(Q_j, \mathbb{R})$. Therefore, on Q_j,

$$\frac{T_1(w)(x)}{w(x)} = \frac{T_1(b)(x)}{b(x)}\,,$$

and so

$$\frac{T_1(b)(x)}{b(x)} \in \mathrm{BMO}(Q_j, d\omega_j)\,.$$

Observe that $b(x)$ is a positive, Hölder continuous function on Q_j with constant $c\,d_j^{-\varepsilon}$.

Then by Stegenga [Sg] we can prove that if

$$\frac{T_1(b)(x)}{b(x)} \in \mathrm{BMO}(Q_j, d\omega_j) \qquad \text{then} \qquad T_1(b)(x) \in \mathrm{BMO}(Q_j, d\omega_j)$$

with constant depending on the BMO constant of $T_1(w)/w$, on $\|b\|_\infty$ and on

$$\alpha = \sup_{Q \subseteq Q_j} \frac{1}{\omega_j(Q)} \left(\log \frac{1}{\omega_j(Q)}\right) \int_Q |b(x) - m_Q b| \, d\omega_j \,,$$

$$m_Q b = \frac{1}{\omega_j(Q)} \int_Q b(x) \, d\omega_j \,.$$

It is easy to see now that since $|b(x) - b(x')| \leq c\,|x - x'|^\varepsilon / d_j^\varepsilon$, we have that $\alpha \leq 1$.

Then $T_1(b) \in \mathrm{BMO}(Q_j, d\omega_j)$ with constant independent of j.

On the other hand,

$$T_1(b) = T_1(b\,\chi_{Q_j}) + T_1(b\,(1 - \chi_{Q_j}))\,.$$

But $T_1(b(1-\chi_{Q_j}))$ is in $\mathrm{BMO}(Q_j, d\omega_j)$ with constant independent of j (this follows from Lemma 2.1 at the end of this section; in fact, $T_1(b(1-\chi_{Q_j}))(x) \sim \log(|Q_j|^{1/2}/d(x, \partial Q_j))$). Then $T_1(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$. In the same way $T_1^*(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$. Therefore,

i) $L_j(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$,

ii) $L_j^*(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$,

iii) L_j has the WBP on Q_j -relative to $d\omega_j$.

Therefore, by the $T1$-Theorem (*cf.* [DJ])

$$L_j : L^2(Q_j, \frac{dx}{x_2^2}) \longrightarrow L^2(Q_j, \frac{dx}{x_2^2})$$

with constant independent on j.

Now we concentrate on E_j. We wish to show that

$$E_j : L^2(N_j, \frac{dx}{x_2^2}) \longrightarrow L^2(N_j, \frac{dx}{x_2^2}),$$

$$E_j(f)(x) = \int \chi_{Q_j}(x) \frac{K(x,y)}{y_2^2} \chi_{N_j \setminus Q_j}(y) f(y)\, dy .$$

If we call

$$e_j(x,y) = \chi_{Q_j}(x) \frac{K(x,y)}{y_2^2} \chi_{N_j \setminus Q_j}(y)$$

then

$$|e_j(x,y)| \le \frac{c}{|x-y|^2}, \qquad \text{for } x \in Q_j,\ y \in N_j \setminus Q_j .$$

We will show that there exist two positive measurable functions $p(x)$ and $q(y)$ and two positive numbers α and β such that

$$(\mathrm{S}_1) \qquad \int |e_j(x,y)|\, q(y)\, dy \le \alpha\, p(x), \qquad \text{for almost every } x,$$

$$(\mathrm{S}_2) \qquad \int |e_j(x,y)|\, p(x)\, dx \le \beta\, q(y), \qquad \text{for almost every } y.$$

Let

$$q(y) = d(y, \partial Q_j)^{-1/2}, \qquad y \in N_j \setminus Q_j,$$
$$p(x) = d(x, \partial Q_j)^{-1/2}, \qquad x \in Q_j .$$

We need to show that

$$(\mathrm{S}_1)\ \chi_{Q_j}(x) \int_{N_j \setminus Q_j} |e_j(x,y)|\, d(y, \partial Q_j)^{-1/2}\, dy \le \alpha\, d(x, \partial Q_j)^{-1/2},$$

$$(\mathrm{S}_2)\ \chi_{N_j \setminus Q_j}(x) \int_{Q_j} |e_j(x,y)|\, d(y, \partial Q_j)^{-1/2}\, dy \le \beta\, d(x, \partial Q_j)^{-1/2}.$$

First we observe that it is enough to prove (S_1) and (S_2) for $\overset{=}{Q}_j$ instead of $N_j \setminus Q_j$ where $\overset{=}{Q}_j$ is the cube with the same center as Q_j and 5 times its side length.

