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On the Admissibility of Singular Perturbations in Cauchy Problems II

By

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Abstract

We shall give a new proof of the admissibility of singular perturbations, which was introduced by the author [2], without the assumption on the characteristic roots assumed in [2] using the method introduced by the author in [5, Appendix].

§ 1. Introduction

Let us consider the following linear partial differential operator of kowalewskian type with constant coefficients containing small positive parameter ε satisfying $0 \leq \varepsilon < 1$:

$$L_\varepsilon(D) = \varepsilon \cdot P_1(D) + P_2(D).$$

Denote by m the order of $P_1(D)$ with respect to D_1 and by m' that of $P_2(D)$. Put $m'' = m - m'$ and assume that $m > m' > 0$. Then the order of L_0 is less than that of L_ε for $\varepsilon \neq 0$. Such an operator as L_ε is called a *singularly perturbed operator*.

We shall study the following so-called *singularly perturbed unilateral Cauchy problems* for $L_\varepsilon(D)$ in $\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n; x_1 > 0\}$:

$$\text{(CP)} \qquad \begin{cases} L_\varepsilon(D)u(x) = 0, & \text{in } \boldsymbol{R}^n_+; \\ \lim_{\delta \downarrow 0} D_1^{j-1} u(\delta, x') = \varphi_j(x'), & j = 1, \cdots, m, \end{cases}$$

and the following so-called *reduced unilateral Cauchy problem* for (CP):

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$$\text{(RCP)} \qquad \begin{cases} L_0(D)u(x)=0, & \text{in } \mathbb{R}^n_+; \\ \lim_{\delta\downarrow 0} D_1^{j-1}u(\delta, x')=\varphi_j(x'), & j=1,\cdots,m'. \end{cases}$$

Put $\varphi'=(\varphi_1,\cdots,\varphi_{m'})$, $\varphi''=(\varphi_{m'+1},\cdots,\varphi_m)$, and $\varphi=(\varphi',\varphi'')$. Denote by A' the space of Cauchy data φ', by A'' the space of Cauchy data φ'', and by $A=A'\times A''$ the space of Cauchy data φ, respectively. Hereafter, we assume that (CP) is uniquely solvable in $C(\mathbb{R}^n_+)$ for every $\varphi\in A$ and (RCP) is uniquely solvable in $C(\mathbb{R}^n_+)$ for every $\varphi'\in A'$. For example, if $A=O(\mathbb{C}^{n-1})^m$, where we denote by $O(\mathbb{C}^{n-1})$ the set of entire functions defined in $\mathbb{C}^{n-1}$, then the Cauchy-Kowalewski theorem implies that the Cauchy problems (CP) are globally uniquely solvable. If $A=F^{-1}(C_0^\infty(\mathbb{R}^{n-1}))^m$, where F^{-1} denotes the inverse Fourier transformation, then the Cauchy problems (CP) can be solved uniquely by the Fourier transformation.

Definition 1.1. The unilateral Cauchy problems (CP) with the Cauchy data space A are said to be *admissible* with respect to the reduced unilateral Cauchy problem (RCP) with the Cauchy data space A' if every Cauchy data φ in A, the solutions $u_\varepsilon(x;\varphi)$ of (CP) converge to the solution $u_0(x;\varphi')$ of (RCP) in $C(\mathbb{R}^n_+)$.

Let the symbols of $P_1(D)$ and $P_2(D)$ be represented as

$$P_1(\xi)=\sum_{j=0}^{m} p_{1,j}(\xi')\xi_1^{m-j},$$

$$P_2(\xi)=\sum_{j=0}^{m'} p_{2,j}(\xi')\xi_1^{m'-j},$$

where $p_{1,0}$ and $p_{2,0}$ are non-zero constants. Put $p=p_{2,0}/p_{1,0}$. The following theorem has already been stated in [2] under the assumption that for the characteristic roots of $P_2(\xi)=0$ with respect to ξ_1, which we denote $\sigma_j(\xi')$, $j=1,\cdots,m'$, there exists a point ξ'_0 in $\mathbb{R}^{n-1}$ such that

$$\sigma_j(\xi'_0)\neq\sigma_k(\xi'_0), \qquad 1\leq j<k\leq m'.$$

But using a new expansion formula of a certain meromorphic function defined as the quotient of two determinants, which will be stated in Proposition, we can remove such an assumption as the above. This formula is proved by calculating the Laplace expansions of these determinants, which was essentially stated in [5, Appendix]. Our aim in this paper is to give a new proof of Theorem (based on this expansion formula).

For a complex number p, we denote by $\Im p$ the imaginary part of p and by $\Re p$ the real part of p.

Theorem. *The unilateral Cauchy problems* (CP) *with* $A=F^{-1}(C_0^\infty(\boldsymbol{R}^{n-1}))^m$ *are asmissible with respect to* (RCP) *if and only if either the following conditions* $(C1)$ *or* $(C2)$ *is satisfied.*

(C1) $\qquad m''=2$ *and* $p<0$;

(C2) $\qquad m''=1$ *and* $\Im p \leq 0$.

§ 2. Preliminaries

We shall state here notation and lemmas without proofs. The proofs are essentially the same as [5, Appendix]. We start with introducing general notations.