Next we observe that it is enough to show

$$(\mathrm{S}_1')\quad \chi_{B_j}(x) \int_{\overset{=}{B}_j \setminus B_j} |e_j(x,y)|\, d(y, \partial B_j)^{-1/2}\, dy \le \alpha\, d(x, \partial B_j)^{-1/2}$$

$$(\mathrm{S}_2')\quad \chi_{\overset{=}{B}_j \setminus B_j}(x) \int_{B_j} |e_j(x,y)|\, d(y, \partial B_j)^{-1/2}\, dy \le \beta\, d(x, \partial B_j)^{-1/2},$$

where B_j is a ball with same center as Q_j and radius comparable to $\ell(Q_j)$.

Indeed, given Q_j and B_j, there is a bilipschitz map $h : \mathbb{R}^2 \to \mathbb{R}^2$ sending B_j to Q_j. The interior of B_j goes to the interior of Q_j and there exists $M > 0$ such that

$$\frac{1}{M} \leq \frac{h(z) - h(w)}{z - w} \leq M .$$

Therefore if Jh is the jacobian of h, $|Jh(z)| \leq 2M^2$ almost everywhere. Moreover, there exists a bilipschitz map $h : \mathbb{R}^2 \to \mathbb{R}^2$ sending $\overline{\overline{B}}_j \setminus B_j$ to $\overline{\overline{Q}}_j \setminus Q_j$. First consider h_1 bilipschitz from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending $\overline{\overline{B}}_j$ to $\overline{\overline{Q}}_j$ and the interior to the interior. Then we consider a neighborhood C_j of $h_1(B_j)$ at a distance proportional to $\partial\overline{\overline{B}}_j$ and ∂B_j and consider $h_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a bilipschitz map such that $h_2(z) = \tilde{h}_2(z)$ if $z \in C_j$ and $h_2(z) = z$ if $z \in \mathbb{R}^2 \setminus C_j$.

Here, $\tilde{h}_2$ is the map that sends C_j to another neighborhood C'_j and maps $h_1(B_j)$ -inside C_j- to Q_j -inside C'_j-. Also, $C_j \setminus h_1(B_j)$ is mapped to $C'_j \setminus Q_j$.

Finally, we take $h = h_2 \circ h_1$, $\quad h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\overline{\overline{B}}_j \setminus B_j$ to $\overline{\overline{Q}}_j \setminus Q_j$ and is bilipschitz with constant M' independent of j, (*cf.* [T], [JK]).

To prove (S'_1) and (S'_2) we can proceed in two different ways. One is elementary, the other more constructive. We choose to prove (S'_1) in the elementary and (S'_2) in the other way. We can assume B_j is centered at 0, that x lies in the real axis, and it is in the interior of B_j.

Let r be the radius of B_j. Then,

$$\begin{aligned}
\int_{\overline{\overline{B}}_j \setminus B_j} & |e_j(x,y)|\, d(y, \partial B_j)^{-1/2}\, dy \\
& \leq \int_{\overline{\overline{B}}_j \setminus B_j} \frac{1}{|x-y|^2}\, d(y,\partial B_j)^{-1/2}\, dy \\
& = \int_{-\pi}^{\pi} \int_r^{3r} \frac{\rho\,(\rho - r)^{-1/2}}{\rho^2 + |x|^2 - 2\rho |x| \cos\,\theta}\, d\rho\, d\theta \\
& = 2\pi \int_r^{3r} \frac{\rho}{(\rho - r)^{1/2} (\rho^2 - |x|^2)}\, d\rho \\
& \leq 2\pi \int_r^{3r} \frac{d\rho}{(\rho - r)^{1/2} (\rho - |x|)}
\end{aligned}$$

$$\begin{aligned}&= 2\pi \int_{r-|x|}^{3r+|x|} \frac{d\rho}{\rho(\rho+|x|-r)^{1/2}}\\ &\leq \frac{\pi}{\sqrt{r-|x|}}\\ &= \pi\, d(x, \partial B_j)^{-1/2},\end{aligned}$$

where we have used

1) if $b^2 > c^2$,

$$\int \frac{dx}{b + c\cos ax} = \frac{2}{a(b^2-c^2)^{1/2}} \arctan\Big(\frac{(b-c)\tan(ax/2)}{(b^2-c^2)^{1/2}}\Big) + c,$$