Let n be an integer. We shall substitute n for m, m', m'', and so on. Let $z=(z_1, \cdots, z_n)$ and $w=(w_1, \cdots, w_n)$ be complex variables. For a non-negative integer l, denote $a(l)(z)=((z_j)^l; j\rightarrow 1, \cdots, n)$ and for non-negative integers $l_1, \cdots, l_n$ satisfying $0\leq l_1<\cdots<l_n$, denote

$$A(l_1, \cdots, l_n)(z)=\det(a(l_i)(z); i\downarrow 1, \cdots, n).$$

Let $i=\sqrt{-1}$ and x_1 be a real parameter. Denote $e(w,x_1)=(e^{iw_jx_1}; j\rightarrow 1, \cdots, n)$ and for non-negative integers $l_1, \cdots, l_{n-1}$ satisfying $0\leq l_1<\cdots<l_{n-1}$, denote

$$B(l_1, \cdots, l_{n-1})(z,w,x_1)=\det{}^t({}^te(w,x_1), {}^ta(l_1)(z), \cdots, {}^ta(l_{n-1})(z)).$$

Expand the determinant $B(l_1,\cdots,l_{n-1})(z,w,x_1)$ with respect to the first row. Then

$$B(l_1, \cdots, l_{n-1})(z,w,x_1)=\sum_{j=1}^{n}(-1)^{1+j}A(l_1, \cdots, l_{n-1})(z(j))e^{iw_jx_1},$$

where $z(j)=(z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n)$. Denote

$$C(l_1, \cdots, l_n)(z)=A(l_1, \cdots, l_n)(z)/A(0, \cdots, n-1)(z)$$

and

$$D(l_1, \cdots, l_{n-1})(z,w,x_1)=B(l_1, \cdots, l_{n-1})(z,w,x_1)/A(0, \cdots, n-1)(z).$$

Then $C(l_1, \cdots, l_n)(z)$ is a homogeneous symmetric polynomial in $\boldsymbol{Z}[z]$ of order $l_1 + \cdots + l_n - (n-1)n/2$ and $D(l_1, \cdots, l_{n-1})(z, w, x_1)$ is meromorphic in z and entire in w. Put

$$E_j(z) = \{(-1)^{n-j} \prod_{k \neq j, 1 \leq k \leq n} (z_j - z_k)\}^{-1}, \quad j = 1, \cdots, n.$$

Then $E_j(z)$, $j = 1, \cdots, n$ are meromorphic in $\boldsymbol{C}^n$ and holomorphic when $z_i \neq z_j$, $1 \leq i < j \leq n$. As a meromorphic function in z, we have

$$D(l_1, \cdots, l_{n-1})(z, w, x_1) = \sum_{j=1}^{n} (-1)^{1+j} C(l_1, \cdots, l_{n-1})(z(j)) e^{iw_j x_1} E_j(z). \tag{2.1}$$

Let m, m', and m'' be positive integers such that $m = m' + m''$. Denote $z' = (z_1, \cdots, z_{m'})$, $z'' = (z_{m'+1}, \cdots, z_m)$, and $z = (z', z'')$. Denote $w = (w', w'')$ in the same manner. Let $l_1, \cdots, l_{m-1}$ be non-negative integers satisfying $0 \leq l_1 < \cdots < l_{m-1}$. Let S_1 be the set of all bijections ρ from $\{1, \cdots, m-1\}$ onto $\{l_1, \cdots, l_{m-1}\}$ satisfying

$$\rho(1) < \cdots < \rho(m'); \rho(m'+1) < \cdots < \rho(m-1)$$

and S_2 be the set of all bijections ρ from $\{1, \cdots, m-1\}$ onto $\{l_1, \cdots, l_{m-1}\}$ satisfying

$$\rho(1) < \cdots < \rho(m'-1); \rho(m') < \cdots < \rho(m-1).$$

Each bijection in S_1 uniquely corresponds to a way of selecting m' objects from $m-1$ objects and each bijection in S_2 uniquely corresponds to a way of selecting $m'-1$ objects from $m-1$ objects, respectively. Define the bijection π from $\{l_1, \cdots, l_{m-1}\}$ onto $\{2, \cdots, m\}$ as $\pi(l_j) = j+1$, $j = 1, \cdots, m-1$. Denote

$$I(\rho) = \sum_{j=1}^{m'} \pi(\rho(j)) + m'(m'+1)/2; \quad J(\rho) = 1 + \sum_{j=1}^{m'-1} \pi(\rho(j)) + m'(m'+1)/2.$$

For $z_i \neq z_j$, $1 \leq i \leq m'$, $m'+1 \leq j \leq m$, denote

$$E(z) = \prod_{1 \leq i \leq m', m'+1 \leq j \leq m} (z_j - z_i)^{-1};$$

$$D_1(l_1, \cdots, l_{m-1})(z, w, x_1)$$

$$= \sum_{\rho \in S_1} (-1)^{I(\rho)} C(\rho(1), \cdots, \rho(m'))(z')$$

$$\times D(\rho(m'+1), \cdots, \rho(m-1))(z'', w'', x_1) E(z);$$

$$D_2(l_1, \cdots, l_{m-1})(z, w, x_1)$$

$$= \sum_{\rho \in S_2} (-1)^{J(\rho)} D(\rho(1), \cdots, \rho(m'-1))(z', w', x_1)$$

$$\times C(\rho(m'), \cdots, \rho(m-1))(z'') E(z).$$

Then we have the following new expansion formula. As the proof is similar as that of [5, Lemma A.1], we omit it.