2) if $b < 0$,

$$\int \frac{dx}{x\,(ax+b)^{1/2}} = \frac{2}{\sqrt{-b}} \arctan \frac{(ax+b)^{1/2}}{\sqrt{-b}} + c.$$

To prove (S_2') we can assume with no loss of generality that $B_j = \Delta$, the unit disk centered at 0. We wish to prove that there exists $\beta > 0$ such that

$$\begin{aligned}&\int_\Delta \frac{1}{|x-y|^2}(1-|y|^2)^{-1/2}\,dy \leq \beta\,(|x|^2-1)^{-1/2}, \qquad \text{for } x \notin \Delta,\\ &\int_\Delta \frac{1}{|x-y|^2}(1-|y|^2)^{-1/2}\,dy \leq \frac{1}{|a|}\int_\Delta \frac{1}{|x-y|^2}(1-|y|^2)^{-1/2}\,dy,\end{aligned}$$

where $\bar{a} = 1/x$ and $|a| < 1$. There is no loss of generality in assuming that x is real and that $|a| \geq 3/4$. So, it is enough to prove that

$$\int_\Delta \frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy \leq c\,(1-|a|^2)^{-1/2}.$$

To do so we look at the level lines for $1/|1-\bar{a}y|$, at the points $c_n = 2^{-n}/(1-|a|^2)$

$$\begin{aligned}C_n &= \Big\{y \in \Delta :\ \frac{1}{|1-\bar{a}y|} = \frac{2^{-n}}{(1-|a|^2)}\Big\}\\ &= \Big\{y \in \Delta :\ \Big|\frac{1}{\bar{a}} - y\Big|^2 = \frac{2^n(1-|a|^2)^2}{|a|^2}\Big\}.\end{aligned}$$

These are circles centered at $1/\bar{a}$ and radius $2^n(1-|a|^2)/|a|$. Observe that if $n=0$, the radius is $(1-|a|^2)/|a| \sim 1/|\bar{a}|-1$. This gives us the first circle whose intersection with Δ occurs in its boundary. As n increases we obtain a sequence of circles each time with double radius. Then

$$\int_\Delta \frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy \le \sum_{n=0}^{\infty}\int_{A_n}\frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy\,,$$

where

$$A_n = \{y\in\Delta:\ 2^{n-1}(1-|a|^2)\le(1-\bar{a}y)\le 2^n(1-|a|^2)\}\,,\quad \Delta\subseteq\bigcup_{n=0}^{\infty}A_n\,.$$

But the last inequality is less than or equal to

$$c\sum_{n=0}^{M_0}\frac{2^{-2n}}{(1-|a|^2)^2}\int_{A_n}\frac{dy}{(1-|y|^2)^{1/2}}+\int_{R_{M_0}}\frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy\,,$$

where $R_{M_0}=\Delta\setminus\bigcup_{n=0}^{M_0}A_n$. Now, A_n is contained in a larger region T_n for $n\le M_0$ where M_0 is the last integer n before C_n touches or includes 0 in A_{n+1}. In polar coordinates, T_n is defined by letting θ vary between 0 and $\ell(\Gamma_n)$ and r between $1-2^n(1-|a|^2)$ and 1. We call $\Gamma_n=\partial(\cup_{j=0}^{n}A_j)\cap\partial\Delta$.