Proposition. *For* $z_i \neq z_j$, $1 \leq i \leq m'$, $m'+1 \leq j \leq m$,

$$D(l_1, \cdots, l_{m-1})(z, w, x_1)$$

$$= D_1(l_1, \cdots, l_{m-1})(z, w, x_1) + D_2(l_1, \cdots, l_{m-1})(z, w, x_1).$$

Hereafter in this section, we assume that $\{l_1, \cdots, l_{m-1}\} = \{0, \cdots, k-1, k+1, \cdots, m-1\}$. For $k = m', \cdots, m-1$, for $j = 1, \cdots, m''$, and for $\rho \in S_1$, denote

$$z''(j) = (z_{m'+1}, \cdots, z_{m'+j-1}, z_{m'+j+1}, \cdots, z_m);$$

$$E_j(z'') = \{(-1)^{m''+j} \prod_{k \neq j, 1 \leq k \leq m''} (z_{m'+j} - z_{m'+k})\}^{-1};$$

$$f_{k,j,\rho}(z) = (-1)^{I(\rho)+1+j} C(\rho(1), \cdots, \rho(m'))(z')$$

$$\times C(\rho(m'+1), \cdots, \rho(m-1))(z''(j)) E_j(z'') E(z);$$

$$F_{k,j}(z) = \sum_{\rho \in S_1} f_{k,j,\rho}(z).$$

These $F_{k,j}$ are rational functions in z and holomorphic if $z_i \neq z_j$ are satisfied for $1 \leq i \leq m'$ and $m'+1 \leq j \leq m$ and for $m'+1 \leq i < j \leq m$.

Lemma 2.1. *Assume that $z_i \neq z_j$ are satisfied for $1 \leq i \leq m'$ and $m'+1 \leq j \leq m$ and for $m'+1 \leq i < j \leq m$. Then, for $k = m', \cdots, m-1$,*

$$D_1(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1) = \sum_{j=1}^{m''} F_{k,j}(z) e^{i w_{m'+j} x_1}.$$

Proof. By (2.1), we have

$$\begin{aligned} (2.2) \qquad & D(\rho(m'+1), \cdots, \rho(m-1))(z'', w'', x_1) \\ &= \sum_{j=1}^{m''} (-1)^{1+j} C(\rho(m'+1), \cdots, \rho(m-1))(z''(j)) e^{i w_{m'+j} x_1} E_j(z''). \end{aligned}$$

On the other hand, by the definition, we have

$$\begin{aligned} (2.3) \qquad & D_1(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1) \\ &= \sum_{\rho \in S_1} C(\rho(1), \cdots, \rho(m'))(z') \\ &\quad \times D(\rho(m'+1), \cdots, \rho(m-1))(z'', w'', x_1) E(z). \end{aligned}$$

Substitute (2.2) for (2.3), and we come to the conclusion. □

For a positive parameter η, put $t_j = \eta z_j, j = 1, \cdots, m$. Denote $t' = \eta z'$, $t'' = \eta z''$, and $t = \eta z$.

Assumption 2.1. There exists positive numbers M, M', c, and η_0 with $\eta_0 \leq 1$ such that for every η satisfying $0 < \eta \leq \eta_0$, the following estimates are satisfied:

$$|z'| \leq M; \quad |t''| \leq M; \quad \inf_{m'+1 \leq i < j \leq m} |t_i - t_j| \geq c; \quad \inf_{1 \leq i \leq m', m'+1 \leq j \leq m} |t_i - t_j| \geq c.$$

Denote

$$M' = M'(w, x_1) = \sum_{j=m'+1}^{m} e^{-\Im w_j x_1}.$$

Then we have a similar results as [5, Lemma A.3] as follows. The proof is omitted.

Lemma 2.2. *Let Assumption* 2.1 *be satisfied. Assume that* $z_i \neq z_j$ *for* $1 \leq i \leq m'$ *and* $m'+1 \leq j \leq m$ *and for* $m'+1 \leq i < j \leq m$. *Then there exist positive numbers* C_1 *and* C_2 *such that for* $k=0, \cdots, m'-1$ *the following three estimates hold*:

$$|D_1(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1)| \leq C_1 M' \eta, \tag{2.4}$$

$$\begin{aligned} &|D_2(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1) \\ &- D(0, \cdots, k-1, k+1, \cdots, m'-1)(z', w', x_1)(t_{m'+1} \cdot \cdots \cdot t_m)^{m'} E(t)| \\ &\leq C_2 \max_{\rho \in S_2} |D(\rho(1), \cdots, \rho(m'-1))(z', w', x_1)| \eta, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &|D(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1) \\ &- D(0, \cdots, k-1, k+1, \cdots, m'-1)(z', w', x_1)(t_{m'+1} \cdot \cdots \cdot t_m)^{m'} E(t)| \\ &\leq (C_1 M' + C_2 \max_{\rho \in S_2} |D(\rho(1), \cdots, \rho(m'-1))(z', w', x_1)|) \eta. \end{aligned} \tag{2.6}$$

For $k = m', \cdots, m-1$, *the following three estimates hold*:

$$|D_1(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1)| \leq C_1 M' \eta^{k-m'+1}, \tag{2.7}$$

$$\begin{aligned} &|D_2(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1)| \\ &\leq C_2 \max_{\rho \in S_2} |D(\rho(1), \cdots, \rho(m'-1))(z', w', x_1)| \eta^{k-m'+1}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &|D(0, \cdots, k-1, k+1, \cdots, m-1)(z, w, x_1)| \\ &\leq (C_1 M' + C_2 \max_{\rho \in S_2} |D(\rho(1), \cdots, \rho(m'-1))(z', w', x_1)|) \eta^{k-m'+1}. \end{aligned} \tag{2.9}$$

The bijection ρ satisfying

$$\{\rho(m'+1), \cdots, \rho(m-1)\} = \{m', \cdots, k-1, k+1, \cdots, m-1\}$$

is uniquely determined. Call this ρ as ρ_{max}. Denote

$$L''(\rho) = \begin{cases} \rho(m'+1) + \cdots + \rho(m-1) - (m''-1)m''/2, & \text{for } \rho \in S_1 \\ \rho(m') + \cdots + \rho(m-1) - (m''-1)m''/2, & \text{for } \rho \in S_2. \end{cases}$$

Lemma 2.3. *Let Assumption* 2.1 *be satisfied. Assume that* $z_i \neq z_j$ *for* $1 \leq i \leq m'$ *and* $m'+1 \leq j \leq m$ *and for* $m'+1 \leq i < j \leq m$. *Then there exist a positive number* C_1 *such that for* $k = m', \cdots, m-1$ *and* $j = 1, \cdots, m''$,

$$|F_{k,j}(z)| \leq C_1 \eta^k; \tag{2.10}$$

$$|F_{k,j}(z) - f_{k,j,\rho_{max}}(z)| \leq C_1 \eta^{k+1}. \tag{2.11}$$

Proof. Note that $C(\rho(m'+1), \cdots, \rho(m-1))(z''(j))$ is a homogeneous polynomial of order $L''(\rho) + m'' - 1$ for $\rho \in S_1$. This implies that in the expression of $F_{k,j}(z)$ the largest order of $C(\rho(m'+1), \cdots, \rho(m-1))(z''(j))$ is $m'm'' + m'' - 1 - k$ and it is attained only by $\rho = \rho_{max}$. On the other hand, for $j = 1, \cdots, m''$ we have

$$E_j(z'')E(z) = \eta^{m''-1+m'm''} E_j(\eta z'')E(\eta z).$$

Hence a similar method as in the proof of (2.7) can be applied to this lemma. For details we refer to [5, Lemma A.3]. □

§3. Proof of Theorem

To prove Theorem we need several steps. Denote the roots of $L_\varepsilon(\xi) = 0$ with respect to ξ_1 by $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ and those of $L_0(\xi) = P_2(\xi) = 0$ with respect to ξ_1 by $\sigma_j(\xi')$, $j = 1, \cdots, m'$, respectively. It is well known that $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ are continuous in (ε, ξ') for $\varepsilon \neq 0$ and $\sigma_j(\xi')$, $j = 1, \cdots, m'$ are continuous in ξ'. Put $B_R = \{|\xi'| \leq R\}$, $p = p_{2,0}/p_{1,0}$, $\theta = \arg(-p)$, $\Theta = e^{i\theta/m''}$, $\zeta = e^{2\pi i/m''}$, and $\tau'_j = \zeta^{j-m'-1}$, $j = m'+1, \cdots, m$.

First let us begin with the following lemma, which can be obtained through a similar argument as in [3, Lemma 2.2].

Lemma 3.1. *For every positive number R, there exist a positive number* ε_R *with* $\varepsilon_R < 1$ *and continuous functions* $\tau_{j,1}(\varepsilon, \xi')$, $j = 1, \cdots, m$ *on* $[0, \varepsilon_R] \times B_R$ *satisfying*

$$\lim_{\varepsilon \downarrow 0} \sup_{\xi' \in B_R} |\tau_{j,1}(\varepsilon, \xi')| = 0, \quad j = 1, \cdots, m$$

such that for $m'+1 \leq i < j \leq m$ *and for* $1 \leq i \leq m'$, $m'+1 \leq j \leq m$

$$\tau_i(\varepsilon, \xi') \neq \tau_j(\varepsilon, \xi'), \qquad \text{on } (0, \varepsilon_R] \times B_R,$$

and

$$\tau_j(\varepsilon,\xi')=\sigma_j(\xi')+\tau_{j,1}(\varepsilon,\xi'),\qquad \text{for } j=1,\cdots,m';$$

$$\varepsilon^{1/m''}\tau_j(\varepsilon,\xi')=\Theta\tau_j'|p|^{1/m''}+\varepsilon^{1/m''}\tau_0(\xi')+\varepsilon^{1/m''}\tau_{j,1}(\varepsilon,\xi'),$$

for $j=m'+1,\cdots,m$. *Here*

$$\tau_0(\xi')=(p_{1,0}p_{2,1}(\xi')-p_{1,1}(\xi')p_{2,0})/(m''p_{1,0}p_{2,0}),$$

which is a polynomial of ξ' *whose order is at most one.*

Remark. Put $\eta=\varepsilon^{1/m''}$, $z=\tau$, and $t=\eta z$, then Lemma 3.1 implies Assumption 2.1.