If $y\in R_{M_0}$ then $|1-\bar{a}y|>1/10$, and

$$\begin{aligned}\int_{R_{M_0}}\frac{1}{|1-\bar{a}y|^2}\frac{1}{(1-|y|^2)^{-1/2}}\,dy &\le \frac{4}{3}10^2\int_\Delta\frac{1}{(1-|y|^2)^{-1/2}}\,dy\\ &\le c_0(1-|a|^2)^{-1/2}\,.\end{aligned}$$

Now,

$$\begin{aligned}\int_{T_n}\frac{1}{(1-|y|^2)^{-1/2}}\,dy &= \int_0^{\ell(\Gamma_n)}d\theta\int_{1-2^n(1-|a|^2)}^{1}\frac{r}{(1-r^2)^{1/2}}\,dr\\ &\le c\,\ell(\Gamma_n)\,2^{n/2}(1-|a|^2)^{1/2}\,.\end{aligned}$$

But circle is chord arc and so we have that $\ell(\Gamma_n)\sim|b_n-d_n|\le c\,2^n(1-|a|^2)$, where c is a positive absolute constant and b_n, d_n are the points where C_n crosses $\partial\Delta$. Therefore,

$$\begin{aligned}c\sum_{n=0}^{M_0}\frac{2^{-2n}}{(1-|a|^2)^2}\,2^{3n/2}(1-|a|^2)^{3/2} &\le c(1-|a|^2)^{-1/2}\sum_{n=0}^{\infty}2^{-n/2}\\ &\le c(1-|a|^2)^{-1/2}\,.\end{aligned}$$

Therefore we have (S_2') and with this we conclude the proof of Theorem 2.1.

Lemma 2.1. *Let S be a singular integral operator associated to a kernel $s(x,y)$ satisfying standard estimates. Then $S(1-\chi_{Q_j})(x)$ is in* $\mathrm{BMO}(Q_j)$ *with constant independent of the size of Q_j.*

PROOF. The proof of this uses the same sort of argument used to show that a Calderón-Zygmund operator maps L^∞ into BMO continuously. We refer the reader to [DJS], [N].

Preliminaries for Section 3.

We would like to recall some of the results necessary for the proof of Theorem 3.1. No proofs are shown but they can be found at the indicated references.

Given a simply connected domain Ω (with nontrivial boundary) the Riemann mapping Theorem tells us we can construct a univalent mapping -that is a conformal homeomorphism- f of the unit disk Δ onto Ω.

Lemma P.1. *Suppose Ω and Ω' are domains on $\bar{\mathbb{C}}$ and $f : \Omega' \longrightarrow \Omega$ is conformal. Then if $G(z,w)$ is a Green's function on Ω, $G'(z,w) \equiv G(f(z), f(w))$ is a Green's function for Ω'.*

Theorem K. (Weak form of Koebe 1/4 Theorem). *Suppose $f \colon \Delta \longrightarrow \mathbb{C}$ is univalent, $f(0) = 0$, and $f'(0) = 1$. Then there exists $c_0 > 0$ (independent of f) such that $D(0, c_0) \subset f(\Delta)$.*

Corollary K. *If $f : \Omega \longrightarrow \Omega'$ is conformal, then for all $z \in \Omega$,*

$$|f'(z)| \sim \frac{\operatorname{dist}\{f(z), \partial\Omega'\}}{\operatorname{dist}\{z, \partial\Omega\}} .$$

If f is a univalent function on Δ, f' never vanishes, so we can write $f' = e^\varphi$ for some holomorphic φ on Δ.

Theorem P.2. *There is a universal constant $c_0 > 0$ such that if f is univalent on Δ and $f' = e^\varphi$ then $|\varphi'(z)| \le c_0 (1-|z|)^{-1}$ for all $z \in \Delta$.*

Theorem P.3. *There is a universal constant $\varepsilon_0 > 0$ such that if $f' = e^{\varphi}$ where φ is holomorphic and $|\varphi'(z)| \leq \varepsilon(1-|z|)^{-1}$ for $\varepsilon \leq \varepsilon_0$ and $z \in \Delta$, then $f : \Delta \longrightarrow f(\Delta)$ is a conformal map onto a Jordan domain bounded by a quasicircle (with constant $\sim 1 + c\varepsilon$).*

And conversely, let Ω be a simply connected domain in $\mathbb{R}^2$ bounded by a Jordan curve that is a K-quasicircle (∞ is fixed), $K = 1 + \varepsilon$. Let f be the Riemann map from Δ to Ω (f has a K^2-quasiconformal extension to $\mathbb{R}^2$, ∞ remains fixed) and let $f' = e^{\varphi}$. Then,

$$\sup_{\Delta} |\varphi'(z)| \, (1 - |z|^2) \leq c_0 \, \varepsilon \, ,$$

where $c_0 > 0$ is an absolute constant (cf. [P], [L]).