Put $b(\tau)=(\tau^{j-1};j\downarrow 1,\cdots,m)$ and $c_j=(\delta_{j,k};k\downarrow 1,\cdots,m)$, where $\delta_{j,k}$ is Kronecker's delta. Denote by $Y_j(\varepsilon,x_1,\xi')$, $j=1,\cdots,m$ the fundamental solutions of the following ordinary differential equation with parameter (ε,ξ'):

$$\text{(ODE)}\qquad \begin{cases} L_\varepsilon(D_1,\xi')Y(\varepsilon,x_1,\xi')=0; \\ D_1^{k-1}Y(\varepsilon,0,\xi')=\delta_{j,k}, \quad j,k=1,\cdots,m. \end{cases}$$

We shall use the following abbreviation: $\tau_j(\varepsilon,\xi')=\tau_j, j=1,\cdots,m$ and $\tau=(\tau_1,\cdots,\tau_m)$. Then Cramer's formula implies that if $\tau_i\neq\tau_j$, $1\leq i<j\leq m$ then

$$Y_j(\varepsilon,x_1,\xi')=\sum_{k=1}^{m}e^{i\tau_k x_1}\frac{\det(b(\tau_1),\cdots,b(\tau_{k-1}),c_j,b(\tau_{k+1}),\cdots,b(\tau_m))}{\det(b(\tau_1),\cdots,b(\tau_m))}$$

$$=\frac{\det{}^t({}^ta(0),\cdots,{}^ta(j-2),{}^te(\tau,x_1),{}^ta(j),\cdots,{}^ta(m-1))}{A(0,\cdots,m-1)}$$

$$=(-1)^{j-1}D(0,\cdots,j-2,j,\cdots,m-1)(\tau,\tau,x_1),\quad j=1,\cdots,m.$$

But the last representations remain valid without any restriction on τ_j, $j=1,\cdots,m$. For, since $B(l_1,\cdots,l_{n-1})(z,z,x_1)$ is an entire function of z and vanishes on the zeros of irreducible polynomials z_j-z_i, $(1\leq i<j\leq n)$, $B(l_1,\cdots,l_{n-1})(z,z,x_1)$ is divided by $A(0,\cdots,n-1)(z)$ in the ring of entire functions and consequently $D(l_1,\cdots,l_{n-1})(z,z,x_1)$ is also an entire function.

Using Lemma 2.2 and Lemma 3.1, we can prove the following two lemmas in a similar manner as in the proof of [5, Lemma 3 and Lemma 4].

Lemma 3.2. *Let ε_R be the same as in Lemma 3.1.*

(1) *For every positive number R, there exists a positive number C_R such that*

$$\sup_{0<\varepsilon\leq\varepsilon_R, 0\leq x_1\leq T, |\xi'|\leq R} |Y_j(\varepsilon, x_1, \xi')| \leq C_R, \tag{3.1}$$

for $j=1, \cdots, m'$.

(2) *Let either (C1) or (C2) be satisfied. Then, for every positive number R, there exists a positive number C_R such that*

$$\sup_{0<\varepsilon\leq\varepsilon_R, 0\leq x_1\leq T, |\xi'|\leq R} \varepsilon^{-(j-m')/m''} |Y_j(\varepsilon, x_1, \xi')| \leq C_R, \tag{3.2}$$

for $j=m'+1, \cdots, m$.

Denote by $y_j(x_1, \xi')$, $j=1, \cdots, m'$ the fundamental solutions of the following ordinary differential equation with parameter ξ':

$$\text{(RODE)} \qquad \begin{cases} L_0(D_1, \xi')y(x_1, \xi')=0; \\ D_1^{k-1}y(0, \xi')=\delta_{j,k}, \quad j, k=1, \cdots, m'. \end{cases}$$

As we have already shown in the case of $Y(\varepsilon, x_1, \xi')$,

$$y_j(x_1, \xi')=(-1)^{j-1}D(0, \cdots, j-2, j, \cdots, m-1)(\sigma, \sigma, x_1), \quad j=1, \cdots, m', \tag{3.3}$$

where $\sigma_j=\sigma_j(\xi')$, $j=1, \cdots, m'$ are the roots appearing in Lemma 3.1 and $\sigma=(\sigma_1, \cdots, \sigma_{m'})$.

Lemma 3.3.

(1) *Let ε_R be the same as in Lemma 3.1. Then*

$$Y_j(\varepsilon, x_1, \xi') \to y_j(x_1, \xi'), \quad j=1, \cdots, m' \tag{3.4}$$

uniformly on $[0, T]\times B_R$ when $\varepsilon\downarrow 0$.

(2) *Let ε_R be the same as in Lemma 3.1 and either (C1) or (C2) be satisfied. Then*

$$Y_j(\varepsilon, x_1, \xi') \to 0, \quad j=m'+1, \cdots, m \tag{3.5}$$

uniformly on $[0, T]\times B_R$ when $\varepsilon\downarrow 0$.

Denote

$$M_+=\max\{\Im(\Theta\tau'_j);j=m'+1,\cdots,m\};\quad M_-=\min\{\Im(\Theta\tau'_j);j=m'+1,\cdots,m\}.$$

Both the maximum and the minimum are attained by one j or two j's. Denote $\boldsymbol{R}_-=\{x\in\boldsymbol{R};x<0\}$. Then the following four cases can be considered:

(1) Case 1: If $m-m'\geq 3$ or if $m-m'=2$ and $p/|p|=-\Theta^2\in\boldsymbol{C}\backslash(\boldsymbol{R}_-\cup\{0\})$, then $M_+>0$ and $M_-<0$.

(2) Case 2: If $m-m'=2$ and $p<0$ or if $m-m'=1$ and $p\in\boldsymbol{R}$, then $M_+=M_-=0$.

(3) Case 3: If $m-m'=1$ and $\Im p=\Im(-\Theta)>0$, then $M_+=M_-<0$.

(4) Case 4: If $m-m'=1$ and $\Im p=\Im(-\Theta)<0$, then $M_+=M_->0$.

The condition that (C1) or (C2) is satisfied corresponds to Case 2 or Case 4. Note that in either case $\Im(\Theta\tau'_j)\geq 0$ holds for $j=m'+1,\cdots,m$.

Lemma 3.4. *Let R and ε_R be the same as in Lemma 3.1. Then*

(1) (*The case that M_- is attained by only one element of $\{\Im(\Theta\tau'_j);j=m'+1,\cdots,m\}$, say, $\Im(\Theta\tau'_l)$.*)

$$\limsup_{\varepsilon_0\downarrow 0}\varepsilon^{-(m-1)/m''}|D_1(0,\cdots,m-2)(\tau,\tau,x_1)e^{\Im(\Theta\tau'_l)(|p|/\varepsilon)^{1/m''}x_1}$$
$$-F_{m-1,l-m'}(\tau)e^{i\tau_l+\Im(\Theta\tau'_l)(|p|/\varepsilon)^{1/m''}x_1}|=0, \tag{3.6}$$

where the supremum is taken for $0<\varepsilon\leq\varepsilon_0$, $\delta\leq x_1\leq T$, $|\xi'|\leq R$ for arbitrary positive numbers ε_0 and δ with $\varepsilon_0\leq\varepsilon_R$ and $\delta\leq T$.

(2) (*The case that M_- is attained by two elements of $\{\Im(\Theta\tau'_j);j=m'+1,\cdots,m\}$. In this case we may assume that these are $\Im(\Theta\tau'_l)$ and $\Im(\Theta\tau'_{l+1})$, and that $\Re(\Theta\tau'_l)=-\Re(\Theta\tau'_{l+1})$.*)

$$\limsup_{\varepsilon_0\downarrow 0}\varepsilon^{-(m-1)/m''}|D_1(0,\cdots,m-2)(\tau,\tau,x_1)e^{\Im(\Theta\tau'_l)(|p|/\varepsilon)^{1/m''}x_1}$$
$$-F_{m-1,l-m'}(\tau)e^{i\tau_l+\Im(\Theta\tau'_l)(|p|/\varepsilon)^{1/m''}x_1}$$
$$-F_{m-1,l+1-m'}(\tau)e^{i\tau_{l+1}+\Im(\Theta\tau'_l)(|p|/\varepsilon)^{1/m''}x_1}|=0, \tag{3.7}$$

where the supremum is taken for $0<\varepsilon\leq\varepsilon_0$, $\delta\leq x_1\leq T$, $|\xi'|\leq R$ for arbitrary positive numbers ε_0 and δ with $\varepsilon_0\leq\varepsilon_R$ and $\delta\leq T$.

Proof. Put $z=w=\tau$ and $k=m-1$ in Lemma 2.1. Then

$$D_1(0,\cdots,m-2)(\tau,\tau,x_1)=\sum_{j=1}^{m''} F_{m-1,j}(\tau)e^{i\tau_{j+m'}x_1}. \tag{3.8}$$

Put $\eta=\varepsilon^{1/m''}$, $\eta_R=\varepsilon_R^{1/m''}$, and $t=\eta\tau$. As Assumption 2.1 is satisfied, (2.10) implies that

$$|F_{m-1,j}(\tau)|\le C_1\varepsilon^{(m-1)/m''},\quad j=1,\cdots,m''.$$

Lemma 3.1 implies that

$$|e^{(i\tau_{j+m'}+\Im(\Theta\tau_l')(|p|/\varepsilon)^{1/m''})x_1}|\le e^{(\Im(\Theta(\tau_l'-\tau_{j+m'}'))(|p|/\varepsilon)^{1/m''}-\Im\tau_0-\Im\tau_{j+m',1})x_1}. \tag{3.9}$$

Assume that either $j+m'\neq l$ in the case of (1) or $j+m'\neq l,l+1$ in the case of (2). Then $\Im(\Theta(\tau_l'-\tau_{j+m'}'))<0$. Since $|\Im\tau_0(\xi')|$ and $|\Im\tau_{j+m',1}(\varepsilon,\xi')|$ are bounded, we have, for $\delta\le x_1\le T$, the right-hand side of (3.9) tends to zero as $\varepsilon_0\downarrow 0$. Thus we come to the conclusion. □

Put, for $j=1,\cdots,m''$,

$$A_j=(-1)^{I(\rho_{max})+1+m''}/\{(\Theta|p|^{1/m''}\zeta^{j-1})^{m-1}\prod_{k=1}^{m''-1}(1-\zeta^k)\},$$

which are non-zero constants depending only on p and j.

Lemma 3.5. *Let R and B_R be the same as in Lemma* 3.1. *Then*

$$\lim_{\varepsilon\downarrow 0} F_{m-1,j}(\tau)\varepsilon^{-(m-1)/m''}=A_j,\quad j=1,\cdots,m'', \tag{3.10}$$

uniformly on B_R.