A quasicircle is a Jordan curve Γ in $\mathbb{R}^2$ that is the image of the unit circle $\mathbb{T}$ under a globally quasiconformal homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Conformal maps are mappings sending small circles to small circles. Quasiconformal mapping take small circles to ellipses of bounded eccentricity, (*cf.* [A], [LV], for an exposition in the subject).

Definition P.4. *For g defined on Δ we define*

$$\|g\|_{\mathcal{B}} = \sup_{\Delta} |g'(z)| \, (1 - |z|^2) \, .$$

The set of holomorphic functions g on Δ with $\|g\|_{\mathcal{B}} < +\infty$ is called the Bloch class $\mathcal{B}$; $\| \cdot \|_{\mathcal{B}}$ is conformally invariant.

Note that on Δ, $1 - |z| \sim 1 - |z|^2$ so if f is univalent and $f' = e^{\varphi}$ then by Theorem P.2, φ is in the Bloch class.

Recall that on Δ the hyperbolic distance is defined by

$$\rho(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|} \, .$$

Lemma P.5. *If φ is holomorphic on Δ then $\varphi \in \mathcal{B}$ if and only if there is an $A > 0$ such that $|\varphi(z) - \varphi(w)| \leq A \, \rho(z, w)$ for all $z, w \in \Delta$. Moreover, if A_0 is the smallest such constant $A_0 \sim \|\varphi\|_{\mathcal{B}}$ (cf.* [P]).

3. Proof of Theorem 3.1.

Set $K = 1+\varepsilon$, then $K^2 \sim 1+3\varepsilon$, ε is very small and it will become clear at the end of the proof how small it should be.

Let $f : \Delta \to \Omega$ univalent. Let $h = f^{-1} : \Omega \to \Delta$ univalent, ρ^* the hyperbolic metric on Ω, $\rho^*(z', w') = \rho(h(z'), h(w')) = \rho(z, w)$, $z' = f(z)$, $w' = f(w)$, ρ the hyperbolic metric on Δ. We want to prove that if $\delta(w) = \operatorname{dist}\{w, \partial\Omega\}$ then there exists $\eta = \eta(\varepsilon)$, $0 < \eta < 1$ and $c > 0$ an absolute constant such that for $z' \in \Omega$,

$$\int R(z', w')\, \delta(w')^{\eta} \,\frac{dw'}{\delta(w')^2} \le c\, \delta(z')^{\eta}\,.$$

Recall that if $A \subseteq \Omega$, $\operatorname{vol}_h(A) = \int_A dw'/\delta(w')^2$,

$$\begin{aligned}\int & R(z', w')\, \delta(w')^{\eta-2}\, dw' \\ &\le \int_{\{w':\rho^*(z',w')>1\}} \frac{(1-|h(z')|^2)^{1/K^2}(1-|h(w')|^2)^{1/K^2}}{|1-h(z')\overline{h(w')}|^{2/K^2}}\, \delta(w')^{\eta-2}\, dw'\,,\end{aligned}$$

$$1-|h(z')|^2 = d(h(z'), \partial\Delta) \qquad \text{and} \qquad f'(z) = \frac{1}{h(z')} \quad \text{if } h(z') = z\,.$$

By Theorem K we have that

$$\begin{aligned} d(w', \partial\Omega) &\sim |f'(w)|\, d(w, \partial\Delta) \\ &\sim \frac{1}{|h'(w')|}\,(1-|h(w')|^2)\,,\end{aligned}$$

h is conformal. Therefore, if $w = h(w')$

$$\frac{dw}{\delta(w)^2} \sim \frac{dw'}{\delta(w')^2}\,.$$

Then, the last inequality is

$$\begin{aligned} &\le c\,\delta(z')^{1/K^2}\,|h'(z')|^{1/K^2} \int_{\{w:\rho(w,z)>1\}} \frac{(1-|w|^2)^{1/K^2+\eta}}{|1-h(z')\bar{w}|^{2/K^2}}\,|f'(w)|^{\eta}\,\frac{dw}{\delta(w)^2} \\ &\le c\,(1-|z|^2)^{1/K^2} \int_{\{w:\rho(z,w)>1\}} \frac{\delta(w)^{1/K^2+\eta}}{|1-z\bar{w}|^{2/K^2}}\,|f'(w)|^{\eta}\,\frac{dw}{\delta(w)^2}\,.\end{aligned}$$