Proof. Since

$$C(m',\cdots,m-2)(z''(j))=(z_{m'+1}\cdot\cdots\cdot z_{m'+j-1}\cdot z_{m'+j+1}\cdot\cdots\cdot z_m)^{m'},$$

we find

$$f_{m-1,j,\rho_{max}}(z)=(-1)^{I(\rho_{max})+1+j}(z_{m'+1}\cdot\cdots\cdot z_{m'+j-1}\cdot z_{m'+j+1}\cdot\cdots\cdot z_m)^{m'}$$
$$\times E_j(z'')E(z).$$

Put $\eta=\varepsilon^{1/m''}$ and $t=\eta z$. Then

$$f_{m-1,j,\rho_{max}}(z)=f_{m-1,j,\rho_{max}}(t)\eta^{m-1}$$
$$=(-1)^{I(\rho_{max})+1+j}\eta^{m-1}(t_{m'+1}\cdot\cdots\cdot t_{m'+j-1}\cdot t_{m'+j+1}\cdot\cdots\cdot t_m)^{m'}E_j(t'')E(t).$$

Hence

$$\lim_{\eta\downarrow 0} f_{m-1,j,\rho_{max}}(\eta\tau)(\Theta|p|^{1/m''})^{m-1}$$
$$=(-1)^{I(\rho_{max})+1+j}(\tau'_{m'+1}\cdot\cdots\cdot\tau'_{m'+j-1}\cdot\tau'_{m'+j+1}\cdot\cdots\cdot\tau'_m)^{m'}$$
$$\times E_j(\tau'_{m'+1},\cdots,\tau'_m)E(0,\cdots,0,\tau'_{m'+1},\cdots,\tau'_m)$$
$$=(-1)^{I(\rho_{max})+1+j}(\tau'_{m'+1}\cdot\cdots\cdot\tau'_m)^{m'}E(0,\cdots,0,\tau'_{m'+1},\cdots,\tau'_m)$$
$$\times(\tau'_{m'+j})^{-m'}E_j(\tau'_{m'+1},\cdots,\tau'_m).$$

Since $(\tau'_{m'+1}\cdot\cdots\cdot\tau'_m)^{m'}E(0,\cdots,0,\tau'_{m'+1},\cdots,\tau'_m)=1$ and

$$(\tau'_{m'+j})^{-m'}E_j(\tau'_{m'+1},\cdots,\tau'_m)$$
$$=(-1)^{m''-j}/\{(\zeta^{j-1})^{m'+m''-1}\prod_{k=1}^{m''-1}(1-\zeta^k)\}$$
$$=(\Theta|p|^{1/m''})^{m-1}A_j/(-1)^{I(\rho_{max})+1+j},$$

(2.11) in Lemma 2.3 implies (3.10). □

Proof of Theorem. It is well known that the solution $u_\varepsilon(x;\varphi)$ of (CP) satisfies

$$u_\varepsilon(x;\varphi)=\sum_{j=1}^{m}F^{-1}_{\xi'\to x'}(Y_j(\varepsilon,x_1,\xi')\hat{\varphi}_j(\xi')) \tag{3.11}$$

and that the solution $u_0(x;\varphi')$ of (RCP) satisfies

$$u_0(x;\varphi')=\sum_{j=1}^{m'}F^{-1}_{\xi'\to x'}(y_j(x_1,\xi')\hat{\varphi}_j(\xi')), \tag{3.12}$$

where ^ denotes the partial Fourier transformation with respect to x', ξ' denotes the dual variable of x', and $F^{-1}_{\xi'\to x'}$ denotes its inverse Fourier transformation. Put

$$u_{1,\varepsilon}(x;(\varphi',0))=\sum_{j=1}^{m'}F_{\xi'\to x'}^{-1}(Y_j(\varepsilon,x_1,\xi')\hat{\varphi}_j(\xi'));$$

$$u_{2,\varepsilon}(x;(0,\varphi''))=\sum_{j=m'+1}^{m}F_{\xi'\to x'}^{-1}(Y_j(\varepsilon,x_1,\xi')\hat{\varphi}_j(\xi')).$$

Then $u_\varepsilon(x;\varphi)=u_{1,\varepsilon}(x;(\varphi',0))+u_{2,\varepsilon}(x;(0,\varphi''))$.

First we shall show the sufficiency of (C1) or (C2) for the admissibility. Since $\varphi\in F^{-1}(C_0^\infty(\boldsymbol{R}^{n-1}))^m$, there exists a positive number R such that $\operatorname{supp}\hat{\varphi}_j\subset B_R$, $j=1,\cdots,m$. By (3.1), Lebesgue's bounded convergence theorem can be applied to (3.11) when $\varepsilon\downarrow 0$. Then (3.4) implies that

$$\lim_{\varepsilon\downarrow 0}u_{1,\varepsilon}(x;(\varphi',0))=\sum_{j=1}^{m'}F_{\xi'\to x'}^{-1}(\lim_{\varepsilon\downarrow 0}Y_j(\varepsilon,x_1,\xi')\hat{\varphi}_j(\xi'))$$

$$=\sum_{j=1}^{m'}F_{\xi'\to x'}^{-1}(y_j(x_1,\xi')\hat{\varphi}_j(\xi'))=u_0(x;\varphi'),$$

which remains true even when neither (C1) nor (C2) is satisfied. Hence (3.2) and (3.5) imply that

$$\lim_{\varepsilon\downarrow 0}u_{2,\varepsilon}(x;(0,\varphi''))=\sum_{j=m'+1}^{m}F_{\xi'\to x'}^{-1}(\lim_{\varepsilon\downarrow 0}Y_j(\varepsilon,x_1,\xi')\hat{\varphi}_j(\xi'))=0.$$

Thus $\lim_{\varepsilon\downarrow 0}u_\varepsilon(x;\varphi)=u_0(x;\varphi')$. Since this convergence remains true in $C(\boldsymbol{R}_+^n)$, (CP) are admissible.