We know that $f' = e^{\varphi}$ for some φ in the Bloch class such that $\|\varphi\|_{\mathcal{B}} \leq c_0\, \varepsilon$; call $\delta = c_0\, \varepsilon$. Then we can make the last expression less than or equal to

$$c \int_{\Delta} e^{-\rho(z,w)/K^2}\, e^{\eta \operatorname{Re}\varphi(w)}\, \delta(w)^{\eta}\, \frac{dw}{\delta(w)^2}$$
$$= c\, e^{\eta u(z)} \int_{\Delta} e^{-\rho(z,w)/K^2}\, e^{\eta(u(w)-u(z))}\, \delta(w)^{\eta}\, \frac{dw}{\delta(w)^2}\,,$$

where $u = \operatorname{Re}\varphi$ and $|f'(w)|^{\eta} = e^{\eta u(w)}$. But,

$$\|\varphi\|_{\mathcal{B}} \leq \delta \qquad \text{implies} \qquad u(w) - u(z) \leq |\varphi(w) - \varphi(z)| \leq \delta\, \rho(w,z)\,.$$

Therefore,

$$\leq c\, e^{\eta u(z)} \int_{\Delta} e^{-(1/K^2 - \eta\delta)\,\rho(w,z)}\, \delta(w)^{\eta-2}\, dw$$
$$\leq c\, e^{\eta u(z)} \int_{\Delta} \frac{(1-|z|^2)^{1/K^2-\eta\delta}\,(1-|w|^2)^{-(1/K^2-\eta+\eta\delta)}}{(|1-z\bar{w}|^2)^{1/K^2-\eta\delta}}\, dw\,.$$

Let $1+\alpha = 1/K^2 - \eta\delta$, $r = -(-1/K^2 + 2 - \eta + \eta\delta)/2$. Then

$$= c\, e^{\eta u(z)}\,(1-|z|^2)^{1+\alpha} \int_{\Delta} \frac{(1-|w|^2)^{2r}}{|1-z\bar{w}|^{2(1+\alpha)}}\, dw\,.$$

Call $B(z,z)^{-r} = (1-|z|^2)^{2r}$. We know that for $r > -1/2$ and $\alpha - r > 0$ (*cf.* [CR]),

$$\int_{\Delta} \frac{B(w,w)^{-r}}{|1-z\bar{w}|^{2(1+\alpha)}}\, dw \leq \frac{1}{(1-|z|^2)^{2(\alpha-r)}}\,.$$

But,

$$r = \frac{1}{2}\Big(\frac{1}{K^2} - 2 - \eta\delta + \eta\Big) > -\frac{1}{2} \quad \text{if and only if} \quad \eta > \Big(1 - \frac{1}{K^2}\Big)\,\frac{1}{1-\delta}\,.$$

And,

$$\alpha - r > 0 \qquad \text{if and only if} \qquad \eta < \frac{1}{(\delta+1)K^2}\,.$$

Recall that $\delta = c_0\, \varepsilon$, and that $K^2 \sim 1 + 3\varepsilon$. Then, for ε sufficiently small, we have that

$$0 < \Big(1 - \frac{1}{K^2}\Big)\,\frac{1}{1-\delta} < \frac{1}{(1+\delta)K^2}\,.$$

Therefore, altogether, we have that

$$c\, e^{\eta u(z)} \frac{(1-|z|^2)^{1+\alpha}}{(1-|z|^2)^{2(\alpha-r)}} \le c\, |f'(z)|^\eta (1-|z|^2)^\eta \le c\, d(z', \partial\Omega)^\eta = c\, \delta(z')^\eta$$

for

$$\Big(1 - \frac{1}{K^2}\Big) \frac{1}{1-\delta} < \eta < \frac{1}{(1+\delta)K^2}\,,$$

since $1 + \alpha - 2\alpha + 2r = \eta$. This concludes the proof of Theorem 3.1.

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