Next we shall show the necessity of (C1) or (C2) for the admissibility. It is enough to show that if $M_-<0$ then there exists a Cauchy data $\tilde{\varphi}$ such that the solutions $u_\varepsilon(x;\tilde{\varphi})$ diverge in $C(\boldsymbol{R}_+^n)$. Choose $\tilde{\varphi}=(0,\cdots,0,\varphi_m)$, where $\varphi_m\neq 0$ and put

$$v_{\varepsilon,l}(x;\varphi_m)=F_{\xi'\to x'}^{-1}((-1)^{m-1}D_l(0,\cdots,m-2)(\tau,\tau,x_1)\hat{\varphi}_m(\xi')),$$

for $l=1$, 2. Then

$$u_\varepsilon(x;\tilde{\varphi})=F_{\xi'\to x'}^{-1}(Y_m(\varepsilon,x_1,\xi')\hat{\varphi}_m(\xi'))=v_{\varepsilon,1}(x;\varphi_m)+v_{\varepsilon,2}(x;\varphi_m).$$

Since (2.8) in Lemma 2.2 implies that $\lim_{\varepsilon\downarrow 0}v_{\varepsilon,2}(x;\varphi_m)=0$ in $C(\boldsymbol{R}_+^n)$, it is enough to show that $v_{\varepsilon,1}(x;\varphi_m)$ diverge when $\varepsilon\downarrow 0$. Put

$$w_\varepsilon(x;\varphi_m)=\sum_{j=1}^{m''}F^{-1}_{\xi'\to x'}((-1)^{m-1}F_{m-1,j}(\tau)\varepsilon^{-(m-1)/m''}$$

$$\times e^{i(\tau_{j+m'}-\Theta\tau_l'(|p|/\varepsilon)^{1/m''})x_1}\hat{\varphi}_m(\xi')).$$

Then Lemma 2.1 implies that

$$v_{\varepsilon,1}(x;\varphi_m)=\sum_{j=1}^{m''}F^{-1}_{\xi'\to x'}((-1)^{m-1}F_{m-1,j}(\tau)e^{i\tau_{j+m'}x_1}\hat{\varphi}_m(\xi'))$$

$$=\varepsilon^{(m-1)/m''}e^{i\Theta\tau_l'(|p|/\varepsilon)^{1/m''}x_1}w_\varepsilon(x;\varphi_m).$$

(1) The case that M_- is attained by only one $\mathscr{F}(\Theta\tau_l')$. Lemma 3.1, 3.4, and 3.5 imply that

$$\lim_{\varepsilon\downarrow 0}w_\varepsilon(x;\varphi_m)=(-1)^{m-1}A_{l-m'}F^{-1}_{\xi'\to x'}(e^{i\tau_0(\xi')x_1}\hat{\varphi}_m(\xi')),$$

in $C(\boldsymbol{R}^n_+)$.

(2) The case that M_- is attained by two $\Im(\Theta\tau_l')$ and $\Im(\Theta\tau_{l+1}')$. Put $L=\Re(\Theta\tau_{l+1}')$ and $\varepsilon_n(x_1)=|p|/\{\pi(n+(m-1)/m'')(Lx_1)^{-1}\}^{m''}$, for every integer n. Then $L>0$, $\Re(\Theta\tau_l')=-L$, and

$$e^{2iL(|p|/\varepsilon_n(x_1))^{1/m''}x_1}=\zeta^{m-1}=A_{l-m'}/A_{l+1-m'}.$$

Hence Lemma 3.1, 3.4, and 3.5 imply that

$$\lim_{n\uparrow\infty}w_{\varepsilon_n}(x;\varphi_m)=(-1)^{m-1}F^{-1}_{\xi'\to x'}(A_{l-m'}e^{i\tau_0(\xi')x_1}\hat{\varphi}_m(\xi')$$

$$+A_{l+1-m'}\zeta^{m-1}e^{i\tau_0(\xi')x_1}\hat{\varphi}_m(\xi'))$$

$$=2(-1)^{m-1}A_{l-m'}F^{-1}_{\xi'\to x'}(e^{i\tau_0(\xi')x_1}\hat{\varphi}_m(\xi')),$$

in $C(\boldsymbol{R}^n_+)$.

In either case, $\lim_{n\uparrow\infty}w_{\varepsilon_n}(x;\varphi_m)$ is a non-trivial real analytic function in $\boldsymbol{R}^n_+$. On the other hand, for $x_1>0$, $\varepsilon^{(m-1)/m''}e^{i\Theta\tau_l'(|p|/\varepsilon)^{1/m''}x_1}$ diverge when $\varepsilon\downarrow 0$. Thus $v_{\varepsilon_n,1}(x;\varphi_m)$ diverge in $C(\boldsymbol{R}^n_+)$ when $n\uparrow\infty$. □

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